ON TRIVIALITY OF THE FUNCTOR $\text{Coker}(K_1(F) \rightarrow K_1(D))$ FOR DIVISION ALGEBRAS

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ABSTRACT. Let D be a cyclic division algebra over its centre F of index n . Consider the group $CK_1(D) = D^*/F^*D'$ where D^* is the group of invertible elements of D and D' is its commutator subgroup. In this note we shall show that the group $CK_1(D)$ is trivial if and only if D is an ordinary quaternion division algebra over a real Pythagorean field F . This in particular shows that if the index of D is an odd prime p , then the exponent of CK_1 is p . We show that the converse does not hold by exhibiting a division algebra D and a division subalgebra $A \subset D$ such that $CK_1(A) \cong CK_1(D)$. Using valuation theory, the group $CK_1(D)$ is computed for some valued division algebras.

1. INTRODUCTION

Let A be an Azumaya algebra over a commutative ring R . Consider the functor $CK_1(A) = \text{Coker}(K_1(R) \xrightarrow{i} K_1(A))$ where i is the inclusion map. If R is semilocal, since the stable rank of R and A are one, $CK_1(A) = A^*/R^*A'$ where A^* and R^* are the group of invertible elements of A and R respectively and A' is the derived subgroup of A^* . A study of this group in the case of central simple algebras is initiated in [6] and further in [5]. It has been established that despite of a “different nature” of this group from the reduced Whitehead group SK_1 , these two groups have the same functorial properties. In [6] this functor is determined for totally ramified division algebras and in particular for any finite abelian group G , a division algebra D such that $CK_1(D) = G \times G$ is constructed. Further in [5], this functor is studied in more cases and examples of cyclic CK_1 (even over non local fields) constructed. It was conjectured in [6] that if the functor CK_1 is trivial then the index of the division algebra is 2. In this note we characterise quaternion division algebras D such that $CK_1(D)$ is trivial. In fact triviality of CK_1 characterises the ground field. We shall show that if $CK_1(D)$, for a quaternion division algebra D , is trivial then D is an ordinary quaternion and the centre of D is a real Pythagorean field. In fact, this is the only class of cyclic division algebras such that CK_1 is trivial (Theorem 2.3). We shall show that some of the notions from the theory of quadratic forms, like rigidity of an element, can be formulated as a property of the group CK_1 . It seems that this group is highly sensitive to the arithmetic property of the ground field. For example we will observe that $CK_1(\frac{x,x}{\mathbb{R}((x))}) \cong \mathbb{Z}_2$ whereas $CK_1(\frac{x,x}{\mathbb{F}_3((x))}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

2. TRIVIALITY OF CK_1

Let F be a field with characteristic not 2. Let $(\frac{a,b}{F})$ be a quaternion division algebra over F . The elements of $D = (\frac{a,b}{F})$ have the form $c_0 + c_1i + c_2j + c_3ij$ and $Nrd_D(c_0 + c_1i + c_2j + c_3ij) = c_0^2 - ac_1^2 - bc_2^2 + abc_3^2$ where Nrd_D is the reduced norm map for D . Note that $CK_1(D) = D^*/F^*D'$ where D^* and F^* are the group of invertible elements of D and F respectively and D' is the commutator subgroup of D^* . If $CK_1(D) = 1$ then $Nrd_D(D) = F^{*2}$. Thus $Nrd_D(i) = -a \in F^2$ and $Nrd_D(j) = -b \in F^2$. Therefore $(\frac{a,b}{F}) \cong (\frac{-1,-1}{F})$.

Our first theorem shows that triviality of $(\frac{a,b}{F})$ forces that F be a real Pythagorean field. In fact this property characterises this field. Recall that F is a real Pythagorean if $-1 \notin F^{*2}$ and sum of any two square elements is a square in F . It follows immediately that F is an ordered field. F is called Euclidean if F^{*2} is an ordering of F .

Theorem 2.1. *Let F be a field of characteristic not 2. Then the following are equivalent.*

- 1) $(\frac{-1,-1}{F})$ is a division ring and $CK_1(\frac{-1,-1}{F})$ is trivial.
- 2) F is the real Pythagorean field.
- 3) $(\frac{-1,-1}{F})$ is a division ring and every maximal subfield of $(\frac{-1,-1}{F})$ is F -isomorphic to $F(\sqrt{-1})$.

Proof. We shall show that 1 and 2 are equivalent. The equivalency of 2 and 3 are known (see [2]).

1) $\Rightarrow$ 2) Since $D = (\frac{-1,-1}{F})$ is a division ring, $-1 \notin F^{*2}$. By induction, if $-1 = f_1^2 + \dots + f_n^2$, then considering $f_1^2 + f_2^2 = Nrd_D(f_1 + f_2i) = f^2$, a contradiction follows. This shows that -1 is not sum of squares plus sum of any two squares is a square

2) $\Rightarrow$ 1) Suppose F is a real Pythagorean. It is easy to see that $D = (\frac{-1,-1}{F})$ is a division ring. Now for any $x \in D^*$, $Nrd_D(x)$ is a sum of four squares, thus $Nrd_D(D^*) = F^{*2}$. Since $SK_1(D)$ is trivial, this forces that CK_1 be trivial. $\square$

Over Euclidean fields, the only quaternion division algebra is the ordinary quaternion division algebra, and from the above Theorem it follows that its CK_1 is trivial.

We shall show that an ordinary quaternion division algebra over a real Pythagorean field is in fact the only type of cyclic division algebras such that its CK_1 is trivial. Before we state the theorem, note that there are examples of infinite dimensional division algebras D such that D^* coincide with D' [7]. In the finite dimensional case, it is not hard to see that $D^* \neq D'$, in fact $K_1(D) = D^*/D'$ is not torsion. In the case of algebraic division algebras, everything remains as a mystery.

We need the following theorem which says that the functor CK_1 has a decomposition property similar to one for the reduced Whitehead group.

Theorem 2.2. [5] *Let A and B be division algebras with centre F such that $(i(A), i(B)) = 1$. Then $CK_1(A \otimes_F B) \cong CK_1(A) \times CK_1(B)$.*

Theorem 2.3. *Let D be a cyclic division algebra finite dimension over its centre F . Then $CK_1(D) = 1$ if and only if D is an ordinary quaternion division algebra over a real Pythagorean field F .*

Proof. The only if part is already proved in Theorem 2.1. For the if part, we shall show that the index of D is 2. Then the claim follows again from Theorem 2.1. First suppose that $i(D) = n$ is odd. Let $\alpha \in D$ such that $F(\alpha)$ is a maximal cyclic subfield of D with a minimal polynomial of the form $x^n - f$ where $f \in F$. Since $D^* = F^*D'$, we can choose $\alpha \in D'$. Now $\alpha^n = N_{F(\alpha)/F}(\alpha) = Nrd_D(\alpha) = 1$. From this it follows that the degree of the minimal polynomial of α is less than n which can not happen. Since $CK_1(D)$ has a decomposition property (Theorem 2.2), the only case which remains, is $n = 2^k$, for some integer k . We shall show that $k = 1$. Again consider α as above. This time $-\alpha^{2^k} = N_{F(\alpha)/F}(\alpha) = Nrd_D(\alpha) = 1$. On the other hand since $1 + N_{F(\alpha)/F}(\alpha) = N_{F(\alpha)/F}(\alpha + 1) \in F^{*2^k}$, it follows that $\sqrt{2} \in F$. Thus if $k > 1$ then $\alpha^{2^k} + 1 = (\alpha^{2^{k-1}} + 1)^2 - 2\alpha^{2^{k-1}}$ can be decomposed further which leads to a contradiction that the minimal polynomial of α has degree less than n . Therefore $n = 2$ and the claim follows from Theorem 2.1 $\square$

Note that in the above theorem we could replace CK_1 with a smaller group $D^*/F^*D^{(1)}$, where $D^{(1)}$ is set of elements with reduced norm 1.

The above theorem states that if in the case of division algebra D of odd prime index, $CK_1(D)$ happens to be trivial, then this forces the well-known conjecture that any division algebra of prime index is cyclic would be false!

It is a celebrated result of Wedderburn that a division algebra of index 3 is cyclic. This together with Theorem 2.2 and Theorem 2.3 implies that

Theorem 2.4. *If D is a division algebra of index $3n$ where $3 \nmid n$, then $CK_1(D) \neq 1$.*

Remark 2.5. Let D be a finite dimensional division algebra with centre F , of index n . Consider the sequence

$$(1) \quad K_1(D) \xrightarrow{Nrd} K_1(F) \xrightarrow{i} K_1(D)$$

Where Nrd_D is the reduced norm map and i is the inclusion map. One can see that the composition $i Nrd_{D/F} = \eta_n$ where $\eta_n(a) = a^n$. (See for example the proof of Lemma 4, p. 157 [1]). From this the formula $a^n = Nrd_{D/F}(a)c_a$ where $a \in D^*$ and $c_a \in D'$ follows. In particular this implies that the exponent of the abelian group $CK_1(D)$ divides n , the index of D .

It is not known if $\exp(CK_1(D)) < i(D)$ what would be imposed on the algebraic structure of D . We mention that if D is a totally ramified division algebra then $\exp(CK_1(D)) = i(D)$ if and only if D is cyclic [5]. In fact from the above theorem it follows that if D is a cyclic division algebra of index p , an odd prime, then the exponent of $CK_1(D)$ is exactly p . The converse is not true, as the following example shows. This is an example of a cyclic decomposable F -division algebra D of index $2p$, p an odd prime, such that it has a proper F -division subalgebra $A \subset D$, where $CK_1(A) \cong CK_1(D)$. This in particular shows that the exponent of $CK_1(D)$ is less than the index of D , but D is cyclic, that is, the phenomena of cyclicity of D and $\exp(CK_1(D))$

does not follow the same pattern as in the case of totally ramified. For this we need the Fein-Schacher-Wadsworth example of a division algebra of index $2p$ over a Pythagorean field F [3]. We briefly recall the construction. Let p be an odd prime and K/F be a pair of real Pythagorean fields such that K is cyclic over F with the generating automorphism σ . Now consider

$$D = \left(\frac{-1, -1}{F((x))} \right) \otimes_{F((x))} \left(K((x))/F((x)), \sigma, x \right).$$

$K((x))/F((x))$ is a cyclic extension where $K((x))$ and $F((x))$ are Laurant power series fields of K and F respectively. D is a division algebra of index $2p$. Since F is real Pythagorean, so does $F((x))$. Now by Theorem 2.2 the primary decomposition of the division algebra D induces a corresponding decomposition for $CK_1(D)$. Thus $CK_1(D) \cong CK_1\left(\frac{-1, -1}{F((x))}\right) \times CK_1(A)$ where $A = (K((x))/F((x)), \sigma, x)$. By Theorem 2.1, $CK_1\left(\frac{-1, -1}{F((x))}\right) = 1$, thus $CK_1(D) \cong CK_1(A)$.

Question. If $\exp(CK_1(D))$ is less than the index $i(D)$, what can be said about the algebraic structure of D .

Recall that $a \in F$ is called *rigid* if $a \notin \pm F^{*2}$ and $F^{*2} + aF^{*2} = F^{*2} \cup aF^{*2}$. The rigidity concept plays a role in the study of the extension of Pythagorean fields. In particular if $K = F(\sqrt{a})$ is a quadratic extension of F , then K is real Pythagorean if and only if F is real Pythagorean and a is rigid [8].

Theorem 2.6. *Let F be a real Pythagorean field. Let $a \notin \pm F^{*2}$. Then a is rigid if and only if $CK_1\left(\frac{-1, -a}{F}\right) = \mathbb{Z}_2$.*

Proof. Suppose a is rigid. Since $-a \notin F^{*2}$, then $D = \left(\frac{-1, -a}{F}\right)$ is a division ring. Since F is Pythagorean, $Nrd_D(D^*) = c_0^2 + ac_1^2$, where $c_0, c_1 \in F^*$. From rigidity of a it follows that $Nrd_D(D^*) = F^{*2} \cup aF^{*2}$. Because $a \notin F^{*2}$, then F^{*2} and $F^{*2}a$ are two distinct elements in the group $Nrd_D(D^*)/F^{*2}$. Since $SK_1(D)$ is trivial, $CK_1(D) \cong Nrd_D(D^*)/F^{*2} \cong \mathbb{Z}_2$. The if part follows the same way. $\square$

Remark 2.7. Let F be a real Pythagorean field and $a \in F^*$ such that $-a \notin F^{*2}$. Then $CK_1\left(\frac{-1, -a}{F}\right) \cong L^*/F^*L^{*2}$ where $L = F(\sqrt{-a})$. For consider the square class exact sequence

$$1 \longrightarrow \{F^{*2}, F^{*2}a\} \longrightarrow F^*/F^{*2} \longrightarrow L^*/L^{*2} \xrightarrow{N} N_{L/F}(L^*)/F^{*2} \longrightarrow 1.$$

In this setting, it is easy to see that $N_{L/F}(L^*) = Nrd_{D/F}(D^*)$ thus the claim.

3. CK_1 FOR VALUED QUATERNION DIVISION ALGEBRAS

In this section we study the valued quaternion division algebras. Valuation theory for division algebras is the main tool for computation of CK_1 . Recall that if D is equipped with a valuation, then $\bar{D}$ and $\bar{F}$ denote the residue division algebra and the residue field of D and F respectively. Γ_D and Γ_F are the value groups of the valuation.

A tame valued quaternion division algebra could be unramified, semi-ramified or totally ramified. This follows from the fact that D is defectless. Recall that D is *tame* if $\text{char}(\bar{F})$ does not divide the index of D . Thus D

is *defectless*, namely $[\bar{D} : \bar{F}][\Gamma_D : \Gamma_F] = [D : F]$. D is called *unramified* if $[\bar{D} : \bar{F}] = [D : F]$ and *totally ramified* if $[\Gamma_D : \Gamma_F] = [D : F]$. D is called *semiramified* if $\bar{D}$ is a field and $[\bar{D} : \bar{F}] = [\Gamma_D : \Gamma_F] = i(D)$. For an excellent survey of the valuation theory of division algebras see [10].

We will see that there are quaternion division algebras of any type mentioned above.

- $D = (\frac{-1, -1}{\mathbb{R}((x))})$ is unramified, and $CK_1(D) = 1$ (see Theorem 2.1).
- $D = (\frac{-1, x}{\mathbb{R}((x))})$ is semiramified, and $CK_1(D) = \mathbb{Z}_2$ (see Theorem 3.2).
- $D = (\frac{x, y}{\mathbb{R}((x, y))})$ is totally ramified, and $CK_1(D) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (see Remark 3.3).

Although all the examples that we are dealing with here have finite CK_1 , let us mention that CK_1 of a quaternion division algebra could be infinite. It is known that if p is a rational prime such that $p \equiv 3 \pmod{4}$ then $(\frac{-1, p}{\mathbb{Q}})$ is a division algebra. It is not hard to see that $CK_1(\frac{-1, p}{\mathbb{Q}}) = \bigoplus \mathbb{Z}_2$ is infinite.

In order to compute the functor CK_1 in the case of a valued division algebra, we need the following Theorem.

Theorem 3.1. [5] *Let D be a Henselian division algebra tame over its centre with index n . Then*

- 1) *If D is unramified, then $CK_1(D) \cong CK_1(\bar{D})$.*
- 2) *If D is totally ramified, then $CK_1(D) = \Gamma_D / \Gamma_F$.*
- 3) *If D is semiramified, and $\bar{D}$ is cyclic over $\bar{F}$, then there is an exact sequence*

$$1 \longrightarrow N_{\bar{D}/\bar{F}}(\bar{D}^*)/\bar{F}^{*n} \longrightarrow CK_1(D) \longrightarrow \Gamma_D / \Gamma_F \longrightarrow 1.$$

It is known that if $Z(\bar{D})$ is separable over $\bar{F}$, then the fundamental homomorphism $\Phi : \Gamma_D / \Gamma_F \rightarrow \text{Gal}(Z(\bar{D})/\bar{F})$ is epimorphism [10]. Since in the above theorem, part 3) D is semiramified over F , Φ is an isomorphism. On the other hand $\bar{D}$ is cyclic over $\bar{F}$, Thus $\Gamma_D / \Gamma_F \cong \mathbb{Z}_n$. So we can rewrite the above exact sequence as

3') *If D is semiramified, and $\bar{D}$ is cyclic over $\bar{F}$, then there is an exact sequence*

$$1 \longrightarrow N_{\bar{D}/\bar{F}}(\bar{D}^*)/\bar{F}^{*n} \longrightarrow CK_1(D) \longrightarrow \mathbb{Z}_n \longrightarrow 1.$$

Theorem 3.1 is our main tool to compute the functor CK_1 for a valued quaternion division algebra.

Let F be a real Pythagorean field. Let N be the set of non rigid elements of F . Recall that if $N = \pm F^{*2}$ then F is called *super-Pythagorean* [8]. The following theorem is a demonstration of how valuation theory enables us to compute CK_1 .

Theorem 3.2. *Let F be a real Pythagorean field and N the set of non rigid elements of F . Then for any $t \in F((x)) \setminus \pm F((x))^2 \cup N$, $CK_1(\frac{-1, -t}{F((x))}) = \mathbb{Z}_2$.*

Proof. The valuation of $F((x))$ extends to $D = (\frac{-1, -1}{F((x))})$ as follows; $v(x) = \frac{1}{2}v(Nrd_D(x))$ where v denotes the valuation map for both F and D . Suppose first that $t \in F$. It is easy to see that

$$(\frac{-1, -t}{F((x))}) \cong (\frac{-1, -t}{F}) \otimes_F F((x))$$

is unramified. Thus by 3.1 1), $CK_1(\frac{-1, -t}{F((x))}) \cong CK_1(\frac{-1, -t}{F})$. Since t is rigid in F by Theorem 2.6, $CK_1(\frac{-1, -t}{F}) \cong \mathbb{Z}_2$ and we are done.

Now if $t \in F((x)) \setminus F$, then it is not hard to see that D is semiramified, $\bar{D} = F(\sqrt{-1})$ and $N_{\bar{D}}(F(\sqrt{-1})) = F^{*2}$. Now from Theorem 3.1 3') it follows that $CK_1(D) = \mathbb{Z}_2$. $\square$

Theorem above is another way to observe that if F is super-Pythagorean then so is $F((x))$.

Remark 3.3. Consider the division algebra $D = (\frac{x, y}{\mathbb{R}((x, y))})$. Since $Nrd_D(i) = -x$ and $Nrd_D(j) = -y$, it can be seen that the extension of the valuation from $\mathbb{R}((x, y))$ to D is totally ramified. From Theorem 3.1 2) follows that $CK_1(D) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

3.1. CK_1 for local quaternion division algebras. Let F be a local field and D a quaternion division algebra over D . We shall use Theorem 3.1 to show that $CK_1(D) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Recall that characteristic of F is not 2. We further assume that $char(\bar{F})$ is not 2.

Theorem 3.4. *Let D be a quaternion division algebra over a local field F . Then $CK_1(D) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

Proof. The valuation of F extends to D . Furthermore one can see that with this valuation D is semiramified over F and $\bar{D}$ is cyclic over $\bar{F}$. Since $\bar{D}$ is finite, $N_{\bar{D}}(\bar{D}^*) = \bar{F}^*$ and the exact sequence of Theorem 3.1 3' has the look

$$1 \longrightarrow \bar{F}^*/\bar{F}^{*2} \longrightarrow CK_1(D) \longrightarrow \mathbb{Z}_2 \longrightarrow 1.$$

Since $char(\bar{F})$ is not 2, the quotient $\bar{F}^*/\bar{F}^{*2}$ is not trivial. Because $\bar{F}^*$ is a cyclic group therefore $\bar{F}^*/\bar{F}^{*2} \cong \mathbb{Z}_2$. Since $exp(CK_1(D))$ divides 2, from the above exact sequence it follows that $CK_1(D) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. $\square$

The above theorem shows that $CK_1(\frac{-1, x}{\mathbb{F}_3((x))}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ as mentioned before.

We shall mention that this result was not unknown. In fact, since $SK_1(D)$ is trivial and the reduced norm is surjective for division algebras over local fields, $CK_1(D) \cong F^*/F^{*2}$. From the theory of quadratic forms for local fields we can thus obtain the same result. Also if $char \bar{F} = 2$, it is known that $F^*/F^{*2} \cong \bigoplus_{2^{n+1}} \mathbb{Z}_2$ (See P.217 [9])

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