A ZERO THEOREM FOR THE TRANSFER OF COHERENT WITT GROUPS

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ABSTRACT. Let R be a Gorenstein ring of finite Krull dimension and $t \in R$ a regular element. We show that if the quotient map $R \rightarrow R/Rt$ has a flat splitting then the transfer morphism of coherent Witt groups $\mathrm{Tr}_{(R/Rt)/R} : \tilde{W}^i(R/Rt) \rightarrow \tilde{W}^{i+1}(R)$ is zero for all $i \in \mathbb{Z}$. As an application we give another proof of the Gersten conjecture for Witt groups in the case of regular semilocal rings essentially of finite type over a field of characteristic not 2.

0. INTRODUCTION

Let $\tilde{W}^i(R)$ be the i -th coherent Witt group of the Gorenstein ring of finite Krull dimension R as defined in [11], and let $t \in R$ be a regular element, i.e. t is not a zero divisor in R . Then R/Rt is a Gorenstein ring, too (see e.g. [8], Proposition 3.1.19), and we have a transfer morphism (cf. [11] and [12]):

$$\mathrm{Tr}_{(R/Rt)/R} : \tilde{W}^i(R/Rt) \longrightarrow \tilde{W}^{i+1}(R)$$

for all $i \in \mathbb{Z}$. The aim of this short note is to prove:

Theorem 0.1. *Let R be a Gorenstein ring of finite Krull dimension, $t \in R$ a regular element, and $\pi : R \rightarrow R/Rt$ the quotient map. Suppose π has a flat splitting, i.e. there exists a flat morphism $q : R/Rt \rightarrow R$ such that $\pi \cdot q = \mathrm{id}_{R/Rt}$. Then the transfer morphism*

$$\mathrm{Tr}_{(R/Rt)/R} : \tilde{W}^i(R/Rt) \longrightarrow \tilde{W}^{i+1}(R)$$

is zero for all $i \in \mathbb{Z}$.

As special as this result sounds it has an interesting application, which we explain now. Balmer and Walter [5] have constructed a so-called augmented Gersten-Witt complex

$$GW_{\bullet}(X) : 0 \longrightarrow W(X) \longrightarrow \bigoplus_{x \in X^{(0)}} W(k(x)) \longrightarrow \bigoplus_{x \in X^{(1)}} W(k(x)) \longrightarrow \dots$$

for any regular scheme X (here $X^{(i)} \subseteq X$ denotes the points of codimension i and $k(x)$ the residue field at $x \in X$). It is conjectured that this complex is exact if X is the spectrum of a regular local ring R . This is known to be true if $\dim R < 5$ by Balmer and Walter [5] and if R contains a field by Balmer, Gille, Panin and Walter [4]. The proof in [4] uses a trick of Panin to go from the finite type case, proved by Balmer [3], to the general case. We give here another proof of Balmer's result, i.e. we show that $GW_{\bullet}(X)$ is exact if X is the spectrum of a semilocal regular ring of finite type over a field.

Our proof contains two new features. First, contrary to [3], we avoid the rather complicated variant of the normalization lemma from [9] (based on the work of Gabber [10]) and need only Quillen's normalization lemma. Second, Quillen uses additivity of the K-functor – which does

not hold for Witt groups – to conclude. We use the above zero theorem instead, which may hence be considered as a weak version of additivity for coherent Witt groups of Gorenstein rings.

1. TERMINOLOGY

We assume that all rings are commutative, noetherian with 1 and have 2 invertible. Let S be a ring. We denote $\mathcal{M}_S$ the category of all S -modules, $\mathcal{M}_S^{fg}$ the category of finitely generated S -modules and $\mathcal{P}_S$ the category of projective S -modules of finite rank. With $D^b(\mathcal{E})$ we denote the bounded derived category of the exact category $\mathcal{E}$, and $P_\bullet \mapsto P_\bullet[1] = P_{\bullet-1}$ denotes the translation functor in $D^b(\mathcal{E})$. Let $D_{fg}^b(\mathcal{M}_S)$ be the full triangulated subcategory of $D^b(\mathcal{M}_S)$ consisting of complexes whose homology modules are in $\mathcal{M}_S^{fg}$. On this triangulated category $D_{fg}^b(\mathcal{M}_S)$ we have a codimension of support filtration:

$$D_{fg}^b(\mathcal{M}_S) = D_S^0 \supseteq D_S^1 \supseteq \dots \supseteq D_S^n \supseteq D_S^{n+1} \supseteq \dots,$$

where we have set

$$D_S^p := \left\{ M_\bullet \in D_{fg}^b(\mathcal{M}_S) \mid \begin{array}{l} H_i(M_\bullet)_\varphi = 0 \text{ for all } \varphi \in \text{Spec } S \text{ with } \text{ht } \varphi \leq p-1 \\ \text{and all } i \in \mathbb{Z}. \end{array} \right\}$$

for $p \geq 0$. We use the following sign conventions for the total Hom and the tensor product of complexes. Let $P_\bullet$, $M_\bullet$, and $I_\bullet$ be bounded complexes of S -modules. Then $d^{P \otimes M}$ is given on $P_r \otimes_S M_s$ by $d_r^P \otimes \text{id}_{M_s} + (-1)^r \text{id}_{P_r} \otimes d_s^M$ and $d^{\text{Hom}(M, I)}$ is defined on $\text{Hom}_S(M_{-i-r}, I_{-r})$ by $f \mapsto f d_{-i-r+1}^M + (-1)^{i+1} d_{-r}^I f$. We have then the following natural isomorphisms:

$$\tau^{(i)} : P_\bullet \otimes (M_\bullet[i]) \xrightarrow{\sim} (P_\bullet \otimes M_\bullet)[i]$$

given by $(-1)^{ij} \text{id}_{P_j \otimes \dots}$ on $P_j \otimes \dots$, and

$$\iota^{(i)} : P_\bullet[i] \otimes M_\bullet \xrightarrow{\sim} (P_\bullet \otimes M_\bullet)[i]$$

which is the identity in every degree.

If S is a Gorenstein ring of finite Krull dimension and

$$0 \longrightarrow S \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow \dots \longrightarrow I_{-\dim S} \longrightarrow 0$$

a finite injective resolution of S , then $\chi_{fg}^S := \text{Hom}_S(-, I_\bullet)$ is a contravariant functor from $D_{fg}^b(\mathcal{M}_S)$ to itself, the so-called canonical duality. Here we consider $I_\bullet$ as an element of $D_{fg}^b(\mathcal{M}_S)$ with I_0 in degree 0 and I_{-1} in degree (-1) . The category $D_{fg}^b(\mathcal{M}_S)$ equipped with this duality is a triangulated category with 1-exact duality in the sense of Balmer [1]. We denote the corresponding triangular Witt groups $\tilde{W}^i(S)$, $i \in \mathbb{Z}$. These groups are called the coherent Witt groups of the ring S . They classify i -spaces in $D_{fg}^b(\mathcal{M}_S)$ for the canonical duality up to spaces having a Lagrangian. Recall that an i -space is a pair $[M_\bullet, \varphi]$ with $M_\bullet \in D_{fg}^b(\mathcal{M}_S)$ and $\varphi : M_\bullet \xrightarrow{\sim} \chi_{fg}^S(M_\bullet)[i]$ an isomorphism, such that $\chi_{fg}^S(\varphi) \cdot \varpi_M = \varphi$ where $\varpi : \text{id}_{D_{fg}^b(\mathcal{M}_S)} \xrightarrow{\sim} \chi_{fg}^S \chi_{fg}^S$ is the natural isomorphism to the bidual (cf. e.g. [11], Sect. 2.5.1). Observe that the full triangular subcategory D_S^p of $D_{fg}^b(\mathcal{M}_S)$ is closed under the canonical duality and hence also a triangulated category with duality for all $p \geq 0$. We denote the corresponding triangular Witt groups by the symbols $W^i(D_S^p)$ ($i \in \mathbb{Z}$, $p \geq 0$).

Another example of a triangulated Witt groups which appears in this work are the derived Witt groups $W^i(S)$ of a ring S . These are by definition the triangular Witt groups of $D^b(\mathcal{P}_S)$

for the duality $\text{Hom}_S(-, S)$. Recall that by a result of Balmer [1] the natural functor $\mathcal{P}_S \rightarrow D^b(\mathcal{P}_S)$ induces an isomorphism $W(S) \xrightarrow{\cong} W^0(S)$ where $W(S)$ denotes the usual Witt group of the ring S .

For a regular ring S , we have an equivalence $D^b(\mathcal{P}_S) \xrightarrow{\cong} D_{fg}^b(\mathcal{M}_S)$ and thus an isomorphism $W^i(S) \cong \tilde{W}^i(S)$ for all $i \in \mathbb{Z}$. These isomorphisms are not natural for the following reason. To get a homomorphism between the Witt groups we have to make the natural embedding $F = F_S : D^b(\mathcal{P}_S) \hookrightarrow D_{fg}^b(\mathcal{M}_S)$ duality preserving, i.e. we need an isomorphism of functors $\eta : F\text{Hom}_S(-, S) \xrightarrow{\cong} \text{Hom}_S(F(-), I_\bullet)$ satisfying two compatibility axioms (cf. e.g. [11], Definition 2.6). These axioms make sure that $[F(P_\bullet), \eta_P[i] \cdot F(\varphi)]$ is an i -form in $D_{fg}^b(\mathcal{M}_S)$ if $[P_\bullet, \varphi]$ is an i -form in $D^b(\mathcal{P}_S)$ (cf. e.g. *loc.cit.* Theorem 2.7). Since $F(S) = S$ (considering S as a complex which is concentrated in degree 0) the existence of such an isomorphism of functors η is equivalent to the existence of a quasi isomorphism $\iota : S \xrightarrow{\cong} I_\bullet$. But there is no canonical choice for ι , and therefore we have several isomorphisms η making F duality preserving which are all equally natural. Any such pair (F, η) is called a duality preserving functor.

2. PROOF OF THEOREM 0.1

We prove a more general result than announced in the introduction. Before we state it recall the following. The transfer $\text{Tr}_{(R/Rt)/R}$ is induced by the restriction of scalars functor $\pi_* : D_{fg}^b(\mathcal{M}_{R/Rt}) \rightarrow D_{fg}^b(\mathcal{M}_R)$. But $\pi_*(D_{R/Rt}^p) \subseteq D_R^{p+1} \subseteq D_R^p$ and so we have also a transfer homomorphism $\text{Tr}_{(R/Rt)/R} : W^i(D_{R/Rt}^p) \rightarrow W^{i+1}(D_R^p)$.

Theorem 2.1. *Let R be a Gorenstein ring of finite Krull dimension, $t \in R$ a regular element, and $\pi : R \rightarrow R/Rt$ the quotient map. Suppose π has a flat splitting, i.e. there exists a flat morphism $q : R/Rt \rightarrow R$ such that $\pi \cdot q = \text{id}_{R/Rt}$. Then the transfer morphism*

$$\text{Tr}_{(R/Rt)/R} : W^i(D_{R/Rt}^p) \rightarrow W^{i+1}(D_R^p)$$

is zero for all $i \in \mathbb{Z}$ and $p \geq 0$.

Remark 2.2. Observe ([12], Remark 6.5) that $\text{Tr}_{(R/Rt)/R} : \tilde{W}^i(R/Rt) \rightarrow \tilde{W}^{i+1}(R)$ as defined in *loc.cit.* factors through the dévissage map $\tilde{W}^i(R/Rt) \rightarrow \tilde{W}_{Rt}^{i+1}(R)$ described in [11], Sect. 4.1.

For the proof of Theorem 2.1 above we need

2.3. Another description of coherent Witt groups.

We recall some results from [11], Sect. 2.5. Let S be a Gorenstein ring of finite Krull dimension and $j : S \rightarrow J_\bullet$ a finite injective resolution of the S -module S . We denote $\mathcal{E}(S)$ the exact category of finitely generated $\text{Hom}_S(-, S)$ -acyclic S -modules. Since S is Gorenstein this category has some useful properties. The contravariant functor $\text{Hom}_S(-, S)$ is a duality on $\mathcal{E}(S)$ making it an exact category with duality (in particular all modules in $\mathcal{E}(S)$ are reflexive). We denote the corresponding derived Witt groups (cf. [2]) $W^i(\mathcal{E}(S))$. The natural functor $D^b(\mathcal{E}(S)) \rightarrow D_{fg}^b(\mathcal{M}_S)$ is an equivalence ([6], Theorem 8.2). It becomes duality preserving using $\text{Hom}_S(-, j)$ as duality transformation and we have ([11], Corollary 2.17) an isomorphism $F_{S,j} : W^i(\mathcal{E}(S)) \xrightarrow{\cong} \tilde{W}^i(S)$ for all $i \in \mathbb{Z}$. We want to point out that

this isomorphism depends on j , a fact which we use below (cf. end of last section). The triangulated category $D^b(\mathcal{E}(S))$ has also a filtration by full triangular subcategories:

$$D^b(\mathcal{E}(S)) = D^b(\mathcal{E}(S))^{(0)} \supseteq D^b(\mathcal{E}(S))^{(1)} \supseteq D^b(\mathcal{E}(S))^{(2)} \supseteq \dots,$$

where $D^b(\mathcal{E}(S))^{(p)} := D^b(\mathcal{E}(S)) \cap D_S^p$. The restriction of $\mathrm{Hom}_S(-, S)$ is a duality on $D^b(\mathcal{E}(S))^{(p)}$ making this full subcategory a triangulated category with duality for all $p \geq 0$. We denote $W^i(D^b(\mathcal{E}(S))^{(p)})$ their triangular Witt groups.

If S' is another Gorenstein ring and $f : S \rightarrow S'$ is a flat morphism, we have a homomorphism of derived Witt groups $f^* : W^i(\mathcal{E}(S)) \rightarrow W^i(\mathcal{E}(S'))$ for all $i \in \mathbb{Z}$ which is induced by the exact functor $M_\bullet \mapsto S' \otimes_S M_\bullet$.

We have a pairing $W^d(S) \times W^m(\mathcal{E}(S)) \rightarrow W^{d+m}(\mathcal{E}(S))$. It is defined by considering $P_\bullet \in D^b(\mathcal{P}_S)$ as duality preserving functor, made duality preserving using a form on it. We explain this briefly and refer to [13] for details and more information. Let $x = [P_\bullet, \varphi] \in W^d(S)$. Then we have the following isomorphism η_M^x :

$$\begin{aligned} P_\bullet \otimes_S \mathrm{Hom}_S(M_\bullet, S) &\xrightarrow{\varphi \otimes \mathrm{id}} \mathrm{Hom}_S(P_\bullet, S)[d] \otimes_S \mathrm{Hom}_S(M_\bullet, S) \\ &\xrightarrow{\mathfrak{l}^{(d)}} (\mathrm{Hom}_S(P_\bullet, S) \otimes_S \mathrm{Hom}_S(M_\bullet, S))[d] \simeq \mathrm{Hom}_S(P_\bullet \otimes_S M_\bullet)[d] \end{aligned}$$

natural in $M_\bullet \in D^b(\mathcal{E}(S))$. It is a duality transformation for the exact functor $P_\bullet \otimes_S -$, and so we get a homomorphism $x \star_l - : W^m(\mathcal{E}(S)) \rightarrow W^{d+m}(\mathcal{E}(S))$, the so-called left product with x . It sends the m -space $[M_\bullet, \psi] \in W^s(\mathcal{E}(S))$ to the following $(d+m)$ -space:

$$P_\bullet \otimes_S M_\bullet \xrightarrow{\mathrm{id} \otimes \psi} P_\bullet \otimes_S (\mathrm{Hom}_S(M_\bullet, S)[m]) \xrightarrow{(-1)^{dm} \eta_M^x[m] \cdot \mathfrak{r}^{(m)}} \mathrm{Hom}_S(P_\bullet \otimes_S M_\bullet, S)[d+m]$$

(cf. [13], Remark 1.9, Theorem 2.6, and Theorem 2.9).

After these preparations we can start with the proof of Theorem 2.1.

2.4. A homomorphism $W^i(\mathcal{E}(R/Rt)) \rightarrow W^{i+1}(\mathcal{E}(R))$.

Denote $K(t)$ the (homological) Koszul complex of t . We consider $K(t)$ as a complex in $D^b(\mathcal{P}_R)$ living in degrees 1 and 0. On it there is the “standard” 1-form ℓ_1 :

$$\begin{array}{ccccccc} \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{\cdot t} & R & \longrightarrow & 0 & \longrightarrow & \dots & & K(t) \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \ell_1 \\ \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{\cdot (-t)} & R & \longrightarrow & 0 & \longrightarrow & \dots & & \mathrm{Hom}_R(K(t), R)[1], \\ & & & \text{deg 1} & & \text{deg 0} & & & & & & \end{array}$$

where we have identified $R = \mathrm{Hom}_R(R, R)$. This space $[K(t), \ell_1]$ is zero in the derived Witt group $W^1(R)$ because

$$\begin{array}{ccccccc} \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & 0 & \longrightarrow & \dots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{\cdot t} & R & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

is a Lagrangian. Therefore the following homomorphism

$$W^i(\mathcal{E}(R/Rt)) \longrightarrow W^{i+1}(\mathcal{E}(R)) \quad [M_\bullet, \varphi] \longmapsto [K(t), \ell_1] \star_l q^*([M_\bullet, \varphi])$$

is zero. Now a space is zero in a triangular Witt group if and only if it has a Lagrangian ([1], Theorem 3.5). We now give a Lagrangian of $[K(t), \ell_1] \star_l q^*([M_\bullet, \varphi])$. By [13], Lemma 1.15 there is an isometry $[K(t), \ell_1] \star_l q^*([M_\bullet, \varphi]) \simeq (-1)^i [K(t), \ell_1] \star_r q^*([M_\bullet, \varphi])$, where $\star_r$ is the right product. This right product is defined analogous to the left product: We make the exact functor $- \otimes q^*(M_\bullet) : D^b(\mathcal{P}_R) \longrightarrow D^b(\mathcal{E}(R))$ duality preserving using the form induced by φ on $q^*(M_\bullet)$ ([13], Sect. 1.3). From the theory of duality preserving functors it follows that $R \otimes_R q^*(M_\bullet)$ is a Lagrangian of $(-1)^i [K(t), \ell_1] \star_r q^*([M_\bullet, \varphi])$ and so of $[K(t), \ell_1] \star_l q^*([M_\bullet, \varphi])$ as well, since R (considered as a complex concentrated in degree 0) is a Lagrangian of $K(t)$. Note that due to the flatness of q we have $q^*(D^b(\mathcal{E}(R/Rt))^{(p)}) \subseteq D^b(\mathcal{E}(R))^{(p)}$. It follows that $[K(t), \ell_1] \star_l q^*([M_\bullet, \varphi])$ is not only zero in $W^{i+1}(\mathcal{E}(R))$, but also in $W^{i+1}(D^b(\mathcal{E}(R))^{(p)})$ if $M_\bullet \in D^b(\mathcal{E}(R/Rt))^{(p)}$. We state this as a Lemma.

Lemma 2.5. *Let $[M_\bullet, \varphi]$ be an i -space with $M_\bullet \in D^b(\mathcal{E}(R/Rt))^{(p)}$ for some $p \geq 0$. Then the space*

$$[K(t), \ell_1] \star_l q^*([M_\bullet, \varphi])$$

is zero in $W^{i+1}(D^b(\mathcal{E}(R))^{(p)})$.

2.6. Compatible injective resolutions.

We want to choose injective resolutions $\kappa : R \rightarrow I_\bullet$ and $\epsilon : R/Rt \rightarrow E_\bullet$ which are compatible in a sense which is explained by Lemma 2.8 below. Let for this

$$0 \longrightarrow R \xrightarrow{\kappa} I_0 \xrightarrow{d_0^I} I_{-1} \xrightarrow{d_{-1}^I} \dots \xrightarrow{d_{-(n-1)}^I} I_{-n} \longrightarrow 0 \quad n = \dim R$$

be a minimal injective resolution of R (cf. [8], Sect. 3.2), and

$$E_{-r} := I_{-(r+1)}^t := \{h \in I_{-(r+1)} \mid t \cdot h = 0\} \quad \text{and} \quad d_{-r}^E := d_{-(r+1)}^I|_{I_{-(r+1)}^t}.$$

It is shown in [11], Sect. 3.1, in particular Theorem 3.5, that $E_\bullet$ is a minimal injective resolution of $\ker d_0^E \simeq R/Rt$. We need a particular isomorphism $R/Rt \xrightarrow{\sim} \ker d_0^E$.

Lemma 2.7. a) *Multiplication by t : $I_0 \rightarrow I_0$, $y \mapsto t \cdot y$ is an isomorphism. We denote the inverse t^{-1} .*

b) *The kernel of the composition*

$$R \xrightarrow{\kappa} I_0 \xrightarrow{t^{-1}} I_0 \xrightarrow{d_0^I} I_{-1}$$

is the ideal Rt and the image is $\ker d_0^E$. In particular we have an isomorphism

$$\epsilon : R/Rt \xrightarrow{\sim} \ker d_0^E \quad \pi(r) \longmapsto d_0^I(t^{-1}\kappa(r)).$$

Proof. a) By definition $R \xrightarrow{\kappa} I_0$ is an injective hull of R and so in particular an essential extension. It follows that $\ker(I_0 \xrightarrow{t} I_0) \neq 0$ if and only if $\ker(R \xrightarrow{t} R) \neq 0$. But t is not a zero divisor and so multiplication by t is an injection. We get a decomposition $I_0 \simeq tI_0 \oplus H$ for some submodule $H \subset I_0$ because I_0 is injective. But this is only possible for $H = 0$ since over a commutative noetherian ring any injective module has a unique decomposition into indecomposable injective modules (cf. e.g. [8], Proposition 3.2.8).

b) It is obvious that the kernel of this composition is the ideal Rt and that the image lies in $\ker d_0^E$. Let $e \in \ker d_0^E \subseteq \ker d_{-1}^I$. Then there is an $e' \in I_0$, such that $d_0^I(e') = e$. Since $t \cdot e = 0$ we have $t \cdot e' \in \ker d_0^I = \text{Im } \kappa$, and so $t \cdot e' = \kappa(r)$ for some $r \in R$. $\square$

Therefore

$$0 \longrightarrow R/Rt \xrightarrow{\epsilon} E_0 \xrightarrow{d_0^E} E_{-1} \longrightarrow \dots \xrightarrow{d_{-(n-1)}^E} E_{-(n-1)} \longrightarrow 0$$

is a minimal injective resolution. Theorem 2.1 now follows from Lemma 2.5 and the following surprising

Lemma 2.8. *Let x be an i -space representing an element of $W^i(\mathcal{E}(R/Rt))$. Then there is an isometry*

$$F_{R,\kappa}([K(t), \ell_1] \star_l q^*(x)) \xrightarrow{\cong} \text{Tr}_{(R/Rt)/R}(F_{R/Rt,\epsilon}(x)).$$

Remark 2.9. If R/Rt is regular one can give a quite simple proof of this lemma using the projection formula [12], Theorem 5.2 (cf. the proof of *loc.cit.* Theorem 9.3).

Proof. For brevity we will use the following notation. Let $M \in \mathcal{M}_{R/Rt}$ and $N \in \mathcal{M}_R$. We set

$$M^R := R \otimes_{R/Rt} M \quad \text{and} \quad D_i(N) := \text{Hom}_R(N, I_{-i}).$$

Let $x = [M_\bullet, \varphi]$. We may assume that φ is a morphism of complexes and not a fraction. This is no restriction: If φ is given by a fraction $M_\bullet \xleftarrow{\alpha} H_\bullet \xrightarrow{\beta} \text{Hom}_{R/Rt}(M_\bullet, R/Rt)[i]$, then $[H_\bullet, \text{Hom}_{R/Rt}(\alpha, R/Rt)[i] \cdot \beta]$ is an i -space in $D^b(\mathcal{E}(R/Rt))$ which is isometric to $[M_\bullet, \varphi]$ via the isomorphism $\alpha : H_\bullet \xrightarrow{\cong} M_\bullet$. Hence assume φ is given in degree l by the following R/Rt -linear map:

$$\varphi_l : M_l \longrightarrow \text{Hom}_{R/Rt}(M_{-l+i}, R/Rt).$$

Then $q^*(x)$ is given in degree l by $\varphi_l^R : M_l^R \longrightarrow \text{Hom}_R(M_{-l+i}^R, R)$ which sends the element $r \otimes m \in M_l^R = R \otimes_{R/Rt} M_l$ to the following R -linear map

$$M_{-l+i}^R \longrightarrow R \quad r' \otimes m' \longmapsto (rr') \cdot q(\varphi_l(m)(m')).$$

It follows that $F_{R,\kappa}([K(t), \ell_1] \star_l q^*(x))$ is given in degree l by (cf. the discussion above):

$$\Theta_l := \begin{pmatrix} 0 & -\kappa\varphi_{l-1}^R \\ (-1)^i \kappa\varphi_l^R & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} :$$

$$M_l^R \oplus M_{l-1}^R \longrightarrow (D_0 M_{-l+i+1}^R \oplus D_0 M_{-l+i}^R) \oplus (D_1 M_{-l+i}^R \oplus D_1 M_{-l+i-1}^R) \oplus \dots$$

The space $\text{Tr}_{(R/Rt)/R}(F_{R/Rt,\epsilon}([M_\bullet, \varphi]))$ on the complex of R -modules $M_\bullet$ is given in degree l by (cf. [11], proof of Theorem 4.2; note that we get a $(-1)^i$ because we use the i -th shift of

the duality transformation, *cf. loc.cit.* proof of Theorem 2.7):

$$\Xi_l := \begin{pmatrix} 0 \\ (-1)^l \epsilon \varphi_l \\ 0 \\ \vdots \\ 0 \end{pmatrix} :$$

$$M_l \longrightarrow D_0 M_{-l+i+1} \oplus D_1 M_{-l+i} \oplus D_2 M_{-l+i-1} \oplus \dots$$

Because the morphism of rings q is flat the following sequence of complexes of R -modules

$$0 \longrightarrow M_\bullet^R \xrightarrow{t \otimes \text{id}} M_\bullet^R \xrightarrow{r \otimes m \mapsto \pi(r) \cdot m} M_\bullet \longrightarrow 0$$

is exact for any $M_\bullet \in D_{fg}^b(\mathcal{M}_{R/Rt})$. The tensor product $K(t) \otimes_R M_\bullet^R$ is isomorphic to the total complex of $M_\bullet^R \xrightarrow{t \otimes \text{id}} M_\bullet^R$. Hence the exact sequence of complexes above yields a quasi-isomorphism of complexes of R -modules $\alpha : K(t) \otimes_R M_\bullet^R \longrightarrow M_\bullet$ given in degree l by

$$(R \otimes_{R/Rt} M_l) \oplus (R \otimes_{R/Rt} M_{l-1}) \longrightarrow M_l \quad (r \otimes m, r' \otimes m') \longmapsto \pi(r) \cdot m.$$

We claim that α is the desired isometry. This follows from the following fact whose straightforward proof is left to the reader (use Lemma 2.7 and the sign conventions of section 1). The morphisms

$$s_l := \begin{pmatrix} t^{-1} \kappa \varphi_l^R & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} :$$

$$M_l^R \oplus M_{l-1}^R \longrightarrow (D_0 M_{-l+i}^R \oplus D_0 M_{-l+i-1}^R) \oplus (D_1 M_{-l+i-1}^R \oplus D_1 M_{-l+i-2}^R) \oplus \dots$$

define a homotopy between Θ and $\text{Hom}_R(\alpha, I_\bullet)[i+1] \cdot \Xi \cdot \alpha$. $\square$

3. THE PROOF OF THE GERSTEN CONJECTURE FOR WITT GROUPS VIA QUILLEN NORMALIZATION

3.1. The spectral sequence.

We now recall the definition of the Gersten-Witt spectral sequence. More details and proofs can be found in [5] (regular schemes) and [11] (Gorenstein schemes of finite Krull dimension). Let S be a Gorenstein ring with $\dim S = n < \infty$, and $I_\bullet$ a finite injective resolution of S . The canonical duality $\text{Hom}_S(-, I_\bullet)$ induces a duality on the full triangulated category D_S^p and so we have an exact sequence of triangulated categories with duality

$$0 \longrightarrow D_S^{p+1} \longrightarrow D_S^p \longrightarrow D_S^p / D_S^{p+1} \longrightarrow 0$$

for all $p \geq 0$. Applying Balmer's [1] localization theorem, we get a long exact sequence of triangular Witt groups

$$\dots \longrightarrow W^i(D_S^{p+1}) \longrightarrow W^i(D_S^p) \longrightarrow W^i(D_S^p/D_S^{p+1}) \xrightarrow{\partial} W^{i+1}(D_S^{p+1}) \longrightarrow$$

for all $p \geq 0$ and $i \in \mathbb{Z}$, and so (using Massey's method of exact couples) a convergent spectral sequence, the so-called Gersten Witt spectral sequence:

$$E_1^{p,q}(S) := W^{p+q}(D_S^p) \implies \tilde{W}^{p+q}(S).$$

It follows from the dévissage result [11], Corollary 3.11 that $E_1^{p,q}(S) = 0$ if $q \not\equiv 0 \pmod{4}$ and that there is a non-canonical isomorphism $E_1^{p,4s}(S) \simeq \bigoplus_{\text{ht } \wp=p} W(k(\wp))$ for all $s \in \mathbb{Z}$, where $k(\wp)$

is the residue field at $\wp \in \text{Spec } S$. The zero line of this spectral sequence composed with the natural homomorphism $W(S) = W^0(D_S^0) \longrightarrow W^0(D_S^0/D_S^1)$ is the augmented Gersten Witt complex $GW_\bullet(S)$:

$$0 \longrightarrow W(S) \longrightarrow \bigoplus_{\text{ht } \wp=0} W(k(\wp)) \longrightarrow \bigoplus_{\text{ht } \wp=1} W(k(\wp)) \longrightarrow \dots$$

3.2. The Gersten conjecture for Witt groups.

This conjecture asserts that $GW_\bullet(A)$ is exact for any regular local ring A . Our aim here is to give another proof of this conjecture for regular semilocal rings of finite type over a field.

Theorem 3.3. *Let A be a regular semilocal ring essentially of finite type over a field k . Then the augmented Gersten-Witt complex $GW_\bullet(A)$:*

$$\begin{aligned} 0 \longrightarrow W(A) \longrightarrow \bigoplus_{\text{ht } \wp=0} W(k(\wp)) \longrightarrow \bigoplus_{\text{ht } \wp=1} W(k(\wp)) \longrightarrow \dots \\ \dots \longrightarrow \bigoplus_{\text{ht } \wp=\dim A-1} W(k(\wp)) \longrightarrow \bigoplus_{\text{ht } \wp=\dim A} W(k(\wp)) \longrightarrow 0 \end{aligned}$$

is exact.

Remark 3.4. As already mentioned in the introduction this has been proved before. Namely, Balmer [3] has proved this if the field is infinite (of course one easily deduces from this the case of a finite field, cf. [4], Sect. 5). Since we can use Quillen's [15] variant of the normalization lemma our proof works at once for any field.

Proof. By well known arguments (e.g. [15], Proposition 5.6) it is enough to show that the natural homomorphism $W^i(D_A^{p+1}) \longrightarrow W^i(D_A^p)$ is zero for all $p \geq 0$ and all $i \in \mathbb{Z}$. Let $x \in W^i(D_A^{p+1})$. Since A is essentially of finite type over the field k , there exists a finitely generated k -algebra R which is smooth at a finite set of points S such that $A = U^{-1}R$, where $U = R \setminus \bigcup_{\wp \in S} \wp$. Hence there exists $g \notin \bigcup_{\wp \in S} \wp$ such that x is in the image of $W^i(D_{R_g}^{p+1}) \longrightarrow W^i(D_A^{p+1})$. Replacing R by $R_{g'}$ for some $g' \notin \bigcup_{\wp \in S} \wp$ we may assume that R is smooth over k

and that $x \in \text{Im}(W^i(D_R^{p+1}) \longrightarrow W^i(D_A^{p+1}))$. Hence the theorem follows from

Lemma 3.5. *Let R be an algebra essentially of finite type over a field k which is smooth over k and $S \subseteq \text{Spec } R$ a finite set. Then for any $x \in W^i(D_R^{p+1})$ there exists an element $f \notin \bigcup_{\wp \in S} \wp$,*

such that the homomorphism

$$W^i(D_R^{p+1}) \longrightarrow W^i(D_{R_f}^p)$$

is zero.

Proof. Let $x = [M_\bullet, \phi] \in W^i(D_R^{p+1})$. Then by definition $\text{codim}(\text{supp } M_\bullet) \geq p+1 \geq 1$, where

$$\text{supp } M_\bullet := \{ \wp \in \text{Spec } R \mid H_i(M_\bullet)_\wp \neq 0 \text{ for some } i \in \mathbb{Z} \}.$$

It follows from dévissage [11], Theorem 4.1 that there exists a regular $t \in R$, such that

$$x \in \text{Im}(\text{Tr}_{(R/Rt)/R} : W^{i-1}(D_{R/Rt}^p) \longrightarrow W^i(D_R^p))$$

Therefore it is enough to show that the composition

$$W^{i-1}(D_{R/Rt}^p) \xrightarrow{\text{Tr}_{(R/Rt)/R}} W^i(D_R^p) \longrightarrow W^i(D_{R_f}^p)$$

is zero for some $f \notin \bigcup_{\wp \in S} \wp$. Let ρ be the quotient map $R \longrightarrow R/Rt$. We now use the following normalization lemma of Quillen ([15], §7, Lemma 5.12 and the discussion thereafter).

Lemma 3.6. *There exists an $f \notin \bigcup_{\wp \in S} \wp$ and a k -algebra C , such that there is*

- a) *a finite morphism $v : R_f \longrightarrow C$,*
- b) *a surjective map $\pi : C \longrightarrow (R/Rt)_f$, such that*
 - i) *$\ker \pi = C \cdot t'$ with t' regular, and*
 - ii) *the following diagram commutes*

$$\begin{array}{ccc} C & \xleftarrow{v} & R_f \\ \pi \downarrow & \nearrow \rho_f & \\ (R/Rt)_f & & \end{array},$$

and

- c) *a smooth morphism $q : (R/Rt)_f \longrightarrow C$ essentially of finite type which splits π , i.e. $\pi \cdot q = \text{id}_{(R/Rt)_f}$.*

□

Since t is regular the ring R/Rt is Gorenstein. Now $q : (R/Rt)_f \longrightarrow C$ is smooth and so in particular flat. Moreover since smoothness is preserved by base change we know that all fibers of q are smooth over the appropriate residue fields (cf. e.g. [7], Proposition 2.4/8). It follows that q has regular fibers (cf. e.g. [7], Theorem 2.2/15) and hence C is Gorenstein by [8], Corollary 3.3.15. Therefore we can apply Theorem 2.1 to see that

$$\text{Tr}_{(R/Rt)_f/C} : W^{i-1}(D_{(R/Rt)_f}^p) \longrightarrow W^i(D_C^p)$$

is the zero map and hence $\text{Tr}_{(R/Rt)_f/R_f} : W^{i-1}(D_{(R/Rt)_f}^p) \longrightarrow W^i(D_{R_f}^p)$, too, since we have $\text{Tr}_{(R/Rt)_f/R_f} = \text{Tr}_{C/R_f} \cdot \text{Tr}_{(R/Rt)_f/C}$ by elementary properties of the transfer for coherent Witt groups stated in [12], Sect. 4. From this reference we also know that the following diagram

commutes:

$$\begin{array}{ccc}
 W^{i-1}(D_{R/Rt}^p) & \xrightarrow{\mathrm{Tr}_{(R/Rt)/R}} & W^i(D_R^p) \\
 \downarrow & & \downarrow \\
 W^{i-1}(D_{(R/Rt)_f}^p) & \xrightarrow{\mathrm{Tr}_{(R/Rt)_f/R_f}} & W^i(D_{R_f}^p),
 \end{array}$$

and so we are done. $\square$

Remark 3.7. Note that this gives also a new proof of an old result of Ojanguren [14] which asserts that for any regular local ring A essentially of finite type over a field k and with quotient field K the natural morphism $W(A) \longrightarrow W(K)$ is injective.

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