Finite dimensional motives and the Conjectures of Beilinson and Murre

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1 Introduction

Let k be a field of characteristic 0 and let $\mathcal{V}_k$ be the category of smooth projective varieties over k. By $\sim$ we denote an *adequate* equivalence relation for algebraic cycles on varieties [Ja00]. For every $X \in \mathcal{V}_k$ let $A^i_{\sim}(X) = (Z^i(X)/\sim)\otimes\mathbb{Q}$ be the Chow group of codimension i cycles on X modulo the chosen relation $\sim$ with coefficients in $\mathbb{Q}$.

Let $X, Y \in \mathcal{V}_k$, let $X = \cup X_i$ be the connected components of X and let $d_i = \dim(X_i)$. Then $Corr^r_{\sim}(X,Y) = \oplus_i A^{d_i+r}_{\sim}(X_i \times Y)$ is called a space of correspondences of degree r from X into Y. For any $f \in Corr^r(X,Y)$ and $g \in Corr^s(Y,Z)$ their composition $g \circ f \in Corr^{r+s}(X,Z)$ is defined by the formula $g \circ f = p_{XZ*}(p^*_{XY}(f) \cdot p^*_{YZ}(g))$ where p_{XZ}, p_{XY} and p_{YZ} are the appropriate projections. In particular, we have a linear action of correspondences $Corr^s(Y,Z) \times A^t(Y) \to A^{s+t}(Z)$ defined by the rule $(\alpha, x) \mapsto p_{Y*}(\alpha \cdot p^*_X(x))$, where p_X and p_Y are the projections.

The category of *pure motives* $\mathcal{M}_{\sim}$ over k with coefficients in $\mathbb{Q}$ with respect to the given equivalence relation $\sim$ can be defined as follows [Sch94]. Its objects are triples $M = (X, p, m)$, where $X \in \mathcal{V}_k$, $p \in Corr^0_{\sim}(X,X)$ is a projector (i.e. $p \circ p = p$) and $m \in \mathbb{Z}$. Morphisms from $M = (X,p,m)$ into $N = (Y,q,n)$ in $\mathcal{M}_{\sim}$ are given by correspondences $f \in Corr^{n-m}_{\sim}(X,Y)$, such that $f \circ p = q \circ f = f$, and compositions of morphisms are induced by compositions of correspondences.

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The category $\mathcal{M}_\sim$ is pseudoabelian and $\mathbb{Q}$-linear. Moreover, it is a tensor category with tensor structure defined by the formula $(X,p,m)\otimes(Y,q,n) = (X\times Y, p\otimes q, m+n)$. The triple $1=(Spec(k), id, 0)$ plays a role of the unite object in $\mathcal{M}_\sim$ and the Lefschetz motive $\mathbb{L}$ is the triple $(Spec(k), id, -1)$. For any motive $M=(X,p,m)$ one defines the Tate twist $M(r)$ to be the motive $M\otimes\mathbb{L}^{-r} = (X,p,m+r)$, where $\mathbb{L}^r = \mathbb{L}^{\otimes r}$ for a positive integer r, $\mathbb{L}^0 = 1$ and $\mathbb{L}^r = \mathbb{L}^{\otimes -r}$ for a negative r, see [Sch94], 1.9. At last, $\mathcal{M}_\sim$ is rigid [Ja00] in the sense that there exists internal Hom's and dual obiects M^* for all $M\in\mathcal{M}_\sim$ satisfying well known axioms [DeMi82].

For any algebraic cycle Γ on $X\times Y$ we will denote by Γ^t its transpose lying on $Y\times X$. By $M_\sim : \mathcal{V}_k^{opp}\to\mathcal{M}_\sim$ we will denote the functor which associates to any $X\in\mathcal{V}_k$ its motive $M_\sim(X) = (X, id, 0)$, where $id=[\Delta_X]$ is the class of the diagonal Δ_X in $Corr^0_\sim(X,X)$, and to a morphism $f: X\to Y$ the correspondence $M(f) = [\Gamma_f^t]\in Corr^0_\sim(Y,X) = Hom_{\mathcal{M}_\sim}(M(Y), M(X))$, where $\Gamma_f = \{(x,f(x)) \mid x\in X\}\subset X\times Y$ is the graph of f.

In the following we fix a *Weil cohomology* theory with L-coefficients H^*, where L is a field of characteristic zero, see [Kl94] for the definition. For example, if k is an arbitrary field one can take the étale cohomology groups $H^*_{et}(\bar{X},\mathbb{Q}_l)$ over the algebraic closure $\bar{k}$, i.e. $\bar{X} = X\times_k\bar{k}$ and $l\neq char(k)$. If $k=\mathbb{C}$ one can take also the usual Betti cohomology. Then one defines a functor $H^i:\mathcal{M}_{rat}\to Vect_L$ for every $i\in\mathbb{Z}$ by $H^i(M) = p_*H^{i+2m}(X)$ where $M=(X,p,m)$. By $cl: A^i_{rat}(X)\to H^{2i}(X)$ we denote the cycle map; then $\alpha\in A^i_{rat}(X)$ is homologically equivalent to zero iff $cl(\alpha)=0$.

If $\sim$ is rational equivalence then $\mathcal{M}_{rat}$ is called the category of *Chow motives* over k with coefficients in $\mathbb{Q}$. In the following we will write A^i for A^i_{rat}, $h(-)$ for the functor $M_{hom}(-):\mathcal{V}_k\to\mathcal{M}_{hom}$ and $A^i(X)_{hom}$ for the kernel of cl, i.e. the subgroup in $A^i(X)$ of cycles which are homologically trivial.

Under these assumptions one may consider the following equivalence relations $\sim$ on cycles: (rat) rational equivalence; (alg) algebraic equivalence; (hom) homological equivalence and (num) numerical equivalence [Ja00]. It is known that

$$(rat)\Rightarrow(alg)\Rightarrow(hom)\Rightarrow(num)$$

Rational equivalence is strictly finer than algebraic equivalence already for divisors on curves; a famous counterexample by Griffiths showed that algebraic equivalence is strictly finer than homological equivalence, even modulo torsion, for codimension 2 cycles on a complex 3-fold. According to Grothendieck's Standard Conjectures on algebraic cycles [Kl94] homological equivalence and numerical equivalence should coincide. By a result of Jannsen [Ja92] the category $\mathcal{M}_\sim$ is abelian semisimple iff $\sim$ is the numerical equivalence.

Now let $X \in \mathcal{V}_k$ and assume, for simplicity, that X is irreducible of dimension d. If we suppose that the conjecture $C(X)$ holds, see [Kl94], p.14, i.e. if the Künneth components $\Delta(i, 2d-i)$ of the diagonal Δ_X are algebraic (which is known to be true for curves, surfaces and abelian varieties), then the idempotent $H^*(X) \to H^i(X) \to H^*(X)$ is represented by an algebraic correspondence σ_i which is an idempotent in $A^d_{hom}(X \times X)$. Therefore we get a natural decomposition:

$$h(X) \simeq \bigoplus_{1 \leq i \leq 2d} h^i(X)$$

where $h^i(X) = (X, \sigma_i, 0)$ and σ_i is, in fact, the Künneth component $\Delta(i, 2d-i)$.

Following [Mu93(1)] we will say that X has a *Chow-Künneth decomposition* if there exist orthogonal idempotents π_i $(0 \leq i \leq 2d)$ in $A^d(X \times X)$, such that $cl(\pi_i) = \Delta(2d-i, i)$ and

$$[\Delta_X] = \sum_{0 \leq i \leq 2d} \pi_i$$

in $A^d(X \times X)$. This implies that in $\mathcal{M}_{rat}$ the motive $M(X)$ decomposes as follows:

$$M(X) = \bigoplus_{0 \leq i \leq 2d} M^i(X)$$

where $M_i(X) = (X, \pi_i, 0)$.

Murre conjectured, see [Mu93(1)] and [Mu93(2)], that every $X \in \mathcal{V}_k$ has a Chow-Künneth decomposition over the algebraic closure $\bar{k}$. This conjecture is true for curves, surfaces, abelian varietes, uniruled threefolds and elliptic modular varieties, see [DMu91] and [dAMSt00] for further references. If X and Y have a Chow-Künneth decompositon, then the same holds for $X \times Y$.

If X is a smooth projective variety of dimension d satisfing conjecture $C(X)$ then, with the above notations, one has the following isomorphisms [Ja94]:

$$A^d_{hom}(X \times X) = \bigoplus_i End_{\mathcal{M}_{hom}}(h^i(X))$$

$$A^d(X \times X) = \bigoplus_i End_{\mathcal{M}_{rat}}(M_i(X))$$

In order to relate rational equivalence with homological equivalence for algebraic cycles it is therefore natural to ask under which conditions the map

$$End_{\mathcal{M}_{rat}}(M_i(X)) \to End_{\mathcal{M}_{hom}}(h^i(X))$$

(induced by the functor h) is an isomorphism. This is in turn strictly related, see [Ja94], Prop 5.8, with the existence of a suitable filtration on the Chow ring of $X \times X$, such that the associated graded groups only depend on the motives $h^i(X)$, or, equivalently, to Murre's Conjecture (see Section 2 below).

In this paper we show how finite dimensionality of the motive $M(X)$ (see Def. 6) is related with the existence of such a filtration.

The paper is organized as follows: in Section 2 we recall the Conjectures of Beilinson and Murre on the existence of a suitable filtration $F^\bullet$ on the Chow ring of a smooth projective variety, and then we relate them with Bloch's Conjecure for surfaces.

In Section 3, after recalling the definitons and properties of finite dimensional motives and some results of [AK02], we prove Theorems 14 and 17 which relate the finite dimensionality of the motive with Murre's Conjecture.

In Section 4 we show that for a smooth projective surface over an algebraically closed field of characteristic 0 with $p_g = 0$ the motive $M(X)$ is finite dimensional iff the Chow group of 0-cycles of X is finite dimensional in the sense of Mumford.

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2 The Conjectures of Beilinson, Bloch and Murre

Beilinson has conjectured the existence of decreasing filtrations on Chow groups of all smooth projective varieties over k which are uniquely determined by the action of correspondences on algebraic cycles [Ja94]:

Conjecture 1 *For every $X \in \mathcal{V}_k$ there exists a decreasing filtration $F^\bullet$ on $A^i(X)$, such that:*

(a) $F^0A^j(X) = A^j(X)$; $F^1A^j(X) = (A^j(X))_{hom}$;

(b) $F^\bullet$ is compatible with the intersection product of cycles;

(c) $F^\bullet$ is compatible with f^* and f_* if $f : X \to Y$ is a morphism;

(d) (if the Künneth components of Δ_X are algebraic) the associated graded group $Gr^\nu_F A^j(X) = F^\nu A^j(X)/F^{\nu+1}A^j(X)$ depends only on the motive $h^{2j-\nu}(X)$ of X in $\mathcal{M}_{hom}$;

(e) $F^{j+1}A^j(X) = 0$ for all j.

If such a filtration exists then it is unique [Ja94]. If the Künneth components of the diagonal are algebraic, then a weaker form (see [Ja00], p. 12) of the condition (d) is:

(d$'$) Let $Y \in \mathcal{V}$ and let $\gamma \in Corr^{j-i}(Y \times X)$. If the induced map γ_* between $H^{2i-\nu}(Y)$ and $H^{2j-\nu}(X)$ is zero, then so is the map $Gr^{\nu}_F\gamma : Gr^{\nu}_F A^i(Y) \to Gr^{\nu}_F A^j(X)$.

Conjecture 1 is in turn equivalent (again assuming that the Künneth components of the diagonal are algebraic) to the following Conjecture of Murre, see [Mu93(1)] and [Ja94]:

Conjecture 2 *For any smooth projective (irreducible, for simplicity) variety X of dimension d:*
(I) there exists a Chow-Künneth decomposition $[\Delta_X] = \sum_{i=0}^{2d} \pi_i$;
(II) the correspondences $\pi_0, \ldots, \pi_{j-1}$ and $\pi_{2j+1}, \ldots, \pi_{2d}$ act as 0 on $A^j(X)$;
(III) if $F^{\bullet}$ is the filtration on $A^j(X)$ defined by $F^{\nu}A^j(X) = ker(\pi_{2j}) \cap ker(\pi_{2j-1}) \cap \ldots \cap ker(\pi_{2j-\nu+1})$, then $F^{\bullet}$ is independent of the choice of the projectors π_i;
(IV) $F^1A^j(X) = A^j(X)_{hom}$.

The status of Conjecture 2 is as follows: it is trivially true for curves. For surfaces and for the product of a surface with a curve Murre has shown the existence of a Chow-Künneth decomposition satsfying (II) and (IV), see [Mu90], [Mu93(1)] and [Mu93(2)]. For surfaces he also shows that there is a filtration which is the natural one, i.e. it coincides with the filtration for 0-cycles considered in [Bl80]. For abelian varieties the existence of a Chow-Künneth decomposition follows from works of Shermenev, Denninger-Murre and Küenneman (see [Kü94] for references): part of (II) is true (see Remark 15) and if (II) is true then (III) is also true for a natural choice of the projectors π_i [Mu93(1)].

Let us consider the case when X is a smooth projective surface over an algebraically closed field k. By results in [Mu90], X has a Chow-Künneth decomposition $[\Delta_X] = \sum_{i=0}^{4} \pi_i$, where $\pi_0 = [x_0 \times X]$ and $\pi_4 = [X \times x_0]$ are the trivial projectors induced by a fixed point $x_0 \in X$, π_1 is the Picard projector (which is closely connected with the Picard variety $Pic^0(X)$ of the surface X), $\pi_3 = \pi_1^t - \pi_1 \circ \pi_1^t$ is the Albanese projector (connected with the Albanese variety $Alb(X)$ of X) and $\pi_2 = \Delta_X - \pi_0 - \pi_1 - \pi_3 - \pi_4$. The projectors π_i yield the motivic decomposition

$$M(X) = \sum_{0 \le i \le 4} M_i(X) ,$$

where $M_i(X) = (X, \pi_i, 0)$ for any $i = \{0, \ldots, 4\}$, and the corresponding Murre's filtration is:

$$F^0A^i(X) = A^i(X) ,$$
$$F^{i+1}A^i(X) = 0 ,$$

$$F^1A^1(X) = A^1(X)_{hom} = A^1(X)_{num} = \ker(\pi_2) ,$$
$$F^1A^2(X) = ker(\pi_4) = A^2(X)_0$$

– the group of zero-cycles of degree 0 on X, and

$$F^2A^2(X) = \ker(\pi_3|_{F^1}) = T(X) ,$$

where $T(X)$ is so called Albanese kernel of the surface X, i.e. the kernel of the Abel-Jacobi map $a_X : A^2(X)_0 \to Alb(X)$. The graded group $Gr^*_F(A^2(X))$ associated to the filtration above is:

$$\mathbb{Q} \oplus Alb(X)_{\mathbb{Q}} \oplus T(X) .$$

A similar (truncated) filtration $F^\bullet$ can be defined for the Chow group of 0-cycles of any smooth variety Y of dimension d. Then one has, in analogy to the case of surfaces:

$$Gr^*_F A^d(Y) = \mathbb{Q} \oplus Alb(Y)_{\mathbb{Q}} \oplus T(Y) .$$

If Beilinson's Conjectural Filtration $F^\bullet$ exists for every smooth projective variety and X is a surface, then any correspondence $\gamma \in A^2(Y \times X)$, where $d = dim(Y)$, respects the filtration; if $\gamma \in A^2(Y \times X)_{hom}$ then γ_* acts as 0 on $Gr^*_F A^d(Y)$. This shows that the Beilinson's Conjecture implies the following conjecture formulated in [Bl80]:

Conjecture 3 *Let X be a smooth projective surface and let Y be a smooth projective variety of dimension d. For any $\gamma \in A^2(Y \times X)$ its action on $Gr^*_F A^d(Y)$:*

$$Gr^*_F\gamma : Gr^*_F A^d(Y) \longrightarrow Gr^*_F A^2(X)$$

depends only upon the cohomology class $cl(\gamma)$ in $H^4(Y \times X)$.

Conjecture 3 implies

Conjecture 4 *If X is a complex surface with geometric genus $p_g = 0$, then the Albanese kernel $T(X)$ vanishes, see [Bl80], 1.11.*

Note that, by a result of [Ro80], if k is algebraically closed then the kernel of the Abel-Jacobi map a_X, considering with coefficients in $\mathbb{Z}$, is torsion free.

Bloch's conjecture on the Albanese kernel holds for surfaces of Kodaira dimension less than 2 [BKL76] and it is still open for complex surfaces of general type with $p_g = 0$, see [InMiz79], [Voi93] and [GP02].

Remark 5 In general Bloch's Conjecture 3 does not imply that the action $\gamma : A^d(Y) \to A^2(X)$ only depends on the cohomology class $cl(\gamma)$. In fact, let X be a complex surface with $p_g(X) > 0$ and $q(X) = 0$ (where $q(X) = dim(H^1(X, \mathcal{O}_X))$ is the irregularity of X) and let C be a generic curve on X. Let $Y = S^2C$ be the symmetric square of the curve C. Then $A^d(Y) \simeq J(C) \oplus T(Y)$ and $A^2(X) \simeq T(X)$, where $J(C)$ is the Jacobian of the curve C, $T(Y)$ and $T(X)$ are the Albanese kernels. The map $f : S^2C \to S^2X$ yields a series of effective 0-cycles of degree 2 on X. Let $\Gamma \subset Y \times X$ be the associated correspondence, i.e.

$$[\Gamma] = \{[(Y, P + Q)] \mid P, Q \in C\} \subset A^2(Y \times X)$$

and let $\gamma = [\Gamma]$. Then the class $cl(\gamma)$ in $H^4(Y \times X)$ has components $\gamma(0,4)$, $\gamma(4,0)$ and $\gamma(2,2)$. By adding constant correspondences to γ we may assume that $\gamma(0,4) = \gamma(4,0) = 0$. Moreover the component $\gamma(2,2)$ in $H^2(Y) \otimes H^2(X)$ belongs to $NS(Y) \otimes NS(X)$. Therefore the action of $\gamma(2,2)$ on 0-cycles is trivial because every 0-cycle can be moved away from a finite number of divisors. The graded map

$$Gr^*_F\gamma : Gr^*_F A^d(Y) \to Gr^*_F A^2(X)$$

is 0. In fact we have $\Gamma \subset Y \times C \subset Y \times X$, whence γ can also be viewed as a correspondence between Y and C. As such it determines a map

$$\gamma' : A^d(Y) \simeq J(C) \oplus T(Y) \to J(C) ,$$

which is just the projection onto the first factor. Since γ factors trough γ', we see that γ is 0 on $T(Y)$ and, therefore, $Gr^*_F\gamma$ is the zero map. However the map

$$\gamma : A^d(Y) \to A^2(X) = T(X)$$

is not zero: in fact, C being a general curve on the surface X with $p_g(X) > 0$, the map induced by γ between $J(C)$ and $T(X)$ is non trivial. This is the consequence of a famous results of Mumford on the group of 0-cycles on surfaces with $p_g > 0$, see [Voi93], pg. 186.

3 Finite dimensional motives and Murre's Conjecture

In this section we first recall the definition and some results on finite dimensional motives, which have been introduced by S.-I. Kimura in [Ki98], and then prove our results relating finite dimensionality with the Conjectures stated in Section 2.

Let $\mathcal{C}$ be a pseudoabelian, $\mathbb{Q}$-linear, tensor category and let X be an object in $\mathcal{C}$. Let Σ_n be the symmetric group of order n. Any $\sigma \in \Sigma_n$ defines an endomorphism $\Gamma_\sigma : (x_1, \dots, x_n) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$ of the n-fold tensor product X^n of X by itself. There is a one-to-one correspondence between all irreducible representations of the group Σ_n (over $\mathbb{Q}$) and all partitions of the integer n. Let V_λ be the irreducible representation corresponding to a partition λ of n and let χ_λ be the character of the representation V_λ. Let

$$d_\lambda = \frac{\dim(V_\lambda)}{n!} \sum_{\sigma \in \Sigma_n} \chi_\lambda(\sigma) \cdot \Gamma_\sigma$$

Then $\{d_\lambda\}$ is a set of pairwise orthogonal idempotents in $End_{\mathcal{C}}(X^n)$, such that $\sum d_\lambda = id_{X^n}$. The category $\mathcal{C}$ being pseudoabelian they give a decomposition of X^n. The n-th symmetric product $S^n X$ of X is then defined to be $im(d_\lambda)$ when λ corresponds to the partition (n), and the n-th exterior power $\wedge^n X$ is $im(d_\lambda)$ when λ corresponds to the partition $(1, \dots, 1)$. In particular, we have symmetric and exterior powers in $\mathcal{M}_\sim$.

The following definition was made in [Ki98], see also [GP02] or [AK02]:

Definition 6 *The object X in $\mathcal{C}$ is said to be evenly (oddly) finite dimensional if $\wedge^n X = 0$ ($S^n X = 0$) for some n. An object X is finite dimensional if it can be decomposed into a direct sum $X_+ \oplus X_-$ where X_+ is evenly finite dimensional and X_- is oddly finite dimensional.*

Now we want to show that, if the motive $M(X)$ is finite dimensional, then X has a Chow-Künneth decomposition. We first recall a result which has been proved in [Ja94], 5.3:

Lemma 7 *Assume X is a smooth projective variety of dimension d, such that $A^d(X \times X)_{hom}$ is a nilpotent ideal of $A^d(X \times X)$. Assume moreover that the Künneth components of the diagonal are algebraic. Then X has a Chow-Künneth decomposition.*

Theorem 8 *Let M be a finite dimensional motive in $\mathcal{M}_{rat}$ and let f be a homologically trivial endomorphism of M, i.e. f induces the 0 map on $H^*(M)$. Then f is nilpotent in $End_{\mathcal{M}_{rat}}(M)$.*

Proof. See [Ki98], 7.2 □

Corollary 9 *Let $M(X)$ be a finite dimensional Chow motive. Assume that the Künneth components of the diagonal of X are algebraic. Then X has a Chow-Künneth decomposition.*

Proof. Apply Theorem 8 and Lemma 7 □

Remark 10 If $M(X)$ has a Chow-Künneth decomposition then the projectors π_i defining the motives $M_i(X)$ are by no means unique: for instance the cycle class of the trivial projector π_0 depends on the choice of a rational point x_0 on X. However the motives $M_0(X)$ and M_{2d} are unique (up isomorphisms in $\mathcal{M}$). Also, for a curve C, uniqueness of the motives $M_i(X)$ for $i = 0, 1, 2$ is easy [Mu90], 5.1. For an arbitrary X of dimension d Murre has shown [Mu90], 5.2, that the motives $M_1(X) = (X, \pi_1, 0)$ and $M_{2d-1}(X) = (X, \pi_{2d-1}, 0)$, where π_1 and π_{2d-1} are respectively the Picard and the Albanese projectors, are, up to isomorphisms, independent of the polarization choosen to construct π_1 and π_{2d-1}.

We will show in Theorem 14 that, if $M(X)$ is finite dimensional, then all the $M_i(X)$ are unique, up to isomorphisms.

The main known properties of finite dimensional objects are:

1) If two objects $X, Y \in \mathcal{C}$ are finite dimensional so is their direct sum $X \oplus Y$ and their tensor product $X \otimes Y$. If X is a subobject of a finite dimensional object Y then X is finite dimensional (equivalently, if X is a quotient object of a finite dimensional object Y, it is finite dimensional). Moreover, a direct summand of an evenly (oddly) finite dimensional motive is evenly (oddly) finite dimensional. Note that these properties were proved by Kimura for Chow motives over a field. But they can be proved in an arbitrary pseudoabelian $\mathbb{Q}$-linear tensor category,see [AK02].

2)In particular the properties in 1) impliy the following .If $f : Y \to X$ is a proper surjective morphism of smooth projective varieties and $M_\sim(Y)$ is finite dimensional then $M_\sim(X)$ is also finite dimensional; the motoive $M_\sim(X) \otimes M_\sim(Y) = M_\sim(X \times Y)$ of the fibered product $X \times Y$ is finite dimensional if $M_\sim(X)$ and $M_\sim(Y)$ are finite dimensional.

3) If a motive M is evenly and oddly finite dimensional then $M = 0$ [Ki98], 6.2.

4) The dual object X^* in a rigid category $\mathcal{C}$ is finite dimensional iff X is finite dimensional.

5) Finite dimensionality is a birational invariant for surfaces, [GP02], Th. 2.8.

The following theorem gives classes of smooth projective varieties whose motives are finite dimensional

Theorem 11 *(i) The motive of a smooth projective curve over a field is finite dimensional. (ii) The motive of a variety which is the quotient of a product* $C_1 \times \cdots C_n$ *of curves under the action of a finite group* G *acting freely on* $C_1 \times \cdots \times C_n$ *is finite dimensional. (iii) If* X *is an abelian variety, then* $M(X)$ *is finite dimensional. (iv) The same result holds if* X *is a Fermat hypersurface of degree* d *in* $\mathbb{P}^n$*.*

Proof. (i) was proved in [Ki98]. (ii) and (iii) follow from (i) and the above properties 1) – 5). For abelian varieties see also [Sch94], 3.4. The proof of the fact that the motive of a Fermat hypersurface is finite dimensional can be found in [GP02]. □

Let $\mathcal{M}_{Kim}$ be the full subcategory of $\mathcal{M}_{rat}$ generated by finite dimensional oblects. From the properties 1) – 5) it follows then that $\mathcal{M}_{Kim}$ is a pseudoabelian, rigid and tensor category. Kimura stated

Conjecture 12 $\mathcal{M}_{Kim} = \mathcal{M}_{rat}$

Evidently, $\mathcal{M}_{Kim}$ contains a subcategory generated by the Chow motives of varieties as in Theorem 11, their products and quotients in $\mathcal{M}_{rat}$.

The relations between finite dimensionality and the Conjectures stated in Section 2 can be made more precise using some recent results from [AK02]. We first recall the definition of *dimension* for an object in a rigid tensor category $\mathcal{C}$, see [AK02] or [DeMi82].

For any $X \in \mathcal{C}$ let $\epsilon_X : X \otimes X^* \to 1$ be the evaluation map, and for any two $X, Y \in \mathcal{C}$ let

$$i_{X,Y} : Hom_{\mathcal{C}}(1, X^* \otimes Y) \xrightarrow{\simeq} Hom_{\mathcal{C}}(X, Y)$$

be the canonical isomorphism. Let $h \in End_{\mathcal{C}}(X)$. Then we define the trace of h to be

$$tr(h) = \epsilon_{X^*} \circ i^{-1}_{X,X}(h) \in Hom_{\mathcal{C}}(1,1) \simeq \mathbb{Q}$$

and define $dim(X) = tr(id_X)$.

If $dim(X) = d$ then

$$dim(\wedge^n M) = \binom{d}{n} = \frac{d(d-1)\cdot \ldots \cdot (d-n+1)}{n!}$$

and

$$dim(S^n A) = \binom{d+n-1}{n} = \frac{d(d+1)\cdot \ldots \cdot (d+n-1}{n!} ,$$

see [AK02], 7.2.4. Therefore, if $dim(X) = d > 0$, then $dim(\wedge^{d+1}X) = 0$; if $dim(X) = -d < 0$ then $dim(S^{d+1}X) = 0$.

This dimension is related with Kimura's finite dimensionality in the following way.

Definition 13 *Let $X \in \mathcal{C}$ be afinite dimensional object. Then $kim(X)$ is the smallest integer n, such that $\wedge^n X = 0$ if X is evenly finite dimensional, and $S^n X = 0$ if X is oddly finite dimensional.*

If X is Kimura finite dimensional, then $dim(X)$ is an integer [AK02], 9.1.5: if X is evenly finite dimensional then $dim(X) = kim(X)$, while if X is oddly finite dimensional then $dim(X) = -kim(X)$.

If H is a Weil cohomology theory (with coefficients in a field L of characteristic zero) on $\mathcal{V}_k$ then for every Chow motive $M \in \mathcal{M}_{rat}$ we have $dim(M) = \sum_{i \in \mathbb{Z}}(-1)^i dim(H^i(M))$. For all $X \in \mathcal{V}_k$ which satisfy the standard conjecture $C(X)$, i.e. the Künneth components of the diagonal Δ_X are algebraic, there exist projectors p^+_M and p^-_M in $End_{\mathcal{M}_{rat}}(M(X))$, such that $H(p^+_M)$ and $H(p^-_M)$ are the projectors corresponding to the splitting of $H(X)$ respectively into the even and the odd part.

Let $\mathcal{A}$ be the full subcategory of $\mathcal{M}_{rat}$ of objects A, such that projectors p^+_A and p^-_A exist in $End_{\mathcal{M}_{rat}}(A)$. Then $\mathcal{A}$ is a rigid, tensor and $\mathbb{Q}$-linear subcategory of $\mathcal{M}_{rat}$ containing all the motives of curves, surfaces, abelian varieties, their products and subobjects. For every object $A \in \mathcal{A}$ the projectors p^+_A and p^-_A induce a decomposition $A = A^+ \oplus A^-$, see [AK02], 8.3.

If $A \in \mathcal{A}$ has a decomposition $A = A^+ \oplus A^-$ then A^+ is evenly finite dimensional and A^- is oddly finite dimensional (and hence A is finite dimensional) iff there exists an integer n, such that :

$$s \wedge^n A = 0 \ ,$$

where $s \wedge^n A = \bigoplus_{i+j=n} \wedge^i A^+ \otimes S^j A^-$. If such n exists then the smallest one is the integer $kim(A^+) + kim(A^-) + 1$, see [AK02], 9.1.11.

If A is finite dimensional then the decomposition $A = A^+ \oplus A^-$ is unique up to isomorphisms, i.e. if $A = \tilde{A}^+ \oplus \tilde{A}^-$, where $\tilde{A}^+$ ($\tilde{A}^-$) is evenly (oddly) finite dimensional, then $A^+ \simeq \tilde{A}^+$ and $A^- \simeq \tilde{A}^-$, see [Ki98], 6.3.

Theorem 14 *Let X be a smooth projective variety over k, such that the Künneth components of the diagonal Δ_X are algebraic. Assume that the motive $M(X) \in \mathcal{M}_{rat}$ is finite dimensional. Then $M(X)$ has a Chow-Künneth decomposition*

$$M(X) = \bigoplus_{0 \le i \le 2d} M_i(X)$$

with $M_i(X) = (X, \pi_i, 0)$, which is independent of the choice of the projectors π_i, i.e., if $\{\tilde{\pi}_i\}$ is another set of orthogonal idempotents lifting the Künneth components of Δ_X, then

$$M_i(X) \simeq \tilde{M}_i(X)$$

in $\mathcal{M}_{rat}$, where $\tilde{M}_i(X) = (X, \tilde{\pi}_i, 0)$.

Proof. By Corollary 9 the motive $M(X)$ has a Chow-Künneth decomposition. Let $M(X) = \bigoplus_{0 \leq i \leq 2d} M_i(X)$, where $M_i(X) = (X, \pi_i, 0)$ and let $\{\tilde{\pi}_i\}$ be another set of orthogonal idempotents lifting the Künneth components of Δ_X.

Let's consider the following composition of projectors, for $i = 0, \ldots, 2d$:

$$M_i(X) \stackrel{\pi_i}{\to} M(X) \stackrel{\tilde{\pi}_i}{\to} M_i(X) \stackrel{\tilde{\pi}_i}{\to} M(X) \stackrel{\pi_i}{\to} M_i(X)$$

and set

$$e_i = \pi_i \circ \tilde{\pi}_i \circ \tilde{\pi}_i \circ \pi_i = \pi_i \circ \tilde{\pi}_i \circ \pi_i \ .$$

Then $e_i \circ \pi_i = \pi_i \circ e_i$, i.e. $e_i \in End_{\mathcal{M}_{rat}}(M_i(X))$.

We claim that $e_i = \pi_i$, i.e. e_i is the identity on $M_i(X)$.

$M(X)$ being finite dimensional from Th. 8 it follows that $I = A^2(X \times X)_{hom}$ is a nilpotent ideal of $A^2(X \times X)$. Therefore there exists an element $\eta \in I$, such that $\tilde{\pi}_i = (1+\eta)^{-1} \circ \pi_i \circ (1+\eta)$ for $i = 0, \ldots, 2d$, see [Ja94], 5.4, and we have:

$$e_i - \pi_i = \pi_i - (\pi_i - \pi_i \circ \eta \circ \pi_i \circ \eta \circ \pi_i) = \pi_i \circ \eta \circ \pi_i \circ \eta \circ \pi_i \ .$$

So we are left to show that $\pi_i \circ \eta \circ \pi_i \circ \eta \circ \pi_i = 0$. By induction on the index of nilpotency of I we may assume that $I^2 = 0$. Then we can take $\eta = \pi_i \circ \epsilon_i - \epsilon_i \circ \pi_i$ where $\tilde{\pi}_i = \pi_i + \epsilon_i$ with $\epsilon_i \in I$ and $\epsilon_i^2 = 0$, see [Mu90], page 203. Expanding $(\pi_i + \epsilon_i)^2$ leads to the equation $\epsilon_i = \pi_i \circ \epsilon_i + \epsilon_i \circ \pi_i$, whence:

$$\pi_i \circ \epsilon_i \circ \pi_i = \epsilon_i \circ \pi_i \epsilon_i = 0 \ .$$

From the equalities above we get:

$$\pi_i \circ \eta \circ \pi_i \circ \eta \circ \pi_i = \pi_i \circ (\pi_i \circ \epsilon_i - \epsilon_i \circ \pi_i) \circ \pi_i \circ (\pi_i \circ \epsilon_i - \epsilon_i \circ \pi_i) \circ \pi_i =$$

$$\pi_i \circ \epsilon_i \circ \pi_i \circ \epsilon_i \circ \pi_i = 0 \ .$$

In a completely similar way one shows that $\tilde{e}_i = \tilde{\pi}_i \circ \pi_i \circ \tilde{\pi}_i$ is the identity on $\tilde{M}_i(X)$. Therefore, $\tilde{\pi}_i \circ \pi_i$ yields an isomorphism $M_i(X) \simeq \tilde{M}_i(X)$. □

Remark 15 *(Abelian varieties)* Let X be an abelian variety of dimension d over an algebraically closed field k of char 0. Then $M(X)$ has a Chow-Künneth decomposition; moreover, there exists a unique decompositon $[\Delta_X] = \sum_i \pi_i \in A^d(X \times X)$, such that

$$n^* \circ \pi_i = \pi_i \circ n^* = n^i \pi_i$$

for every $n \in \mathbb{Z}$, where $n^* = (id_X \times n)^*$ and n is the multiplication by n on X. The correspondenses $\{\pi_i\}$ are orthogonal projectors, such that $\pi_0, \ldots, \pi_{j-1}$ and $\pi_{j+d+1}, \ldots, \pi_{2d}$ operates as 0 on $A^j(X)$, see [Mu93(1)], 2.5.2. The corresponding decomposition $M(X) = \sum_{0 \leq i \leq 2d} M_i(X)$ satisfies a part of conditon (II) in Conjecture 2. The motive $M(X)$ is finite dimensional: from Theorem 14 it follows that this decomposition is unique (up to isomorphism). Therefore, if there exists a Chow-Künneth decompositon satisfing the rest of the condition (II), i.e. such that also π_i operates as 0 on $A^j(X)$ for $2j+1 \leq i \leq j+d$, then it is isomorphic to the one above. This condition is in turn equivalent to *Beauville's Conjecture*, see [Mu93(1)], 2.5.3, and [Be86], on the vanishing of the groups $A^j_s(X) = \{\alpha \in A^j(X) \mid n^*\alpha = n^{2j-s}\alpha\}$ for $s < 0$.

Beauville's Conjecture being true for cycles of codimension $j = 0, 1, d-2, d-1$ it follows that conditon (II) is in particular satisfied for all abelian varieties of dimension at most 4. Therefore, for all abelian varieties which satisfy Beauville's Conjecture, the filtration associated to a Chow -Künneth decomposition is independent of the choices of the projectors, in the sense that it only depends on the isomorphism classes of the motives $M_i(X)$. This proves that Beauville's Conjecture implies Murre's conjecture for an abelian variety.

In [AK02], 9.2.4, it has been remarked that if Beilinson's Conjecture or, equivalently, Conjecture 2 is true for all varieties X and also the Standard Conjectures hold, then all Chow motives of smooth projective varieties are finite dimensional, i.e. Kimura's Conjecture holds. The following Theorem 17 avoids the assumption about the Standard Conjectures.

We first prove a lemma which is a direct consequence of a result in [Ja94], 5.8.

Lemma 16 *Let Y be a smooth projective variety of dimension d, such that Y has a Chow-Künneth decomposition, say $[\Delta_Y] = \sum_{i=0}^{2d} \pi_i$, and $Y \times Y$ satisfies the Murre Conjecture. Let $N_i = (Y, \pi_i, 0)$: then*

$$Hom_{\mathcal{M}_{rat}}(N_s, N_t) = 0 \ \ \textit{if} \ \ s \neq t$$

and

$$Hom_{\mathcal{M}_{rat}}(N_s, N_s) = Hom_{\mathcal{M}_{hom}}(h(N_s), h(N_s))$$

for any $s \in \{0, 1, \ldots, 2d\}$.

Proof. Let $\tilde{\pi}_i = \pi^t_{2d-i}$ be the transpose of π_{2d-i} and let $\Pi_r = \sum_{i+j=r} \tilde{\pi}_i \times \pi_j$. By the same argument as in [Ja94], 5.8, the projector Π_r is a lifting of the rth Künneth component of the diagonal $\Delta_{Y\times Y}(4d-r, r)$. Since $Y \times Y$ satisfies the conditon (II) in Conjecture 2, it follows that Π_r acts as 0 on $A^d(Y \times Y)$ for $r > 2d$, whence we get, for all pairs (i, j) with $i + j > 2d$:

$$0 = (\tilde{\pi}_i \times \pi_j)A^d(Y \times Y) = \pi_j \circ Corr^0(Y, Y) \circ \tilde{\pi}^t_i =$$

$$= \pi_j \circ Corr^0(Y, Y) \circ \pi_{2d-i} = Hom_{\mathcal{M}_{rat}}(N_{2d-i}, N_j) \ .$$

This shows that $Hom_{\mathcal{M}_{rat}}(N_s, N_t) = 0$ for $s < t$.

If we take $\tilde{\pi}_i = \pi^t_i$ and $\tilde{\pi}_j = \pi_{2d-j}$, the projector $\Pi_r = \sum_{i+j=r} \tilde{\pi}_i \times \tilde{\pi}_j$ is (up to an isomorphism of $H^*(Y \times Y \times Y \times Y)$) again a lifting of $\Delta_{Y\times Y}(4d-r, r)$. As such Π_r acts as 0 on $A^d(Y \times Y)$ for $r > 2d$. Just as before we get, for all pairs (i, j) with $i + j > 2d$:

$$0 = (\tilde{\pi}_i \times \tilde{\pi}_j)A^d(Y \times Y) = \pi_{2d-j} \circ Corr^0(Y, Y)\tilde{\pi}^t_i =$$

$$= \pi_{2d-j} \circ Corr^0(Y, Y) \circ \pi_i = Hom_{\mathcal{M}_{rat}}(N_i, N_{2d-j}) \ .$$

Therefore, $Hom_{\mathcal{M}_{rat}}(N_s, N_t) = 0$ for $s > t$.

The proof of the equality $Hom_{\mathcal{M}_{rat}}(N_s, N_s) = Hom_{\mathcal{M}_{hom}}(h(N_s), h(N_s))$ follows from the same argument as in [Ja94], 5.8: one takes projectors $\Pi_r = \sum_{i+j=r} \tilde{\pi}_i \times \pi_j$ where $\tilde{\pi}_i = \pi^t_{2d-i}$ and applies conditon (IV) in Murre's Conjecture. Then $A^d(Y \times Y)_{hom} = F^1A^d(Y \times Y) = ker(\Pi_{2d})$ and we obtain

$$(\tilde{\pi}_{2d-s} \times \pi_s)A^d(Y \times Y) = (\tilde{\pi}_{2d-s} \times \pi_s)Corr^0_{hom}(Y, Y) =$$

$$= \Delta_Y(2d-s, s) \circ Corr^0_{hom}(Y, Y) \circ \Delta_Y(2d-s, s) =$$

$$= Hom_{\mathcal{M}_{hom}}(h(N_s), h(N_s)) \ .$$

This proves that $Hom_{\mathcal{M}_{rat}}(N_s, N_s) = Hom_{\mathcal{M}_{hom}}(h(N_s), h(N_s))$. □

Theorem 17 *Let X be a smooth projective variety of dimension d over k. Let $n = \sum_i dim(H^i(X))$ and let $m = n + 1$. Assume that X has a Chow-Künneth decomposition and $X^m \times X^m$ satisfies Murre's Conjecture. Then the motive $M(X)$ is finite dimensional.*

Proof. There exist projectors p_+ and p_- splitting the motive $M = M(X)$ into M^+ and M^-, such that the cohomology of M^+ is $H^+(X) = \sum_{i\in\mathbb{Z}} H^{2i}(X)$ and the cohomology of M^- is $H^-(X) = \sum_{i\in\mathbb{Z}} H^{2i+1}(X)$. Therefore, $M(X)$ is finite dimensional iff M^+ is evenly finite dimensional and M^- is oddly finite dimensional. We have: $dim(M^+) = B_+ = dim(H^+(X))$ and $dim(M^-) = -B_- = -dim(H^-(X))$. Therefore,

$$dim(s \wedge^m M) = dim\left(\sum_{i+j=m} \wedge^i M \otimes S^j M\right) = 0$$

if $m = B_+ + B_- + 1$. So, in order to show that M is finite dimensional, it is enough to prove that

$$s \wedge^m M = \bigoplus_{i+j=m} \wedge^i M^+ \otimes S^j M_- = 0 \ .$$

The functor $H : \mathcal{M}_{hom} \to Vect_L$ being faithful, from $dim(s \wedge^m M) = 0$ we get: $s \wedge^m h(M) = 0$, see [AK02], 8.3.1. Let q_i^+ and q_j^- be the projectors which define respectively $\wedge^i M^+$ and $S^i M^-$. Then the projector $q = \sum_{i+j=m} q_i^+ \otimes q_j^-$, which defines $s \wedge^m M$, belongs to $End_{\mathcal{M}_{rat}}(M(X^m))$ and is homologically trivial, i.e. $h(q) = 0$.

We claim that $q = 0$, i.e. $s \wedge^m M = 0$.

Let $Y = X^m$. Since X has a Chow-Künneth decomposition also $Y = X^m$ has a Chow-Künneth decomposition, see [Mu93(2)], 5.1. Moreover, $Y \times Y$ satisfies Murre's Conjecture by assumptions. Let $M(Y) = \sum_{0 \le s \le 2md} N_i$ where $N_i = (Y, \pi_i, 0)$ be a Chow-Künneth decomposition for Y. From Lemma 16 it follows that:

$$Hom_{\mathcal{M}_{rat}}(N_s, N_t) = \begin{cases} 0 & \text{if } s \neq t, \\ Hom_{\mathcal{M}_{hom}}(h(N_s), h(N_s)) & \text{if } s = t \end{cases} \qquad (1)$$

Let $f_{s,t} = \pi_t \circ q \circ \pi_s \in Hom_{\mathcal{M}_{rat}}(N_s, N_s)$ be the composition map:

$$N_s \xrightarrow{\pi_s} M(Y) \xrightarrow{q} M(Y) \xrightarrow{\pi_t} N_t \ .$$

Then $\sum_s \pi_i \circ q = \sum_s q \circ \pi_s = q$ and $\sum_{s,t} f_{s,t} = \sum_t \pi_t \circ \sum_s q \circ \pi_s = \sum_t \pi_t \circ q = q$. Therefore we get:

$$q = \sum_{s \neq t} (\pi_t \circ q \circ \pi_s) + \sum_s \pi_s \circ q \circ \pi_s \ .$$

From (1) it follows that $\sum_{s \neq t} (\pi_t \circ q \circ \pi_s) = 0$ which yields:

$$q = \sum_{0 \le s \le 2md} \pi_s \circ q \circ \pi_s \in Hom_{\mathcal{M}_{rat}}(N_s, N_s)$$

with $h(q) = 0$. From the second equality in (1) it follows that $q = 0$. This proves that $s \wedge^m M = 0$. □

Definition 18 *Let X be a smooth projective variety over k and let $c_1, \ldots, c_n$ be 0-cycles on X. We define their wedge product to be the following:*

$$c_1 \wedge \ldots \wedge c_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} sgn(\sigma) c_{\sigma(1)} \times \ldots \times c_{\sigma(n)}$$

where $c_{\sigma(1)} \times \ldots \times c_{\sigma(n)}$ is the exterior product of cycles, see [Ful84], Ch.5.

In [Ki98], 5.14, it is proved that, if a surface X is the product of 2 curves, then there exists an integer N, such that the product $c_1 \wedge \ldots \wedge c_N = 0$, where c_i are 0-cycles in the Albanese kernel $T(X)$. The following theorem extends this result to any surface X whose motive is finite dimensional.

Theorem 19 *Let X be a smooth projective surface over k. If the motive $M(X) \in \mathcal{M}_{rat}$ is finite dimensional then $c_1 \wedge \ldots \wedge c_{d+1} = 0$, where c_i are 0-cycles in the Albanese kernel $T(X)$, $d = b_2 - \rho$, $b_2 = dim(H^2(X))$ and $\rho = dim(NS(X)_{\mathbb{Q}})$.*

Proof. The motive $M(X)$ has a Chow-Künneth decomposition as follows:

$$M(X) = \sum_{0 \le i \le 4} M_i(X) \ .$$

Since $M(X)$ is finite dimensional, $M_2(X)$ is also finite dimensional. From [Mu90] it follows that $A^1(M_2(X)) = NS(X)_{\mathbb{Q}}$ and from [Sch94], 2.2:

$$Hom_{\mathcal{M}_{rat}}(M_2(X), \mathbb{L}) \simeq Hom_{\mathcal{M}_{rat}}(\mathbb{L}, M_2(X)) \simeq NS(X)_{\mathbb{Q}} \ ,$$

where $\mathbb{L} = (Spec(k), id, -1)$ is the Lefschetz motive. $NS(X)_{\mathbb{Q}}$ is a finite dimensional $\mathbb{Q}$-vector space of dimension ρ (the corank of $Pic(X)_{\mathbb{Q}}$). Let $[e_i]$, for $1 \le i \le \rho$, be a base for $NS(X)_{\mathbb{Q}}$ and let $\alpha = \sum q_i[e_i] \in NS(X)_{\mathbb{Q}}$. Let $f_\alpha : \mathbb{L} \to M_2(X)$ be the corresponding morphism in $\mathcal{M}_{rat}$. Then $f_\alpha = \sum q_i[Spec(k) \times e_i]$. The transpose f_α^t is a morphism $M_2(X) \to \mathbb{L}$ and

$$f_\alpha^t \circ f_\alpha \in Hom_{\mathcal{M}_{rat}}(\mathbb{L}, \mathbb{L}) \simeq \mathbb{Q} \ .$$

Therefore, for every $i \le \rho$, $f_{[e_i]} : \mathbb{L} \to M_2(X)$ is an injective map. Let $f = \sum f_{[e_i]}$. Then f defines an injective map:

$$\mathbb{L} \oplus \ldots \oplus \mathbb{L} \ (\rho \text{ times}) \to M_2(X) \ .$$

This yields a splitting in $\mathcal{M}_{rat}$:

$$M_2(X) = \rho\mathbb{L} \oplus N \ .$$

We have: $H^i(N) = 0$ for $i \neq 2$ and $H^2(M_2(X)) = H^2(X) = \rho H^2(\mathbb{L}) \oplus H^2(N)$ where $H^2(\mathbb{L}) = \mathbb{Q}$. Therefore, $H^2(N) = (b_2 - \rho) \cdot \mathbb{Q}$.

$M_2(X)$ being finite dimensional N is finite dimensional too. N is evenly finite dimensional because it does not have any odd cohomology, see [Ki98], 3.9. Therefore, $dim(N) = dim(H^2(N)) = kim(N) = d = (b_2 - \rho)$, so that $\wedge^{d+1} N = 0$. We also have $A^2(N) = A^2(M_2(X)) = T(X)$. If $N = (Y, q, n) \in \mathcal{M}_{rat}$, then it follows from the definition of $\wedge$ that, for any r, $\wedge^r N$ is the image of the motive N^r under the projector $(1/r!)(\sum_{\sigma \in \Sigma_r} sgn(\sigma)\Gamma_\sigma \circ q^r)$. Therefore, if $c_1, \dots, c_{d+1}$ are 0-cycles in $T(X)$, then the cycle $c = c_1 \wedge \dots \wedge c_{d+1}$ belongs to $A^2(\wedge^{d+1} N) = 0$. This proves that $c = 0$. $\square$

4 Surfaces with $p_g = 0$

From Th. 19 it follows that, if X is a smooth projective surface with $p_g = 0$ (a condition which is equivalent to $b_2 = \rho$), then the finite dimensionality of the motive $M(X)$ implies $T(X) = 0$. In this section we prove (Theorem 27) that the converse also holds.

We first recall, see [BV99], the definition of a *balanced* variety:

Definition 20 *Let X be a reduced separated and equidimensional scheme of finite type over k and $d = dim(X)$. X is said to be balanced of weight w if there exist cycles Γ_1 and Γ_2 of codimension d on $X \times X$, such that*

$$[\Delta_X] = [\Gamma_1] + [\Gamma_2]$$

in $A^d(X \times X)$, where Γ_1 is supported on $Z_1 \times X$, Γ_2 is supported on $X \times Z_2$, Z_1 and Z_2 are equidimensional closed subschemes of X and

$$w = min\{dim(Z_1), dim(Z_2)\} .$$

Lemma 21 *Let $\mathcal{C}$ be a pseudoabelian category and let X be an object in $\mathcal{C}$. Assume that there exists a finite collection of objects Y_i and morphisms $X \stackrel{a_i}{\to} Y_i \stackrel{b_i}{\to} X$ in $\mathcal{C}$, $i = 1, \dots, n$, such that $id_X = \sum_{i=1}^n b_i a_i$ in $End_{\mathcal{C}}(X)$. Let*

$$X \xrightarrow{f} \bigoplus_{i=1}^{n} Y_i \xrightarrow{g} X$$

be the morphisms induced by $\{a_i\}$ and $\{b_i\}$ respectively. Then $gf = 1_X$ and therefore X is isomorphic to a direct summand of $\oplus_{i=1}^n Y_i$.

Proof. Let $\pi_i : \oplus_{i=1}^n Y_i \to Y_i$ and $\iota_i : Y_i \to \oplus_{i=1}^n Y_i$ be the canonical projection and the canonical embedding. Then

$$gf = g \circ 1_{\oplus_{i=1}^n Y_i} \circ f = g \circ \left(\sum_{i=1}^n \iota_i \pi_i \right) \circ f = \sum_{i=1}^n g \iota_i \pi_i f = \sum_{i=1}^n b_i a_i = 1_X \ .$$

□

Theorem 22 *Let X be a smooth projective (equidimensional) variety balanced by subschemes Z_1 and Z_2. Let $\tilde{Z}_i$ be a desingularization of Z_i, $i = 1, 2$. Assume that the motives $M(\tilde{Z}_1)$ and $M(\tilde{Z}_2)$ are finite dimensional. Then the motive $M(X)$ is finite dimensional.*

Proof. For any $Y \in \mathcal{V}_k$ and $\alpha \in Corr^0(Y, X)$ let $f_\alpha : M(Y) \to M(X)$ be the corresponding morphism from $M(Y)$ to $M(X)$. By assumptions, $[\Delta_X] = [\Gamma_1] + [\Gamma_2]$ in $Corr^0(X, X)$, where $\Gamma_1 \subset Z_1 \times X$ and $\Gamma_2 \subset X \times Z_2$. In other words, $id_{M(X)} = f_{[\Gamma_1]} + f_{[\Gamma_2]}$ in $End_{\mathcal{M}_{rat}}(M(X))$.

Let $s_i : \tilde{Z}_1 \to X$ be a composition of the closed embedding $Z_i \hookrightarrow X$ with a blow up desingularization $v_i : \tilde{Z}_i \to Z_i$ of Z_i, $i = 1, 2$. Let also $w_i = dim(Z_i)$.

Since Γ_1 lies on $Z_1 \times X$, we may consider its class $[\Gamma_1]$ in the Chow group $A^{w_1}(Z_1 \times X)$ of the scheme $Z_1 \times X$. Let $[\tilde{\Gamma}_1]$ be the pull back of $[\Gamma_1]$ with respect to the morphism $v_1 \times id_X : \tilde{Z}_1 \times X \to Z_1 \times X$, see [Ful84]. The variety $\tilde{Z}_1$ is smooth projective and $dim(\tilde{Z}_1) = w_1$. It follows that $[\tilde{\Gamma}_1]$ lies in $A^{w_1}(\tilde{Z}_1 \times X) = Corr^0(\tilde{Z}_1, X)$. Consider the corresponding morphism $f_{[\tilde{\Gamma}_1]} : M(\tilde{Z}_1) \to M(X)$ in the category $\mathcal{M}_{rat}$. Since $v_1 \times id_X$ is a blow up, it follows that $(v_1 \times id_X)_*(v_1 \times id_X)^*([\Gamma_1]) = [\Gamma_1]$ [Ful84]. Therefore we get: $f_{[\Gamma_1]} = f_{[\tilde{\Gamma}_1]} \circ M(s_1)$, whence the morphism $f_{[\Gamma_1]} : M(X) \to M(X)$ factors through the motive $M(\tilde{Z}_1)$.

Similarly one shows, by applying duality in $\mathcal{M}_{rat}$, that the morphism $f_{[\Gamma_2]} : M(X) \to M(X)$ factors through the motive $M(\tilde{Z}_2)(w_2 - d)$ where $d = dim(X)$. Indeed, let Γ_2^t be the transposition of the cycle Γ_2. Let $[\tilde{\Gamma}_2^t]$ be a pull back of its class $[\Gamma_2^t]$ (in the Chow group $A^{w_2}(Z_2 \times X)$) with respect to the blow up $v_2 \times id_X : \tilde{Z}_2 \times X \to Z_2 \times X$. As above we get: $f_{[\Gamma_2^t]} = f_{[\tilde{\Gamma}_2^t]} \circ M(j_2)$. Considering $f_{[\Gamma_2^t]}$ as an endomorphism of the motive $M(X)(d) = M(X) \otimes \mathbb{L}^{-d}$ it factors through $M(\tilde{Z}_2)(d)$. By dualizing we see that $f_{[\Gamma_2]} : M(X) \to M(X)$ factors through the motive $M(\tilde{Z}_2)(w_2 - d)$.

By Lemma 21 we have that $M(X)$ is isomorphic to a direct summand of the motive $M(\tilde{Z}_1) \oplus M(\tilde{Z}_2)(w_2 - d)$. Since both motives $M(\tilde{Z}_1)$ and $M(\tilde{Z}_2)(w_2 - d)$ are finite dimensional, their direct sum is finite dimensional. Therefore $M(X)$ is also finite dimensional. □

Corollary 23 *Let X a smooth projective surface. Assume that either X or a Zariski open dense subset U of X are balanced. Then $M(X)$ is finite dimensional.*

Proof. Let $Z = X - U$. Then $codim_X(Z) \leq 1$. Let U be balanced over closed subschemes Z_1 and Z_2 of codimension ≤ 1. By a result of Barbieri-Viale, [BV99], X is balanced of weight ≤ 1. The motives of points and curves are finite dimensional. Therefore Theorem 22 implies that $M(X)$ is finite dimensional. □

Remark 24 For any field k of characteristic 0 V.Voevodsky has constructed in [Voe00] a triangulated category of motives $DM(k)$ over k and a functor $M : Sm/k \to DM(k)$ from the category Sm/k of smooth separated schemes over k into $DM(k)$. This triangulated category contains a full subcategory, generated by motives $M(X)$ of smooth projective varieties X, which is equivalent to $\mathcal{M}_{rat}$. Moreover, it is pseudo-abelian and, if we consider finite correspondences on schemes with coefficients in $\mathbb{Q}$ to construct $DM(k)$, it is $\mathbb{Q}$-linear. Therefore we can define, according to Def. 6, finite dimensionality of the motive $M(V)$ for every $V \in Sm/k$. Moreover, if U is an open subset of a smooth projective variety X one has the following distinguished triangle in $DM(k)$ [Voe00], 3.5.4:

$$M(U) \to M(X) \to M(Z)(i)[2i] \to M(U)[1]$$

where $Z = X - U$ and i is the codimension of Z in X. If X is a surface and U an open subset of X then Z has dimension ≤ 1, so that $M(Z)$ is finite dimensional. This implies that also $M(Z)(i)[2i]$ is finite dimensional. Therefore Corollary 23 naturally suggests the following question: assuming that $M(U)$ is finite dimensional, is $M(X)$ also finite dimensional?

The next result (Theorem 27) shows that, for a surface with $p_g = 0$ finite dimensionality of the motive $M(X)$ is equivalent to the finite dimensionality of the Chow group of 0-cycles in the sense of Mumford. Here is the definition:

Definition 25 *Let X a smooth projective variety of dimension d over an algebraically closed field k and let $A^d(X)_0$ be the group of 0-cycles of degree 0 on X. Then $A^d(X)_0$ is finite dimensional if there exists an integer n, such that the natural map*

$$s_n : S^nX \times S^nX \to A^d(X)_0$$

is surjective, where $s_n(A, B) = A - B$ and S^nX is the n-th symmetric power of X.

Remark 26 Note that, if $p_g > 0$, then finite dimensionality of the motive $M(X)$ does not, in general, imply the finite dimensionality of the Chow group, as it can be shown by taking products of curves C_i of genus > 1. If X is a complex surface with $p_g = 0$, then $A^2(X)_0$ is finite dimensional iff Conjecture 4 holds for X.

Theorem 27 *Let X be a smooth projective surface over an algebraically closed field k of characteristic 0 with $p_g(X) = 0$. Then the motive $M(X)$ is finite dimensional if and only if the group $A^2(X)_0$ is finite dimensional (i.e. Bloch's conjecture on Albanese kernel is true for X).*

Proof. If $M(X)$ is finite dimensional then by [GP02], Theorem 2.11, we have $ker(\pi_3) = T(X) = 0$ where $T(X)$ is the Albanese kernel. This implies that $A^2(X)_0$ is finite dimensional, see [Ja94], 1.6.

Conversely, assume that $A^2(X)_0$ is finite dimensional. Then there exists, [Ja94], 1.6, a closed subscheme Y of dimension ≤ 1, such that $A^2(X-Y) = 0$. By results of [BS83] X is balanced of weight ≤ 1. From Theorem 22 it follows that $M(X)$ is finite dimensional. □

Remark 28 *(Relations with K-theory)* Let X be a smooth projective surface over $\mathbb{C}$. Then one has the following description for the K-groups $K_n(X)$, for $n > 0$, see [PW01], 6.7:

$$K_n(X) \simeq \begin{cases} B \oplus (\mathbb{Q}/\mathbb{Z})^{2+b_2} \oplus V_n & \text{if } n \geq 1 \text{ odd} \\ A \oplus (\mathbb{Q}/\mathbb{Z})^{b_1+b_3} \oplus V_n & \text{if } n \geq 2 \text{ even} \end{cases}$$

where $A = H^2(X(\mathbb{C}), \mathbb{Z})_{tors}$, $B = H^3(X(\mathbb{C}), \mathbb{Z})_{tors}$, b_i are the Betti numbers and V_n are uniquely divisible groups.

A similar result also holds for any smooth variety over $\mathbb{C}$ [PW00] if one assumes the so called *norm residue Conjecture* which asserts that the norm residue map: $K_n^M(F)/m \to H^n_{et}(F, \mathbb{Z}/m)$ is an isomorphism for all m, where F is the function field of X and K_*^M is Milnor's K-theory.

It follows that, for a surface X, $K_n(X)_{tors}$ depends only upon the topological invariants of the manifold $X(\mathbb{C})$. On the other hand the groups $K_n(X)_{\mathbb{Q}} = K_n(X) \otimes \mathbb{Q}$ depend on the motive $M(X)$ via the Bloch-Lichtenbaum spectral sequence $E_2^{p,q} = H_{\mathcal{M}}^{p-q}(X, \mathbb{Q}(-q))$ which converges to $K_{-p-q}(X)_{\mathbb{Q}}$. Here

$$H^{2i}_{\mathcal{M}}(X, \mathbb{Q}(i)) = Hom_{DM(k)}(M(X), \mathbb{Q}(i)[2i])$$

is the motivic cohomology of X and $\mathbb{Q}(i)[2i]$ plays a role of the power $\mathbb{L}^i$ in $DM(k)$ [Voe00].

Now let X be a smooth projective surface with $p_g = q = 0$. If $M(X)$ is finite dimensional then, by [GP02], 2.14, the motive $M(X)$ is "trivial" in the sense that it is a direct sum of the unit motive 1, of $\mathbb{L}^2$ and of a finite number of copies of $\mathbb{L}$. From Th. 27 it follows that the Albanese kernel $T(X)$ vanishes and this, by [Pe00], Th.0.1, implies

$$K_n(X)_{\mathbb{Q}} \simeq K_0(X) \otimes K_n(\mathbb{C})_{\mathbb{Q}}$$

So also the higher K-theory of X is "trivial".

Note that, if either p_g or q do not vanish, then the above isomorphism is, in general, not true, see [Pe00].

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