On motivic cohomology with $\mathbf{Z}/l$-coefficients

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Preliminary version, June 16 2003

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1 Introduction

In this paper we prove the Bloch-Kato conjecture relating the Milnor K-theory and etale cohomology. It is a continuation of [5] where the particular case of $\mathbf{Z}/2$-coefficients ("Milnor's conjecture") was established and we refer to the introduction to [5] for general discussion about the Bloch-Kato conjecture.

The proof given here uses two results which have not so far been published. One is the information about the motivic cohomology of the motivic Eilenberg-Maclane spaces contained in Lemmas 2.2 and 2.3. We plan to consider these results in a sequel to the paper [6] on the motivic Steenrod operations. Another result is Theorem 6.3 which was announced by Markus Rost and we hope that its proof will be available in the written form soon.

The general idea of the approach to the Bloch-Kato conjecture for $l > 2$ used in the present paper appeared in the fall of 1996 as a way to reduce the Bloch-Kato conjecture to the existence of "splitting varieties" with certain properties. Since we could not find a way to show that such varieties exist and since this approach required a lot of general results which were not firmly established at that time it remained undeveloped for all these years. The situation changed when Markus Rost announced that he knows how to construct varieties with required properties and enough of the motivic homotopy theory got written up to make the necessary computations.

The goal of the first two sections of the paper is to prove Theorem 3.8 which relates two types of cohomological operations in motivic cohomology. One of the operations appearing in the theorem is defined in terms of symmetric power functors in the categories of relative Tate motives and another one in terms of the motivic reduced power operations introduced in [6]. While we do not make direct use of its results, the proof of this theorem is inspired by [1]. It is very likely that a more direct proof of Theorem 3.8 exists. Finding such a proof would eliminate the need for Lemmas 2.2 and 2.3.

In the third section we consider motives over a special class of simplicial schemes which are called "embedded simplicial schemes" (see [7]). Up to an equivalence, embedded simplicial schemes correspond to subsheaves of the constant one point sheaf on Sm/k i.e. with classes of smooth varieties such that

1. if X is in the class and $Hom(Y, X) \neq \emptyset$ then Y is in the class, and
2. if $U \to X$ is a Nisnevich covering and U is in the class then X is in the class.

In particular for a symbol $\underline{a} = (a_1, \ldots, a_n)$ we have an embedded simplicial scheme $\mathcal{X}_{\underline{a}}$ associated with the class all splitting varieties for $\underline{a}$ and the motivic cohomology of $\mathcal{X}_{\underline{a}}$ plays a key role in our proof of the Bloch-Kato conjecture.

The goal of the third section is to prove a technical result - Theorem 4.4, which is used in the next section to establish the purity of the generalized Rost motives. We call this result "a motivic degree theorem" because it is analogous to the simplest degree formula for varieties which asserts that a morphism from a ν_n-variety to a variety without zero cycles of degree prime to l has degree prime to l. The main difference between the standard degree formula and our result is that the target of the morphism in our case is a motive rather than a variety. As a consequence of this higer generality we also require stronger conditions on the target than simply the absence of zero cycles of degree prime to l.

In the fourth section we introduce the construction which represents the key difference between the case of $\mathbf{Z}/2$-coefficients and $\mathbf{Z}/l$-coefficients for $l > 2$. In the $\mathbf{Z}/2$-coefficients case the Pfister quadrics provide canonical ν_{n-1}-splitting varieties for symbols of length n. The explicit nature of these varieties made it possible for Markus Rost to invented a simple geometric construction which showed that the motive of a Pfister quadric splits as a direct sum of an "essential part" (which we called the Rost motive in [5]) and a “non essential part” which can be ignored as far as our goals are concerned. The fact that the Rost motive is a direct summand of the motive of a smooth projective variety and at the same time has a description in terms of Tate motives over the embedded simplicial scheme $\mathcal{X}_{\underline{a}}$ defined by the symbol puts strong restrictions on the motivic cohomology of $\mathcal{X}_{\underline{a}}$. Combined with the vanishing of the motivic Margolis homology for $\mathcal{X}_{\underline{a}}$ these restrictions allow one to reformulate the vanishing result needed for the proof of the Milnor conjecture in terms of a motivic homology group of the Pfister quadric which can be analyzed geometrically.

A direct extention of these arguments to the $l > 2$ case fails for two main reasons. On the one hand we do not have nice geometric models for ν_{n-1}-splitting varieties for symbols of length n. On the other hand the argument which for $l = 2$ moves the problem from a motivic cohomology group to a motivic homolopgy group having a geometric description fails to produce the same result for $l > 2$ ending in a group which is not any easier to understand than the original one. This forces us to use a different line of arguments.

We show in the fourth section that any embedded simplicial scheme $\mathcal{X}$ which has a non-trivial motivic cohomology class of certain bidegree and such that the corresponding class of varieties contains a ν_n-variety defines a *generalized Rost motive*. This motive is constructed from the Tate motives over $\mathcal{X}$ and we use the motivic degree theorem of the previous section to prove that it is a direct summand of the motive of a smooth projective variety over k. The key ingredient of the proof is the relation between the $(l-1)$-st symmetric powers and Milnor operations Q_i provided by Theorem 3.8 and Lemma 5.13.

Generalized Rost motives unify two previously known families of motives - the Rost motives for $l = 2$ discussed above and the motives of cyclic field extensions of prime degree. The generalized Rost motives correspond to motivic cohomology classes which have ν_n-splitting varieties in the same way as the motives of the cyclic field extensions correspond to the motivic cohomology classes in $H^{1,1}(k, \mathbf{Z}/l)$. Together with the motives of quadrics and motives of general finite field extensions they belong to the class of Artin-Tate motives.

In the fifth section we give a proof of the Bloch-Kato conjecture based on the results of the previous sections, [5] and Theorem 6.3.

I would like to specially thank several people who helped me to understand things used in this paper. Pierre Deligne for explaining to me how to define sheaves over simplicial schemes and for help with the computa-

tion of $H^*(B\mathbf{G}_a, \mathbf{G}_a)$. Peter May for general remarks on tensor triangulated categories. Fabien Morel for helping me to figure out the relation (5.9).

2 Computations with cohomological operations

For the purpose of this section a pointed smooth simplicial scheme is a pointed simplicial scheme such that its terms are disjoint unions of smooth schemes of finite type over k pointed by a disjoint point. For a pointed smooth simplicial scheme $\mathcal{X}$ the simplicial suspension $S^1_s \wedge \mathcal{X}$ is again a pointed smooth simplicial scheme. For a motivic cohomology class

$$\alpha \in H^{p,q}(\mathcal{X}, R)$$

of a pointed smooth simplicial scheme $\mathcal{X}$ we let

$$\sigma_s \alpha \in H^{p,q}(S^1_s \wedge \mathcal{X}, R)$$

denote the simplicial suspension of α. The goal of this section is to prove the following uniqueness result.

Theorem 2.1 *Let k be a field of characteristic zero. Let ϕ_i, $i = 1, 2$ be two cohomological operations on the motivic cohomology of pointed smooth simplicial schemes of the form*

$$\tilde{H}^{2n+1,n}(-, \mathbf{Z}/l) \to \tilde{H}^{2nl+2,nl}(-, \mathbf{Z}/l)$$

such that:

1. *for $b \in \mathbf{Z}/l$ one has $\phi_i(b\alpha) = b\phi_i(\alpha)$*
2. *for any $\alpha \in H^{2n,n}(\mathcal{X}, \mathbf{Z}/l)$ one has $\phi(\sigma_s\alpha) = 0$*

Then there exists $c \in \mathbf{Z}/l$ such that $\phi_1 = c\phi_2$.

Observe first that since motivic cohomology respect local equivalences and any pointed simplicial sheaf is locally equivalent to a pointed smooth simplicial scheme, operations ϕ_i extend canonically to operations on the motivic cohomology of pointed simplicial sheaves.

Let K_m, $m = 2n, 2n+1$ be a pointed simplicial sheaf which represents on the pointed motivic homotopy category the functor $\tilde{H}^{m,n}(-, \mathbf{Z}/l)$. Let

α_m be the canonical class in $\tilde{H}^{m,n}(K_m, \mathbf{Z}/l)$. Consider the motivic Steenrod algebra $\mathcal{A}^{*,*}$ over

$$H^{*,*} = H^{*,*}(Spec(k), \mathbf{Z}/l)$$

generated by the reduced power operations and the Bockstein homomorphisms (see [6]). In order to prove our theorem we will have to use the following two results which will be proved in a sequel to [6].

Lemma 2.2 *Let k be a field of characteristic zero. Then motivic cohomology of K_{2n} are generated as an algebra over $H^{*,*}$ by elements of the form $u(\alpha_{2n})$ where $u \in \mathcal{A}^{*,*}$.*

Lemma 2.3 *Let k be a field of characteristic zero. Then the Kunnet homomorphism*

$$\tilde{H}^{*,*}(K_{2n}, \mathbf{Z}/l) \otimes_{H^{*,*}} \ldots \otimes_{H^{*,*}} \tilde{H}^{*,*}(K_{2n}, \mathbf{Z}/l) \to \tilde{H}^{*,*}(K_{2n}^{\wedge i}, \mathbf{Z}/l)$$

is an isomorphism for all $i \geq 0$.

We will also use the following result.

Lemma 2.4 *For all $i > 0$ one has $\alpha_{2n}^i \neq 0$.*

Proof: Since K_{2n} represents the functor $\tilde{H}^{2n,n}(-, \mathbf{Z}/l)$ the condition $\alpha_{2n}^i = 0$ would imply that for any X and any $\alpha \in H^{2n,n}(X, \mathbf{Z}/l)$ one has $\alpha^i = 0$. Taking X to be $\mathbf{P}^N$ for N large enough and α to be a generator of $H^{2n,n}(\mathbf{P}^N, \mathbf{Z}/l)$ we get a contradiction.

Since both operations ϕ_i are natural for morphisms of pointed smooth simplicial schemes and any morphism in the motivic homotopy category can be represented by a hat of morphisms of pointed smooth simplicial schemes it is enough to show that

$$\phi_1(\alpha_{2n+1}) = c\phi_2(\alpha_{2n+1}).$$

for an element $c \in \mathbf{Z}/l$.

Choosing K_m to be a sheaf of $\mathbf{Z}/l$ vector spaces we get an action of $Aut(\mathbf{Z}/l) = (\mathbf{Z}/l)^*$ by automorphisms on K_m. This action defines an action on the motivic cohomology of K_m with $\mathbf{Z}/l$-coefficients which gives a canonical splitting of these cohomology groups into the direct sum of subspaces of weights $w = 0, \ldots, l-2$. To distinguish the weight in this sense from the weight as the second index of motivic cohomology we will call the former one the scalar weight and specify it by a third index such that $H^{p,q,r}(K_m, \mathbf{Z}/l)$ is the subgroup of elelemnts of scalar weight r in $H^{p,q}(K_m, \mathbf{Z}/l)$. A class γ is in this subgroup if for any $a \in (\mathbf{Z}/l)^*$ the automorphism f_a defined by a takes γ to $a^i\gamma$.

The first condition of the theorem means that

$$\phi_i(\alpha_{2n+1}) \in \tilde{H}^{2nl+2,nl,1}(K_{2n+1}, \mathbf{Z}/l)$$

The second condition says that $\phi_i(\alpha_{2n+1})$ lie in the kernel of the homomorphism

$$\tilde{H}^{2nl+2,nl}(K_{2n+1}, \mathbf{Z}/l) \to \tilde{H}^{2nl+2,nl}(\Sigma_s^1 K_{2n}, \mathbf{Z}/l))$$

defined by the obvious morphism

$$i : \Sigma_s^1 K_{2n} \to K_{2n+1}. \tag{2.1}$$

The statement of the theorem follows now from the proposition below.

Proposition 2.5 *The kernel of the homomorphism*

$$\tilde{H}^{2nl+2,nl,1}(K_{2n+1}, \mathbf{Z}/l) \to \tilde{H}^{2nl+2,nl}(\Sigma_s^1 K_{2n}, \mathbf{Z}/l) \tag{2.2}$$

is generated by one element.

Proof: We can choose K_{2n} to be a sheaf of abelian groups. Then we may realize K_{2n+1} as the simplicial sheaf $B_\bullet K_{2n}$ where $B_\bullet$ refers to the standard simplicial classifying space of a group space such that

$$B_p(K_{2n}) = K_{2n}^p.$$

Let $M(w)$ be fibrant (injective) model for the complex $\mathbf{Z}/l(w)$. The complexes $\tilde{H}^0(B_p K_{2n}, M(w))$ form a cosimplicial complex and we let

$$N\tilde{H}^0(B_* K_{2n}, M(w))$$

denote the corresponding normalized bicomplex. Note that its terms along the former cosimplicial dimension are of the form $\tilde{H}^0(K_{2n}^{\wedge p}, M(w))$. Then we have

$$\tilde{H}^{d,w}(K_{2n+1}, \mathbf{Z}/l) = H^d(Tot(N\tilde{H}^0(B_\bullet K_{2n}, M(w))))$$

where Tot refers to the total complex of our bicomplex. Hence we have a standard spectral sequence of a bicomplex with the E_1 term of the form

$$E_1^{p,q} = H^q(N\tilde{H}^0(B_*K_{2n}, M(w))_p) = \tilde{H}^{q,w}(K_{2n}^{\wedge p}, \mathbf{Z}/l) \qquad (2.3)$$

which tries to converge to $\tilde{H}^{p+q,w}(K_{2n+1}, \mathbf{Z}/l)$. To keep track of the motivic weight of our cohomology groups we will use a third index $E_r^{p,q,w}$ for the terms of this spectral sequence.

One can easily see that this spectral sequence coincides with the spectral sequence defined by the skeletal filtration

$$sk_0(B_\bullet K_{2n}) \subset sk_1(B_\bullet K_{2n}) \subset \dots \subset sk_p(B_\bullet K_{2n}) \subset \dots \qquad (2.4)$$

on the simplicial sheaf $B_\bullet K_{2n}$. Note that the first term of this filtration $sk_1 B_\bullet K_n$ is $\Sigma_s^1 K_n$ and the morphism (2.1) is the natural inclusion

$$i : sk_1 B_\bullet K_n \to B_\bullet K_n.$$

Lemma 2.6 *The spectral sequence (2.3) converges to* $\tilde{H}^{p+q,w}(K_{2n+1}, \mathbf{Z}/l)$.

Proof: Interpreting (2.3) as the spectral sequence associated with the filtration (2.4) we see that to prove the convergence it is enough to show that for a given w there exists N such that for all $p > N$ one has

$$\tilde{H}^{*,w}(sk_p(B_\bullet K_{2n})/sk_{p-1}(B_\bullet K_{2n}), \mathbf{Z}/l) = 0.$$

It is easy to see that we have

$$sk_p(B_\bullet K_{2n})/sk_{p-1}(B_\bullet K_{2n}) = \Sigma_s^p K_{2n}^{\wedge p}$$

where Σ_s is the simplicial suspension. On the other hand by [6, Cor. 3.5] we know that K_{2n} is n-fold T-connected and therefore $K_{2n}^{\wedge p}$ is np-fold T-connected and its motivic cohomology of weight $< np$ are zero.

Let us consider now what the spectral sequence (2.3) says about the group $A = \tilde{H}^{2nl+2,nl,1}(K_{2n+1}, \mathbf{Z}/l)$. Note first that since the spectral sequence is constructed out of a filtration which respects the action of $Aut(\mathbf{Z}/l)$ it splits into a direct sum of spctral sequences $E_r^{p,q,w,s}$ for individual scalar weights $s = 0, \dots, l-2$. Hence the groups which contribute to A are of the form

$$E_1^{p,2nl+2-p,nl,1} = \tilde{H}^{2nl+2-p,nl,1}(K_{2n}^{\wedge p}, \mathbf{Z}/l) \qquad (2.5)$$

For an element x in $H^{*,*}(K_m, \mathbf{Z}/l)$ we let $s(x)$ (resp. $w(x)$, $d(x)$) denote its scalar weight (resp. its motivic weight, its dimension) if it is well defined. We will say that an element x in $H^{*,*}(K_{2n}, \mathbf{Z}/l)$ is simple if it is of the form $x = \prod_i u_i(\alpha_{2n})$ where u_i are products of Steenrod operations of the form β and P^i. Note that for a simple x one has

$$w(x) \geq ns(x) \tag{2.6}$$

$$w(x) \geq n(l-1) \text{ if } s(x) = 0 \tag{2.7}$$

$$\begin{aligned} w(x) &\geq n \\ d(x) &\geq 2w(x) \end{aligned} \tag{2.8}$$

Lemma 2.7 *For any $p > 1$, $q < nl$ one has*

$$\tilde{H}^{*,q,1}(K_{2n}^{\wedge p}, \mathbf{Z}/l) = 0$$

Proof: By Lemma 2.3 it is enough to consider elements of the form $x = x_1 \otimes \ldots \otimes x_p$ where x_i are simple elements of $\tilde{H}^{*,<nl}(K_{2n}, \mathbf{Z}/l)$. Suppose that we have such an element x of weight 1. Since $p > 1$ there are two possibilities. Either $s(x_i) = 0$ for some i or

$$s(x_1) + \cdots + s(x_p) \geq l \tag{2.9}$$

The first possibility can be excluded since Lemma 2.2 implies that if $s(x_i) = 0$ then $w(x_i) \geq n(l-1)$ and then $w(x) \geq nl$ since $p > 1$. The second possibility can be excluded since by the same lemma we have $w(x_i) \geq ns(x_i)$ which implies by (2.9) that $w(x) \geq nl$.

Lemma 2.8 *For any $p \geq 3$ one has*

$$\tilde{H}^{2nl+2-p,nl,1}(K^{\wedge p}, \mathbf{Z}/l) = 0$$

Proof: By Lemma 2.3 an element in this group can be written as a formal linear combination $\sum a_i x_i$ where $a_i \in H^{*,v_i}$ and

$$x_i = x_{i1} \otimes \ldots \otimes x_{ip} \in H^{*,nl-v_i,1}(K_{2n}^{\wedge p}, \mathbf{Z}/l)$$

where x_{ij} are simple elements. By Lemma 2.7 we conclude that for $v_i > 0$ we have $x_i = 0$. Since $H^{m,0} = 0$ for $m > 0$ it is enough to consider elements of the form

$$y_1 \otimes \ldots \otimes y_p \in H^{2nl+2-p,nl,1}(K_{2n}^{\wedge p}, \mathbf{Z}/l).$$

where y_j are simple. For $p > 2$ there are no such element since for simple y_j one has $d(y_j) \geq 2w(y_j)$.

Lemma 2.8 together with (2.5) show that there is a short exact sequence

$$0 \to E_\infty^{2,2nl,nl,1} \to \tilde{H}^{2nl+2,nl,1}(K_{2n+1}, \mathbf{Z}/l) \to E_\infty^{1,2nl+1,nl,1} \to 0$$

For $p = 1$ the incoming differentials are zero starting with d_1 and hence E_∞ is contained in E_1 and we have an exact sequence

$$0 \to E_\infty^{2,2nl,nl,1} \to \tilde{H}^{2nl+2,nl,1}(K_{2n+1}, \mathbf{Z}/l) \to \tilde{H}^{2nl+1,nl,1}(K_{2n}, \mathbf{Z}/l)$$

where the last arrow is exactly (2.2). It remains to show that $E_\infty^{2,2nl,nl,1}$ is generated by one element. Since this is a subgroup of the corresponding E_2 term it is sufficient to show that this E_2 term is generated by one element.

The $E_2^{p,q,nl,s}$ term is the cohomology of the complex

$$\tilde{H}^{q,nl,s}(K_{2n}^{\wedge(p-1)}, \mathbf{Z}/l) \to \tilde{H}^{q,nl,s}(K_{2n}^{\wedge p}, \mathbf{Z}/l) \to \tilde{H}^{q,nl,s}(K_{2n}^{\wedge(p+1)}, \mathbf{Z}/l)$$

where the differential is defined by the differential in the normalized complex corresponding to $B_\bullet K_{2n}$.

Lemma 2.9 *For $p > 1$ the group*

$$D_p = \tilde{H}^{2nl,nl,1}(K_{2n}^{\wedge p}, \mathbf{Z}/l)$$

is a free $\mathbf{Z}/l$ module generated by monomials of the form

$$\alpha_{2n}^{i_1} \wedge \cdots \wedge \alpha_{2n}^{i_p}$$

where $i_j > 0$ and $\sum_j i_j = l$.

Proof: Note first that these monomials are linearly independent by Lemmas 2.3 and 2.4. It remains to show that they generate D_p as a $\mathbf{Z}/l$-module. Lemma 2.7 implies that D_p is generated over $\mathbf{Z}/l$ by elements of the form $x = x_1 \wedge \cdots \wedge x_p$ where x_i are simple elements. The inequality (2.7) shows that at most one x_i may have scalar weight 0. For $p > 2$, (2.7) together with (2.6) shows that all $s(x_i)$ must be > 0. If $\sum_i s(x_i) > l$ then by (2.6) we would have $w(x) > nl$. Hence, $\sum_i s(x_i) = l$ and $w(x_i) = ns(x_i)$ for all i and hence $d(x_i) = 2w(x_i)$ for all i. Any simple x_i satisfying these conditions is of the form

$$x_i = \alpha_{2n}^{s(x_i)}$$

which proves the lemma for $p > 2$. The same argument works for $p = 2$ if $s(x_i) > 0$ for all i. Assume that $p = 2$ and $s(x_1) = 0$ and $s(x_2) = 1$. Then by (2.7) and (2.7) we must have $w(x_0) = n(l-1)$ and $w(x_1) = n$. Hence by (2.8) we have $d(x_0) = 2n(l-1)$ and $d(x_1) = 2n$. Since x_i are simple this again implies that $x_0 = \alpha_{2n}^{l-1}$ and $x_1 = \alpha_{2n}$.

To proceed further we will use a techique which allows one to obtain elements in the E_2 term of the spectral sequence associated with the skeletal filtration on $B_\bullet G$ for any sheaf of groups G. Let $v : G \times G \to G$ be the morphism given by $(g_1, g_2) \mapsto g_1 g_2^{-1}$. Note that the face map

$$\partial_i : G^{p+1} \to G^p$$

in $B_\bullet G$ is of the form

$$\partial_i(g_0, \ldots, g_p) = \begin{cases} (g_0, \ldots, \hat{g}_i, \ldots, g_p) & \text{for } i \le p \\ (g_0 g_p^{-1}, \ldots, g_{p-1} g_p^{-1}) & \text{for } i = p \end{cases}$$

Let γ be an element in $H^{d,w}(G, \mathbf{Z}/l)$ such that

$$v^*(\gamma) = \gamma \wedge 1 - 1 \wedge \gamma. \tag{2.10}$$

Consider the pointed simplicial scheme $B_\bullet \mathbf{G}_a$ over $\mathbf{Z}/l$ and let

$$C^\bullet = \mathcal{O}(B_\bullet \mathbf{G}_a)$$

be the corresponding (reduced) cosimplicial abelian group. Then $C^0 = 0$ and for $p > 0$ the terms of $C^\bullet$ are polynomial rings

$$C^p = \mathbf{Z}/l[x_1, \ldots, x_p]$$

and the face maps are given by obvious explicit formulas. Note that the face map are homogenious in x_i of degree 1 and therefore we may consider $C^\bullet$ as a graded simplicial abelian group. We will write this grading by degrees in x_i's as the second index.

Define homomorphisms

$$C^{p,q} \to H^{dq,wq}(G^p, \mathbf{Z}/l)$$

by the rule $x_i \mapsto 1 \wedge \cdots \wedge \gamma \wedge \cdots \wedge 1$ where γ is on the i-th place. One verifies immediately that our condition on $v^*(\gamma)$ implies that these homomorphisms define a homomorphism of complexes

$$\tilde{C}^{*,q} \to E_1^{*,dq,wq} \tag{2.11}$$

where $\tilde{C}^*$ is the normalized complex defined by the cosimplicial abelian group $C^\bullet$ and $E_1^{*,dq,wq}$ is the appropriate row of our spectral sequence for $B_\bullet G$ with d_1 as the differential. The cohomology of $\tilde{C}^*$ are the cohomology groups $H^*(B\mathbf{G}_a, \mathbf{G}_a)$ over $\mathbf{Z}/l$. Hence, any γ as above defines a homomorphism

$$H^{p,q}(B\mathbf{G}_a, \mathbf{G}_a) \to E_2^{p,dq,wq} \tag{2.12}$$

where the second grading on the left hand side is defined by the polynomial degree of the cocycles.

Let us return now to the case when $G = K_{2n}$ and $\gamma = \alpha_{2n}$. Note that the condition (2.10) is satisfied since α_{2n} is defined by the identity homomorphism of the abelian group K_{2n} and hence its composition with $v : K_{2n} \times K_{2n} \to K_{2n}$ is exactly $\alpha_{2n} \otimes 1 - 1 \otimes \alpha_{2n}$. Since γ is homogenious of degree 1 with respect to the scalar weight the homomorphism (2.11) in this case is of the form

$$\tilde{C}^{*,q} \to E_1^{*,2nq,nq,q \, mod\, (l-1)} \tag{2.13}$$

The part of this homomorphism we are interested in at the moment is

$$\tilde{C}^{p,l} \to E_1^{p,2nl,nl,1} \tag{2.14}$$

Lemma 2.9 implies immedialtely that (2.14) is an isomorphism for $p > 1$. Therefore, the corresponding map

$$H^{p,l}(B\mathbf{G}_a, \mathbf{G}_a) \to E_2^{p,2nl,nl,1} \tag{2.15}$$

is surjective for $p = 2$ and is an isomorphism for $p > 2$. It remains to show that for $p = 2$ the left hand side of (2.15) is generated by one element. This follows immediately from the computation of $H^*(B\mathbf{G}_a, \mathbf{G}_a)$ given in [2, Th.12.1, p.375].

Remark 2.10 Note that if (2.10) is satisfied for an element γ then it is also satisfied for $u(\gamma)$ for any motivic Steenrod operation u. Hence we can extend homomorphism (2.12) to a homomorphism

$$\mathcal{A}^{a,b} \otimes_{\mathbf{Z}/l} H^{p,q}(B\mathbf{G}_a, \mathbf{G}_a) \to E_2^{p,a+dq,b+wq} \tag{2.16}$$

3 Computations with symmetric powers

In this section we fix a prime l and consider the categories of motives with coefficients in R where R is a commutative ring such that all primes but l are invertible in R. For our applications we will need the cases of $R = \mathbf{Z}_{(l)}$ and $R = \mathbf{Z}/l$. Our goal is to prove several results about the structure of the symmetric powers $S^i(M)$ for $i < l$ when M is a Tate motives of the form

$$R(p)[2q] \to M \to R \to R(p)[2q+1]$$

and to use these results to define a cohomological operation

$$\phi_{l-1} : H^{2q+1,p}(-, R) \to H^{2ql+2,pl}(-, R)$$

Let us first consider an arbitrary tensor additive category C which is R-linear and Karoubian (has images of projectors). For any $i < l$ and any M in C define the symmetric power $S^i(M)$ as follows. The symmetric group S_i acts by permutations on $M^{\otimes i}$. Since $i!$ is invertible in our coefficients ring we may consider the averaging projector $p : M^{\otimes i} \to M^{\otimes i}$ given by

$$p = (1/i!) \sum_{\sigma \in S_i} \sigma$$

We set $S^i(M) := Im(p)$. We will use morphisms

$$a : S^i(M) \to S^{i-1}(M) \otimes M$$

and

$$b : S^{i-1}(M) \otimes M \to S^i(M)$$

where a is defined as the quotient of the morphism $\tilde{a} : M^{\otimes i} \to M^{\otimes i}$ given by

$$\tilde{a}(m_1 \otimes \ldots \otimes m_i) = \sum_{j=1}^{i} (m_1 \otimes \ldots \otimes \hat{m}_j \otimes \ldots \otimes m_i) \otimes m_j$$

and b is the quotient of the identity morphism.

Let us consider now the case when $C = DT(\mathcal{X}, R)$ for a smooth simplicial scheme $\mathcal{X}$ and M is a motive which is given together with a distinguished triangle of the form

$$R(p)[2q] \xrightarrow{x} M \xrightarrow{y} R \xrightarrow{\alpha} R(p)[2q+1]$$

where $p, q \geq 0$. Composing a with the morphism defined by y we get a morphism

$$u : S^i(M) \to S^{i-1}(M)$$

and composing b with the morphism defined by x we get a morphism

$$v : S^{i-1}(M)(p)[2q] \to S^i(M)$$

Lemma 3.1 *There exist unique morphisms*

$$r : S^{i-1}(M) \to R(ip)[2iq+1]$$

and

$$s : R \to S^{i-1}(M)(p)[2q+1]$$

such that the sequences

$$R(ip)[2iq] \xrightarrow{x^i} S^i(M) \xrightarrow{u} S^{i-1}(M) \xrightarrow{r} R(ip)[2iq+1] \qquad (3.1)$$

$$S^{i-1}(M)(p)[2q] \xrightarrow{v} S^i(M) \xrightarrow{y^i} R \xrightarrow{s} S^{i-1}(M)(p)[2q+1] \qquad (3.2)$$

are distinguished triangles. If $p > 0$ then these triangles are isomorphic to the triangles

$$\Pi_{\geq ip}(S^i(M)) \to S^i(M) \to \Pi_{<ip}(S^i(M)) \to \Pi_{\geq ip}(S^i(M))[1]$$

and

$$\Pi_{\geq p}(S^i(M)) \to S^i(M) \to \Pi_{<p}(S^i(M)) \to \Pi_{\geq p}(S^i(M))[1]$$

Proof: Assume first that $p > 0$. Since the category of Tate motives is closed under tensor products and direct summands the symmetric power of a Tate motive is a Tate motive. Therefore it is sufficient to verify that the first three terms of the sequences (3.1) and (3.2) satisfy the conditions of [7, Lemma 5.18]for $n = ip$ and $n = p$ respectively.

By [7, Lemma 5.15]one has

$$s_*(M^{\otimes i}) = s_*(M)^{\otimes i}$$

which immediately implies that

$$s_*(S^i(M)) = S^i(s_*(M))$$

and that these isomorphisms are compatible with the maps a, b. Since $p > 0$ we have $s_*(M) = R \oplus R(p)[2q]$ and therefore

$$s_*(S^i(M)) = \oplus_{j=0}^{i} R(pj)[2qj].$$

where the morphism $R(pj)[2qj] \to s_*(S^i(M))$ is $s_*(x^j)$. We denote this morphism by t^j. Computing the slices of the morphisms involved in (3.1) and (3.2) one gets:

$$s_*(u)(t^j) = (i-j)t^j \tag{3.3}$$

$$s_*(v)(t^j) = t^{j+1} \tag{3.4}$$

The morphism x^i is t^i. Since $i-j$ are invertible for all $j = 0, \ldots, i-1$ this implies together with (3.3) that (3.1) satisfies the conditions of [7, Lemma 5.18]. The morphism y^i takes t^j to 0 for $j \neq 0$ and takes 1 to 1. This implies together with (3.4) that (3.2) satisfies the conditions of [7, Lemma 5.18].

Consider now the case of $p = 0$. Using [7, Prop. 5.20]we can identify DT_0 with a full sybcategory in $DLC(\mathcal{X}, R)$. If $q > 0$ consider the homology of $S^i(M)$ with respect to the standard t-structure on $DLC(\mathcal{X}, R)$. One can easily see that x^i defines an isomorphism of $R[2iq]$ with $\tau_{\geq 2iq}(S^i(M))$ and u defines an isomorphism of $S^{i-1}(M)$ with $\tau_{<2iq}(S^i(M))$ where τ refers to the canonical filtration with respect to our t-structure. The standard argument shows now that there exists a unique r with the required property. A similar argument shows the existence and uniqueness of s.

Conisder now the case $p = q = 0$. Then the original triangle comes from an exact sequence of the form

$$0 \to R \to M \to R \to 0 \tag{3.5}$$

in $LC(\mathcal{X})$ and for all $i < l$ we have $S^i(M) \in LC$. To prove the existence and uniqueness of r and s it is sufficient to show that the sequences defined by x^i and u and by v and y^i are exact. We can verify the exactness on each term of $\mathcal{X}$ individually. On a smooth scheme the constant presheaf with transfers is a projective object and therefore the restrictions of (3.5) to each term of $\mathcal{X}$ are split exact. The exactness of the sequences defined by x^i and u and by v and y^i follows by an easy computation.

Consider the composition

$$(r \otimes Id_{R(p)[2q+1]}) \circ s : R \to R((i+1)p)[2(i+1)q+2]$$

Since the morphism $\alpha : R \to R(p)[2q+1]$ determines M up to an isomorphism which commutes with x and y and our construction is natural with respect to such morphisms in M, this composition depends only on α. We denote it by $\phi_i(\alpha)$. Note that it is defined only for $i < l$. Since our construction is natural in M and the inverse image functors commute with tensor product we get the following result.

Lemma 3.2 *For any $\alpha \in H^{2q+1,p}(\mathcal{X}, R)$ and any morphism of simplicial schemes $f : \mathcal{Y} \to \mathcal{X}$ one has*

$$f^*(\phi_i(\alpha)) = \phi_i(f^*(\alpha))$$

Remark 3.3 One observes easily that $\phi_1(\alpha) = \alpha^2$. One can aslo show that $\phi_i(\alpha) = 0$ for $i < l-1$. We will see in Lemma 3.7 that for $R = \mathbf{Z}/l$ and any $n \geq 0$ the operation ϕ_{l-1} is not identically zero.

Proposition 3.4 *Let γ be a morphism of the form $R \to R(r)[2s]$ and σ a morphism of the form $R \to R(p)[2q+1]$. Then one has*

$$\phi_i(\gamma\sigma) = \gamma^{i+1}\phi_i(\sigma)$$

Proof: Set $\alpha = \gamma\sigma$. For simplicity of notations we will write $\{n\}$ instead of $(r)[2s]$ and $\{m\}$ instead of $(p)[2q+1]$. For example $X\{i(n+m)\}$ is $X(i(r+p))[i(2s+2q+1)]$.

Let M_γ and M_σ be objects defined (up to an isomorphism) by distinguished triangles

$$R\{n\}[-1] \to M_\gamma \to R \xrightarrow{\gamma} R\{n\}$$

$$R\{m\}[-1] \to M_\sigma \to R \xrightarrow{\sigma} R\{m\}$$

The octahedral axiom applied to the representation of α as compositions

$$R \xrightarrow{\gamma} R\{n\} \xrightarrow{\sigma\{m\}} R\{n+m\}$$

and

$$R \xrightarrow{\sigma} R\{m\} \xrightarrow{\gamma\{n\}} R\{n+m\}$$

shows that there are morphisms

$$f : M_\sigma \to M_\alpha$$

$$g : M_\alpha \to M_\sigma\{n\}$$

which fit into morphisms of distinguished triangles of the form

$$\begin{array}{ccccccc} R\{m\}[-1] & \longrightarrow & M_\sigma & \longrightarrow & R & \xrightarrow{\sigma} & R\{m\} \\ \downarrow{\scriptstyle \gamma\{m\}[-1]} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \gamma\{m\}} \\ R\{m+n\}[-1] & \longrightarrow & M_\alpha & \longrightarrow & R & \xrightarrow{\alpha} & R\{m+n\} \end{array} \tag{3.6}$$

$$\begin{array}{ccccccc} R\{m+n\}[-1] & \longrightarrow & M_\alpha & \longrightarrow & R & \xrightarrow{\alpha} & R\{m+n\} \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle Id} \\ R\{m+n\}[-1] & \longrightarrow & M_\sigma\{n\} & \longrightarrow & R\{n\} & \xrightarrow{\sigma\{n\}} & R\{m+n\} \end{array} \tag{3.7}$$

Applying May's axiom [3, Axiom TC3] to these two triangles we conclude that morphisms f and g can be chosen in such a way that

$$g \circ f = Id \otimes \gamma \tag{3.8}$$

Consider now the diagrams

$$\begin{array}{ccccccc} S^i(M_\sigma) & \longrightarrow & R & \longrightarrow & S^{i-1}(M_\sigma)\{m\} & \longrightarrow & S^i(M_\sigma)[1] \\ \downarrow{\scriptstyle S^i(f)} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle S^{i-1}(f)\otimes\gamma\{m\}} & & \downarrow{\scriptstyle S^i(f)[1]} \\ S^i(M_\alpha) & \longrightarrow & R & \longrightarrow & S^{i-1}(M_\alpha)\{n+m\} & \longrightarrow & S^i(M_\alpha)[1] \end{array}$$

and

$$\begin{array}{ccccccc} S^i(M_\alpha) & \longrightarrow & S^{i-1}(M_\alpha) & \longrightarrow & R\{i(m+n)\}[1-i] & \longrightarrow & \dots \\ \downarrow{\scriptstyle S^i(g)} & & \downarrow{\scriptstyle S^{i-1}(g)\otimes\gamma} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle S^i(g)[1]} \\ S^i(M_\sigma)\{in\} & \longrightarrow & S^{i-1}(M_\sigma)\{in\} & \longrightarrow & R\{i(m+n)\}[1-i] & \longrightarrow & \dots \end{array}$$

Where:

1. the upper row in the first diagram is (3.2) for M_σ
2. the lower row in the first diagram is (3.2) for M_α
3. the upper row in the second disgram is (3.1) for M_α
4. the lower row in the second diagram is (3.1) for M_σ twisted by $\{in\}$

Let us show that these diagrams commute. The commutativity of the right square in the first diagram is an immediatel corollary of the commutativity of the left square in (3.6). Since both rows are dsitinguished triangles we conclude that there is a morphism

$$\psi : R \to R$$

which makes two other squares commute. Applying the slice functor we conclude that the commutativity of the left square implies that $\psi = 1$.

The commutativity of the left square in the second diagram is an immediatel corollary of the commutativity of the middle square in (3.7). Since both rows are dsitinguished triangles we conclude that there is a morphism

$$\psi : R\{i(m+n)\}[-i] \to R\{i(m+n)\}[-i]$$

which makes two other squares commute. Applying the slice functor we conclude that the commutativity of the middle square implies that $\psi = 1$.

We see now that $\phi_i(\alpha)$ is the composition:

$$\begin{array}{ccc} R \xrightarrow{(1)} & S^{i-1}(M_\sigma)\{m\} & \\ & \downarrow{\scriptstyle S^{i-1}(f)\{m\}} & \\ & S^{i-1}(M_\alpha)\{n+m\} & \\ & \downarrow{\scriptstyle S^{i-1}(g)\otimes\gamma\{m+n\}} & \\ & S^{i-1}(M_\sigma)\{(i+1)n+m\} & \xrightarrow{(2)\{(i+1)n+m\}} R\{(i+1)(m+n)\}[1-i] \end{array}$$

We further have by definition

$$\phi_i(\sigma) = (2) \circ (1)$$

and by (3.8) we have

$$S^{i-1}(g) \circ S^{i-1}(f) = S^{i-1}(g \circ f) = Id \otimes S^{i-1}(\gamma) = Id \otimes \gamma^{i-1}$$

Taking the composition we get

$$\phi_i(\alpha) = \gamma^{i+1}\phi_i(\sigma)$$

Corollary 3.5 *For any $\alpha : R \to R(p)[2q+1]$ and any $c \in \mathbf{Z}$ one has*

$$\phi_i(c\alpha) = c^{i+1}\phi_i(\alpha)$$

Since operations ϕ_i are natural in $\mathcal{X}$ we can extend them to reduced motivic cohomology groups of pointed simplicial schemes in the usual way. We can further extend then to the reduced motivic cohomology of pointed simplicial sheaves using the fact that any simplicial sheaf has a weakly equivalent replacement by a smooth simplicial scheme.

Corollary 3.6 *Let α be a class in $\tilde{H}^{2q,p}(\mathcal{X}, \mathbf{Z}/l)$. Then*

$$\phi_i(\sigma_s\alpha) = 0$$

Proof: The pull-back of $\sigma_s\alpha$ with respect to the projection

$$(S^1_s \times \mathcal{X})_+ \to \Sigma^1_s\mathcal{X}$$

is the class $\sigma \wedge \alpha$ where σ is the canonical class in $H^{1,0}(S^1_s, \mathbf{Z}/l)$. Since the restriction homomorphism is a monomorphism it is enough to show that $\phi_i(\sigma \wedge \alpha) = 0$. By Proposition 3.4 we have

$$\phi_i(\sigma \wedge \alpha) = \phi_i(\sigma) \wedge \alpha^{i+1}$$

The class $\phi_i(\sigma)$ lies in the group $H^{2,0}(S^1_s) = 0$ which proves the corollary.

Lemma 3.7 *For any $n \geq 0$ there exists $\mathcal{X}$ and $\alpha \in H^{2n+1,n}(\mathcal{X}, \mathbf{Z}/l)$ such that $\phi_{l-1}(\alpha) \neq 0$.*

Proof: To show that there exists $\alpha \in H^{2n+1,n}$ such that $\phi_{l-1}(\alpha) \neq 0$ it is sufficient in view of Proposition 3.4 to show that there exists $\alpha \in H^{1,0}$ such that $\phi_{l-1}(\alpha) \neq 0$ and then consider $\alpha\gamma$ for an appropriate γ i.e. we may assume that $n = 0$. In this case one can take α to be a generator of

$$H^{1,0}(K(\mathbf{Z}/l, 1), \mathbf{Z}/l) = \mathbf{Z}/l$$

this generator is represented by the canonical extension

$$0 \to \mathbf{Z}/l \to M \to \mathbf{Z}/l \to 0$$

which corresponds to the standard 2-dimensional representation of $\mathbf{Z}/l$ over $\mathbf{Z}/l$. The symmetric power $S^{l-1}(M)$ is given by the regular representation $\mathbf{Z}/l[\mathbf{Z}/l]$ of $\mathbf{Z}/l$ over $\mathbf{Z}/l$ and $\phi_{l-1}(\alpha)$ is the second extension represented by the exact sequence

$$0 \to \mathbf{Z}/l \to \mathbf{Z}/l[\mathbf{Z}/l] \stackrel{g}{\to} \mathbf{Z}/l[\mathbf{Z}/l] \to \mathbf{Z}/l \to 0 \tag{3.9}$$

where the middle arrow is the multiplication by the generator of $\mathbf{Z}/l$. Let K be the complex given by the middle two terms of (3.9) with the last one placed in degree 0. Then we have a distinguished triangle

$$\mathbf{Z}/l[1] \to K \to \mathbf{Z}/l \stackrel{\phi_{l-1}(\alpha)}{\to} \mathbf{Z}/l[2] \tag{3.10}$$

Since $\mathbf{Z}/l[\mathbf{Z}/l]$ is a projective $\mathbf{Z}/l$-module we have

$$Hom(K, \mathbf{Z}/l[2]) = 0$$

where the morphisms are in the derived category. From the long exact sequence associated with (3.10) we conclude that the map

$$H^0(\mathbf{Z}/l, \mathbf{Z}/l) \to H^2(\mathbf{Z}/l, \mathbf{Z}/l) \tag{3.11}$$

defined by $\phi_{l-1}(\alpha)$ is surjective. Since the right hand side of (3.11) is not zero we conclude that $\phi_{l-1}(\alpha) \neq 0$.

Theorem 3.8 *Let $\alpha \in \tilde{H}^{2n+1,n}(\mathcal{X}, \mathbf{Z}/l)$ be a motivic cohomology class. Then there exists $c \in (\mathbf{Z}/l)^*$ such that*

$$\phi_{l-1}(\alpha) = c\beta P^n(\alpha) \tag{3.12}$$

where β is the Bockstein homomorphism and P^n is the motivic reduced power operation.

Proof: The operation ϕ_{l-1} satisfies the conditions of Theorem 2.1 by Lemma 3.2, Corollary 3.5 and Corollary 3.6. The operation βP^n satisfies the first condition of Theorem 2.1 because the motivic Steenrod operations are additive. It satisfies the second condition since for $\alpha \in H^{2n,n}$ one has

$$\beta P^n(\sigma_s \alpha) = \sigma_s \beta P^n(\alpha) = \sigma_s \beta \alpha^l = 0$$

where the first equality follows from [6, Lemma 9.2], the second equality from [6, Lemma 9.7] and the third from [6, Eq. (8.1)]. We conclude that (3.12) holds for $c \in \mathbf{Z}/l$. Since $\beta P^n \neq 0$ by [6, Cor. 11.5] and $\phi_{l-1} \neq 0$ by Lemma 3.7 we conclude that $c \neq 0$.

4 Motivic degree theorem

In this section we fix a prime l and unless the opposite is explicitly specified we always assume that all other primes are invertible in the coefficient ring. In particular $\mathbf{Z}$ always means $\mathbf{Z}_{(l)}$ - the localization of $\mathbf{Z}$ in l.

Recall from [5] that we let $s_d(X)$ denote the d-th Milnor class of a smooth variety X. This class lies in $H^{2d,d}(X, \mathbf{Z})$ and if $dim(X) = d$ one may consider the number $deg(s_d(X))$. We say that a smooth projective variety X is a ν_n-variety if $dim(X) = l^n - 1$ and

$$deg(s_{l^n-1}(X)) \neq 0 (mod\, l^2)$$

In [5] we constructed for any smooth projective variety X a stable normal bundle V on X and a morphism

$$\tau : T^N \to Th_X(V) \tag{4.1}$$

in the pointed $\mathbf{A}^1$-homotopy category which defines the degree map on the motivic cohomology. Consider the cofibration sequence

$$T^N \stackrel{\tau}{\to} Th_X(V) \stackrel{p}{\to} Th_X(V)/T^N \stackrel{\partial}{\to} \Sigma^1_s T^N \tag{4.2}$$

For $d = dim(X) > 0$ the Thom class

$$t \in \tilde{H}^{2N-2d,N-d}(Th_X(V), \mathbf{Z})$$

restricts to zero on T^N for the weight reasons and there exists a unique class

$$\tilde{t} \in \tilde{H}^{2N-2d,N-d}(Th_X(V)/T^N, \mathbf{Z})$$

such that $p^*(\tilde{t}) = t$. On the other hand the pull-back of the tautological class in $H^{2N+1,N}(\Sigma^1_s T^N, \mathbf{Z})$ with respect to ∂ defines a class

$$v \in \tilde{H}^{2N+1,N}(Th_X(V)/T^N, \mathbf{Z})$$

Lemma 4.1 *Let X be a smooth projective variety of dimension $d = l^n - 1$ where $n > 0$. Then one has*

$$Q_n(\tilde{t}) = (deg(s_{l^n-1}(X))/l)v \ mod \ l \tag{4.3}$$

Proof: Recall from [6] that $Q_n = \beta q_n \pm q_n \beta$ where β is the Bockstein homomorphism. Since $\tilde{t}$ is the reduction of an integral class $\beta(\tilde{t}) = 0$ and it is sufficient to show that

$$\beta q_n(\tilde{t}) = (deg(s_{l^n-1}(X))/l)v \ mod \ l \tag{4.4}$$

The image of (4.2) in DM is an appropriate twist of a sequence of the form

$$\mathbf{Z}(d)[2d] \xrightarrow{\tau'} M(X) \to cone(\tau') \xrightarrow{v} \mathbf{Z}(d)[2d+1] \tag{4.5}$$

By [6, Cor. 14.3] we have $q_n(t) = s_{l^n-1}(X)t$ and therefore there is a commutative square in the motivic category of the form

$$\begin{array}{ccc} M(X) & \longrightarrow & cone(\tau') \\ {\scriptstyle s_{l^n-1}(X)}\downarrow & & \downarrow{\scriptstyle q_n(\tilde{t})} \\ \mathbf{Z}/l^2(d)[2d] & \longrightarrow & \mathbf{Z}/l(d)[2d] \end{array}$$

This square extends to a morphism of distinguished triangles

$$\begin{array}{ccccccc} \mathbf{Z}(d)[2d] & \xrightarrow{\tau'} & M(X) & \longrightarrow & cone(\tau') & \xrightarrow{v} & \mathbf{Z}(d)[2d+1] \\ {\scriptstyle u}\downarrow & & {\scriptstyle s_{l^n-1}(X)}\downarrow & & \downarrow{\scriptstyle q_n(\tilde{t})} & & \downarrow{\scriptstyle u} \\ \mathbf{Z}/l(d)[2d] & \longrightarrow & \mathbf{Z}/l^2(d)[2d] & \longrightarrow & \mathbf{Z}/l(d)[2d] & \xrightarrow{\beta} & \mathbf{Z}/l(d)[2d+1] \end{array}$$

for some morphism u. If u sends 1 to c then the commutativity of the left square means that we have

$$deg(s_{l^n-1}(X)) = lc \ mod \ l^2$$

and the commutativity of the right square means that we have

$$cv = \beta q_n(\tilde{t}) \ mod \ l$$

multiplying the second equality by l and combining with the first one we get

$$deg(s_{l^n-1}(X))v = l\beta q_n(\tilde{t}) \ mod \ l^2$$

which is equivalent to (4.4).

Remark 4.2 The intermediate statement (4.4) of Lemma 4.1 actually holds for any motivic Steenrod operation ϕ if one replaces s_{l^n-1} by an appropriate characteristic class c_ϕ as described in [6, Th. 14.2].

From this point until the end of the section we consider all our motives with $\mathbf{Z}/l$-coefficients. In particular "an embedded simplicial scheme" means a simplicial scheme embedded with respect to $\mathbf{Z}/l$-coefficients.

Recall that the Milnor operations Q_i have the property that $Q_i^2 = 0$ and we define for any pointed simplicial scheme $\mathcal{X}$ and any $i \geq 0$ the motivic Margolis homology $\widetilde{MH}_i^{*,*}(\mathcal{X}, \mathbf{Z}/l)$ of $\mathcal{X}$ as homology of the complex $(\tilde{H}^{*,*}(\mathcal{X}, \mathbf{Z}/l), Q_i)$. Our first application of Lemma 4.1 is the following result which is a slight generalization of [5, Th. 3.2].

Lemma 4.3 *Let $\mathcal{X}$ be an embedded (with respect to $\mathbf{Z}/l$-coefficients) simplicial scheme such that there exists a ν_n-variety X with $M(X, \mathbf{Z}/l)$ in $DM_{\mathcal{X}}$. Let further*

$$\tilde{\mathcal{X}} = cone(\mathcal{X}_+ \to S^0)$$

be the unreduced suspension of $\mathcal{X}$. Then

$$\widetilde{MH}_n^{*,*}(\tilde{\mathcal{X}}, \mathbf{Z}/l) = 0.$$

Proof: Our proof is a version of the proof given in [5]. We will assume that $n > 0$. The case $n = 0$ has a similar (easier) proof. We will use the notations established in the proof of Lemma 4.1. Let $cone(\tau')$ be the motive defined by (4.5). Consider the morphisms in DM with $\mathbf{Z}/l$ coefficients of the form

$$M(\tilde{X})(d)[2d+1] \stackrel{Id\otimes v}{\longleftarrow} M(\tilde{X}) \otimes cone(\tau') \stackrel{Id\otimes \tilde{t}}{\longrightarrow} M(\tilde{X})$$

Since $M(X)$ is in $DM_{\mathcal{X}}$, [7, Lemma 6.9]shows that $M(\tilde{X}) \otimes M(X) = 0$ and therefore sequence (4.5) implies that the first arrow is an isomorphism. Consider the homomorphism

$$\phi : H^{*,*}(\tilde{X}, \mathbf{Z}/l) \to H^{*-2d-1,*-d}(\tilde{X}, \mathbf{Z}/l)$$

defined by $(Id \otimes \tilde{t}) \circ (Id \otimes v)^{-1}$. We claim that for any motivic cohomology class x of $\tilde{\mathcal{X}}$ one has

$$\phi Q_n(x) - Q_n \phi(x) = -(-1)^{deg(x)} s_{l^n-1}(X) x$$

which clearly implies the statement of the lemma. Since $Id \otimes v$ is an isomorphism it is sufficient to check that both sides become the same after multiplication with v. Since v is the image of a morphism in the homotopy category it commutes with cohomological operations and we have to check that

$$Q_n(x) \wedge \tilde{t} - Q_n(x \wedge \tilde{t}) = -(-1)^{deg(x)} s_{l^n-1}(X) x \wedge v \tag{4.6}$$

For $l > 2$ we have

$$Q_n(x \wedge \tilde{t}) = Q_n(x) \wedge \tilde{t} + (-1)^{deg(x)} x \wedge Q_n(\tilde{t})$$

by [6, Prop. 13.3] and the same holds for $l = 2$ by [6, Prop. 13.4] since $Q_i(\tilde{t}) = 0$ for $i < n$ by weight reasons. Applying Lemma 4.1 we further get

$$Q_n(x \wedge \tilde{t}) = Q_n(x) \wedge \tilde{t} + (-1)^{deg(x)} x \wedge v$$

which implies (4.6).

Let $\mathcal{X}$ be an embedded simplicial scheme, $n > 0$ an integer and X be a ν_n-variety such that $M(X) = M(X, \mathbf{Z}/l)$ lies in $DM_{\mathcal{X}}(\mathbf{Z}/l)$.

Let $\mathbf{Z}/l_{\mathcal{X}}(i)[j]$ denote the Tate motives over $\mathcal{X}$ which we identify with $M(\mathcal{X}, \mathbf{Z}/l)(i)[j]$. The image of (4.1) in $DM(k, \mathbf{Z}/l)$ is a morphism of the form

$$\mathbf{Z}/l(d)[2d] \to M(X)$$

and its composition with the morphism $\mathbf{Z}/l_{\mathcal{X}}(d)[2d] \to \mathbf{Z}/l(d)[2d]$ gives us relative fundamental class

$$\tau_{\mathcal{X}} : \mathbf{Z}/l_{\mathcal{X}}(d)[2d] \to M(X)$$

On the other hand [7, Lemma 6.11]implies that the structure morphism $\pi : M(X) \to \mathbf{Z}/l$ is the composition of a unique morphism

$$\pi_{\mathcal{X}} : M(X) \to \mathbf{Z}/l_{\mathcal{X}}$$

with the morphism $\mathbf{Z}/l_{\mathcal{X}} \to \mathbf{Z}/l$.

Theorem 4.4 *Consider a commutative diagram in $DM_{\mathcal{X}}(\mathbf{Z}/l)$ of the form*

$$\begin{array}{ccc} M(X, \mathbf{Z}/l) & \xrightarrow{s} & N \\ {\scriptstyle \pi_{\mathcal{X}}}\downarrow & & \downarrow{\scriptstyle r} \\ \mathbf{Z}/l_{\mathcal{X}} & \xrightarrow{Id} & \mathbf{Z}/l_{\mathcal{X}}. \end{array}$$

Assume that there exists a class $\alpha \in H^{p,q}(\mathcal{X}, \mathbf{Z}/l)$ *such that the following conditions hold:*

1. $p > q$ *and* $\alpha \neq 0$
2. $\alpha \circ r = 0$
3. $Q_n(\alpha) = 0$

Then $s \circ \tau_{\mathcal{X}} : \mathbf{Z}/l_{\mathcal{X}}(d)[2d] \to N$ *is not zero.*

Proof: Let N' be the motive defined by the distinguished triangle

$$\mathbf{Z}/l_{\mathcal{X}}(q)[p-1] \to N' \to \mathbf{Z}/l_{\mathcal{X}} \stackrel{\alpha}{\to} \mathbf{Z}/l_{\mathcal{X}}(q)[p]$$

Our assumption that $\alpha \circ r = 0$ is equivalent to the assumption that there is a morphism $N \to N'$ which makes the diagram

$$\begin{array}{ccc} N & \longrightarrow & N' \\ {\scriptstyle r}\downarrow & & \downarrow \\ \mathbf{Z}/l_{\mathcal{X}} & \xrightarrow{Id} & \mathbf{Z}/l_{\mathcal{X}} \end{array}$$

commutative. Therefore to prove the proposition it is sufficient to show that the composition

$$g : \mathbf{Z}_{\mathcal{X}}(d)[2d] \to M(X) \to N \to N' \tag{4.7}$$

is non zero. We may now forget about the original N and consider only N'.

The composition $\pi_{\mathcal{X}}\tau_{\mathcal{X}}$ is zero and there exists a unique morphism

$$\tilde{\pi}_{\mathcal{X}} : cone(\tau_{\mathcal{X}}) \to \mathbf{Z}/l_{\mathcal{X}}$$

which restricts to $\pi_{\mathcal{X}}$ on $M(X)$. If the composition (4.7) is zero then

$$\alpha \circ \tilde{\pi_{\mathcal{X}}} : cone(\tau_{\mathcal{X}}) \to \mathbf{Z}/l_{\mathcal{X}}(q)[p]$$

is zero. To finish the proof of the proposition it remains to show that it is non-zero. Smashing the sequence (4.2) with $\mathcal{X}_+$ we get a cofibration sequence

$$T^N \wedge \mathcal{X}_+ \to Th_X(V) \wedge \mathcal{X}_+ \to (Th_X(V)/T^N) \wedge \mathcal{X}_+ \stackrel{\partial_{\mathcal{X}}}{\to} \Sigma^1_s T^N \wedge \mathcal{X}_+$$

Up to the shift of the bidegree by $(2N-2d, N-d)$, the motivic cohomology of $(Th_X(V)/T^N) \wedge \mathcal{X}_+$ coincide as the module over the motivic cohomology of $\mathcal{X}$ with the motivic cohomology of $cone(\tau_{\mathcal{X}})$ such that $\tilde{\pi}_{\mathcal{X}}$ corresponds to the pull-back of $\tilde{t}$.

Hence all we need to show that $\tilde{t}\alpha \neq 0$. We are going to show that $Q_n(\tilde{t}\alpha) \neq 0$. For $l > 2$ one has by [6, Prop. 13.3]

$$Q_n(u \wedge v) = Q_n(u) \wedge v \pm u \wedge Q_n(v) \tag{4.8}$$

and since $Q_n(\alpha) = 0$ we get that

$$Q_n(\tilde{t}\alpha) = Q_n(\tilde{t})\alpha. \tag{4.9}$$

For $l = 2$ we have additional terms in (4.8) which depend on $Q_i(\tilde{t})$ for $i < n$. It follows from the simple weight considerations that $Q_i(\tilde{t}) = 0$ for $i < n$ and therefore (4.9) holds for $l = 2$ as well.

Lemma 4.1 shows that the right hand side of (4.9) equals $cv\alpha$ where $c = s_{l^n-1}(X)/l$. Since X is a ν_n-variety, c is an invertible element of $\mathbf{Z}/l$. Hence it remains to check that $v\alpha \neq 0$. Since $v = \partial^*(u)$ where u is the generator of

$$\mathbf{Z}/l = H^{2N+1,N}(\Sigma_s^1 T^N, \mathbf{Z}/l)$$

we have $v\alpha = \partial^*_{\mathcal{X}}(u\alpha)$. The element $u\alpha$ lies in the bidegree $(p+2N+1, q+N)$. The kernel of $\partial^*_{\mathcal{X}}$ in this bidegree is covered by the group

$$H^{p+2N+1,q+N}(\Sigma_s^1 Th_X(V) \wedge \mathcal{X}_+, \mathbf{Z}/l) = H^{p+2N,q+N}(Th_X(V) \wedge \mathcal{X}_+, \mathbf{Z}/l) \tag{4.10}$$

The image of the projection $pr : Th_X(V) \wedge \mathcal{X}_+ \to Th_X(V)$ in DM is an appropriate twist of the morphism

$$M(X) \otimes \mathbf{Z}_{\mathcal{X}} \to M(X)$$

which is an isomorphism by [7, Lemma 6.9]. Therefore, pr defines an isomorphism on the motivic cohomology with $\mathbf{Z}/l$-coefficients and we conclude that (4.10) is isomorphic to the group

$$H^{p+2N,q+N}(Th_X(V), \mathbf{Z}/l) = H^{p+2d,q+d}(X, \mathbf{Z}/l)$$

which is zero for $p > q$ by the cohomological dimension theorem.

Remark 4.5 The end of the proof of Theorem 4.4 shows that the first condition of the theorem can be replaced by the condition that α does not belong to the image of the homomorphism

$$H_{-p,-q}(X, \mathbf{Z}/l) \to H^{p,q}(\mathcal{X}, \mathbf{Z}/l).$$

5 Generalized Rost motives

In this section we work over fields of characteristic zero to be able to use the results of Section 2 and the motivic duality. All motives are with $\mathbf{Z}_{(l)}$-coefficients. We consider $n > 0$ and an embedded smooth simplicial scheme $\mathcal{X}$ which satisfies the following conditions:

1. There exists a ν_n-variety X such that $M(X)$ lies in $DM_{\mathcal{X}}$
2. There exists an element δ in $H^{n+1,n}(\mathcal{X}, \mathbf{Z}/l)$ such that

$$Q_0 Q_1 \dots Q_n(\delta) \neq 0 \tag{5.1}$$

   where Q_i are the Milnor operations introduced in [6, Sec.13].

Under these conditions we will show that there exists a Tate motive M_{l-1} in $DM_{\mathcal{X}}$ which is a direct summand of $M(X)$. Using the construction of M_{l-1} we will show among other things that

$$M(\mathcal{X}) = M(\check{C}(X)).$$

Remark 5.1 Note that our assumptions imply in particular that X has no zero cycles of degree prime to l.

Remark 5.2 Modulo the Bloch-Kato conjecture in weight $\leq n$ and Conjecture 1 (or assuming that for all $i \leq n$ there exist a ν_i-variety X_i such that $M(X_i)$ is in $DM_{\mathcal{X}}$), the condition (5.1) is equivalent to the condition $\delta \neq 0$ (see the proof of Lemma 6.7).

Remark 5.3 Let $\mathcal{X}_0$ be the zero term of $\mathcal{X}$. Then, modulo the Bloch-Kato conjecture in weight $\leq n$ one has

$$H^{n+1,n}(\mathcal{X}, \mathbf{Z}/l) =$$

$$= \bigcap_{\alpha} ker(H^{n+1}_{et}(k, \mu_l^{\otimes n}) \to H^{n+1}_{et}(k(X_\alpha), \mu_l^{\otimes n}))$$

where X_α are the connected components of $\mathcal{X}_0$ (see the proof of Lemma 6.5). Therefore, our conditions on $\mathcal{X}$ can be reformulate by saying that there exist ν_i-varieties in $DM_{\mathcal{X}}$ for all $i \leq n$ and

$$ker(H^{n+1}_{et}(k, \mu_l^{\otimes n}) \to H^{n+1}_{et}(k(X_\alpha), \mu_l^{\otimes n})) \neq 0$$

i.e. $\mathcal{X}_0$ splits a non-zero element in $H^{n+1}_{et}(k, \mu_l^{\otimes n})$.

Remark 5.4 Extending the previous remark we see that if k contains a primitive l-th root of unity (such that $\mu_l \cong \mathbf{Z}/l$) the results of this section are applicable to all non-zero elements in $H^{n+1,n+1}(k, \mathbf{Z}/l)$ which can be split by a ν_n-variety. Theorem 6.3 shows that any pure symbol i.e. the product of $n+1$ elements from $H^{1,1}$ is such an element. It seems natural to conjecture that the inverse implication also holds i.e. that an element in $H^{n+1,n+1}(k, \mathbf{Z}/l)$ which can be split by a ν_n-variety is a pure symbol.

Set

$$\mu = \tilde{Q}_0 Q_1 \ldots Q_{n-1}(\delta) \tag{5.2}$$

where $\tilde{Q}_0$ is the intergal-valued Bockstein homomorphism

$$H^{*,*}(-, \mathbf{Z}/l) \to H^{*+1,*}(-, \mathbf{Z})$$

Then

$$\mu \in H^{2b+1,b}(\mathcal{X}, \mathbf{Z})$$

where $b = (l^n - 1)/(l-1)$.

Consider μ as a morphism in the category of Tate motives over $\mathcal{X}$ and define $M = M_\mu$ by the distinguished triangle in $DM_{\mathcal{X}}$ of the form

$$\mathbf{Z}_{\mathcal{X}}(b)[2b] \xrightarrow{x} M \xrightarrow{y} \mathbf{Z}_{\mathcal{X}} \xrightarrow{\mu} \mathbf{Z}_{\mathcal{X}}(b)[2b+1] \tag{5.3}$$

For any $i < l$ let

$$M_i = S^i M \tag{5.4}$$

be the i-th symmetric power of M. The motive M_{l-1} is called the *generalized Rost motive* defined by X and δ. Note that μ is an l-torsion element and therefore we have

$$M_i \otimes \mathbf{Q} = \oplus_{j=0}^{i} \mathbf{Q}(jb)[2jb].$$

With integral coefficients M_i does not split into a direct sum. Instead the distinguished triangles of the form (3.1) and (3.2) give us distingished triangles

$$M_{i-1}(b)[2b] \to M_i \to \mathbf{Z}_{\mathcal{X}} \to M_{i-1}(b)[2b+1] \tag{5.5}$$

and

$$\mathbf{Z}_{\mathcal{X}}(bi)[2bi] \to M_i \to M_{i-1} \to \mathbf{Z}_{\mathcal{X}}(bi)[2bi+1] \tag{5.6}$$

which describe M_i in terms of Tate motives

$$\mathbf{Z}_{\mathcal{X}}(jb)[2jb] = M(\mathcal{X})(jb)[2jb]$$

over $\mathcal{X}$. The main goal of this section is to show that M_{l-1} is a pure motive which is essentially self-dual and which splits as a direct summand from $M(X)$. It can be shown that this property is special to M_{l-1} and does not hold for M_i where $i < l-1$.

Example 5.5 For $l = 2$ the Pfister quadric $Q_{\underline{a}}$ defined by a sequence of invertible elements $(a_1, \ldots, a_{n+1})$ of k is a ν_n-variety. There is a unique non-zero class δ in $H^{n+1,n}(\check{C}(Q_{\underline{a}}), \mathbf{Z}/2)$ and it satisfies the condition (5.1). The corresponding motive $M_1 = M$ is the standard Rost motive considered in [5].

Example 5.6 Everywhere below we consider the case $n > 0$. The case $n = 0$ gives a good motivating example but the construction of M has to be modified slightly since (5.2) clearly makes no sense in this case. A ν_0-variety is a variety of dimension zero and degree non divisble by l^2. The simpliest interesting example is $X = Spec(E)$ where E is an extension of degree l. In order to have $H^{1,0}(\check{C}(X), \mathbf{Z}/l) \neq 0$, k must contain a primitive l-th root of unity. In that case we may set $\mu = \delta$ and define M as a motive with $\mathbf{Z}/l$-coefficients given by

$$\mathbf{Z}/l \to M \to \mathbf{Z}/l \stackrel{\delta}{\to} \mathbf{Z}/l[1]$$

over $\check{C}(X)$. Then M_{l-1} is the motive of $Spec(E)$ with $\mathbf{Z}/l$-coefficients.

We start with several results about the motives M_i which do not depend on any subtle properties of X or μ. For the proof of these results it will be convenient to consider our motives as relative Tate motives over $\mathcal{X}$.

Lemma 5.7 *For any $i = 1, \ldots, l-1$ there exists a morphism*

$$e_i : M_i \otimes M_i \to \mathbf{Z}_{\mathcal{X}}(bi)[2bi]$$

such that (M_i, e_i) is an internal Hom-object from M_i to $\mathbf{Z}_{\mathcal{X}}(bi)[2bi]$ in $DM_{\mathcal{X}}$.

Proof: Consider first the case $i = 1$. Since the Tate objects are quasi-invertible there exist internal Hom-objects $(\mathbf{Z}_{\mathcal{X}}, u)$ (resp. $(\mathbf{Z}_{\mathcal{X}}(b)[2b], v)$) from $\mathbf{Z}_{\mathcal{X}}(b)[2b]$ (resp. $\mathbf{Z}_{\mathcal{X}}$) to $\mathbf{Z}_{\mathcal{X}}(b)[2b]$. The dual $D\mu$ is again μ and applying [7, Th. 8.3]to the distinguished triangle defining M we conclude that there exists e_1 with the required property.

We can now define e_i for $i > 1$ as the morphism

$$e_i : M_i \otimes M_i \cong S^i(M \otimes M) \stackrel{S^i e_1}{\to} S^i(\mathbf{Z}_{\mathcal{X}}(b)[2b]) = \mathbf{Z}_{\mathcal{X}}(bi)[2bi]$$

[7, Lemma 5.17]implies immediately that (M_i, e_i) is an internal Hom-object from M_i to $\mathbf{Z}_{\mathcal{X}}(bi)[2bi]$.

Consider the homomorphism

$$End(M_i) \to \oplus_{j=0}^{i} \mathbf{Z} \tag{5.7}$$

defined by the slice functor over $\mathcal{X}$ and the identifications

$$End(s_{bj}(M_i)) = End(\mathbf{Z}) = \mathbf{Z}, \;\; j = 0, \ldots, i$$

Lemma 5.8 *The image of (5.7) is contained in the subgroup of elements* $(c_0, \ldots, c_i)$ *such that* $c_k = c_j \, mod \, l$ *for all* k, j.

Proof: Let w be an endomorphism of M_i, c_j be the j-th slice of w and c_{j+1} the $(j+1)$-st slice of w. We need to show that $c_j = c_{j+1} \, mod \, l$. Consider the object $\Pi \geq jb \Pi_{<jb+2}(M_i)$. By Lemma 3.1 we have

$$\Pi_{\geq jb}(M_i) = M_j(jb)[2jb]$$

$$\Pi_{<jb+2}(M_j(jb)[2jb]) = (\Pi_{<2}(M_j))(jb)[2jb] = M(jb)[2jb]$$

This reduces the problem to the case $j = 0$ and $i = 1$ i.e. to an endomorphism

$$M \to M.$$

Since the defining triangle for M coincides with one of the triangles of the slice tower of M it is natural in M. This fact together with the fact that μ is non-zero modulo l implies the result we need.

Remark 5.9 It is easy to see that the image of (5.7) in fact coincides with the subgroup of Lemma 5.8.

Corollary 5.10 *Let* $w : M_i \to M_i$ *be a morphism such that the square*

$$\begin{array}{ccc} M_i & \xrightarrow{w} & M_i \\ \downarrow & & \downarrow \\ \mathbf{Z}_{\mathcal{X}} & \xrightarrow{c} & \mathbf{Z}_{\mathcal{X}} \end{array}$$

commutes for an integer c *prime to* l. *Then* w *is an isomorphism.*

Proof: Since the slice functor is conservative on Tate motives it is sufficient to show that w is an isomorphism on each slice. Our assumption implies that w is c on the zero slice and since c is prime to l it is an isomorphism there. We conlude that w is also prime to l and hence an isomorphism on the other slices by Lemma 5.8.

Let

$$\pi_{\mathcal{X}} : M(X) \to \mathbf{Z}_{\mathcal{X}}$$

be the unique morphism such that the composition

$$M(X) \to \mathbf{Z}_{\mathcal{X}} \to \mathbf{Z}$$

is the structure morphism $\pi : M(X) \to \mathbf{Z}$.

Lemma 5.11 *For any smooth X such that $M(X)$ is in $DM_{\mathcal{X}}$ there exists λ which makes the diagram*

$$\begin{array}{ccc} M(X) & \xrightarrow{\lambda} & M_i \\ {\scriptstyle \pi_{\mathcal{X}}}\downarrow & & \downarrow{\scriptstyle S^i(y)} \\ \mathbf{Z}_{\mathcal{X}} & \xrightarrow{Id} & \mathbf{Z}_{\mathcal{X}} \end{array} \qquad (5.8)$$

commutative.

Proof: The distinguished triangle of the form (5.5) for M_i shows that the obstruction to the existence of λ lies in the group of morphisms

$$Hom(M(X), M_{i-1}(b)[2b+1])$$

Using induction on i and the sequences (5.6) to compute these group we see that it is build out of the groups

$$Hom(M(X), \mathbf{Z}_{\mathcal{X}}(bj)[2bj+1]) = H^{2bj+1,bj}(X, \mathbf{Z})$$

where the equality holds by [7, Lemma 6.11]. Since X is smooth these groups are zero.

Let us now consider the motive M_i for $i = l-1$. To simplify the notations we set $d = b(l-1) = l^n - 1$.

Proposition 5.12 *For any λ which makes the square (5.8) commutative (for $i = l-1$) the composition*

$$\lambda\tau_{\mathcal{X}} : \mathbf{Z}_{\mathcal{X}}(d)[2d] \to M_{l-1}$$

is not divisible by l.

Proof: In view of Theorem 4.4 it is enough to construct a non-zero motivic cohomology class α in $H^{p,q}(\mathcal{X}, \mathbf{Z}/l)$ for some $p > q$ such that α vanishes on M_{l-1} and such that $Q_n(\alpha) = 0$. We set $\alpha = Q_n(\mu \, mod \, l)$. Let us verify that all the required conditions hold. The bidegree of α is $(2b + 2d + 2, b + d) = (lb + 2, lb)$. In particular the dimension is greater than weight. By Lemma 4.3 the n-th motivic Margolis homology of the unreduced suspension $\tilde{\mathcal{X}}$ of $\mathcal{X}$ is zero. Hence if $Q_n(\mu) = 0$ then $\mu = Q_n(\gamma)$ where

$$\gamma \in H^{2b-2d+1,b-d}(\tilde{\mathcal{X}}, \mathbf{Z}/l)$$

For $l > 2$ and $n > 0$ we have $b - d < 0$ and this group is zero. For $l = 2$ we have $b = d$ and the group $H^{1,0}(\tilde{\mathcal{X}}, \mathbf{Z}/2)$ is zero from the long exact sequence relating the motivic cohomology of $\tilde{\mathcal{X}}$ and the motivic cohomology of $\mathcal{X}$. Since $\mu \neq 0$ by our assumption (5.1) we conclude that $\alpha \neq 0$.

The condition $Q_n(\alpha) = 0$ follows immediately from the fact that $Q_n^2 = 0$ (see [6, Prop. 13.3, 13.4]). It remains to check that α vanishes on M_{l-1}. In view of Theorem 3.8 and the definition of the operation ϕ_{l-1} the class $\beta P^b(\mu)$ vanishes on M_{l-1}. Since $Q_i(\mu) = 0$ for $i < n$ we conclude by Lemma 5.13 that

$$Q_n(\mu) = \beta P^b(\mu)$$

which finishes the proof of the proposition for $l > 2$. The proof for $l = 2$ will be added in a later version.

Lemma 5.13 *One has the following equality in the motivic Steenrod algebra for $l > 2$:*

$$Q_0 P^b = P^b Q_0 + P^{b-1} Q_1 + P^{b-l-1} Q_2 + \cdots + P^0 Q_n \tag{5.9}$$

Proof: Since $l > 2$ the subalgebra of the motivic Steenrod algebra generated by operations β, P^i is isomorphic to the usual topological Steenrod algebra. In the topological Steenrod algebra the equation follows by easy induction on n from the commutation relation for the Milnor basis given in [4, Theorem 4a].

Let $\Delta^* : M(X) \otimes M(X) \to \mathbf{Z}(d)[2d]$ be the morphism defined by the diagonal and

$$e_X = \Delta^*_{\mathcal{X}} : M(X) \otimes M(X) \to \mathbf{Z}_{\mathcal{X}}(d)[2d]$$

the morphism which corresponds to Δ^* by [7, Lemma 6.11].

Proposition 5.14 *The pair $(M(X), e_X)$ is an internal Hom-object from $M(X)$ to $\mathbf{Z}_{\mathcal{X}}(d)[2d]$ in $DM_{\mathcal{X}}$.*

Proof: It follows from [7, Lemma 6.14]and [7, Lemma 6.12].

Define $D\lambda$ as the dual of λ with respect to e_X and e_M.

Lemma 5.15 *There exists c prime to l such that the diagram*

$$\begin{array}{ccc} M_{l-1} & \xrightarrow{\lambda D\lambda} & M_{l-1} \\ \downarrow & & \downarrow \\ \mathbf{Z}_{\mathcal{X}} & \xrightarrow{c} & \mathbf{Z}_{\mathcal{X}} \end{array}$$

commutes. In particular, λ is a split epimorphism.

Proof: We will show that there is c such that the diagram

$$\begin{array}{ccccc} M_{l-1} & \xrightarrow{D\lambda} & M(X) & \xrightarrow{\lambda} & M_{l-1} \\ \downarrow{\scriptstyle S^{l-1}(y)} & & \downarrow{\scriptstyle \pi_{\mathcal{X}}} & & \downarrow{\scriptstyle S^{l-1}(y)} \\ \mathbf{Z}_{\mathcal{X}} & \xrightarrow{c} & \mathbf{Z}_{\mathcal{X}} & \xrightarrow{Id} & \mathbf{Z}_{\mathcal{X}} \end{array} \tag{5.10}$$

commutes. Since the right hand side square commutes by definition of λ we only have to consider the left hand side square. Onserve first that

$$\pi_{\mathcal{X}} = D\tau_{\mathcal{X}}.$$

On the other hand

$$S^{l-1}(y) = DS^{l-1}(x)$$

Using the fact that $D(gf) = D(f)D(g)$ we see that to show that the left hand side square commutes it is enough to show that there is c prime to l such that the square

$$\begin{array}{ccc} \mathbf{Z}_{\mathcal{X}}(d)[2d] & \xrightarrow{c} & \mathbf{Z}_{\mathcal{X}}(d)[2d] \\ {\scriptstyle \tau_{\mathcal{X}}}\downarrow & & \downarrow{\scriptstyle S^{l-1}(x)} \\ M(X) & \xrightarrow{\lambda} & M_{l-1} \end{array}$$

commutes. The fact that there exists $c \in \mathbf{Z}$ which makes this diagram commutative follows immediately from the distinguished triangles (3.1) and the fact that Tate objects of higher weight admit no nontrivial morphisms to Tate objects of lower weight. The fact that c must be prime to l follows from Proposition 5.12.

Combining Lemma 5.15 with Corollary 5.10 we conclude that $\lambda D\lambda$ is an isomorphism. Let ϕ be its inverse. Then the composition

$$p : D\lambda \circ \phi \circ \lambda : M(X) \to M(X)$$

is a projector i.e. $p^2 = p$ and its image is M_{l-1}. We conclude that M_{l-1} is a direct summand of $M(X)$. Together with [7, Lemma 6.15]this implies the following important result.

Theorem 5.16 *The motive M_{l-1} is restricted.*

Combining Theorem 5.16 with Lemmas 5.7 and [7, Lemma 6.12]we get the following duality theorem for M_{l-1}.

Corollary 5.17 *Let e'_M be the composition*

$$M_{l-1} \otimes M_{l-1} \stackrel{e_M}{\to} \mathbf{Z}_{\mathcal{X}}(d)[2d] \to \mathbf{Z}(d)[2d]$$

Then (M_{l-1}, e'_M) is an internal Hom-object from M_{l-1} to $\mathbf{Z}(d)[2d]$ in the category $DM_{-}^{eff}(k)$.

Proposition 5.18 *Under the assumptions of this section one has*

$$M(\mathcal{X}) \cong M(\check{C}(X))$$

where the motives are considered with $\mathbf{Z}_{(l)}$-coefficients.

Proof: By [7, Lemma 6.23]it is sufficient to show that for any smooth Y in $DM_{\mathcal{X}}$ there exists a morphism $M(Y) \to M(X)$ over $\mathbf{Z}$. Diagram (5.10) shows that $c^{-1}D\lambda$ is a morphism $M_{l-1} \to M(X)$ over $\mathbf{Z}$. On the other hand Lemma 5.11 shows that there is a morphism $M(Y) \to M_{l-1}$ over $\mathbf{Z}$. The statement of the proposition follows.

6 The Bloch-Kato conjecture

In this section we use the techniques developed above to prove the following theorem.

Theorem 6.1 *Let k be a field of characteristic zero which contains a primitive l-th root of unity. Then the norm residue homomorphisms*

$$K_n^M(k)/l \to H_{et}^n(k, \mu_l^{\otimes n})$$

are isomorphisms for all n.

In the next section we will extend this theorem to all fields of characteristic not equal to l. The statement of Theorem 6.1 is know as the Bloch-Kato conjecture (see [5]).

As was shown in [5, p.36], in order to prove Theorem 6.1 it is sufficient to construct for any k of characteristic zero and any sequence of invertible elements $\underline{a} = (a_1, \ldots, a_n)$ of k, a field extension $K_{\underline{a}}$ of k such that the following two conditions hold:

1. the image of $\underline{a}$ in $K_n^M(K_{\underline{a}})$ is divisible by l,
2. the homomorphism of the Lichtenbaum (etale) motivic cohomology groups
$$H_{et}^{n+1,n}(K, \mathbf{Z}_{(l)}) \to H_{et}^{n+1,n}(K_{\underline{a}}, \mathbf{Z}_{(l)})$$
is a monomorphism.

We say that a smooth connected scheme X splits $\underline{a}$ modulo l if $\underline{a}$ becomes zero in $K_n^M(k(X))/l$ where $k(X)$ is the function field of X. We use the notation $H_{-1,-1}(X, \mathbf{Z})$ for the motivic homology group

$$H_{-1,-1}(X, \mathbf{Z}) = Hom_{DM}(\mathbf{Z}, M(X)(1)[1])$$

For $X = Spec(k)$ this group is k^* and for a general X it has a description in terms of cycles with coefficients in K_*^M. If X is smooth projective of dimension d over a field of characteristic zero then the motivic duality theorem implies that

$$H_{-1,-1}(X, \mathbf{Z}) = H^{2d+1,d+1}(X, \mathbf{Z})$$

Definition 6.2 *A smooth projective variety X over k is called a $\nu_{\leq n}$-variety if X is a ν_n-variety and for all $i < n$ there exists a ν_i-variety X_i and a morphism $X_i \to X$.*

It seems likely that the following conjecture holds.

Conjecture 1 *Any ν_n-variety is a $\nu_{\leq n}$-variety.*

A key point in our proof of Theorem 6.1 is the following result announced by Markus Rost.

Theorem 6.3 *For any $\underline{a} = (a_1, \ldots, a_n)$ there exists a $\nu_{\leq(n-1)}$-variety X such that:*

1. *X splits $\underline{a}$*
2. *the sequence*

$$H_{-1,-1}(X \times X, \mathbf{Z}) \stackrel{pr_1 - pr_2}{\longrightarrow} H_{-1,-1}(X, \mathbf{Z}) \to k^*$$

is exact.

In order to prove Theorem 6.1 we will show that for any X satisfying the conditions of Theorem 6.3 the homomorphism

$$H_{et}^{n+1,n}(k, \mathbf{Z}_{(l)}) \to H_{et}^{n+1,n}(k(X), \mathbf{Z}_{(l)})$$

is injective. We will have to assume during the proof that Theorem 6.1 holds in degrees $\leq (n-1)$.

Lemma 6.4 *Assume that Theorem 6.1 holds in degrees $\leq n-1$ and the $\underline{a} = (a_1, \ldots, a_n)$ is a symbol which is not zero in $K_n^M(k)/l$. Then the image of $\underline{a}$ in $H_{et}^n(k, \mu_l^{\otimes n})$ is not zero.*

Proof: By standard transfer argument it is enough to prove the lemma for fields k which have no extensions of degree prime to l. In particular $\mu_l \cong \mathbf{Z}/l$. We proceed by induction on n. We know the statement for $n = 1$. Let

$$E = k[t]/(t^l = a_n)$$

be the cyclic extension of degree l corresponding to a_n and α the class in H_{et}^1 corresponding to a_n. Let γ be the image of $(a_1, \ldots, a_{n-1})$ in H_{et}^{n-1}. By induction we may assume that $\gamma \neq 0$. By [5, Proposition 5.2] we have an exact sequence

$$H_{et}^{n-1}(E, \mathbf{Z}/l) \stackrel{N_{E/k}}{\to} H_{et}^{n-1}(k, \mathbf{Z}/l) \stackrel{\alpha}{\to} H_{et}^n(k, \mathbf{Z}/l) \to H_{et}^n(E, \mathbf{Z}/l)$$

and therefore if $\gamma\alpha = 0$ then $\gamma = N_{E/k}(\gamma')$. In the weight $n-1$ etale cohomology are isomorphic to the Milnor K-theory by our assumption. Therefore $(a_1, \dots, a_{n-1})$ is the norm of an element in $K^M_{n-1}(E)$ and we conclude that

$$(a_1, \dots, a_{n-1}, a_n) = (a_1, \dots, a_{n-1}) \wedge (a_n) = 0$$

Lemma 6.5 *Assume that Theorem 6.1 holds in degrees* $\leq (n-1)$, $\underline{a}$ *is not zero in* $K^M_n(k)/l$ *and* X *is a disjoint union of smooth schemes such that each component of* X *splits* $\underline{a}$. *Then there exists a non-zero element* δ *in* $H^{n,n-1}(\check{C}(X), \mathbf{Z}/l)$.

Proof: Since we assumed the Bloch-Kato conjecture in weight $\leq (n-1)$ we know by [5] that

$$H^{*,n-1}(-, \mathbf{Z}/l) = \mathbf{H}^*_{Nis}(-, B/l(n-1))$$

where $B/l(n-1)$ is the truncation $\tau^{\leq(n-1)}$ of the total direct image of the sheaf $\mu_l^{\otimes(n-1)}$ from the etale to the Nisnevich topology. In particular for any $\mathcal{X}$ one has

$$H^{n,n-1}(\mathcal{X}, \mathbf{Z}/l) = ker(H^n_{et}(\mathcal{X}, \mu_l^{\otimes(n-1)}) \to H^0(\mathcal{X}, \underline{H}^n_{et}(\mathcal{X}, \mu_l^{\otimes(n-1)})))$$

where $\underline{H}^n_{et}$ is the Nisnevich sheaf associated with the presheaf H^n_{et}. For a simplicial scheme $\mathcal{X}$ and any sheaf F we have $H^0(\mathcal{X}, F) \subset H^0(\mathcal{X}_0, F)$ where $\mathcal{X}_0$ is the sero term of $\mathcal{X}$. If $\mathcal{X}_0$ is a disjoint union of smooth schemes and F is a homotopy invariant Nisnevich sheaf with transfers we further have

$$H^0(\mathcal{X}_0, F) \subset \prod_\alpha H^0(Spec(k(X_\alpha)), F)$$

where X_α are the connected components of $\mathcal{X}_0$. Therefore for $\mathcal{X} = \check{C}(X)$ we get

$$H^{n,n-1}(\mathcal{X}, \mathbf{Z}/l) = ker(H^n_{et}(\mathcal{X}, \mu_l^{\otimes(n-1)}) \to \prod_\alpha H^n_{et}(Spec(k(X_\alpha)), \mu_l^{\otimes(n-1)}))$$

where X_α are the connected components of X. If $X \neq \emptyset$ and F is an etale sheaf we have (cf. [5, Lemma 7.2])

$$H^n_{et}(\check{C}, F) = H^n_{et}(Spec(k), F)$$

therefore

$$H^{n,n-1}(\mathcal{X}, \mathbf{Z}/l) =$$
$$= ker(H^n_{et}(Spec(k), \mu_l^{\otimes(n-1)}) \to \prod_\alpha H^n_{et}(Spec(k(X_\alpha)), \mu_l^{\otimes(n-1)})).$$

Recall now that we assumed that k contains a primitive l-th root of unity. Therefore we can replace $\mu_l^{\otimes(n-1)}$ by $\mu_l^{\otimes n}$ and we conclude that $H^{n,n-1}(\mathcal{X}, \mathbf{Z}/l)$ contains

$$ker(H^n_{et}(Spec(k), \mu_l^{\otimes n}) \to \prod_\alpha H^n_{et}(Spec(k(X_\alpha)), \mu_l^{\otimes n}))$$

which is non zero by our condition that each X_α splits $\underline{a}$ and Lemma 6.4.

Set $\mathcal{X} = \check{C}(Y)$ where Y is the disjoint union of all (up to an isomorphism) smooth schemes which split $\underline{a}$ and let $\tilde{\mathcal{X}}$ be the unreduced suspension of $\mathcal{X}$. Note that for a smooth connected variety X one has $M(X) \in DM_{\mathcal{X}}$ if and only if X splits $\underline{a}$.

Lemma 6.6 *Under the assumption that Theorem 6.1 holds in weights $< n$ one has*

$$\tilde{H}^{p,q}(\tilde{\mathcal{X}}, \mathbf{Z}/l) = 0$$

for all $q \leq n-1$ and $p \leq q+1$.

Proof: By [5, Cor. 6.7] and our assumption that Theorem 6.1 holds in weights $< n$ we conclude that for $q \leq n-1$ and $p \leq q+1$ we have

$$H^{p,q}(\tilde{\mathcal{X}}, \mathbf{Z}/l) \subset H^{p,q}_{et}(\tilde{\mathcal{X}}, \mathbf{Z}/l).$$

The right hand side group is zero for all p and q by [5, Lemma 7.2].

Lemma 6.7 *Let δ be as in Lemma 6.5. Then*

$$Q_{n-1} \dots Q_0(\delta) \neq 0$$

Proof: The cofibration sequence which defines $\tilde{\mathcal{X}}$ gives us a homomorphism $H^{p,q}(\mathcal{X}) \to H^{p+1,q}(\tilde{\mathcal{X}})$ which is a monomorphism for $p > q$. Let $\tilde{\delta}$ be the image of δ in $H^{n+1,n-1}(\tilde{\mathcal{X}})$. Since $\delta \neq 0$ we have $\tilde{\delta} \neq 0$. Let us show that

$$Q_i \dots Q_0(\tilde{\delta}) \neq 0$$

for all $i < n$. Assume by induction that

$$Q_{i-1} \dots Q_0(\tilde{\delta}) \neq 0$$

By Theorem 6.3 there exists a $\nu_{\leq(n-1)}$-variety X which splits $\underline{a}$. By our construction we have $M(X) \in DM_{\mathcal{X}}$. Therefore by Lemma 4.3 the motivic Margolis homology $\tilde{MH}_i^{*,*}$ of $\tilde{\mathcal{X}}$ are zero for all $i < n$. Hence $Q_i \dots Q_0(\tilde{\delta}) = 0$ if and only if there exists u such that

$$Q_i(u) = Q_{i-1} \dots Q_0(\tilde{\delta}) \tag{6.1}$$

Let us make some degree computations which will also be useful below. The composition $Q_{i-1} \dots Q_0$ shifts dimension by

$$1 + 2l - 1 + \dots + 2l^{i-1} - 1 = -i + 2l(l^{i-1} - 1)/(l - 1) + 2$$

and weight by

$$0 + l - 1 + \dots + l^i - 1 = -i + l(l^{i-1} - 1)/(l - 1) + 1$$

Therefore the kernel of Q_i on $Q_{i-1} \dots Q_0(\tilde{H}^{p,q}(-,-))$ is covered by the group of dimension

$$-i + 2l(l^{i-1} - 1)/(l - 1) + 2 - 2l^i + 1 = -i + 2lw + 3$$

and weight

$$-i + l(l^{i-1} - 1)/(l - 1) + 1 - l^i + 1 = -i + lw + 2$$

where $w = (l^{i-1} - 1)/(l - 1) - l^i$. Note that $w \leq -1$ and $lw \leq -2$. Therefore the bidegree of u in (6.1) is $(n+1-i+2lw+3, n-1-i+lw+2)$. We conclude that the weight of u is $\leq n - 1$ and the difference between the dimension and the weight is

$$n + 1 - i + 2lw + 3 - (n - 1 - i + lw + 2) = 3 + lw \leq 1$$

By Lemma 6.6 we conclude that $u = 0$ which contradicts our inductive assumption that $Q_{i-1} \dots Q_0(\tilde{\delta}) \neq 0$.

Define μ as in (5.2) starting with δ and let M_i be the motive defined by (5.4). In view of Lemma 6.7 the results of the previous section are applicable. In particular Proposition 5.18 implies the following.

Lemma 6.8 *Let X be a ν_{n-1}-variety which splits $\underline{a}$. Then*

$$M(\mathcal{X}) = M(\check{C}(X)).$$

Lemma 6.9 *Let X be a ν_{n-1}-variety which splits $\underline{a}$. Then there is an exact sequence*

$$H^{n+1,n}(\mathcal{X}, \mathbf{Z}_{(l)}) \to H^{n+1,n}_{et}(k, \mathbf{Z}_{(l)}) \to H^{n+1,n}_{et}(k(X), \mathbf{Z}_{(l)})$$

Proof: The morphism $Spec(k(X)) \to Spec(k)$ admits a decomposition

$$Spec(k(X)) \to X \to \mathcal{X} \to Spec(k)$$

where the middle arrow is the natural morphism from X to $\mathcal{X}$. By [5, Lemma 7.2] the last arrow defines an isomorphism on $H^{n+1,n}_{et}(-, \mathbf{Z}_{(l)})$. Therefore it is sufficient to show that the sequence

$$H^{n+1,n}(\mathcal{X}, \mathbf{Z}_{(l)}) \to H^{n+1,n}_{et}(\mathcal{X}, \mathbf{Z}_{(l)}) \to H^{n+1,n}_{et}(k(X), \mathbf{Z}_{(l)})$$

is exact. The composition of two morphisms is zero because it factors through

$$H^{n+1,n}(k(X), \mathbf{Z}_{(l)}) = 0$$

Let $\mathbf{Z}^{et}_{(l)}(n)$ be the object in $DM^{eff}_{-}(k)$ which represents the etale motivic cohomology of weight n and let $L(n)$ be its canonical truncation at the level $n+1$ (see [5, p.32]). Consider a distinguished triangle of the form

$$\mathbf{Z}_{(l)}(n) \to L(n) \to K(n) \to \mathbf{Z}_{(l)}(n)[1]$$

where the first arrow corresponds to the natural morphism

$$\mathbf{Z}_{(l)}(n) \to \mathbf{Z}^{et}_{(l)}(n).$$

Let x be an element in

$$H^{n+1,n}_{et}(\mathcal{X}, \mathbf{Z}_{(l)}) = \mathbf{H}^{n+1}(\mathcal{X}, L(n))$$

which goes to zero in

$$H^{n+1,n}_{et}(k(X), \mathbf{Z}_{(l)}) = \mathbf{H}^{n+1}(k(X), L(n)).$$

We have to show that the image x' of x in $\mathbf{H}^{n+1}(\mathcal{X}, K(n))$ is zero. By [5, Lemma 7.4] x' maps to zero in $\mathbf{H}^{n+1}(X, K(n))$. By Lemma 5.11 we know that the morphism from $M(X)$ to $M(\mathcal{X})$ factors as

$$M(X_{\underline{a}}) \xrightarrow{\lambda} M_{l-1} \to M(\mathcal{X}) \tag{6.2}$$

where the first arrow is a split epimorphism by Lemma 5.15. By [5, Lemma 6.5], $L(n)$ and $K(n)$ are complexes of sheaves with transfers with homotopy invariant cohomology sheaves. Therefore $Hom_{DM}(M_{l-1}, K(n)[n+1])$ is defined and (6.2) shows that the image of x' in $Hom_{DM}(M_{l-1}, K(n)[n+1])$ is zero. We conclude that $x' = 0$ from (5.5) and the following lemma.

Lemma 6.10 $Hom_{DM}(M_{l-2}(b)[2b], K(n)[n+1]) = 0.$

Proof: Using the distinguished triangles for M_i it is sufficient to show that

$$Hom_{DM}(M(\mathcal{X})(q)[2q], K(n)[n+1]) = 0$$

for all $q > 0$. This is an immediate corollary of [5, Lemma 7.3] and our assumption that Theorem 6.1 holds in weights $< n$.

In view of Lemma 6.9 in order to finish the proof of Theorem 6.1 it remains to prove the following result.

Proposition 6.11 $H^{n+1,n}(\mathcal{X}, \mathbf{Z}_{(l)}) = 0$

The proof is given in Lemmas 6.12-6.15 below.

Lemma 6.12 *There is a monomorphism*

$$H^{n+1,n}(\mathcal{X}, \mathbf{Z}_{(l)}) \to H^{2lb+2,lb+1}(\mathcal{X}, \mathbf{Z}_{(l)}) \tag{6.3}$$

Proof: The cofibration sequence which defines $\tilde{\mathcal{X}}$ implies that it is enough to show that there is a monomorphism

$$H^{n+2,n}(\tilde{\mathcal{X}}, \mathbf{Z}_{(l)}) \to H^{2lb+3,lb+1}(\mathcal{X}, \mathbf{Z}_{(l)})$$

Let X be a $\nu_{\leq(n-1)}$ variety which splits $\underline{a}$. Since X is a $\nu_{\leq 0}$-variety it has a point over a finite field extension of degree not divisible by l^2. Therefore, the motivic cohomology of $\tilde{\mathcal{X}}$ are of exponent l by [5, Lemma 9.3]. Therefore the projection from the motivic cohomology with the $\mathbf{Z}_{(l)}$ coefficients to the motivic cohomology with the $\mathbf{Z}/l$ coefficients is injective. Therefore it is sufficient to show that there is a monomorphism

$$H^{n+2,n}(\tilde{\mathcal{X}}, \mathbf{Z}/l) \to H^{2lb+3,lb+1}(\mathcal{X}, \mathbf{Z}/l) \tag{6.4}$$

which takes the images of the integral classes to the images of the integral classes. Consider the composition of cohomological operations

$$Q_i \dots Q_1 : H^{n+2,n}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbf{Z}/l) \to H^{2l(l^i-1)/(l-1)+n+2-i,l(l^i-1)/(l-1)+n-i}(\tilde{\mathcal{X}}_{\underline{a}}, \mathbf{Z}/l) \tag{6.5}$$

For $i = n-1$ it is of the form (6.4) and we know by [5, Lemma 7.5] that Q_i take the images of integral classes to the images of integral classes. Let us show that it is a mono for all $i \leq n-1$. By Lemma 4.3 we know that the motivic Margolis homology of $\tilde{\mathcal{X}}$ are zero. The computations made in the proof of Lemma 6.7 show that the kernel of Q_i on $Q_{i-1} \dots Q_1(H^{n+2,n})$ is covered by the group of bidegree (p, q) where

$$p = 4 + 2lw + n - i$$

$$q = 2 + lw + n - i$$

$$w = (l^{i-1} - 1)/(l-1) - l^{i-1}.$$

We have $w \leq -1$ and therefore $q \leq n - i$ and $p \leq q$. We conclude that the covering group is zero by Lemma 6.6.

Lemma 6.13 *There is an epimorphism*

$$ker(H^{2b(l-1)+1,b(l-1)+1}(M_{l-1}, \mathbf{Z}_{(l)}) \to H^{1,1}(\mathcal{X}, \mathbf{Z}_{(l)})) \to H^{2lb+2,lb+1}(\mathcal{X}, \mathbf{Z}_{(l)})$$

Proof: Let X be a ν_{n-1}-variety which splits $\underline{a}$. Consider the sequences (5.5) and (5.6) for $i = l-1$. By Lemma 5.15, the motivic cohomology of M_{l-1} embed into the motivic cohomology of X and in particular vanish where the motivic cohomology of X vanish.

From the first sequence and the fact that $lb + 1 > (l-1)b = dim(X)$ we conclude that there is an epimorphism

$$H^{2b(l-1)+1,b(l-1)+1}(M_{l-2}, \mathbf{Z}_{(l)}) \to H^{2lb+2,lb+1}(\mathcal{X}, \mathbf{Z}_{(l)}) \tag{6.6}$$

From the second sequence and the fact that $H^{0,1}(\mathcal{X}, \mathbf{Z}_{(l)}) = 0$ we conclude that the left hand side of (6.6) is the kernel of the homomorphism

$$H^{2b(l-1)+1,b(l-1)+1}(M_{l-1}, \mathbf{Z}_{(l)}) \to H^{1,1}(\mathcal{X}, \mathbf{Z}_{(l)}).$$

Lemma 6.14 *One has:*

$$ker(H^{2b(l-1)+1,b(l-1)+1}(M_{l-1},\mathbf{Z}_{(l)})\to H^{1,1}(\mathcal{X},\mathbf{Z}_{(l)}))=$$
$$=ker(Hom(\mathbf{Z}_{(l)},M_{l-1}(1)[1])\to Hom(\mathbf{Z}_{(l)},\mathbf{Z}_{(l)}(1)[1]))$$

Proof: Since the motivic cohomology in the bidegree $(1,1)$ in the Zariski and the etale topologies coincide and the etale motivic cohomology of $\mathcal{X}$ coincide with the etale motivic cohomology of the point we have

$$H^{1,1}(\mathcal{X},\mathbf{Z}_{(l)})=H^{1,1}(Spec(k),\mathbf{Z}_{(l)})$$

By duality estabilished in Corollary 5.17 we have

$$H^{2b(l-1)+1,b(l-1)+1}(M_{l-1},\mathbf{Z}_{(l)})=Hom(\mathbf{Z}_{(l)},M_{l-1}(1)[1])$$

and one verifies easily that the dual of the morphism

$$\tau_M:\mathbf{Z}(d)[2d]\to M_{l-1}$$

is the morphism $\pi_M:M_{l-1}\to\mathbf{Z}$. The statement of the lemma follows.

Lemma 6.15 *The homomorphism*

$$Hom(\mathbf{Z}_{(l)},M_{l-1}(1)[1])\to Hom(\mathbf{Z}_{(l)},\mathbf{Z}_{(l)}(1)[1])$$

is a monomorphism.

Proof: The distinguished triangle (5.5) together with the obviuous fact that

$$Hom(\mathbf{Z},M(\mathcal{X}(bj)[2bj]))=0$$

for $j>0$, implies that the homomorphism

$$Hom(\mathbf{Z}_{(l)},M_{l-1}(1)[1])\to Hom(\mathbf{Z}_{(l)},M(\mathcal{X})(1)[1])$$

is a monomorphism. It remains to see that

$$Hom(\mathbf{Z}_{(l)},M(\mathcal{X})(1)[1])\to Hom(\mathbf{Z},\mathbf{Z}(1)[1])=k^* \tag{6.7}$$

is a monomorphism. By Lemma 6.8 we may assume that $\mathcal{X}=\check{C}(X)$ where X is a smooth variety satisfying the conditions of Theorem 6.3. The spectral sequence which starts from motivic homology of X and converges to the motivic homology of $\mathcal{X}$ shows that

$$Hom(\mathbf{Z}_{(l)},M(\mathcal{X})(1)[1])=coker(H_{-1,-1}(X^2,\mathbf{Z})\overset{pr_1-pr_2}{\to}H_{-1,-1}(X,\mathbf{Z}))$$

We conclude that (6.7) is a mono by Theorem 6.3.

7 Generalizations and corollaries

This section contains a number of corollaries of the Theorem 6.1. The standard proofs are ommited and will be added in a later version.

Theorem 7.1 *Let k be a field of characteristic $\neq l$. Then the norm residue homomorphisms*

$$K_n^M(k)/l \to H_{et}^n(k, \mu_l^{\otimes n})$$

are isomorphisms for all n.

Theorem 7.2 *Let k be a field and $\mathcal{X}$ a pointed smooth simplicial scheme over k. Then one has:*

1. *for any $n > 0$ the homomorphisms*

$$\tilde{H}^{p,q}(\mathcal{X}, \mathbf{Z}/n) \to \tilde{H}_{et}^{p,q}(\mathcal{X}, \mathbf{Z}/n)$$

   are isomorphisms for $p \leq q$ and monomorphisms for $p = q + 1$

2. *the homomorphisms*

$$\tilde{H}^{p,q}(\mathcal{X}, \mathbf{Z}) \to \tilde{H}_{et}^{p,q}(\mathcal{X}, \mathbf{Z})$$

   are isomorphisms for $p \leq q + 1$ and monomorphisms for $p = q + 2$

Let X be a splitting variety for a symbol $\underline{a}$. Recall that X is called a generic splitting variety if for any field E over k such that $\underline{a} = 0$ in $K_n^M(E)/l$ there exists a zero cycle on X of degree prime to l.

Theorem 7.3 *Let l be a prime and k be a field of chracteristic zero. Let further $\underline{a} = (a_1, \ldots, a_n)$ be a sequence of invertible elements of k and X be ν_{n-1}-variety which splits $\underline{a}$. Then X is a generic splitting variety for $\underline{a}$.*

Proof: It is a reformulation of Lemma 6.8.

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