Finite dimensional objects in distinguished triangles

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Abstract

Let T be a $\mathbb{Q}$-localized closed symmetric monoidal triangulated category in the sense of [11]. We prove an additivity for evenly and oddly finite dimensional vertices of distinguished triangles in the category T (Theorem 1). In particular, the result holds in Voevodsky's motivic stable homotopy category MSH and triangulated category DM of motives over a field, both localized with rational numbers. As a corollary, we get motivic finite dimensionality for schemes of dimension one (Theorem 2). Analogous results have been independently obtained by C. Mazza in DM, see [23]. Theorem 3 extends Mazza's 2-of-3 property to arbitrary triangulated categories T.

1 Introduction

Let C be a $\mathbb{Q}$-linear, pseudoabelian and symmetric monoidal category with a product $\otimes$. Let n be a natural number and let Σ_n be the symmetric group of permutations of n elements. For any $X \in \mathsf{C}$ one can define its wedge $X^{[n)}$ and symmetric $X^{(n]}$ powers as images of the idempotents in $End_{\mathsf{C}}(X^{\otimes n})$ corresponding to the "vertical" and "horizontal" irreducible representations of Σ_n over $\mathbb{Q}$. These powers generalize usual wedge and symmetric powers of vector spaces over a field of characteristic zero. Then X is called to be evenly (oddly) finite dimensional if $X^{[n)}$ (correspondingly $X^{(n]}$) is a zero object for some natural n. In general, X is called to be finite dimensional if $X \cong X_+ \oplus X_-$ where X_+ is evenly and X_- is oddly finite dimensional.

The theory of finite dimensional Chow motives was originated by S.-I. Kimura in [19], and then considered in [7] and [8]. The abstract setting of the theory was developed in [2]. It turns out that finite dimensionality is closely connected with some important problems in algebraic geometry. In particular, if X is a smooth projective complex surface without non-trivial globally holomorphic 2-forms, then Bloch's conjecture on Albanese kernel (see [3], [14] and [39] for its formulation and motivations) for X holds if and only if the motive $M(X)$ of the surface X is finite dimensional, [8], Th. 7. For surfaces of general type (the unknown and hard part of Bloch's conjecture) it holds iff $M(X)$ is evenly finite dimensional, i.e. $M(X)^{[n)} = 0$ for some n, loc. cit.

Finite dimensional objects have good properties with respect to tensor products and quotients (in the sense of a pseudoabelian category). The motive of a smooth projective curve over a field is finite dimensional, [19], Prop. 5.10, 6.9 and Th. 4.2. Objects of a full tensor pseudoabelian subcategory in the category of Chow motives generated by motives of smooth projective curves are finite dimensional. For example, motives of abelian varieties and Fermat hypersurfaces are finite dimensional, and motivic finite dimensionality is a birational invariant for surfaces, [7].

To find further properties of finite dimensional objects and apply them to the geometry of varieties we consider Voevodsky's triangulated category DM of motives over a field k with coefficients in $\mathbb{Q}$, see [37], [35] and [24]. Being a triangulated category it provides a new tool – distinguished triangles – to study various properties of algebraic varieties. Let X be a smooth scheme of finite type over k, Z a closed smooth subscheme of codimension n in X, and let $U = X - Z$. Then we have Gyzin distinguished triangle

$$M(U) \longrightarrow M(X) \longrightarrow M(Z)(n)[2n] \longrightarrow M(U)[1]$$

in DM, [37]. If X is a surface, then Z is just a union of curves or points on X. Assuming motivic finite dimensionality for curves, one may ask: how the finite dimensionality of $M(X)$ could be connected with finite dimensionality of $M(U)$? Or, more generally: is a full subcategory, generated by finite dimensional objects, thick triangulated in DM? As it was pointed out to me by B. Kahn and C. Weibel, the answer is negative in an extremely general setting of a pseudoabelian $\mathbb{Q}$-linear symmetric monoidal triangulated category. Most probably, it is negative also in DM. However, we prove here an additivity property (Theorem 1) for evenly and oddly finite dimensional vertices in distinguished triangles in pseudoabelian $\mathbb{Q}$-localized closed symmetric monoidal triangulated categories in Hovey's sense, [11].

Our method is based on two general ideas. The first one is due to Uwe Jannsen and consists of the expectation of a nice filtration on wedge (correspondingly, symmetric) powers of an object Y, inserted into a dist. triangle $X \to Y \to Z \to \Sigma X$. It should be determined by powers of objects X and Y, similarly to the filtration for a short exact sequence of locally free sheaves of modules on a manifold, see [10], p. 127. Without further assumptions, however, there arise problems to show the required compatibilities in diagrams of distinguished triangles related to the above filtration. The second idea is then to use a triangulated category T, which is the homotopy category of a pointed simplicial model monoidal category C, with the monoidal structure on T induced by the monoidal structure on C , i.e. the localization functor $\mathsf{C} \to \mathsf{T} = Ho(\mathsf{C})$ is monoidal (from now on, for short of notation, a monoidal category is always symmetric and closed monoidal, and one can also use the word "tensor" instead of the word "monoidal"). Since T is a simplicial homotopy category, [11], so that we naturally assume that the shift functor Σ in T is a suspension $\Sigma X = X \wedge S^1$ by the simplicial circle S^1. It turns out that in so structured category T it is possible to control powers of vertices in distinguished triangles using cofiber sequences in the underlying category C. This second idea takes its roots in P. May's paper [22]. Our main result then is:

Theorem 1 *Let* T *be a pseudoabelian,* $\mathbb{Q}$*-linear, monoidal and triangulated category, which is, at the same time, the homotopy category of a pointed simplicial model monoidal category* C*. Assume, furthermore, that the monoidal structure on* T *is induced by the moniodal structure on* C *and that the shift functor in* T *is a simplicial suspension* $\Sigma X = X \wedge S^1$*. Then, for any distinguished triangle*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

in T*, if* X *and* Z *are evenly (oddly) finite dimensional, it follows that* Y *is also evenly (oddly) finite dimensional. Equivalently: if* X *is evenly (oddly) finite dimensional and* Y *is oddly (evenly) finite dimensional, it follows that* Z *is oddly (evenly) finite dimensional.*

$\mathbb{Q}$-localized Voevodsky's motivic stable homotopy category MSH, see [36], [38], [25] and [25], is an important example of the above category T. The underlying category C is the category of motivic symmetric spectra, predicted by Voevodsky in [36] and constructed by Jardine in [15]. The recent result due to Morel [27] (see also [40]) asserts that, if $char(k) = 0$ and -1 is a sum of squares in k, then there exists an exact and monoidal equivalence between MSH and DM over the ground field k. Applying this and Theorem 1 to schemes of dimension one we obtain the following generalization of Kimura's result:

Theorem 2 *Let* k *be a field of characteristic zero and let* X *be a scheme of dimension one, separated and of finite type over* k*. Then its motive* $M(X)$*, considered in Voevodsky's category* DM*, is finite dimensional.*

After publishing of the first version of this paper at Internet I was informed that the same result as in Theorem 1 has been independently obtained by C. Mazza in the category DM, see [23], and he also deduced Theorem 2. Mazza works with a more flexible notion of finiteness based on Schur endofunctors in tensor categories introduced by P. Deligne in [4]. Here we show that our method to prove Theorem 1 works well also with Schur finiteness and provides the generalization of Prop. 5.3 in [23]:

Theorem 3 *Let* T *be a triangulated category satisfying the assumtions of Theorem 1. Then Schur finiteness has 2-of-3 property for distinguished triangles in* T*.*

The paper is organized as follows. For a convenience of the reader, in the second section we recall definitions and basic results on finite dimensional objects, Hovey's triangulated categories (after [11] and [22]) and mention that MSH is an example of such a category. In Section 3 we develop a homotopy technique to deal with finite dimensionality of vertices in distinguished triangles and show the existence of the above filtration on $Y^{[n)}$ with graded pieces $Z^{[p)} \wedge X^{[q)}$ where $p+q=n$ (and the same for symmetric powers), and then prove Theorem 1. In Section 4 we prove Theorem 2. The last section is devoted to the 2-of-3 property for Schur finite vertices in triangles in Hovey's categories. We also show there how to apply this property to motives of threefolds.

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2 Preliminary results

2.1 *Basics on finite dimensional objects*

Let C be a monoidal $\mathbb{Q}$-linear and pseudoabelian category with a monoidal product $\otimes$. For any natural number n and any object X in C by $X^{(n)}$ we denote the n-fold product $X^{\otimes n}$ in C. If $f : X \to Y$ is a morphism in C then let $f^{(n)} : X^{(n)} \to X^{(n)}$ be the n-fold product of f.

Let n be a natural number and let Σ_n be the group of permutations of a finite set consisting of n elements. Let also $A = \mathbb{Q}\Sigma_n$ be the group algebra (over $\mathbb{Q}$) of the group Σ_n. A classical result asserts that the set of all irreducible representations of Σ_n over $\mathbb{Q}$ is in one-to-one correspondence with the set P_n of all partitions λ of the integer n, and that there exists a finite collection $\{e_\lambda\}$ of pairwise orthogonal idempotents in A, such that $\sum_{\lambda \in P_n} e_\lambda = 1_A$, and each e_λ induces the corresponding irreducible representation of Σ_n up to an isomorphism.

For any n and $X \in \mathsf{C}$ let $\Gamma : A \to End_{\mathsf{C}}(X^{(n)})$ be a homomorphism sending any $\sigma \in \Sigma_n$ into its "graph", i.e. the endomorphism of $X^{(n)}$ permuting factors according to σ and the commutativity and associativity constrains for the product $\otimes$ (see [5] or [11]). For each $\lambda \in P_n$ let d_λ be the graph of the idempotent e_λ. Since $\sum_{\lambda \in P_n} e_\lambda = 1$ in A, it follows that $\sum_{\lambda \in P_n} d_\lambda = 1$ in $End_{\mathsf{C}}(X^{(n)})$. The category C being pseudoabelian, it follows that $X^{(n)}$ is a direct sum of images $im(d_\lambda)$ of the idempotents d_λ.

Let n be a natural number. Let d_n^+ be the projector $d_{(\lambda)}$ when λ is the partition $(1, \dots, 1)$ of n, and let d_n^- be the projector $d_{(\lambda)}$ when λ is the partition (n) of n. In other words,

$$d_n^+ = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} sgn(\sigma)\Gamma_\sigma \qquad \text{and} \qquad d_n^- = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \Gamma_\sigma \ .$$

Then we define wedge and symmetric powers of X by the formulas $X^{[n)} = im(d_n^+)$ and $X^{(n]} = im(d_n^-)$ respectively. If X is a vector space over a field of characteristic zero, then $X^{[n)}$ is a usual wedge and $X^{(n]}$ is a usual symmetric powers of X, [9], §B.2.

Then we say that X is evenly (oddly) finite dimensional if $X^{[n)} = 0$ (correspondingly, $X_-^{(n]} = 0$) for some natural number n. In general, X is said to be finite dimensional if it can be decomposed into a direct sum $X = X_+ \oplus X_-$, such that X_+ is evenly and X_- is oddly finite dimensional.

The dimension $dim(X)$, in the sense of S.-I. Kimura, of an evenly (oddly) finite dimensional object X is, by definition, the smallest natural number n, such that $X^{[n)} \neq 0$ (correspondingly, $X^{(n]} \neq 0$). For example, the motive $\mathbb{Q}(d)[2d]$ is evenly one dimensional in DM for any $d \in \mathbb{Z}$. In the mixed case: $dim(X) = dim(X_+) + dim(X_-)$. It's clear that $dim(X \oplus Y) = dim(X) + dim(Y)$, see [2].

The following result involves just properties of pseudoabelian monoidal categories and the theory of representations of symmetric groups:

Proposition 4 *The tensor product of two finite dimensional objects in* C *is finite dimensional and a subobject (i.e. a kernel of an idempotent) of a finite dimensional object is also finite dimensional. Moreover, if* X *and* Y *are two purely finite dimensional objects with the same parity, then* $X \otimes Y$ *is evenly finite dimensional, and if* X *and* Y *have different parity, then* $X \otimes Y$ *is oddly finite dimensional.*

Proof. [19], Prop. 5.10 and Cor. 5.11, and [2], §9. □

Remark 5 Geometrically it means that the motive of a fibered product of two varieties with finite dimensional motives is finite dimensional, and, if $X \to Y$ is a surjective regular map, then $dim(M(X)) < \infty$ implies $dim(M(Y)) < \infty$.

Proposition 6 *Let* X *be a finite dimensional object in* C *and let* $X \cong X_+ \oplus X_- \cong Y_+ \oplus Y_-$ *be two decompositions of* X *into its even and odd parts. Then it follows that* $X_+ \cong Y_+$ *and* $X_- \cong Y_-$*.*

Proof. [19], Prop. 6.3 and [2], Prop. 9.1.10. □

Theorem 7 *The motive of a smooth projective curve* X *over a field is finite dimensional. If* $M(X) = \mathbb{Q} \oplus M^1(X) \oplus \mathbb{Q}(1)[2]$ *is the decomposition (see [29] or [33]) of* $M(X)$ *given by a* k*-rational point on* X*, then* $M^1(X)$ *is oddly finite dimensional of dimension* $2g$*, where* g *is the genus of* X*.*

Proof. See [32] and [19]. The proof essentially involves three items: (i) a big enough symmetric power of a smooth projective curve is a projective bundle over its Jacobian variety by Riemann-Roch theorem; (ii) the description of $c_1(\mathcal{O}(1))$ on such bundles from [34]; (iii) the projective bundle theorem for Chow groups. □

Remark 8 Let CHM be the category of Chow motives over k with coefficients in $\mathbb{Q}$, see [21], [13], [29] and [33] for the definition, and let CHM_1 be a full tensor pseudoabelian subcategory in CHM generated by motives of smooth projective curves. Then Theorem 7 joint with Proposition 4 follow that objects in CHM_1 are finite dimensional. For example, motives of abelian varieties and Fermat hypersurfaces are finite dimensional. Moreover, motivic finite dimensionality is a birational invariant for surfaces, [7].

Remark 9 Motivic finite dimensionality controls "phantom motives": if M is a Chow-motive and $\dim(M) < \infty$, then $H^*(M) = 0$ implies $M = 0$, see [19], Propositions 7.2 and 7.5, as well as Ex. 9.2.4 in [2].

2.2 *Homotopy category of a pointed model monoidal category*

Let C be a pointed model and monoidal category with a monoidal product $\wedge : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ and the unite object S. The coproduct of two objects X and Y in C will be denoted by $X \vee Y$. Let $f : X \to Y$ and $f' : X' \to Y'$ be two maps in C. Consider the coproduct $X \wedge Y' \coprod_{X \wedge X'} Y \wedge X'$, that is colimit of the diagram

$$\begin{array}{ccc} X \wedge X' & \xrightarrow{f \wedge 1} & Y \wedge X' \\ \downarrow{\scriptstyle 1 \wedge f'} & & \\ X \wedge Y' & & \end{array}$$

Then so called pushout smash product of f and f' is a unique map

$$f \square f' : X \wedge Y' \coprod_{X \wedge X'} Y \wedge X' \longrightarrow Y \wedge Y'$$

determined by the above colimit. The connection between model and monoidal structures can be then expressed by the following axioms, [11], 4.2:

- If f and f' are cofibrations then $f \square f'$ is also a cofibration. If, in addition, one of two maps f and f' is a weak equivalence, then so is $f \square f'$.

- If $q : QS \to S$ is a cofibrant replacement for the unite object S, then the maps $q \wedge 1 : QS \wedge X \to S \wedge X$ and $1 \wedge q : X \wedge QS \to X \wedge S$ are weak equivalences for all cofibrant X.

Now we need a series of sophisticated definitions. Let C be a monoidal model category. A model category D is called to be a C-model category if (i) it is a right C-module, (ii) the action $\otimes : \mathsf{D} \times \mathsf{C} \to \mathsf{D}$ is a Quillen bifunctor, [11], p. 107, and (iii) for an cofibrant $X \in Ob(\mathsf{D})$ and for cofibrant replacement $q : QS \to S$ the map $1 \otimes q : X \otimes QS \to X \otimes S$ is a weak equivalence, see [11], p. 114.

Let Δ be the category of ordered finite sets and order preserving maps between them. Let S be the category of sets and let $\mathsf{SS} = \Delta^{op}\mathsf{S}$ be the category of simplicial sets. We consider SS with a standard model and monoidal structures on it. Correspondingly, one has a pointed model category of simplicial sets SS_*. Then an SS-model category is called a simplicial model category (compare with the equivalent definition from Ch. II, §2 and 3 of [16]). A pointed simplicial model category can be then defined as a SS_*-model category, see Prop. 4.2.19 in [11].

The homotopy category $Ho(\mathsf{C})$ of any simplicial model category C (pointed simplicial model category C) is a closed SS-module (SS_*-module). Moreover, this fact can be generalized to an arbitrary model (pointed model) category C, [11], Ch.5. If C is monoidal (pointed monoidal), then the monoidal structure on C induces a monoidal structure on $Ho(\mathsf{C})$, [11], Th.4.3.2, and $Ho(\mathsf{C})$ is an algebra over $Ho(\mathsf{SS})$ (over $Ho(\mathsf{SS}_*)$), loc. cit., Ch.5.

Let C be a pointed simplicial model monoidal category. Since it is pointed, we denote its product by $\wedge$. The category $Ho(\mathsf{C})$, being an algebra over $Ho(\mathsf{SS}_*)$, admits smash products of objects $X \in \mathsf{C}$ with pointed simplicial sets, in particular, with the simplicial interval I and simplicial circle S^1. Therefore, for any X we have a natural notion of a suspension $\Sigma X = X \wedge S^1$ by the simplicial circle S^1 and a cone $CX = X \wedge I$. For any cofibration $f : X \to Y$ between cofibrant objects, a mapping cone $C(f)$ is defined as a colimit of the diagram

$$\begin{array}{ccc} X & \longrightarrow & CX \\ {\scriptstyle f}\downarrow & & \\ Y & & \end{array}$$

Assume that Σ is a Quillen equivalence with an adjoint loop functor Ω. Then $Ho(\mathsf{C})$ is naturally a triangulated category with an endofunctor Σ, [11], 6.5, 6.6, 7.1, and the triangulated structure is compatible with the monoidal structure, see [24], A8 for axioms of compatibility. To be more precise, $Ho(\mathsf{C})$ is a triangulated category in Hovey's sense, i.e. it is a pre-triangulated category, [11], 6.5, and the suspension functor Σ is an autoequivalence on $Ho(\mathsf{C})$. It can be shown that any triangulated category in Hovey's sense is classically triangulated, [11], Prop. 7.1.6.

The category $\mathsf{T} = Ho(\mathsf{C})$ has at least two important advantages: strong compatibility of monoidal and triangulated structures in T, [22], §4, and the possibility to describe distinguished triangles in terms of cofiber sequences in C, [11] and [22], pp. 18-19. If $f : X \to Y$ is a map in the category T, then, using cofibre replacement in C, one can assume that f is a cofibration between cofibrant objects X and Y. Then we have a cofiber distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow C(f) \longrightarrow \Sigma X$$

in T. If $Z = Y/X$ is a quotient object, i.e. colimit of the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \\ * & & \end{array}$$

where $*$ is a initial-terminal object in the pointed category C, then Z is weakly equivalent to $C(f)$, Lemma 5.3 in [22]. The suspension ΣX, being a cogroup object, coacts on Z, see [11], Th. 6.2.1. In particular, one can define a standard boundary map $\partial : Z \to \Sigma X$ as a composition of the coaction $Z \to Z \coprod \Sigma X$ with the evident map $Z \coprod \Sigma X \to \Sigma X$, [11], 6.2. Then we have a cofiber distinguished triangle in the form

$$X \xrightarrow{f} Y \longrightarrow Z \xrightarrow{\partial} \Sigma X .$$

Remark 10 Any distinguished triangle in $\mathsf{T} = Ho(\mathsf{C})$ is isomorphic to a cofiber dist. triangle of the above type, see [11], 6.2-7.1, as well as [22], Section 5.

Lemma 11 *Let $f : X \to Y$ and $f' : X' \to Y'$ be two cofibrations of cofibrant objects in C with cofibers Z and Z' correspondingly. Let $a : X \to X'$ and $b : Y \to Y'$ be maps, such that $bf = f'a$, and let $c : Z \to Z'$ be an induced map on cofibers. Then c is equivariant in T with respect to the cogroup homomorphism Σa, so that we have the corresponding map of distinguished triangles in T:*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \xrightarrow{\partial} & \Sigma X \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow \Sigma a \\ X' & \xrightarrow{f'} & Y' & \longrightarrow & Z' & \xrightarrow{\partial} & \Sigma X' \end{array}$$

Proof. See [11], Prop. 6.2.5. □

2.3 *Motivic stable homotopy category*

Let's consider now an important particular case of the above abstract situation. Let k be a field and let Sm be the category of all smooth schemes, separated and of finite type over k (further called simply smooth schemes). Let Spc be the category of simplicial sheaves in Nisnevich topology on Sm, called spaces, and let Spc_* be the corresponding pointed category with an evident notion of a terminal-initial object $*$, [36]. The model structure on Spc (and, therefore, on Spc_*) depends on the Nisnevich topology on the category Sm, [17], [15], [36], [25] and [26]. Note also that the homotopy categories of simplicial sheaves and presheaves are canonically isomorphic via the forgetfull functor, [15], Th. 1.2 (2), p. 453.

The composition of the Yoneda embedding with the functor from (pre)sheaves into simplicial (pre)sheaves allows to identify a smooth scheme with the corresponding representable simplicial (pre)sheaf, [36], [25] and [26]. Then, since Spc is cocomplete, one can consider colimits in Spc, for example, quotiens, contractions, glueings, etc. In particular, let

$$T = \mathbb{A}^1/(\mathbb{A}^1 - 0)$$

be the quotient of $\mathbb{A}^1$ by $\mathbb{A}^1 - 0$. Here $\mathbb{A}^1$ and $\mathbb{A}^1 - 0$ are pointed by 1. In the homotopy category $Ho(\mathsf{Spc}_*)$ one has

$$T \cong \mathbb{P}^1 \cong S^1 \wedge (\mathbb{A}^1 - 0) ,$$

where S^1 is a simplicial circle, coming from simplicial sets, and $\mathbb{P}^1$ is pointed at ∞.

Now a T-spectrum X (or a motivic spectrum) is a sequence of objects $X^n \in \mathsf{Spc}_*$ and bonding maps $T \wedge X^n \to X^{n+1}$ for each n. A map of spectra $f : X \to Y$ consists of maps $f^n : X^n \to Y^n$ commuting with bonding maps. A motivic symmetric spectrum X is a motivic spectrum X with an extra (left) action of the symmetric group Σ_n on each X^n and with $\Sigma_m \times \Sigma_n$-equivariant compositions of bonding maps $T^m \wedge X^n \to X^{m+n}$, where $T^m = T^{(m)}$. A map of motivic symmetric spectra should be equivariant for the symmetric group action. All of these can be found in [15], Section 4, and should be compared with the theory in [12].

Let Spc_T^Σ be the category of motivic symmetric spectra. In [15] Jardine described a structure of a pointed simplicial model monoidal category on Spc_T^Σ. As we have seen in Section 2.2, there exists a structure of a triangulated monoidal category on the corresponding homotopy category $Ho(\mathsf{Spc}_T^\Sigma)$, such that its shift functor is the simplicial suspension and the localization functor is monoidal. Th. 4.30 in [15] asserts that $Ho(\mathsf{Spc}_T^\Sigma)$ is the desired motivic stable homotopy category MSH on the Nisnevich site on Sm, [36], [25], [26]. So, the category MSH is a typical example of a category satisfying the assumptions of Theorem 1.

It's important to connect now the motivic stable homotopy category MSH with the triangulated category DM of motives over the ground field k, built in [37]. One can construct another monoidal and triangulated category $\tilde{\mathsf{DM}}$, see [26], Sections 4.3 - 5.2, of, so called, $\mathbb{P}^1$-motivic unbounded complexes (or $\mathbb{P}^1$-stabilized category of unbounded complexes) of abelian sheaves on the Nisnevich site. Taking free presheaves of abelian groups, generated by presheaves of sets, and associating normalized chain complexes to free generated simplicial presheaves, one can construct a monoidal triangulated functor

$$C_* : \mathsf{MSH} \longrightarrow \tilde{\mathsf{DM}} ,$$

which is a monoidal triangulated equivalence after the localization by $\mathbb{Q}$, see [26], p. 32. Moreover, there is a canonical triangulated and monoidal functor $\Phi : \tilde{\mathsf{DM}} \to \mathsf{DM}$, which is not an equivalence in general.

Theorem 12 *Let k be a field, such that $char(k) = 0$ and -1 is a sum of squares in k. Then*

$$\Phi C_* : \mathsf{MSH} \longrightarrow \mathsf{DM}$$

is a monoidal and triangulated equivalence of $\mathbb{Q}$-localized categories.

Proof. See [26], [27] + [28], and compare also with the material from [40]. □

Remark 13 If $F : \mathsf{T} \to \mathsf{T}'$ is an additive and monoidal equivalence of $\mathbb{Q}$-linear pseudoabelian monoidal categories, then $X \in Ob(\mathsf{T})$ is (evenly, oddly) finite dimensional in T iff $FX \in Ob(\mathsf{T}')$ is (evenly, oddly) finite dimensional in T'. Thus we see now that Theorem 1 implies the same additivity for finite dimensional objects in distinguished triangles in the category DM, constructed over an arbitrary field k of characteristic zero. Note that the condition -1 to be a sum of squares is not important for our goals because we consider categories localized by rational numbers.

Remark 14 Let X be a smooth projective complex surface with $p_g = 0$. Let $M(X)$ be its motive in DM and let $E^{\Sigma}(X)$ be its symmetric spectrum in MSH. It is known, [8], Theorem 27, that Bloch's conjecture holds for X if and only if $M(X)$ is finite dimensional in DM. Applying Theorem 12, we have that Bloch's conjecture holds for X if and only if $E^{\Sigma}(X)$ is finite dimensional in MSH.

3 Finite dimensional objects in dist. triangles

3.1 *Cofiber sequences and combinatorics of powers*

Let T be a monoidal and triangulated category reinforced by a pointed simplicial model and monoidal category C in the sense of Section 2.2 (but not necessary $\mathbb{Q}$-linear). The monoidal product in C we denote by $\wedge$ and the coproduct by $\vee$. The monoidal product in T will be denoted by $\otimes$ and the direct sum by the symbol $\oplus$. The canonical (localization) functor $\mathsf{C} \to \mathsf{T}$ is monoidal, i.e. it carries an object $X \wedge Y$ in C into the object $X \otimes Y$ in T. The endofunctor $X \mapsto \Sigma X = X \wedge S^1$ is a suspension by a simplicial circle S^1. Let's also recall that "monoidal" always means "closed and symmetric monoidal". In particular, for any fibrant $X \in \mathsf{C}$ both functors $- \wedge X$ and $X \wedge -$ preserve colimits in C.

Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\partial} \Sigma X$$

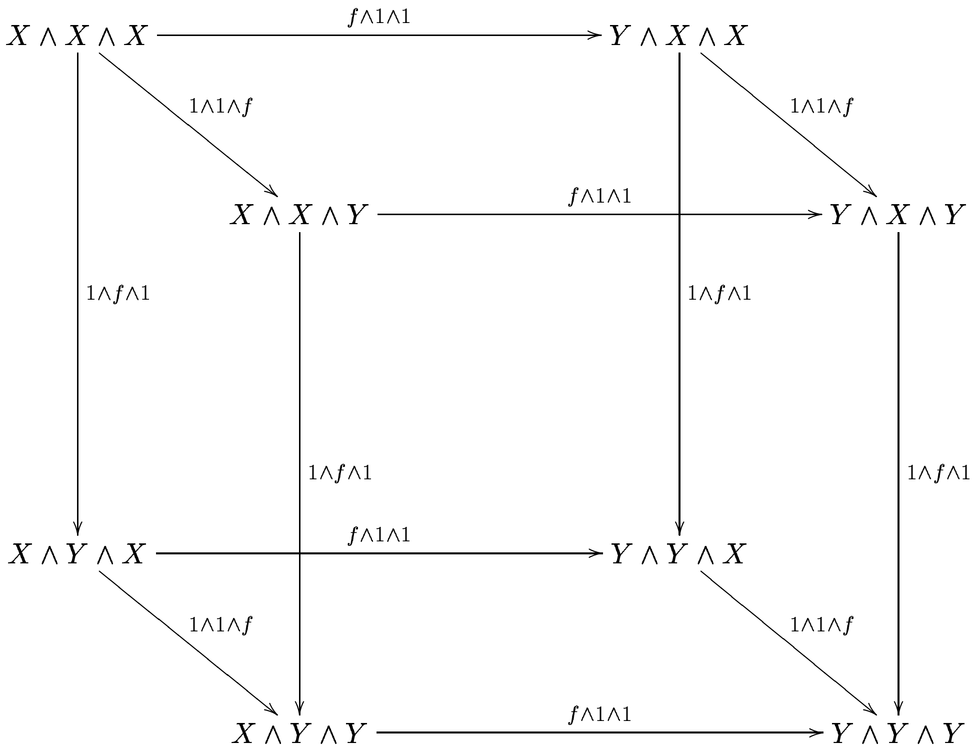
be a distinguished triangle in T. Our goal is to study wedge and symmetric powers of vertices in the triangle XYZ. Without loss of generality, applying cofibrant replacement, we assume that both X and Y are cofibrant and the above distinguished triangle is a cofibration triangle, so that $Z = Y/X$.

Let m be a natural number and let V_m be the collection of all ordered sets $\underline{v} = (v_1, \dots, v_m)$, such that $v_j \in \{0, 1\}$ for each $1 \leq j \leq m$. In particular, we have vectors $\underline{0} = (0, \dots, 0)$ and $\underline{1} = (1, \dots, 1)$. Elements of V_m can be considered as vertices of the unit cube K_m in $\mathbb{R}^m$. Let $D_{\underline{v}}$ be a smash-product $A_1 \wedge \dots \wedge A_m$ in C, such that $A_j = X$ if $v_j = 0$ and $A_j = Y$ if $v_j = 1$. Evidently, $D_{\underline{0}} = X^{(m)}$ and $D_{\underline{1}} = Y^{(m)}$. Place $D_{\underline{v}}$ on the vertex $\underline{v}$ and interpret morphisms between vertices induced by the cofibration $f : X \to Y$ as oriented edges of the cube K_m. Then K_m can be considered as a commutative diagram involving all mixed powers of X and Y of degree m. For example, K_2 is the commutative diagram

$$\begin{array}{ccc} X \wedge X & \xrightarrow{1 \wedge f} & X \wedge Y \\ \downarrow{\scriptstyle f \wedge 1} & & \downarrow{\scriptstyle f \wedge 1} \\ Y \wedge X & \xrightarrow{1 \wedge f} & Y \wedge Y \end{array}$$

where the objects $X \wedge Y$ and $Y \wedge X$ correspond to vertices $(0, 1)$ and $(1, 0)$ respectively.

The commutative diagram K_3 looks as follows:



Here the objects $X \wedge Y \wedge X$, $X \wedge X \wedge Y$ and $Y \wedge X \wedge X$ correspond to vertices $(0,1,0)$, $(0,0,1)$ and $(1,0,0)$ respectively. The objects $X \wedge Y \wedge Y$, $Y \wedge X \wedge Y$ and $Y \wedge Y \wedge X$ correspond to vertices $(0,1,1)$, $(1,0,1)$ and $(1,1,0)$ respectively.

For any $0 \leq i \leq m$ let V_m^i be a subset in V_m consisting of all the vertices $\underline{v}$, such that the the number of units in $\underline{v}$ is less or equal to i. Let K_m^i be a commutative subdiagram in K_m generated by vertices from V_m^i. We will show how this filtration

$$K_m^0 \subset K_m^1 \subset \ldots \subset K_m^m = K_m$$

leads us to the desired filtration on wedge and symmetric m-powers of the object Y.

Let

$$D_m^i = colim\ K_m^i$$

be a colimit of the diagram K_m^i in the category C. Then for each i there exists a universal map

$$w_{m,i} : D_m^i \longrightarrow D_m^{i+1} \ ,$$

induced on colimits in C. Clearly, $D^0 = X^{(m)}$ and $D^m = Y^{(m)}$.

For each $\underline{v} \in V_m$ let $E_{\underline{v}}$ be a product $A_1 \otimes \ldots \otimes A_m$ in T, such that $A_j = X$ if $v_j = 0$ and $A_j = Z$ if $v_j = 1$. For any i let $U_m^i = V_m^i - V_m^{i-1}$ – the set of vertices in K_m containing exactly i units. Then let E_m^i be a direct sum in the category T of all the objects $E_{\underline{v}}$ where $\underline{v}$ runs U_m^i.

Proposition 15 *The morphism $w_{m,i}$ is a cofibration for any i. Moreover, the corresponding quotient object D^{i+1}_m / D^i_m is canonically isomorphic to E^{i+1}_m, so that we have a cofibration distinguished triangle*

$$D^i_m \stackrel{w_{m,i}}{\longrightarrow} D^{i+1}_m \longrightarrow E^{i+1}_m \longrightarrow \Sigma D^i_m$$

in the category T.

Proof. Since C is a closed monoidal model category, it follows that, for any cofibrant object B in C, both endofunctors $- \wedge B$ and $B \wedge -$ on C are Quillen functors, see [11], 4.2. In particular, they preserve cofibrations. By our assumtion, X and Y are cofibrant, so that all edges in the commutative diagram K_m are cofibrations. Then w^i_m's are cofibrations as well. This can be checked in the same way like that cofibrations are closed under pushouts, see, for instance, [11], Cor. 1.1.11. To be more precise, let

$$\begin{array}{ccc} D^i_m & \longrightarrow & A \\ {\scriptstyle w_{m,i}}\downarrow & & \downarrow{\scriptstyle t} \\ D^{i+1}_m & \longrightarrow & B \end{array}$$

be a commutative diagram in C where t is a trivial fibration. We have to show that there exists a map $h : D^{i+1}_m \to A$ preserving the commutativity of the diagram.

Let $\underline{u}$ be any vertex from U^{i+1}_m. Let $\underline{v}$ be a vertex from U^i_m, such that there is an edge $w_{\underline{v},\underline{u}} : D_{\underline{v}} \to D_{\underline{u}}$ in the commutative diagram K_m. Then we have a commutative square

$$\begin{array}{ccc} D_{\underline{v}} & \longrightarrow & D^i_m \\ {\scriptstyle w_{\underline{v},\underline{u}}}\downarrow & & \downarrow{\scriptstyle w_{m,i}} \\ D_{\underline{u}} & \longrightarrow & D^{i+1}_m \end{array}$$

in the category C. Composing the last two commutative squares we have that, by left lifting property for the cofibrations $w_{\underline{v},\underline{u}}$, there exists a map $h_{\underline{u}}$ making the diagram

$$\begin{array}{ccc} D_{\underline{v}} & \longrightarrow & A \\ {\scriptstyle w_{\underline{v},\underline{u}}}\downarrow & \nearrow{\scriptstyle h_{\underline{u}}} & \downarrow{\scriptstyle t} \\ D_{\underline{u}} & \longrightarrow & B \end{array}$$

to be commutative. It is not hard to check that the system of morphisms $\{h_{\underline{u}}\}_{\underline{u} \in U^{i+1}_m}$ gives rise to a cone over the diagram K^{i+1}_m, so that we have a canonical map h :

$D_m^{i+1} \to A$. Using the uniqueness of the universal map for colimits one can easily show that this h is a desired map.

To show the second assertion of the lemma let's recall that C is a pointed category, so that, for any two objects A and B in C, their coproduct $A \vee B$ is a colimit of the diagram

$$\begin{array}{ccc} * & \longrightarrow & B \\ \big\downarrow & & \\ A & & \end{array}$$

If $A \to B$ is a cofibration, then the quotient object A/B may be thought as a contraction of the subobject A in B into the fixed point $* \to B$ in B. Then it is not hard to see that the quotient object D_m^{i+1}/D_m^i is exactly the coproduct $E_{\underline{v}}$ in C, where $\underline{v}$ runs U_m^i (for the proof one has just to represent colimit diagrams carefully). Therefore, as an object it T, it is equal to the direct sum E_m^i. Since $w_{m,i}$ is a cofibration, we obtain the desired cofibration distinguished triangle from the statment of the lemma. □

3.2 *Mixed idempotents and their images*

Let $\underline{v} \in V_m$. Each element $\sigma \in \Sigma_m$ permutes the coordinates $(v_1, \dots, v_m)$ of $\underline{v}$. We say that the vertex $\underline{v}$ is of an inner type with respect to σ if σ carries zeros to zeros and units to units, and we say that $\underline{v}$ is of an outer type otherwise.

If $\underline{v}$ is of the inner type with respect to σ, then it induces an automorphism

$$\Gamma_{\sigma,\underline{v}} : D_{\underline{v}} \longrightarrow D_{\underline{v}}$$

using commutativity and associativity constrains in C. We may say again that $\Gamma_{\sigma,\underline{v}}$ is a graph of the permutation σ on the object $D_{\underline{v}} \in \mathsf{C}$. If $\underline{v}$ is of the outer type with respect to σ, then we still can define a graph of σ as a uniquely defined isomorphism

$$\Gamma_{\sigma,\underline{v}} : D_{\underline{v}} \longrightarrow D_{\sigma(\underline{v})} ,$$

again induced by σ using commutativity and associativity in C.

Let

$$\Gamma_{\sigma,i} : D_m^i \longrightarrow D_m^i$$

be a morphism on colimits induced by all maps $\Gamma_{\sigma,\underline{v}}$ with fixed σ. Then, for any $i \in \{1, \dots, m\}$, we have the commutative diagram

$$\begin{array}{ccc} D_m^i & \xrightarrow{w_{m,i}} & D_m^{i+1} \\ \big\downarrow{\scriptstyle \Gamma_{\sigma,i}} & & \big\downarrow{\scriptstyle \Gamma_{\sigma,i+1}} \\ D_m^i & \xrightarrow{w_{m,i}} & D_m^{i+1} \end{array}$$

In the same fashion, any permutation σ induces a graph of σ on E_m^i:

$$\Xi_{\sigma,i} : E_m^i \longrightarrow E_m^i \ .$$

Then, for any $i \in \{1, \dots, m\}$ we have the morphism of cofibered sequences

$$\begin{array}{ccccc}
D_m^i & \xrightarrow{w_{m,i}} & D_m^{i+1} & \longrightarrow & E_m^{i+1} \\
\downarrow{\scriptstyle \Gamma_{\sigma,i}} & & \downarrow{\scriptstyle \Gamma_{\sigma,i+1}} & & \downarrow{\scriptstyle \Xi_{\sigma,i+1}} \\
D_m^i & \xrightarrow{w_{m,i}} & D_m^{i+1} & \longrightarrow & E_m^{i+1}
\end{array}$$

Applying Lemma 11, we claim that for any i the triple $(\Gamma_{\sigma,i}, \Gamma_{\sigma,i+1}, \Xi_{\sigma,i+1})$ is, in fact, an automorphism of the distinguished triangle from Prop. 15.

From now we assume that T is $\mathbb{Q}$-linear. Let

$$d_{m,i}^{+} = \frac{1}{m!} \sum_{\sigma \in S_m} sgn(\sigma)\Gamma_{\sigma,i} \ ,$$

$$d_{m,i}^{-} = \frac{1}{m!} \sum_{\sigma \in S_m} \Gamma_{\sigma,i} \ ,$$

$$e_{m,i}^{+} = \frac{1}{m!} \sum_{\sigma \in S_m} sgn(\sigma)\Xi_{\sigma,i}$$

and

$$e_{m,i}^{-} = \frac{1}{m!} \sum_{\sigma \in S_m} \Xi_{\sigma,i}$$

for any $0 \le i \le m$. It is not hard to see that all of these maps are idempotents in T. Note that $d_{m,0}^{\pm} = d_m^{\pm}$ for the power $X^{(m)}$, and $d_{m,m}^{\pm} = d_m^{\pm}$ for $Y^{(m)}$, where $d_m^{\pm}$ are the idempotents defined in Section 2.1. Similarly, $e_{m,0}^{\pm} = d_m^{\pm}$ for $X^{(m)}$, and $e_{m,m}^{\pm} = d_m^{\pm}$ for $Z^{(m)}$. Therefore we say that $d_m^{\pm}$ are pure idempotents and that $d_{m,i}^{\pm}$ and $e_{m,i}^{\pm}$ are mixed ones.

Summing vertical maps in the last commutative diagram, we obtain two mixed idempotents of distinguished triangles

$$\begin{array}{ccccccc}
D_m^i & \xrightarrow{w_{m,i}} & D_m^{i+1} & \longrightarrow & E_m^{i+1} & \longrightarrow & \Sigma D_m^i \\
\downarrow{\scriptstyle d_{m,i}^{\pm}} & & \downarrow{\scriptstyle d_{m,i+1}^{\pm}} & & \downarrow{\scriptstyle e_{m,i-1}^{\pm}} & & \downarrow{\scriptstyle \Sigma d_{m,i}^{\pm}} \\
D_m^i & \xrightarrow{w_{m,i}} & D_m^{i+1} & \longrightarrow & E_m^{i+1} & \longrightarrow & \Sigma D_m^i
\end{array}$$

(one for $+$ and one for $-$). We will denote them by $(d_{m,i}^{\pm}, d_{m,i-1}^{\pm}, e_{m,i-1}^{\pm})$.

Let X be a category and let $f : X \to X$ be an idempotent in X, i.e. $f^2 = f$. The typical example: if $g : X \to I$ and $h : I \to X$ are two morphisms in X, such that $gh = 1_Y$, then $f = hg$ is an idempotent. In such a case f is called to be a splitting idempotent and I is considered as an image $im(f)$ of f. In a pseudoabelian category any idempotent splits.

Lemma 16 *Let $f : X \to X$ and $g : Y \to Y$ be splitting idempotents in X and let $t : X \to Y$ be a morphism from f to g, i.e. $gt = tf$. Then there exists a unique morphism $q : im(f) \to im(g)$, such that the diagram*

$$\begin{array}{ccccc} X & \xrightarrow{a} & im(f) & \xrightarrow{b} & X \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle t} \\ Y & \xrightarrow{c} & im(g) & \xrightarrow{d} & Y \end{array}$$

commutes.

Proof. To see the uniqueness assume that q exists. Then $qa = ct$. Since $ab = 1_X$, it follows that q is uniquely determined as $q = ctb$. Defining q as ctb we get: $qa = ctba = ctf = cgt = cdct = ct$ and $dq = dctb = gtb = tfb = tbab = tb$. □

Lemma 16 shows that images of idempotents have good functorial properties and they are determined up to canonical isomorphisms.

Assume that X is a triangulated category with an endofunctor Σ. Let X^{Δ} be the category of distinguished triangles in X with evident morphisms between them.

Lemma 17 *Let (a, b, c) be an endomorphism in X^{Δ}, i.e. we have a commutative diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \Sigma a} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \end{array}$$

where both rows are the same distinguished triangle in X. Assume that a, b and c are splitting idempotents in X. Then (a, b, c) is a splitting idempotent in X^{Δ} with the image $im(a, b, c)$ given by a distinguished triangle

$$im(a) \xrightarrow{f'} im(b) \xrightarrow{g'} im(c) \xrightarrow{h'} \Sigma\, im(a) ,$$

where f', g' and h' are morphisms induced on images by Lemma 16.

Proof. The chain of morphisms

$$im(a) \xrightarrow{f'} im(b) \xrightarrow{g'} im(c) \xrightarrow{h'} \Sigma\, im(a)$$

is a candidate triangle in X, [31], Def. 1.1.1. By symmetry we also have a candidate distinguished triangle

$$im(1_X - a) \xrightarrow{f''} im(1_X - b) \xrightarrow{g''} im(1_X - c) \xrightarrow{h''} \Sigma\, im(1_X - a)$$

in X. At the same time, the distinguished triangle XYZ is a direct sum of these two candidate triangles. Therefore, both candidate triangles are distinguished, [31], Prop. 1.2.3. □

Proposition 18 *For each m and i there exists a distinguished triangle*

$$I^{\pm}_{m,i} \xrightarrow{w^{\pm}_{m,i}} I^{\pm}_{m,i+1} \longrightarrow J^{\pm}_{m,i+1} \longrightarrow \Sigma I^{\pm}_{m,i}$$

in the category T, where

$$I^{\pm}_{m,i} = im(d^{\pm}_{m,i})$$

and

$$J^{\pm}_{m,i} = im(e^{\pm}_{m,i})$$

are images of the mixed idempotents, and the maps $w^{\pm}_{m,i}$ are induced on images of idempotents by the map $w_{m,i}$. This triangle is an image of the splitting idempotent $(d^{\pm}_{m,i}, d^{\pm}_{m,i+1}, e^{\pm}_{m,i+1})$ in the category T^{Δ}.

Proof. Apply Lemma 17 to $(d^{\pm}_{m,i}, d^{\pm}_{m,i+1}, e^{\pm}_{m,i+1})$. □

To compute the images $J^{\pm}_{m,i}$ of idempotents $e^{\pm}_{m,i}$ in terms of the objects X and Z we again need an abstract lemma.

Lemma 19 *Let X be a $\mathbb{Q}$-linear category with splitting idempotents, and let*

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ A & \xleftarrow[d]{} & B \end{array}$$

be a diagram in X, commutative up to a scalar α, i.e. $\alpha a = dbu$ for some $\alpha \in \mathbb{Q}$. Assume, furthermore, that $b^2 = b$ and $ud = \alpha$ (to be more precise, $ud = \alpha 1_B$). Then $a^2 = a$, $ua = bu$, $ad = db$ and the map $im(u) : im(a) \to im(b)$ (which exists because $ua = bu$, see Lemma 16) is an isomorphism.

Proof. Indeed, since $\alpha a = dbu$ and $ud = \alpha$, it follows that $\alpha ua = \alpha bu$, whence $ua = bu$. Similarly, multiplying $\alpha a = dbu$ on d from the right hand side, we have that $\alpha ad = dbud$. Since $ud = \alpha$, we get $\alpha ad = db\alpha$, whence $ad = db$.

Further, $a^2 = \alpha^{-2}(dbu)(dbu) = \alpha^{-2}db(ud)bu = \alpha^{-2}db\alpha bu = \alpha^{-1}dbbu$. Since $b^2 = b$ by assumtions, we get $a^2 = \alpha^{-1}dbu = a$.

Now let's consider the commutative diagram

$$\begin{array}{ccccccc}
B & \xrightarrow{d} & A & \xrightarrow{u} & B & \xrightarrow{d} & A \\
\downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle \pi_A} \\
I_B & \xrightarrow{I_d} & I_A & \xrightarrow{I_u} & I_B & \xrightarrow{I_d} & I_A \\
\downarrow{\scriptstyle \iota_B} & & \downarrow{\scriptstyle \iota_A} & & \downarrow{\scriptstyle \iota_B} & & \downarrow{\scriptstyle \iota_A} \\
B & \xrightarrow{d} & A & \xrightarrow{u} & B & \xrightarrow{d} & A
\end{array}$$

where the columns are splittings of the idempotents a and b, $I_A = im(a)$, $I_B = im(b)$, etc. Easy chasing on this diagram shows that $\iota_B I_u I_d \pi_B = bud = \alpha \iota_B \pi_B$, whence $\iota_B(\alpha^{-1} I_u I_d)\pi_B = \iota_B \pi_B$. Since ι_B is a left inverse for π_B, it follows that $\alpha^{-1} I_u I_d = 1_{I_B}$. Further, $I_d I_u = (\pi_A d \iota_B)(\pi_B u \iota_A) = \pi_A d(\iota_B \pi_B) u \iota_A = \pi_A dbu \iota_A = \pi_A(\alpha a)\iota_A = \alpha \pi_A a \iota_A = \alpha \pi_A \iota_A \pi_A \iota_A = \alpha$, whence $\alpha^{-1} I_d I_u = 1_{I_A}$. □

Proposition 20

$$J^+_{m,i} \cong Z^{[i)} \otimes X^{[m-i)}$$

$$J^-_{m,i} \cong Z^{(i]} \otimes X^{(m-i]}$$

Proof. Let i be an integer, such that $1 \le i \le m$. The group Σ_i may be considered as a subgroup in Σ_m consisting of permutations which preserve the set $\{1,\ldots,i\}$, and Σ_{m-i} as a subgroup in Σ_m of permutations preserving the set $\{i+1,\ldots,m\}$. Then $\Sigma_i \times \Sigma_{m-i}$ is a subgroup in Σ_m consisting of the product of permutations acting on the sets $\{1,\ldots,i\}$ and $\{i+1,\ldots,m\}$ in an inner fashion.

The number of vertices in U^i_m is equal to $C^r_m = \frac{m!}{i!(m-i)!}$, so that it coincides with the number of left cosets in Σ_m modulo the subgroup $\Sigma_i \times \Sigma_{m-i}$. For any $\underline{v} \in U^i_m$ fix an isomorphism

$$u_{\underline{v}} : E_{\underline{v}} \xrightarrow{\cong} Z^{(i)} \otimes X^{(m-i)}$$

fixing, in fact, a representative $\varsigma_{\underline{v}}$ of the corresponding right coset in Σ_m modulo the subgroup $\Sigma_i \times \Sigma_{m-i}$. Let

$$u : E^i_m \longrightarrow Z^{(i)} \otimes X^{(m-i)}$$

be a sum of isomorphisms $u_{\underline{v}}$, $\underline{v} \in U^i_m$, and let

$$d : Z^{(i)} \otimes X^{(m-i)} \longrightarrow E^i_m$$

be a morphism, such that its composition with any projection

$$p_{\underline{v}} : E^i_m \longrightarrow E_{\underline{v}}$$

coincides with the corresponding inverse isomorphism $u_{\underline{v}}^{-1}$. Let's consider the diagram

$$\begin{array}{ccc} E_m^i & \xrightarrow{\quad u \quad} & Z^{(i)} \otimes X^{(m-i)} \\ \Big\downarrow{\scriptstyle e_{m,i}^{\pm}} & & \Big\downarrow{\scriptstyle d_i^{\pm} \otimes d_{m-i}^{\pm}} \\ E_m^i & \xleftarrow[\quad d \quad]{} & Z^{(i)} \otimes X^{(m-i)} \end{array}$$

Since $\{\varsigma_{\underline{v}}\}_{\underline{v} \in U_m^i}$ is a set of representatives of the right cosets in Σ_m modulo the subgroup $\Sigma_i \times \Sigma_{m-i}$, the set $\{\varsigma_{\underline{v}}^{-1}\}_{\underline{v} \in U_m^i}$ is a collection of representatives of the left cosets Σ_m modulo $\Sigma_i \times \Sigma_{m-i}$. Then

$$\Sigma_m = \Sigma_m \varsigma_{\underline{v}} = \cup_{\underline{t} \in U_m^i} \varsigma_{\underline{t}}^{-1} (\Sigma_i \times \Sigma_{m-i}) \varsigma_{\underline{v}} \ ,$$

so that

$$i!(m-i)!(p_{\underline{t}} \circ d \circ (d_i^{\pm} \otimes d_{m-i}^{\pm}) \circ u_{\underline{v}}) = m!(p_{\underline{t}} \circ e_{m,i}^{\pm} \circ u_{\underline{v}}) \ ,$$

or, equivalently,

$$p_{\underline{t}} \circ d \circ (d_i^{\pm} \otimes d_{m-i}^{\pm}) \circ u_{\underline{v}} = \frac{m!}{i!(m-i)!} p_{\underline{t}} \circ e_{m,i}^{\pm} \circ u_{\underline{v}}$$

for any $\underline{v}$ and $\underline{t}$ from U_m^i. This shows that the above diagram is commutative modulo the scalar $\frac{m!}{i!(m-i)!}$, i.e.

$$\frac{m!}{i!(m-i)!} \cdot e_{m,i}^{\pm} = d \circ (d_i^{\pm} \otimes d_{m-i}^{\pm}) \circ u \ .$$

Moreover, the composition ud, evidently, coincides with the multiplication by $\frac{m!}{i!(m-i)!}$. Now it remains just to apply Lemma 19 and observe that

$$im(d_i^{\pm} \otimes d_{m-i}^{\pm}) = im(d_i^{\pm}) \otimes im(d_{m-i}^{\pm}) \ .$$

□

3.3 *Triangle filtrations: the proof of Theorem 1*

Let us recall the following approach to Theorem 1 suggested by Uwe Jannsen: build a filtration on wedge (symmetric) powers of vertices in a distinguished triangle, similar to the filtration for powers in a short exact sequence of locally free sheaves of modules on a manifold. Now it remains just to claim that the maps $w_{m,i}^{\pm}$ from Proposition 18 provide the desired filtration.

Let X be an arbitrary triangulated category and let X be an object in X. The following definition appears in [18], p. 152 - 153. A finite (increasing) filtration $F^{\bullet}$ on X is a sequence of objects

$$0 \stackrel{F^{-1}X}{\longrightarrow} A_0 \stackrel{F^0X}{\longrightarrow} A_1 \stackrel{F^1X}{\longrightarrow} \dots \stackrel{F^{m-1}X}{\longrightarrow} A_m = X$$

in X. The graded pieces of such a filtration are defined by the formula

$$Gr^i_F X = cone(F^{i-1}X)$$

Proposition 21 *Let $\mathsf{T} = Ho(\mathsf{C})$ be a trinagulated category, reinforced by an underlying pointed model and monoidal category C in the sense of Section 2.2, and let*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

be a distinguished triangle in T. Then, for any natural m, there exist a finite increasing filtration $F^\bullet Y^{[m)}$ on $Y^{[m)}$ and a finite increasing filtration $F^\bullet Y^{(m]}$ on $Y^{(m]}$, such that

$$Gr^i_F Y^{[m)} \cong Z^{[i)} \otimes X^{[m-i)}$$

and

$$Gr^i_F Y^{(m]} \cong Z^{(i]} \otimes X^{(m-i]}$$

for any i.

Proof. Put $A_i = I^{\pm}_{m,i}$ for any $0 \le i \le m$, $F^i = w^{\pm}_{m,i}$ for any $0 \le i \le m-1$ and apply Prop. 18 and Prop. 20. □

Now we can finish the proof of Theorem 1. Assume that X and Z in the triangle from Theorem 1 are evenly finite dimensional. It means that $X^{[t)} = 0$ and $Z^{[t)} = 0$ for some natural t. Then, for a big enough m, all graded pieces $Gr^i_F Y^{[m)} \cong Z^{[i)} \otimes X^{[m-i)}$ of the even filtration are equal to zero. Therefore, $Y^{[m)} = 0$. And similarly in the odd case.

As to the the second part of Theorem 1, it is equivalent to the first one. To see this we need just to observe that the shift functor Σ in any $\mathbb{Q}$-linear monoidal triangulated category T carries evenly (oddly) finite dimensional objects into oddly (evenly) finite dimensional objects. This is due to the axioms coding the compatibility of the monoidal and the triangulated structures in T, see A8 in [24].

4 Finite dimensionality of motives of curves over a field

In this section we prove Theorem 2. Let k be a field, $char(k) = 0$. The word "scheme" means a separated scheme of finite type over k, and the word "curve" means a scheme whose irreducible components have dimension one. We will work in $\mathbb{Q}$-localized Voevodsky's triangulated category DM of motives over k with the shift functor denoted by $M \mapsto M[1]$.

4.1 *Scalar extension and a splitting lemma*

Let X be a curve and assume, for simplicity, that X is integral over k (it will be clear from the below arguments how to extend them to the case when X is irreducible or not necessary reduced). If k is algebraically closed, then X can be considered as a Zariski open subset in a projective curve Y, see [10], p. 105, and [30]. Let $p : W \to Y$ be a resolution of singularities of Y and let $U = p^{-1}(X)$, so that we have a commutative square

$$\begin{array}{ccc} W & \xrightarrow{p} & Y \\ \uparrow & & \uparrow \\ U & \xrightarrow{p} & X \end{array} \qquad (1)$$

Let also $Z = Y - X$ and $V = W - U$ be complements of Zariski open subsets Y and U in the projective curves X and W respectively.

If k is not algebraically closed, then we may consider the square (1) first over an algebraic closure of k and then take a finite extension L/k, such that all varieties and maps in (1), as well as Z and V, are defined over L, and L contains $\sqrt{-1}$. But, since we work with motives with coefficients in $\mathbb{Q}$, one can use transfer arguments to show that finite dimensionality of the motive $M(X_L)$ in $\mathsf{DM}(L)$ implies finite dimensionality of $M(X)$ in $\mathsf{DM} = \mathsf{DM}(k)$. It means that, proving Theorem 2, we may assume, without loss of generality, that all the data in the square (1) is rational over k.

We will also need the following useful

Lemma 22 *Let X be a triangulated category with the shift functor Σ. Assume that we have a distinguished triangle*

$$A \oplus B \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} A \oplus C \longrightarrow D \longrightarrow \Sigma(A \oplus B)$$

in X, where a is an automorphism of the object A. Then this triangle is isomorphic to the direct sum of triangles

$$A \xrightarrow{1} A \longrightarrow 0 \longrightarrow \Sigma A$$

and

$$B \xrightarrow{t} C \longrightarrow D \longrightarrow \Sigma B$$

where $t = d - ca^{-1}b$.

Proof. This is just a reformulation of Lemma 1.2.4 from [31]. □

4.2 *The proof of Theorem 2*

We will consider three cases separately: (a) when X is projective, but not necessary smooth; (b) when it is not projective, but smooth, and (c) when X is not projective and, probably, not smooth. Then, to prove Theorem 2, we will have just to join all together.

Proposition 23 *Assume that X is projective. Then $M(X)$ is finite dimensional and, moreover,*

$$M(X) \cong \mathbb{Q} \oplus G \oplus \mathbb{Q}(1)[2] \ ,$$

where G is an oddly finite dimensional object in DM.

Proof. If X is projective and smooth, then the proposition holds by Th. 7. Assume that X is singular. For simplicity, we will consider the case when X has only one singular point (the other case can be proved by the same methods, but with more cumbrous formulas). Then $p : U \to X$ contracts points $\{u_1, \dots, u_n\}$ onto a singular point in X. Let

$$\mathbb{Q}^{\oplus n} \longrightarrow \mathbb{Q} \oplus M(U) \longrightarrow M(X) \longrightarrow \mathbb{Q}^{\oplus n}[1] \tag{2}$$

be a blow up distinguished triangle corresponding to the map $p : U \to X$, [35], Th. 5.2, where $\mathbb{Q}^{\oplus n}$ is just a motive of the finite set $\{u_1, \dots, u_n\}$.

For any i the composition

$$\mathbb{Q} \to \mathbb{Q}^{\oplus n} \longrightarrow \mathbb{Q} \oplus M(U) \to \mathbb{Q} \ ,$$

corresponding to the point u_i, is an isomorphism. By Lemma 22 the triangle (2) is isomorphic to a direct sum of two distinguished triangles

$$\mathbb{Q}^{\oplus n-1} \xrightarrow{t} M(U) \longrightarrow M(X) \longrightarrow \mathbb{Q}[1]^{\oplus n-1} \tag{3}$$

and

$$\mathbb{Q} \longrightarrow \mathbb{Q} \longrightarrow 0 \longrightarrow \mathbb{Q}[1] \ ,$$

In other words, we split the isomorphism $\mathbb{Q} \stackrel{\cong}{\to} \mathbb{Q}$ corresponding to a point from $\{u_1, \dots, u_n\}$, say, to u_1.

Since U is smooth projective, we have a decomposition

$$M(U) = \mathbb{Q} \oplus M^1(U) \oplus \mathbb{Q}(1)[2]$$

induced by some k-rational point on U. Then we rewrite (3) as follows:

$$\mathbb{Q}^{\oplus n-1} \xrightarrow{t} \mathbb{Q} \oplus M^1(U) \oplus \mathbb{Q}(1)[2] \longrightarrow M(X) \longrightarrow \mathbb{Q}[1]^{\oplus n-1} \tag{4}$$

For any i, let $\nu_i : Spec(k) \longrightarrow U$ be the map corresponding to the point u_i. Let also $\gamma : U \to Spec(k)$ be the structure map for U. If $i > 1$, then the composition

$$\mathbb{Q} \to \mathbb{Q}^{\oplus n-1} \xrightarrow{t} M(U) \ ,$$

corresponding to the point u_i, coincides with the difference $M(\nu_i) - M(\nu_1)$ (here we use the general expression for the morphism t given by Lemma 22). The projection

$$M(U) = \mathbb{Q} \oplus M^1(U) \oplus \mathbb{Q}(1)[2] \longrightarrow \mathbb{Q}$$

is, in fact, the morphism $M(\gamma) : M(U) \to M(Spec(k))$. Therefore, for any u_i, $i > 1$, the composition

$$\mathbb{Q} \to \mathbb{Q}^{\oplus n-1} \stackrel{t}{\longrightarrow} M(U) \longrightarrow \mathbb{Q}$$

is equal to the difference $M(\gamma\nu_i) - M(\gamma\nu_1)$, which is equal to zero. And, in addition, any map from $\mathbb{Q}$ to $\mathbb{Q}(1)[2]$ is zero. This shows that the triangle (4) is a direct sum of two distinguished triangles

$$\mathbb{Q}^{\oplus n-1} \stackrel{t}{\longrightarrow} M^1(U) \longrightarrow G \longrightarrow \mathbb{Q}[1]^{\oplus n-1} \tag{5}$$

and

$$0 \longrightarrow \mathbb{Q} \oplus \mathbb{Q}(1)[2] \longrightarrow \mathbb{Q} \oplus \mathbb{Q}(1)[2] \longrightarrow 0 \, .$$

In particular,

$$M(X) = \mathbb{Q} \oplus G \oplus \mathbb{Q}(1)[2] \, .$$

Now recall that $M^1(U)$ is oddly finite dimensional by Th. 7. Then G is oddly finite dimensional by Theorem 1 and Theorem 12. □

Proposition 24 *Assume that U is not projective, i.e. $V \neq \emptyset$. Then $M(U)$ is finite dimensional and, for any decomposition*

$$M(U) = \mathbb{Q} \oplus H \, ,$$

the motive H is oddly finite dimensional.

Proof. Since $V \neq \emptyset$, we have the canonical distinguished triangle

$$M(V) \longrightarrow M(W) \longrightarrow M^c(U) \longrightarrow M(V)[1] \tag{6}$$

in DM, where the motive $M(V)$ is a direct sum of n copies of $\mathbb{Q}$. Let $M(W) = \mathbb{Q} \oplus M^1(W) \oplus \mathbb{Q}(1)[2]$ be a decomposition determined by a point from V, say $v \in V$. Then we rewrite (6) as follows:

$$\mathbb{Q}^{\oplus n} \longrightarrow \mathbb{Q} \oplus M^1(W) \oplus \mathbb{Q}(1)[2] \longrightarrow M^c(U) \longrightarrow \mathbb{Q}[1]^{\oplus n} \tag{7}$$

Splitting the isomorphism $\mathbb{Q} \stackrel{\cong}{\to} \mathbb{Q}$ induced by the point v (Lemma 22) we get the triangle

$$\mathbb{Q}^{\oplus n-1} \longrightarrow M^1(W) \oplus \mathbb{Q}(1)[2] \longrightarrow M^c(U) \longrightarrow \mathbb{Q}[1]^{\oplus n-1} \, , \tag{8}$$

which is a direct sum of two triangles

$$\mathbb{Q}^{\oplus n-1} \longrightarrow M^1(W) \longrightarrow N \longrightarrow \mathbb{Q}[1]^{\oplus n-1} \tag{9}$$

and

$$0 \longrightarrow \mathbb{Q}(1)[2] \longrightarrow \mathbb{Q}(1)[2] \longrightarrow 0 \, ,$$

so that

$$M^c(U) \cong N \oplus \mathbb{Q}(1)[2] \, .$$

The motive $M^1(W)$ is oddly finite dimensional by Th. 7. Applying Theorem 1 joint with Theorem 12 to (9) we claim that N is oddly finite dimensional.

Since U is a smooth scheme of pure dimension one,

$$M(U) \cong N^*(1)[2] \oplus \mathbb{Q}$$

by [37], Th. 4.3.7 (3), where N^* is a motive dual to N. Recall that DM is rigid and that the dualization is a tensor endofunctor of DM, whence N^* is oddly finite dimensional because N is so. The motive $N^*(1)[2]$ is oddly finite dimensional as a product of motives with different parities, Th. 4.

Assume now that we have a splitting $M(U) = H \oplus \mathbb{Q}$. Since $M(U)$ is finite dimensional as a sum of finite dimensional motives $N^*(1)[2]$ and $\mathbb{Q}$, one can use Proposition 6 to show that $H \cong N^*(1)[2]$. □

Proposition 25 *Assume that X is not projective. Then $M(X)$ is finite dimensional, and, moreover,*

$$M(X) = \mathbb{Q} \oplus D \, ,$$

where D is oddly finite dimensional in DM.

Proof. If X is smooth then $U = X$ and the proposition follows from Prop. 24. Assume that X is singular. Again, for simplicity, we assume that $p : U \to X$ contracts n points in U onto one singular point in X. Due to Prop. 24, the present proof is almost the same like the proof of Prop. 23. Starting from a blow up distinguished triangle associated with the contraction $p : U \to X$ and using Lemma 22 we arrive to the distinguished triangle

$$\mathbb{Q}^{\oplus n-1} \stackrel{t}{\longrightarrow} M(U) \longrightarrow M(X) \longrightarrow \mathbb{Q}[1]^{\oplus n-1} \qquad (10)$$

If $M(U) = \mathbb{Q} \oplus \tilde{M}(U)$ is a splitting induced by a k-rational point on U, then (10) gives rise to the distinguished triangle

$$\mathbb{Q}^{\oplus n-1} \stackrel{t}{\longrightarrow} \tilde{M}(U) \longrightarrow D \longrightarrow \mathbb{Q}[1]^{\oplus n-1} \, , \qquad (11)$$

where D is a motive, such that $M(X) = \mathbb{Q} \oplus D$. Note that $\tilde{M}(U)$ is oddly finite dimensional by Prop. 24. Then, again, applying Theorem 1 together with Theorem 12 to the triangle (11) we have that D is oddly finite dimensional because $\tilde{M}(U)$ is so. □

5 Schur finiteness in distinguished triangles

Here we prove Theorem 3 generalizing Mazza's 2-of-3 property for Schur finiteness in DM. The method is, in fact, the same as in the proof of Theorem 1.

5.1 *Basics on Schur functors*

Let again C be a monoidal $\mathbb{Q}$-linear and pseudoabelian category with a product $\otimes$, n be a natural number and let $A = \mathbb{Q}\Sigma_m$ be the group algebra of Σ_m over $\mathbb{Q}$. For any $X \in \mathsf{C}$ we consider the homomorphism of algebras $\Gamma : A \longrightarrow End(X^{(n)})$ sending a permutation $\sigma \in \Sigma_m$ into the endomorphism Γ_σ of $X^{(n)}$ permuting factors according to σ, see Subsection 2.1. Let e_λ be the idempotent in A associated with a Young diagram λ and let $\Gamma_\lambda = \Gamma_{e_\lambda}$. Then define a Schur endofunctor

$$S_\lambda : \mathsf{C} \longrightarrow \mathsf{C}$$

by the formula

$$S_\lambda X = im(\Gamma_\lambda) ,$$

see [4]. Following Mazza, [23], we say that an object X is Schur-finite if there exists a Young diagram λ, such that $S_\lambda X = 0$.

Schur finiteness is closed under direct sums, summands and monoidal products, Prop. 1.4, [23] or 1.6 - 1.8, [4]. It satisfies 2-of-3 property in Voevodsky's category DM, Prop. 5.3 in [23] (and our goal now is to extend this property to stable homotopy categories T). If $\mu \subset \lambda$ is a Young subdiagram, then $S_\mu X = 0$ implies $S_\lambda X = 0$.

An object $X \in \mathsf{C}$ is finite in Kimura's sense if $X = Y \oplus Z$, where $S_{(n)}Y = Y^{[n)} = 0$ and $S_{(1,\dots,1)}Z = Z^{(n]} = 0$. Since Schur-finiteness is closed under direct sums, it follows that any Kimura-finite object is Schur-finite, [23], Cor. 1.5.

5.2 *The poof of Theorem 3*

Let T be a triangulated category satisfying the assumptions of Theorem 1 and let

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

be a distinguished triangle in T. Given a natural number m we return to the automorphism $(\Gamma_{\sigma,i}, \Gamma_{\sigma,i+1}, \Xi_{\sigma,i+1})$ of the distinguished triangle

$$D^i_m \stackrel{w_{m,i}}{\longrightarrow} D^{i+1}_m \longrightarrow E^{i+1}_m \longrightarrow \Sigma D^i_m ,$$

constructed in Section 3. Let λ be a Young diagram for the number m, and let V_λ be the corresponding irreducible representation, given up to an isomorphism. For each i let

$$\Gamma_{\lambda,i} = \frac{dim(V_\lambda)}{m!} \sum_{\sigma \in \Sigma_m} \chi_\lambda(\sigma)\Gamma_{\sigma,i} ,$$

and the same for Ξ. Then one has an idempotent of distinguished triangles

$$\begin{array}{ccccccc}
D^i_m & \xrightarrow{w_{m,i}} & D^{i+1}_m & \longrightarrow & E^{i+1}_m & \longrightarrow & \Sigma D^i_m \\
\downarrow{\scriptstyle \Gamma_{\lambda,i}} & & \downarrow{\scriptstyle \Gamma_{\lambda,i+1}} & & \downarrow{\scriptstyle \Xi_{\lambda,i+1}} & & \downarrow{\scriptstyle \Sigma\Gamma_{\lambda,i}} \\
D^i_m & \xrightarrow{w_{m,i}} & D^{i+1}_m & \longrightarrow & E^{i+1}_m & \longrightarrow & \Sigma D^i_m
\end{array}$$

corresponding to the Young diagram λ. The category T being pseudoabelian we have the distinguished triangle

$$im(\Gamma_{\lambda,i}) \longrightarrow im(\Gamma_{\lambda,i+1}) \longrightarrow im(\Xi_{\lambda,i+1}) \longrightarrow \Sigma\, im(\Gamma_{\lambda,i})$$

by Lemma 17.

Now assume that X and Z are Schur-finite. To prove Theorem 3 we need to show that $X \oplus Z$ is also Schur-finite. Choose a Young diagram λ, such that $S_\lambda X = 0$, $S_\lambda Z = 0$ and $S_\lambda(X \oplus Z) = 0$. It is easy to see (we now intend to use the trick from the proof of Theorem 2.4 in [23]) that the direct sum $\oplus_i E^i_m$ coincides with the power $(Z \oplus X)^{(m)}$ and the corresponding direct sum of idempotents $\Gamma'_\lambda = \oplus_i \Gamma'_{\lambda,i}$ is exactly the idempotent determining $S_\lambda(Z \oplus X)$. Since the last object is zero, it follows that all the cones $im(\Gamma_{\lambda,i})$ are also trivial, whence we obtain the chain of isomorphisms

$$S_\lambda X = im(\Gamma_{\lambda,0}) \to im(\Gamma_{\lambda,1}) \to \ldots \to im(\Gamma_{\lambda,m-1}) \to im(\Gamma_{\lambda,m}) = S_\lambda Y$$

in T. Since $S_\lambda X = 0$, we get $S_\lambda Y = 0$. To finish the proof it remains just to mention that the shift endofunctor in T preserves Schur finite dimensionality. This is because the triangulated structure in T is compatible with the tensor structure in the sense of Appendix 8A in [24].

Remark 26 Certainly, one can also compute the cones $im(\Gamma_{\lambda,i})$ for each i and λ:

$$im(\Gamma_{\lambda,i}) \cong \oplus(S_\mu Z \otimes S_\nu X)^{\oplus[\lambda:\mu,\nu]}$$

where the summands are indexed by Young diagrams μ and ν, such that $|\mu| = m - i$ and $|\nu| = i$.

Corollary 27 *Schur-finiteness is a birational invariant for smooth projective threefolds over an algebraically closed field of characteristic zero.*

Proof. Let X and Y be two birationally equivalent smooth projective threefolds over an algebraically closed field k, $char(k) = 0$. Then they can be connected by a chain of blow ups and blow downs with centers of dimension at most one (see [1] for references on factorizations of birational maps). Any such a birational transformation is detected by the corresponding blow up distinguished triangle in DM. Since motives of points and curves are Kimura-finite, and therefore they are Schur-finite, we apply Theorem 3 to each triangle in the above chain. Then it is easy to see that X is Schur-finite if and only if Y is Schur-finite. $\square$

Remark 28 Let $X(d)$ be a smooth hypersurface of degree d in $\mathbb{P}^4$. Applying Corollary 27 to $X(d)$ one can show that the motive $M(X(d))$ is Schur-finite if $d \leq 4$. By the nilpotency result from [6] we have then that all numerically trivial correspondeces on $X(d)$ are nilpotent. The detailed exposition of applications of Corollary 27 to threefolds will be done in forcecoming papers.

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