GROTHENDIECK-WITT GROUPS OF PROJECTIVE BUNDLES

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Abstract. The derived Witt and Grothendieck-Witt groups of a projective-space bundle over a noetherian scheme are calculated in terms of those of the base using derived-category methods. All shifts and twists are treated. The formulas obtained differ sharply from the classical projective bundle formula for Chow groups and other oriented cohomology theories.

Introduction

The 'usual' projective bundle formula for cohomology theories in algebraic geometry and in K-theory says that if F is a vector bundle of rank $r+1$ over a scheme X, and if $P := \mathbb{P}(F)$ is the associated bundle of projective r-spaces, then for any 'standard' cohomology theory H there is an additive isomorphism which is roughly $H(P) \cong H(X)^{\oplus(r+1)}$. Often the $r+1$ copies of $H(X)$ are not completely identical because of a shift in the grading or some other sort of twisting, so one might prefer to write $H(P) \cong \bigoplus_{i=0}^{r} H(X) \cdot \gamma^i$ with γ some specified class. The multiplicative structure is then determined by a relation $\gamma^{r+1} + a_1\gamma^r + \cdots + a_r\gamma + a_{r+1} = 0$. Cohomology theories satisfying these formulas include K_0 (Grothendieck-Berthelot [10]), higher algebraic K-theory (Quillen [23]), Chow groups, singular homology (for complex analytic varieties), and many others. Recent results of Levine and Morel [21] show the existence of an algebraic cobordism theory which is universal among 'oriented' theories satisfying the standard projective bundle formula and a handful of other standard formulas. But in any case, the standard formula is basically $H(P) \cong H(X)^{\oplus(r+1)}$.

The simplest cohomology theories dealing with symmetric or skew-symmetric bilinear forms are the Grothendieck-Witt groups and their quotients the Witt groups. These parametrize vector bundles over X with nonsingular symmetric or skew-symmetric bilinear forms, or more generally chain complexes $\mathcal{E}$ of vector bundles on X with a quasi-isomorphism into a twisted shifted dual $\phi : \mathcal{E} \xrightarrow{\sim} \mathcal{E}^\vee \otimes L[i]$ which is symmetric ($\phi = \phi^t$). The Witt groups $W^i(X, L)$ and the Grothendieck-Witt groups $GW^i(X, L)$ thus depend not only on the scheme X but also on a line bundle L and an integer i which twist and shift the duality of chain complexes. They are periodic modulo 4 in i and modulo 2 in L, i.e. there are natural isomorphisms $W^i(X, L) \cong W^{i+4}(X, L \otimes M^{\otimes 2})$ and $GW^i(X, L) \cong GW^{i+4}(X, L \otimes M^{\otimes 2})$. Twenty-some years ago Arason [1] showed that for projective space over a field one has $W^0(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r}) \cong W(k)$. This is a case with $W(P) \cong W(X)$, so untwisted, unshifted Witt groups are not an oriented cohomology theory. Some further work has been done on Witt groups of projective spaces and projective bundles over the years (Arason [2], Gille [15][16], Schmid [25], Szyjewski [26]), but the picture has remained incomplete. In this paper we attack the problem systematically and calculate the Witt and Grothendieck-Witt groups of projective bundles over a scheme.

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Even taking shifts and twists into account, the formulas we get remain quite different from the usual projective bundle formulas.

For Witt groups we have the following results. Suppose X is a noetherian scheme with $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$. When the dimension r of the fibers of $\pi : P = \mathbb{P}(F) \to X$ is even, we have isomorphisms of derived Witt groups

$$\pi^* : W^i(X, L) \xrightarrow{\cong} W^i(P, \pi^* L), \qquad \pi_* : W^i(P, \pi^* L \otimes \omega_{P/X}) \xrightarrow{\cong} W^{i-r}(X, L)$$

induced by the inverse image π^* and the derived direct image $\mathbf{R}\pi_*$ (Theorems 1.1 and 1.2). If the fiber dimension r is odd, then when the fiberwise degree of the twist is odd we get $W^i(P, \pi^* L \otimes \mathcal{O}_P(2\ell + 1)) = 0$ (Theorem 1.3), but when the fiberwise degree is even we get a long exact sequence (Theorem 1.4)

$$(1) \qquad \cdots \to W^i(X, L) \xrightarrow{\pi^*} W^i(P, \pi^* L) \xrightarrow{\pi_* \circ \mathrm{per}} W^{i-r}(X, L \otimes \det F^\vee) \xrightarrow{\theta} W^{i+1}(X, L) \to \cdots$$

where per is a periodicity isomorphism $W^i(P, \pi^* L) \cong W^i(P, \pi^*(L \otimes \det F^\vee) \otimes \omega_{P/X})$ due to the fact that $\pi^* \det F^\vee \otimes \omega_{P/X} \cong \mathcal{O}_P(-r-1)$ is divisible by 2 in $\mathrm{Pic}(P)$, while θ is essentially multiplication with the class in $W^{r+1}(X, \det F)$ of the $(-1)^{(r+1)/2}$-symmetric bilinear form $\Lambda^{(r+1)/2} F \times \Lambda^{(r+1)/2} F \to \det F$ given by the exterior multiplication. If $\theta = 0$ then the exact sequence splits, and

$$W^i(P, \pi^* L) \cong W^i(X, L) \oplus W^{i-r}(X, L \otimes \det F^\vee)$$

(Theorem 1.5). Defining the total derived Witt group as $W^{\mathrm{tot}}(X) = \bigoplus_{i,L} W^i(X, L)$ with the sum running over $i \in \mathbb{Z}/4$ and $L \in \mathrm{Pic}(X)/2$, we find that if r is even or $\theta = 0$ then $W^{\mathrm{tot}}(P)$ is a $W^{\mathrm{tot}}(X)$-algebra (via π^*) of the form

$$(2) \qquad W^{\mathrm{tot}}(P) = W^{\mathrm{tot}}(X) \oplus W^{\mathrm{tot}}(X) \cdot \xi$$

with $\xi \in W^r(P, \omega_{P/X})$ satisfying $\pi_*(\xi) = 1 \in W^0(X, \mathcal{O}_X)$.

Under certain circumstances we know that the exact sequence splits. For instance if $\pi : P \to X$ has a global section, or more generally if it has a linear subbundle of even relative dimension over X, then we have $\theta = 0$ (Proposition 8.1). If $\pi : P \to X$ has two disjoint global sections, then in addition we have $\xi \cdot \xi = 0$ (Theorem 8.3). This completely determines the multiplication under these extra hypotheses. These extra hypotheses hold for $P = \mathbb{P}^r_X$, but also for projective bundles of the form $P = \mathbb{P}(F_0 \oplus L' \oplus L'')$ with L' and L'' line bundles on X, and also whenever X is quasi-projective over an infinite field and $r > \dim X$.

If R is a noetherian local ring containing $\frac{1}{2}$, then $\mathrm{Pic}(\mathbb{P}^r_R) \cong \mathbb{Z}$, so there are eight derived Witt groups $W^i(\mathbb{P}^r_R, \mathcal{O}_{\mathbb{P}^r}(\ell))$ for $(i, \ell) \in \mathbb{Z}/4 \times \mathbb{Z}/2$. Since Balmer ([5] Theorem 5.6) showed that $W^i(R) = 0$ for $i \not\equiv 0 \pmod 4$, and since $\mathbb{P}^r_R$ has two disjoint global sections, the formulas above yield two isomorphisms

$$(3) \qquad W^0(\mathbb{P}^r_R, \mathcal{O}_{\mathbb{P}^r}) \underset{\cong}{\xleftarrow{\pi^*}} W(R), \qquad W^r(\mathbb{P}^r_R, \mathcal{O}_{\mathbb{P}^r}(-r-1)) \underset{\cong}{\xrightarrow{\pi_*}} W(R)$$

and six vanishings $W^i(\mathbb{P}^r_R, \mathcal{O}_{\mathbb{P}^r}(\ell)) = 0$ for $(i, \ell) \not\equiv (0, 0)$ or $(r, -r-1)$ in $\mathbb{Z}/4 \times \mathbb{Z}/2$. So when r is odd, two untwisted derived Witt groups $W^0(\mathbb{P}^r_R)$ and $W^r(\mathbb{P}^r_R)$ are nonzero. When r is even, only one untwisted derived Witt group $W^0(\mathbb{P}^r_R)$ is nonzero because the second nonzero group $W^r(\mathbb{P}^r_R, \mathcal{O}_{\mathbb{P}^r}(-r-1))$ has an odd twist.

The long exact sequence does not always split. We give examples in §9 where P is a $\mathbb{P}^1$ bundle over the projective plane $X = \mathbb{P}^2_k$, and $L = \omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$, and $\theta \in W^2(\mathbb{P}^2_k, \omega_{\mathbb{P}^2}) \cong W(k)$ is the Witt class of an arbitrary nondegenerate quadratic form over k of dimension 0, 1, and 2. In the dimension 0 example, we have $\theta = 0$, and P is the blowup of $\mathbb{P}^3_k$ at a rational point. In the dimension 1 example, θ is an isomorphism, and P is the flag variety for $\mathrm{GL}_3(k)$. In the dimension 2 examples, θ is not an isomorphism but need not be 0.

We also compute the derived Grothendieck-Witt groups $GW^i(P, \pi^*L \otimes \mathcal{O}_P(\ell))$. These are the most important calculations in the paper, because they give a better indication of what should happen in higher hermitian K-theory than the formulas for Witt groups do. The results obtained are similar to but rather more complicated than those for Witt groups, so we will describe them in detail in §1.

The results of this paper are all stated for the "usual" derived Witt and Grothendieck-Witt groups of bounded complexes of vector bundles, but they are also valid for derived Witt and Grothendieck-Witt groups of perfect complexes and for Gille's [16] coherent derived Witt groups (and their Grothendieck-Witt analogues) of complexes of quasi-coherent sheaves with bounded coherent cohomology with duality with respect to a dualizing complex ω_X or some $L \otimes \omega_X[i]$ with L a line bundle and i an integer shift. The only restriction is that for the coherent Witt and Grothendieck-Witt groups one must work on noetherian schemes with $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$ which are of finite Krull dimension and have a dualizing complex ω_X.

The contents of the paper are as follows. In §1 we state the main theorems describing in detail the derived Witt and Grothendieck-Witt groups of a projective bundle. Then §2 contains a review of triangulated Grothendieck-Witt theory, while §3 describes the Bernstein-Gelfand-Gelfand correspondence and then treats its compatibility with duality. After this we prove two of our main theorems in §4. The other three main theorems require more work in §5 on the dévissage of triangulated categories with duality and a finer analysis of the duality in the Bernstein-Gelfand-Gelfand correspondence in §6. We prove the remaining main theorems in §7. The ring structure of $W^{\mathrm{tot}}(\mathbb{P}^r_X)$ is studied in §8, and the examples of $\mathbb{P}^1$ bundles over $\mathbb{P}^2$ are in §9. Finally in §11 we give a proof of the critical derived equivalence $\mathrm{D^b}(\mathbf{VB}_P) \simeq \mathrm{D^b}(\mathbf{LC}^{[-r,0]})$ between chain complexes of vector bundles on P and chain complexes of linear complexes, using a Beilinson-style resolution of the diagonal of $P \times_X P$. We use an adjunction between categories of chain complexes and chain maps which becomes an adjoint equivalence after localization. This should be useful for higher quadratic K-theory, or even for Witt groups in characteristic 2 (i.e. symmetric and quadratic L-theory). In principle the same adjunction exists even for projective bundles which are étale but not Zariski locally trivial, but the categories involved would have a different description.

1. The main theorems

We wish to compute the Grothendieck-Witt groups $GW^n(P, \pi^*L \otimes \mathcal{O}_P(\ell))$ and the Witt groups $W^n(P, \pi^*L \otimes \mathcal{O}_P(\ell))$. Because these groups are periodic modulo 2 in the line-bundle argument, we need to compute these groups for each P and L for one even twist ℓ and one odd twist ℓ. The maps $\mathbf{H} : K_0(P) \to GW^n(P, \pi^*L \otimes \mathcal{O}_P(\ell))$ are the maps sending $[\mathcal{E}] \mapsto [\![\mathcal{E} \oplus \mathcal{E}^*(\ell)[n], \left(\begin{smallmatrix} 0 & 1 \\ \varpi_n & 0 \end{smallmatrix}\right)]\!]$ where $\varpi_n = (-1)^{\lceil n/2 \rceil} \varpi$ is the shifted biduality map. Our first main theorem considers r even and ℓ even, e.g. $\ell = 0$..

Theorem 1.1. *If r is even, then the maps $\pi^* : W^n(X, L) \cong W^n(P, \pi^*L)$ are isomorphisms as are the maps*

$$GW^n(X, L) \oplus K_0(X)^{r/2} \xrightarrow{(\pi^*, H)\ \cong} GW^n(P, \pi^*L)$$

where $H(e_1, \ldots, e_{r/2}) = \sum_{i=1}^{r/2} \mathbf{H}(\pi^* e_i \cdot [\mathcal{O}_P(i)])$.

When r is even but ℓ is odd, e.g. $\ell = -r-1$, let $\Omega \in GW^r(P, \det F \otimes \mathcal{O}_P(-r-1))$ be the class of the nonsingular $(-1)^{r/2}$-symmetric form $\Omega^{r/2}_{P/X} \times \Omega^{r/2}_{P/X} \to \omega_{P/X} \cong \det F \otimes \mathcal{O}_P(-r-1)$ given by exterior multiplication.

Theorem 1.2. *If r is even, then there are inverse isomorphisms and a split exact sequence*

$$W^n(P, \pi^* L \otimes \mathcal{O}_P(-r-1)) \underset{\pi_* \ \cong}{\overset{\Omega\ \cong}{\rightleftarrows}} W^{n-r}(X, L \otimes \det F^\vee),$$

$$0 \longrightarrow K_0(X)^{r/2} \xrightarrow{H} GW^n(P, \pi^* L \otimes \mathcal{O}_P(-r-1)) \underset{\pi_*}{\overset{\Omega}{\rightleftarrows}} GW^{n-r}(X, L \otimes \det F^\vee) \longrightarrow 0$$

with $H(e_1, \dots, e_{r/2}) = \sum_{i=1}^{r/2} \mathbf{H}(\pi^* e_i \cdot [\mathcal{O}_P(-i)])$, with π_* coming from a duality-preserving functor whose functor component is the derived direct image $\mathbf{R}\pi_* : \mathrm{D}^{\mathrm{b}}(\mathbf{VB}_P) \to \mathrm{D}^{\mathrm{b}}(\mathbf{VB}_X)$, and with the arrow labeled Ω given by $x \mapsto \Omega \cdot \pi^* x$.

When r is odd and ℓ is odd, e.g. $\ell = -r$, we have the next theorem.

Theorem 1.3. *For r odd we have $W^n(P, \pi^* L \otimes \mathcal{O}_P(-r)) = 0$, and we have isomorphisms*

$$K_0(X)^{(r+1)/2} \xrightarrow{H\ \cong} GW^n(P, \pi^* L \otimes \mathcal{O}_P(-r))$$

given by $H(e_0, \dots, e_{(r-1)/2}) = \sum_{i=0}^{(r-1)/2} \mathbf{H}(\pi^* e_i \cdot [\mathcal{O}_P(-i)])$.

If r is odd, and ℓ is even, e.g. $\ell = 0$, then let $u : \Lambda^{(r+1)/2} F \times \Lambda^{(r+1)/2} F \to \Lambda^{r+1} F = \det F$ be the $(-1)^{(r+1)/2}$-symmetric bilinear form given by the multiplication, and let $\Theta = [\Lambda^{(r+1)/2} F, (-1)^{\lfloor (r-3)/4 \rfloor} u] \in W^{r+1}(X, \det F)$.

Theorem 1.4. *If r is odd, then there is a long exact sequence of Witt groups*

$$\cdots \to W^n(X, L) \xrightarrow{\pi^*} W^n(P, \pi^* L) \xrightarrow{\pi_* \circ \mathrm{per}} W^{n-r}(X, L \otimes \det F^\vee) \xrightarrow{\theta} W^{n+1}(X, L) \to \cdots$$

which can be extended to the "left" as

$$\begin{array}{ccccc} GW^{n-r}(X, L \otimes \det F^\vee) & \xleftarrow[\pi_* \circ \mathrm{per}]{} & GW^n(P, \pi^* L) & \xleftarrow[(\pi^*, H)]{} & GW^n(X, L) \oplus K_0(X)^{(r-1)/2} \\ \downarrow \boldsymbol{\theta} & & & & \\ W^{n+1}(X, L) & \longrightarrow & W^{n+1}(P, \pi^* L) & \longrightarrow & W^{n-r+1}(X, L \otimes \det F^\vee) \longrightarrow \cdots \end{array}$$

Here π^* is the pullback, and H is $(e_1, \dots, e_{(r-1)/2}) \mapsto \sum_{i=1}^{(r-1)/2} \mathbf{H}(\pi^* e_i \cdot [\mathcal{O}_P(i)])$. The arrow $\pi_* \circ \mathrm{per}$ is the composition

$$GW^n(P, \pi^* L) \cong GW^n(P, \pi^* L \otimes \mathcal{O}_P(-r-1)) \xrightarrow{\pi_*} GW^{n-r}(X, L \otimes \det F^\vee)$$

of a periodicity isomorphism with a map coming from a duality-preserving functor whose functor component is the derived direct image $\mathbf{R}\pi_* : \mathrm{D}^{\mathrm{b}}(\mathbf{VB}_P) \to \mathrm{D}^{\mathrm{b}}(\mathbf{VB}_X)$. The map θ is given by multiplication by Θ, while the map marked $\boldsymbol{\theta}$ is the composition of the natural surjection $GW^{n-r} \twoheadrightarrow W^{n-r}$ with θ. In particular, if $\Theta \neq 0$, then the maps θ and $\boldsymbol{\theta}$ do not vanish when $n = r$ and $L = \det F$.

Theorem 1.5. *In the situation of Theorem 1.4 if we have $\Theta = 0$ in $W^{r+1}(X, \det F)$, then the exact sequences split*

$$0 \to W^n(X, L) \xrightarrow{\pi^*} W^n(P, \pi^* L) \underset{\pi_* \mathrm{oper}}{\overset{\Psi}{\leftrightarrows}} W^{n-r}(X, L \otimes \det F^\vee) \to 0,$$

$$0 \to GW^n(X, L) \times K_0(X)^{(r-1)/2} \xrightarrow{(\pi^*, H)} GW^n(P, \pi^* L) \underset{\pi_* \mathrm{oper}}{\overset{\Psi}{\leftrightarrows}} GW^{n-r}(X, L \otimes \det F^\vee) \to 0,$$

with the maps labeled Ψ having the form $x \mapsto \pi^* x \cdot \Psi$ for a class $\Psi \in GW^r(P, \pi^* \det F)$.

Theorems 1.1 and 1.3 are proven in §4. Theorems 1.2, 1.4, and 1.5 are proven in §7. The proof of Theorem 1.5 constructs an explicit Ψ, but it depends nontrivially on certain choices.

2. Review of basic theory

In this section we review the theory of triangulated categories with duality and their Grothendieck-Witt groups developed in Walter [27]. The corresponding theory of Witt groups was developed by Balmer [3] [4] [5] [7].

An additive category *contains* $\frac{1}{2}$ if its Hom groups are uniquely 2-divisible. A *duality* on a triangulated category $\boldsymbol{D}$ containing $\frac{1}{2}$ is a triple $(^*, \delta, \varpi)$ where $\delta = \pm 1$, $^* : \boldsymbol{D}^{\mathrm{op}} \to \boldsymbol{D}$ a δ-exact contravariant functor, and $\varpi : 1_{\boldsymbol{D}} \cong {}^{**}$ an isomorphism of functors which commutes with the translation and such that for any Y in $\boldsymbol{D}$ the composition $\varpi_Y^* \varpi_{Y^*} : Y^* \to Y^{***} \to Y^*$ is the identity 1_{Y^*}. Given one duality $(^*, \delta, \varpi)$, there are shifted dualities $(^*, \delta, \varpi)[n] = (^*[n], (-1)^n \delta, (-1)^{\lceil n/2 \rceil} \delta^n \varpi)$. Any of these dualities may be used to define transposition operations $\mathrm{Hom}_{\boldsymbol{D}}(Y, Z^*[n]) \cong \mathrm{Hom}_{\boldsymbol{D}}(Z, Y^*[n])$ sending $\phi : Y \to Z^*[n]$ to $\phi^{\mathrm{t}} : Z \to Y^*[n]$ such that $\phi^{\mathrm{tt}} = \phi$. A *symmetric object* in $\underline{\boldsymbol{D}}[n] = (\boldsymbol{D}, (^*, \delta, \varpi)[n])$ is a pair (Y, ϕ) with $\phi : Y \cong Y^*[n]$ an isomorphism satisfying $\phi = \phi^{\mathrm{t}}$. A *hyperbolic* symmetric object is one of the form $\mathbf{H}(U) = (U \oplus U^*[n], \left(\begin{smallmatrix} 0 & 1 \\ \varpi_n & 0 \end{smallmatrix}\right))$ where $\varpi_n = (-1)^{\lceil n/2 \rceil} \delta^n \varpi$ is the shifted biduality map with the sign given above.

The triangulated Grothendieck-Witt and Witt groups of $\underline{\boldsymbol{D}}$, written $GW^n(\underline{\boldsymbol{D}})$ and $W^n(\underline{\boldsymbol{D}})$ classify symmetric objects in $\underline{\boldsymbol{D}}[n]$ modulo certain relations (see [27] §2 for the definitions). The Witt groups are quotients of the Grothendieck-Witt groups modulo the classes of the hyperbolic symmetric forms.

The triangulated Grothendieck-Witt and Witt groups are functorial for duality-preserving functors between triangulated categories with duality containing $\frac{1}{2}$, with isomorphic duality-preserving functors sent to the same morphism of groups. Therefore duality-preserving equivalences of triangulated categories with duality induce isomorphisms of Grothendieck-Witt and Witt groups ([27] Proposition 2.1, [7] Lemma 4.1).

The Localization Theorem for triangulated Witt groups was proven by Balmer [4] Theorem 6.2 and extended to triangulated Grothendieck-Witt groups by Walter [27] Theorem 2.4. A (TR4+) triangulated category is a triangulated category satisfying an enhanced octahedral axiom. All triangulated categories appearing in this paper do.

Theorem 2.1 (Localization)**.** *Let $\underline{\boldsymbol{D}} \subset \underline{\boldsymbol{C}}$ be a thick invariant subcategory of a* (TR4+) *triangulated category with duality containing $\frac{1}{2}$. Then there is a long exact sequence of triangulated Witt groups*

$$\cdots \to W^n(\underline{\boldsymbol{D}}) \to W^n(\underline{\boldsymbol{C}}) \to W^n(\underline{\boldsymbol{C}}/\underline{\boldsymbol{D}}) \to W^{n+1}(\underline{\boldsymbol{D}}) \to \cdots.$$

which extends to an exact sequence

$$GW^n(\underline{\boldsymbol{D}}) \to GW^n(\underline{\boldsymbol{C}}) \to GW^n(\underline{\boldsymbol{C}}/\underline{\boldsymbol{D}}) \to W^{n+1}(\underline{\boldsymbol{D}}) \to W^{n+1}(\underline{\boldsymbol{C}}) \to \cdots .$$

The next theorem may be found in [27] Theorem 2.6.

Theorem 2.2 (Fundamental Theorem)**.** *Let $\underline{\boldsymbol{C}} = (\boldsymbol{C}, {}^*, \delta, \varpi)$ be a* (TR4+) *triangulated category with duality containing $\frac{1}{2}$. Then there are exact sequences*

$$GW^{n-1}(\underline{\boldsymbol{C}}) \xrightarrow{\text{forget}} K_0(\boldsymbol{C}) \xrightarrow{\mathbf{H}} GW^n(\underline{\boldsymbol{C}}) \to W^n(\underline{\boldsymbol{C}}) \to 0.$$

An *admissible n-tuple* $(\boldsymbol{A}_1, \boldsymbol{A}_2, \dots, \boldsymbol{A}_n)$ of subcategories of a triangulated category $\boldsymbol{D}$ is an n-tuple of strictly full triangulated subcategories which satisfy $\boldsymbol{A}_i \perp \boldsymbol{A}_j$ for all $i < j$ and which together generate $\boldsymbol{D}$. (Here $\boldsymbol{A}_i \perp \boldsymbol{A}_j$ means that $\mathrm{Hom}_{\boldsymbol{D}}(A_i, A_j) = 0$ for all $A_i \in \boldsymbol{A}_i$ and $A_j \in \boldsymbol{A}_j$.) Admissible pairs of categories can be characterized by the following proposition.

Proposition 2.3 ([11] Lemma 3.1)**.** *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be strictly full triangulated subcategories of a small triangulated category $\boldsymbol{D}$ with $\boldsymbol{A} \perp \boldsymbol{B}$. Then $(\boldsymbol{A}, \boldsymbol{B})$ is an admissible pair of subcategories of $\boldsymbol{D}$ if and only if any of the following equivalent conditions hold:*

(a) *$\boldsymbol{A}$ and $\boldsymbol{B}$ generate $\boldsymbol{D}$.*

(b) *For every $X \in \boldsymbol{D}$ there exists an exact triangle $\ell_{\boldsymbol{B}}X[-1] \to r_{\boldsymbol{A}}X \to X \to \ell_{\boldsymbol{B}}X$ with $r_{\boldsymbol{A}}X \in \boldsymbol{A}$ and $\ell_{\boldsymbol{B}}X \in \boldsymbol{B}$.*

(c) *The inclusion functor $\boldsymbol{A} \hookrightarrow \boldsymbol{D}$ has a right adjoint $r_{\boldsymbol{A}} : \boldsymbol{D} \to \boldsymbol{A}$, and $\boldsymbol{A}^{\perp} = \boldsymbol{B}$.*

(d) *The inclusion functor $\boldsymbol{B} \hookrightarrow \boldsymbol{D}$ has a left adjoint $\ell_{\boldsymbol{B}} : \boldsymbol{D} \to \boldsymbol{B}$, and $\boldsymbol{A} = {}^{\perp}\boldsymbol{B}$.*

Using this proposition we may see that if $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ is an admissible triple of subcategories of $\boldsymbol{D}$, then we have

$$\boldsymbol{A}^{\perp} = \langle \boldsymbol{B}, \boldsymbol{C} \rangle, \qquad \langle \boldsymbol{A}, \boldsymbol{B} \rangle = {}^{\perp}\boldsymbol{C}, \qquad \boldsymbol{A}^{\perp} \cap {}^{\perp}\boldsymbol{C} = \boldsymbol{B}. \tag{4}$$

The first equality follows from part (c) of Proposition 2.3 because $(\boldsymbol{A}, \langle \boldsymbol{B}, \boldsymbol{C} \rangle)$ is an admissible pair of subcategories of $\boldsymbol{D}$ by part (a). For the third equality observe that $(\boldsymbol{B}, \boldsymbol{C})$ is an admissible pair of subcategories of $\langle \boldsymbol{B}, \boldsymbol{C} \rangle$ by part (a), so by part (d) the left orthogonal of $\boldsymbol{C}$ within $\langle \boldsymbol{B}, \boldsymbol{C} \rangle$ is $\boldsymbol{B}$. But this left orthogonal is also $\langle \boldsymbol{B}, \boldsymbol{C} \rangle \cap {}^{\perp}\boldsymbol{C} = \boldsymbol{A}^{\perp} \cap {}^{\perp}\boldsymbol{C}$.

Admissible n-tuples of subcategories behave well under quotienting.

Proposition 2.4 (cf. [12] Proposition 1.6)**.** *Let $(\boldsymbol{A}_1, \boldsymbol{A}_2, \dots, \boldsymbol{A}_n)$ be an admissible n-tuple of subcategories of $\boldsymbol{D}$. Then for each j:*

(a) *$\boldsymbol{A}_j$ is a thick subcategory of $\boldsymbol{D}$.*

(b) *For any $i \neq j$, the composition $\boldsymbol{A}_i \hookrightarrow \boldsymbol{D} \twoheadrightarrow \boldsymbol{D}/\boldsymbol{A}_j$ is fully faithful with essential image $\boldsymbol{B}_i := \langle \boldsymbol{A}_i, \boldsymbol{A}_j \rangle / \boldsymbol{A}_j$.*

(c) *$(\boldsymbol{B}_1, \dots, \boldsymbol{B}_{j-1}, \boldsymbol{B}_{j+1}, \dots, \boldsymbol{B}_n)$ is an admissible $(n-1)$-tuple of subcategories of $\boldsymbol{D}/\boldsymbol{A}_j$.*

Additivity Theorems for triangulated Witt and Grothendieck-Witt groups were proven by Walter [27] Theorems 3.4 and 3.6. They are summarized by the following statement.

Theorem 2.5 (Additivity)**.** *Let $(\boldsymbol{A}_1, \boldsymbol{A}_2, \dots, \boldsymbol{A}_n)$ be an admissible n-tuple of subcategories of a small* (TR4+) *triangulated category $\boldsymbol{D}$ containing $\frac{1}{2}$ with a duality exchanging $\boldsymbol{A}_k$ and $\boldsymbol{A}_{n+1-k}$ for all k.*

(a) *The duality induces equivalences $\boldsymbol{A}_k^{\mathrm{op}} \simeq \boldsymbol{A}_{n+1-k}$ and isomorphisms $K_0\boldsymbol{A}_k \cong K_0\boldsymbol{A}_{n+1-k}$.*

(b) *If n is even, then we have $W^i\underline{\boldsymbol{D}} = 0$ for all i, there are isomorphisms $\prod_{k=1}^{n/2} K_0\boldsymbol{A}_k \cong GW^i\underline{\boldsymbol{D}}$ sending $([A_1], \dots, [A_{n/2}]) \mapsto \sum_{k=1}^{n/2} [\![\mathbf{H}(A_k)]\!]$, and the exact sequences $0 \to GW^{i-1}\underline{\boldsymbol{D}} \to K_0\boldsymbol{D} \to GW^i\underline{\boldsymbol{D}} \to 0$ of the Fundamental Theorem 2.2 are split exact.*

(c) *If n is odd, then there are isomorphisms $W^i\underline{\boldsymbol{A}}_{(n+1)/2} \cong W^i\underline{\boldsymbol{D}}$ sending $[B_{(n+1)/2},\phi] \mapsto [B_{(n+1)/2},\phi]$ and isomorphisms $\prod_{k=1}^{(n-1)/2} K_0\boldsymbol{A}_k \times GW^i\underline{\boldsymbol{A}}_{(n+1)/2} \cong GW^i\underline{\boldsymbol{D}}$ sending*

$$\big([A_1],\dots,[A_{(n-1)/2}],[\![B_{(n+1)/2},\phi]\!]\big) \mapsto \sum\nolimits_{k=1}^{(n-1)/2} [\![\mathbf{H}(A_k)]\!] + [\![B_{(n+1)/2},\phi]\!],$$

and they identify the kernels of the forgetful maps $GW^i\underline{\boldsymbol{D}} \to K_0\boldsymbol{D}$ and $GW^i\underline{\boldsymbol{A}}_{(n+1)/2} \to K_0\boldsymbol{A}_{(n+1)/2}$.

There is an analogous theorem for K_0 ([27] Theorem 3.3).

Theorem 2.6 (Additivity for K_0)**.** *Let $(\boldsymbol{A}_1,\boldsymbol{A}_2,\dots,\boldsymbol{A}_n)$ be an admissible n-tuple of subcategories of a small triangulated category $\boldsymbol{D}$. Then the direct-sum functor $\prod_{i=1}^n \boldsymbol{A}_i \to \boldsymbol{D}$ which sends $(A_1,\dots,A_n) \mapsto \bigoplus_{i=1}^n A_i$ induces an isomorphism $\prod_{i=1}^n K_0(\boldsymbol{A}_i) \cong K_0(\boldsymbol{D})$.*

A *complicial category with weak equivalences* $(\boldsymbol{C},\boldsymbol{w})$ is given by an additive category of chains $\boldsymbol{E}$ (often mentioned only implicitly), a full subcategory $\boldsymbol{C} \subset \mathrm{Ch}(\boldsymbol{E})$ with the structure of an exact category, and a class of weak equivalences $\boldsymbol{w} \subset \mathrm{Mor}\,\boldsymbol{C}$ satisfying a number of axioms (e.g. the weak equivalences contain at least the homotopy equivalences). A *duality* $(^*,\delta,\varpi)$ is given by a δ-exact contravariant differential graded functor * plus functorial weak equivalences $\varpi_Y : Y \to Y^{**}$ satisfying several properties. See [27] §4 for the details.

Theorem 2.7. *The localization $\boldsymbol{C}[\boldsymbol{w}^{-1}]$ of a complicial category with weak equivalences $(\boldsymbol{C},\boldsymbol{w})$ is a* (TR4+) *triangulated category. A differential graded duality on a complicial category with weak equivalences containing $\frac{1}{2}$ induces a duality on the triangulated category $\boldsymbol{C}[\boldsymbol{w}^{-1}]$.*

An *admissible n-tuple of subcategories* of the complicial category with weak equivalences $(\boldsymbol{C},\boldsymbol{w})$ is an n-tuple $(\boldsymbol{A}_1,\boldsymbol{A}_2,\dots,\boldsymbol{A}_n)$ of strictly full exact subcategories of $\boldsymbol{C}$ plus additive functors $\mathrm{gr}_i : \boldsymbol{C} \to \boldsymbol{A}_i$ such that (i) $\boldsymbol{A}_i \perp \boldsymbol{A}_j$ for all $i<j$, (ii) any object Y of $\boldsymbol{C}$ has a filtration by admissible monomorphisms

$$0 = F_0Y \rightarrowtail F_1Y \rightarrowtail \cdots \rightarrowtail F_{n-1}Y \rightarrowtail F_nY = Y$$

with $\mathrm{gr}_iY := F_iY/F_{i-1}Y$ in $\boldsymbol{A}_i$ for each i, (iii) the $\boldsymbol{A}_i$ are translation-invariant, and (iv) the functors $\mathrm{gr}_i : \boldsymbol{C} \to \boldsymbol{A}_i$ preserve weak equivalences.

Theorem 2.8 ([27] Theorem 4.5)**.** *Let $(\boldsymbol{A}_1,\dots,\boldsymbol{A}_n)$ be an admissible n-tuple of subcategories of the complicial category with weak equivalences $(\boldsymbol{C},\boldsymbol{w})$. The the localized functors $\boldsymbol{A}_i[(\boldsymbol{w}\cap\boldsymbol{A}_i)^{-1}] \to \boldsymbol{C}[\boldsymbol{w}^{-1}]$ induced by the inclusions $\boldsymbol{A}_i \subset \boldsymbol{C}$ are fully faithful, and the essential images $\boldsymbol{B}_i$ form an admissible n-tuple $(\boldsymbol{B}_1,\dots,\boldsymbol{B}_n)$ of subcategories of the triangulated category $\boldsymbol{C}[\boldsymbol{w}^{-1}]$.*

If X is a scheme with $\frac{1}{2} \in \Gamma(X,\mathcal{O}_X)$, and if L is a line bundle on X, then the exact category $\mathbf{VB}_X$ of algebraic vector bundles on X has a duality (the "L-twisted duality") $(^\sharp,\varpi)$ consisting of the duality functor $\mathcal{E} \mapsto \mathcal{E}^\sharp = \mathcal{H}om(\mathcal{E},L)$ plus the natural isomorphisms $E \cong E^{\sharp\sharp}$. The duality on the exact category induces a duality $(^\sharp,1,\varpi)$ on the derived category $\mathrm{D^b}(\mathbf{VB}_X)$, and the triangulated Witt and Grothendieck-Witt groups are denoted by $W^n(X,L)$ and $GW^n(X,L)$ and called the *derived Witt and Grothendieck-Witt groups* of (X,L).

The products defined on triangulated Witt groups by Gille and Nenashev [17] can be extended to triangulated Grothendieck-Witt groups. In our context this amounts to saying that the tensor product of chain complexes of vector bundles induces natural morphisms

$$GW^i(X,L_1) \times GW^j(X,L_2) \to GW^{i+j}(X,L_1\otimes L_2).$$

The discussion about two different products in [17] reduces in our context to the product being twisted graded-commutative, i.e. $\xi_i\eta_j = \langle -1\rangle^{ij}\eta_j\xi_i$ for $\xi_i \in GW^i$ and $\eta_j \in GW^j$, where $\langle -1\rangle \in GW^0(X, \mathcal{O}_X)$ is the class of the symmetric bilinear form $\mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$ given by $(f,g) \mapsto -fg$. These products will be used at several points in the paper without a further citation of [17].

3. Linear complexes of vector bundles

We discuss linear complexes of vector bundles on a projective bundle, and the dualities on and admissible subcategories of the derived categories. The use of linear complexes of vector spaces to study coherent sheaves on projective space goes back to Bernstein-Gelfand-Gelfand [9], and a version of the duality may be found in Gille [15] §3.

Let F be a vector bundle of rank $r+1$ over a noetherian scheme X, and let $\pi : P = \mathbb{P}(F) \to X$ be the associated projective bundle. A *linear complex of vector bundles on P* is a complex

$$\cdots \to \pi^*\mathcal{E}_{-1} \otimes \mathcal{O}_P(-1) \to \pi^*\mathcal{E}_0 \otimes \mathcal{O}_P \to \pi^*\mathcal{E}_1 \otimes \mathcal{O}_P(1) \to \cdots \tag{5}$$

where each $\mathcal{E}_i$ is a vector bundle on X, and the term $\pi^*\mathcal{E}_i \otimes \mathcal{O}_P(i)$ occurs in cochain degree i. Such complexes, with degreewise split exact sequences, form an exact category $\mathbf{LC}$.

If $[a,b] \subset \mathbb{Z}$ is a finite interval, then let $\mathbf{LC}^{[a,b]}$ be the full subcategory of $\mathbf{LC}$ of complexes such that $\mathcal{E}_i = 0$ for $i \notin [a,b]$. Let $\rho^{[a,b]} : \mathbf{LC} \to \mathbf{LC}^{[a,b]}$ be the truncation functor sending the complex (5) to

$$0 \to \pi^*\mathcal{E}_a \otimes \mathcal{O}_P(a) \to \cdots \to \pi^*\mathcal{E}_b \otimes \mathcal{O}_P(b) \to 0, \tag{6}$$

suppressing the $\mathcal{E}_i$ with $i \notin [a,b]$.

For each i there is an isomorphism of categories $\mathbf{VB}_X \cong \mathbf{LC}^{[i,i]}$ sending a vector bundle $\mathcal{G}$ on X to the linear complex of vector bundles $\cdots \to 0 \to \pi^*\mathcal{G} \otimes \mathcal{O}_P(i) \to 0 \to \cdots$ on P. Composing the inverse isomorphism with $\rho^{[i,i]}$ gives a functor $\mathrm{gr}_i : \mathbf{LC}^{[a,b]} \to \mathbf{VB}_X$ sending the linear complex (6) to $\mathcal{E}_i$.

The category $\mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{[a,b]})$ of bounded chain complexes in $\mathbf{LC}^{[a,b]}$ is actually a category of bounded double complexes with commuting squares

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \xrightarrow{d_h} & \pi^*\mathcal{E}_{b,j} \otimes \mathcal{O}_P(b) & \xrightarrow{d_h} & \pi^*\mathcal{E}_{b,j+1} \otimes \mathcal{O}_P(b) & \xrightarrow{d_h} & \cdots \\
 & & d_v\uparrow & & d_v\uparrow & & \\
\cdots & \xrightarrow{d_h} & \pi^*\mathcal{E}_{b-1,j} \otimes \mathcal{O}_P(b-1) & \xrightarrow{d_h} & \pi^*\mathcal{E}_{b-1,j+1} \otimes \mathcal{O}_P(b-1) & \xrightarrow{d_h} & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \\
 & & d_v\uparrow & & d_v\uparrow & & \\
\cdots & \xrightarrow{d_h} & \pi^*\mathcal{E}_{a,j} \otimes \mathcal{O}_P(a) & \xrightarrow{d_h} & \pi^*\mathcal{E}_{a,j+1} \otimes \mathcal{O}_P(a) & \xrightarrow{d_h} & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array} \tag{7}$$

The gr_i above extend to functors $\mathrm{gr}_i : \mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{[a,b]}) \to \mathrm{Ch}(\mathbf{VB}_X)$ sending the double complex above to the complex $\cdots \to \mathcal{E}_{i,j} \to \mathcal{E}_{i,j+1} \to \cdots$ of vector bundles on X obtained from the i-th row.

We will consider $\mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{[a,b]})$ as an exact category with exact sequences which are split in every bidegree. It has translations given by translating the double complexes horizontally and changing the signs of the horizontal differentials. We let the weak equivalences $\boldsymbol{w}$ be the row-wise quasi-isomorphisms, i.e. the morphisms $E \to G$ of double complexes such that $\mathrm{gr}_i E \to \mathrm{gr}_i G$ is a quasi-isomorphism in $\mathbf{VB}_X \subset \mathbf{Qcoh}_X$ for all $i \in [a,b]$. Then $(\mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{[a,b]}), \boldsymbol{w})$ becomes a complicial category with weak equivalences.

Similarly $(\mathrm{Ch}^{\mathrm{b}}(\mathbf{VB}_P), \boldsymbol{v})$ is a complicial category with weak equivalences. The exact sequences are the degreewise split exact sequences, and the weak equivalences $\boldsymbol{v}$ are the quasi-isomorphisms.

Let $\mathrm{Tot} : \mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{[a,b]}) \to \mathrm{Ch}^{\mathrm{b}}(\mathbf{VB}_P)$ be the functor sending the double complex E above to the complex

$$\mathrm{Tot}\, E : \quad \cdots \xrightarrow{d} (\mathrm{Tot}\, E)_{n-1} \xrightarrow{d} (\mathrm{Tot}\, E)_n \xrightarrow{d} (\mathrm{Tot}\, E)_{n+1} \xrightarrow{d} \cdots$$

with $(\mathrm{Tot}\, E)_n := \bigoplus_{i+j=n} \pi^* \mathcal{E}_{i,j} \otimes \mathcal{O}_P(i)$ and $d|_{\pi^*\mathcal{E}_{i,j}\otimes \mathcal{O}_P(i)} := d_h + (-1)^j d_v$.

Proposition 3.1. *This* Tot : $(\mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{[a,b]}), \boldsymbol{w}) \to (\mathrm{Ch}^{\mathrm{b}}(\mathbf{VB}_P), \boldsymbol{v})$ *is an exact functor of complicial exact categories with weak equivalences.*

Proof. The functor Tot sends bidegreewise split exact sequences to degreewise split exact sequences, and it sends row-wise quasi-isomorphisms to quasi-isomorphisms. The sign convention for the total coboundary has been chosen so that Tot commutes strictly with (horizontal) translation, and it is compatible with mapping cones without changing any signs. □

By localizing we get an exact functor of triangulated categories $\mathrm{Tot} : \mathrm{D}^{\mathrm{b}}(\mathbf{LC}^{[a,b]}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{VB}_P)$. The next theorem is a variant version of a well-known result of the Russian school. The first versions of this theorem are Bernstein-Gelfand-Gelfand [9] and Beilinson [8]. The later version of Bondal ([11] Theorem 6.2) is closer to what we use, but it is not exactly what we need, so we will include a proof below (Theorem 11.3). The theorem applies when the interval $[a,b]$ is of the form $[a, a+r]$, where $r = \mathrm{rk}\, F - 1$ is the dimension of the fibers of the projective bundle $\pi : P = \mathbb{P}(F) \to X$.

Theorem 3.2. *For any integer a the localization of the functor* Tot *gives an equivalence of triangulated categories* $\mathrm{D}^{\mathrm{b}}(\mathbf{LC}^{[a,a+r]}) \simeq \mathrm{D}^{\mathrm{b}}(\mathbf{VB}_P)$.

The normal untwisted duality of vector bundles on P is given by the duality functor $\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_P}(\mathcal{E}, \mathcal{O}_P)$ plus the standard evaluation maps $\varpi_{P,\mathcal{E}} : \mathcal{E} \to \mathcal{E}^{\vee\vee}$. The untwisted unshifted duality on $\mathrm{D}^{\mathrm{b}}(\mathbf{VB}_P)$ is then $({}^\vee, 1, \varpi_P)$, while the L-twisted n-shifted duality will be written $({}^\vee \otimes L, 1, \varpi_P)[n]$. We will use more or less the same notations for $\mathbf{VB}_X$ and $\mathrm{D}^{\mathrm{b}}(\mathbf{VB}_X)$. For brevity we will often refer to "the triangulated category with duality $(\mathrm{D}^{\mathrm{b}}(\mathbf{VB}_X), {}^\vee \otimes L)$" instead "the triangulated category with duality $(\mathrm{D}^{\mathrm{b}}(\mathbf{VB}_X), {}^\vee \otimes L, 1, \varpi_X)$," but this is an abuse of notation: to specify an object of **TriCatD** one must specify not only the category and the duality functor but also the biduality maps.

On $\mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{[a,b]})$ we wish to use a duality $(^{\natural}, 1, \varpi_{\mathbf{LC}})$ in which the duality functor $E \mapsto E^{\natural}$ sends the double complex (7) above to the double complex

(8)
$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \xrightarrow{d_h^\vee} & \pi^*\mathcal{E}^\vee_{a,-a-b-j}\otimes\mathcal{O}_P(b) & \xrightarrow{d_h^\vee} & \pi^*\mathcal{E}^\vee_{a,-a-b-j-1}\otimes\mathcal{O}_P(b) & \xrightarrow{d_h^\vee} & \cdots \\
 & & {\scriptstyle (-1)^{a+b}d_v^\vee}\uparrow & & {\scriptstyle (-1)^{a+b}d_v^\vee}\uparrow & & \\
\cdots & \xrightarrow{d_h^\vee} & \pi^*\mathcal{E}^\vee_{a+1,-a-b-j}\otimes\mathcal{O}_P(b-1) & \xrightarrow{d_h^\vee} & \pi^*\mathcal{E}^\vee_{a+1,-a-b-j-1}\otimes\mathcal{O}_P(b-1) & \xrightarrow{d_h^\vee} & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \\
 & & {\scriptstyle (-1)^{a+b}d_v^\vee}\uparrow & & {\scriptstyle (-1)^{a+b}d_v^\vee}\uparrow & & \\
\cdots & \xrightarrow{d_h^\vee} & \pi^*\mathcal{E}^\vee_{b,-a-b-j}\otimes\mathcal{O}_P(a) & \xrightarrow{d_h^\vee} & \pi^*\mathcal{E}^\vee_{b,-a-b-j-1}\otimes\mathcal{O}_P(a) & \xrightarrow{d_h^\vee} & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

and the biduality maps $\varpi_{\mathbf{LC}} : E \cong E^{\natural\natural}$ are the natural sign-free maps. (The displayed columns are in horizontal degrees j and $j+1$ in both diagrams of double complexes.) Such a duality can also be twisted by π^*L. The duality is 1-exact essentially because in $\mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{[a,b]})$ one translates in the horizontal direction, and the horizontal differentials have been dualized without a sign. The signs have been chosen to get the following result.

Proposition 3.3. *Let L be a line bundle on X, and let $t : \mathrm{Tot}(E^{\natural} \otimes \pi^*L) \cong (\mathrm{Tot}\, E)^\vee \otimes \pi^*L \otimes \mathcal{O}_P(a+b)$ be the natural sign-free identification. Then*

$$(\mathrm{Tot}, t) : \big(\mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{[a,b]}), \boldsymbol{w}, {}^{\natural} \otimes \pi^*L\big) \to \big(\mathrm{Ch}^{\mathrm{b}}(\mathbf{VB}_P), \boldsymbol{v}, {}^{\vee} \otimes \pi^*L \otimes \mathcal{O}_P(a+b)\big)$$

is a duality-preserving exact functor of complicial exact categories with weak equivalences and duality.

Proposition 3.4. *Let $I_1 \cup I_2 \cup \cdots \cup I_n$ be a partition of $[a,b]$ into a disjoint union of consecutive subintervals. Then the complicial category with weak equivalences $(\mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{[a,b]}), \boldsymbol{w})$ has an admissible n-tuple of subcategories $(\mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{I_n}), \dots, \mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{I_2}), \mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{I_1}))$.*

Formally, the partition is given by integers $a = a_0 < a_1 < \cdots < a_n = b+1$, and we have $I_k = [a_{k-1}, a_k - 1]$.

Proof. The strictly full exact subcategories $\mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{I_k})$ and the truncation functors $\rho^{I_k} : \mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{[a,b]}) \to \mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{I_k})$ satisfy the properties of the definition: (i) for $j \neq k$ the sets of bidegrees in which double complexes in $\mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{I_j})$ and $\mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{I_k})$ can be nonzero are disjoint, so we have $\mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{I_j}) \perp \mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{I_k})$; (ii) any double complex E in $\mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{[a,b]})$ has a filtration

$$0 \rightarrowtail \rho^{[a_{n-1},b]}E \rightarrowtail \rho^{[a_{n-2},b]}E \rightarrowtail \cdots \rightarrowtail \rho^{[a_1,b]}E \rightarrowtail E$$

by bidegreewise split monomorphisms with successive quotients $\rho^{I_k}E$ for $k = n, \dots, 2, 1$; (iii) the $\mathrm{Ch}^{\mathrm{b}}(\mathbf{LC}^{I_k})$ are translation-invariant; and (iv) the ρ^{I_k} preserve weak equivalences. □

Combining Proposition 3.4 with Theorem 2.8 and observation of the action of the duality yields the following corollary.

Corollary 3.5. (a) *If $I_1 \cup I_2 \cup \cdots \cup I_n$ is a partition of $[a,b]$ into a disjoint union of consecutive subintervals, then the triangulated category $\mathrm{D^b}(\mathbf{LC}^{[a,b]})$ has an admissible n-tuple of subcategories $(\mathrm{D^b}(\mathbf{LC}^{I_n}),\dots,\mathrm{D^b}(\mathbf{LC}^{I_2}),\mathrm{D^b}(\mathbf{LC}^{I_1}))$.*

(b) *The duality $\natural \otimes \pi^*L$ exchanges $\mathrm{D^b}(\mathbf{LC}^{[b-i,b-j]})$ with $\mathrm{D^b}(\mathbf{LC}^{[a+j,a+i]})$, and if $a \equiv b \pmod 2$, then the duality's action on $\mathrm{D^b}(\mathbf{LC}^{[(a+b)/2,(a+b)/2]})$ corresponds to the duality $^\vee \otimes L$ on $\mathrm{D^b}(\mathbf{VB}_X)$.*

4. Proofs of Theorems 1.1 and 1.3

In this section we prove Theorems 1.1 and 1.3. Our proof is based on duality-preserving equivalences between derived categories of vector bundles on P and derived categories of linear complexes, followed by an application of the Additivity Theorem 2.5 to the categories of linear complexes. There are no complications.

Proof of Theorem 1.1. Applying Theorem 3.2 and Proposition 3.3 with the interval $[a,b] = [-r/2, r/2]$ gives a equivalence of triangulated categories with duality

$$\big(\mathrm{D^b}\big(\mathbf{LC}^{[-r/2,r/2]}\big), \natural \otimes \pi^*L\big) \simeq \big(\mathrm{D^b}(\mathbf{VB}_P), {}^\vee \otimes \pi^*L\big)$$

So their triangulated Grothendieck-Witt and Witt groups are isomorphic $GW^n(P,\pi^*L) \cong GW^n(\mathrm{D^b}(\mathbf{LC}^{[-r/2,r/2]}), \natural \otimes \pi^*L)$.

By Corollary 3.5 $\mathrm{D^b}(\mathbf{LC}^{[-r/2,r/2]})$ has an admissible $(r+1)$-tuple of subcategories

$$\mathrm{D^b}(\mathbf{LC}^{[r/2,r/2]}),\dots,\mathrm{D^b}(\mathbf{LC}^{[0,0]}),\dots,\mathrm{D^b}(\mathbf{LC}^{[-r/2,-r/2]}),$$

each equivalent to $\mathrm{D^b}(\mathbf{VB}_X)$. The duality $\natural \otimes \pi^*L$ exchanges $\mathrm{D^b}(\mathbf{LC}^{[i,i]})$ and $\mathrm{D^b}(\mathbf{LC}^{[-i,-i]})$, and on $\mathrm{D^b}(\mathbf{LC}^{[0,0]})$ it corresponds to the unshifted L-twisted duality on $\mathrm{D^b}(\mathbf{VB}_X)$. So by the Additivity Theorem 2.5(c) and the various identifications of equivalent categories, the maps $GW^n(X,L) \times K_0(X)^{r/2} \to GW^n(P,\pi^*L)$ given by

$$([\![\mathcal{G},\phi]\!],[\mathcal{E}_1],\dots,[\mathcal{E}_{r/2}]) \mapsto [\![\pi^*\mathcal{G},\pi^*\phi]\!] + \sum_{i=1}^{r/2} [\![\mathbf{H}(\pi^*\mathcal{E}_i \otimes \mathcal{O}_P(i))]\!]$$

are isomorphisms, as are the maps $\pi^* : W^n(X,L) \to W^n(P,\pi^*L)$. □

Proof of Theorem 1.3. Applying the same argument to the interval $[a,b] = [-r,0]$ gives equivalences of triangulated categories with duality

$$\big(\mathrm{D^b}\big(\mathbf{LC}^{[-r,0]}\big), \natural \otimes \pi^*L\big) \simeq \big(\mathrm{D^b}(\mathbf{VB}_P), {}^\vee \otimes \pi^*L \otimes \mathcal{O}_P(-r)\big)$$

with the former having an admissible $(r+1)$-tuple of subcategories

$$\mathrm{D^b}(\mathbf{LC}^{[0,0]}),\dots,\mathrm{D^b}(\mathbf{LC}^{[-(r-1)/2,-(r-1)/2]}),\mathrm{D^b}(\mathbf{LC}^{[-(r+1)/2,-(r+1)/2]}),\dots,\mathrm{D^b}(\mathbf{LC}^{[-r,-r]}),$$

each equivalent to $\mathrm{D^b}(\mathbf{VB}_X)$, with the duality $\natural\otimes\pi^*L\otimes\mathcal{O}_P(-r)$ exchanging $\mathrm{D^b}(\mathbf{LC}^{[-i,-i]})$ and $\mathrm{D^b}(\mathbf{LC}^{[-(r-i),-(r-i)]})$. So by the Additivity Theorem 2.5(b) and the various identifications of equivalent categories, the maps $K_0(X)^{(r+1)/2} \to GW^n(P,\pi^*L \otimes \mathcal{O}_P(-r))$ given by

$$([\mathcal{E}_0],\dots,[\mathcal{E}_{(r-1)/2}]) \mapsto \sum_{i=0}^{(r-1)/2} [\![\mathbf{H}(\pi^*\mathcal{E}_i \otimes \mathcal{O}_P(-i))]\!]$$

are isomorphisms, while we have $W^n(P,\pi^*L \otimes \mathcal{O}_P(-r)) = 0$. □

5. Mutations and duality

Theorems 1.2, 1.4, and 1.5 are more complicated than the theorems which we have just proved because there are no duality-preserving equivalences between derived categories of vector bundles on P with the dualities corresponding to Theorems 1.2, 1.4, and 1.5 and derived categories of linear complexes with the dualities of §3. As a result we will realize the derived categories of vector bundles on P with the right dualities as quotients of derived categories of 'too long' linear complexes. The analysis of the quotienting operation gets quite technical. In this section we present its more formal aspects. Essentially we are dealing with an interaction of the categorical mutations of Bondal [11] (also Bondal-Kapranov [12]) with the duality. We begin by reviewing these categorical mutations.

Proposition 5.1 ([12] Proposition 1.6). *If $(\boldsymbol{A}, \boldsymbol{B})$ is an admissible pair of subcategories of $\boldsymbol{D}$, then $\boldsymbol{A}$ and $\boldsymbol{B}$ are thick subcategories of $\boldsymbol{D}$, and there are triangulated equivalences $\boldsymbol{A} \simeq \boldsymbol{D}/\boldsymbol{B}$ and $\boldsymbol{B} \simeq \boldsymbol{D}/\boldsymbol{A}$.*

In one direction the compositions $\boldsymbol{A} \hookrightarrow \boldsymbol{D} \twoheadrightarrow \boldsymbol{D}/\boldsymbol{B}$ and $\boldsymbol{B} \hookrightarrow \boldsymbol{D} \twoheadrightarrow \boldsymbol{D}/\boldsymbol{A}$ are fully faithful and essentially surjective. The inverse equivalences are obtained by factoring the adjoints $r_{\boldsymbol{A}}$ and $\ell_{\boldsymbol{B}}$ of the inclusions through the quotient maps $\boldsymbol{D} \twoheadrightarrow \boldsymbol{D}/\boldsymbol{B} \simeq \boldsymbol{A}$, and $\boldsymbol{D} \twoheadrightarrow \boldsymbol{D}/\boldsymbol{A} \simeq \boldsymbol{B}$. So the inverse equivalences are $Y \mapsto r_{\boldsymbol{A}} Y$ and $Z \mapsto \ell_{\boldsymbol{B}} Z$, respectively.

Corollary 5.2 ([12] Lemma 1.9). *Let $\boldsymbol{B} \subset \boldsymbol{D}$ be a right and left admissible subcategory, let $r_{^\perp \boldsymbol{B}} : \boldsymbol{D} \to {}^\perp\boldsymbol{B}$ be the right adjoint of the inclusion ${}^\perp\boldsymbol{B} \hookrightarrow \boldsymbol{D}$, and let $\ell_{\boldsymbol{B}^\perp}$ be the left adjoint of the inclusion $\boldsymbol{B}^\perp \hookrightarrow \boldsymbol{D}$. Then the compositions*

$$\mathbf{rmut}_{\boldsymbol{B}} : \boldsymbol{B}^\perp \hookrightarrow \boldsymbol{D} \xrightarrow{r_{^\perp\boldsymbol{B}}} {}^\perp\boldsymbol{B}, \qquad \mathbf{lmut}_{\boldsymbol{B}} : {}^\perp\boldsymbol{B} \hookrightarrow \boldsymbol{D} \xrightarrow{\ell_{\boldsymbol{B}^\perp}} \boldsymbol{B}^\perp$$

are inverse equivalences.

Essentially, because $({}^\perp\boldsymbol{B}, \boldsymbol{B})$ is an admissible pair of subcategories, any Y in $\boldsymbol{B}^\perp$ fits into an exact triangle $B[-1] \to Y' \to Y \to B$ with $B \in \boldsymbol{B}$, and $Y' \in {}^\perp\boldsymbol{B}$. Then $\mathbf{rmut}_{\boldsymbol{B}}(Y) = Y'$, and $\mathbf{lmut}_{\boldsymbol{B}}(Y') = Y$. The functors $\mathbf{rmut}_{\boldsymbol{B}}$ and $\mathbf{lmut}_{\boldsymbol{B}}$ are *right* and *left mutation* with respect to $\boldsymbol{B}$.

The next lemma, used for Theorems 5.5 and 5.6 among other results, will imply that certain mutations are duality-preserving functors. But in order to use it in other contexts as well we give it a more general formulation.

Let $\underline{\boldsymbol{A}} = (\boldsymbol{A}, {}^\sharp, \varepsilon, \varpi)$ and $\underline{\boldsymbol{D}} = (\boldsymbol{D}, {}^*, \delta, \bar{\varpi})$ be triangulated categories with duality, let $g[-1] \xrightarrow{k} f \xrightarrow{i} h \xrightarrow{j} g$ be an exact triangle of exact functors from $\boldsymbol{A}$ to $\boldsymbol{D}$, and let

$$(9) \qquad \begin{array}{ccccccc} g(Y^\sharp)[-1] & \xrightarrow{k(Y^\sharp)} & f(Y^\sharp) & \xrightarrow{i(Y^\sharp)} & h(Y^\sharp) & \xrightarrow{j(Y^\sharp)} & g(Y^\sharp) \\ {\scriptstyle \beta_Y[-1]}\downarrow\cong & \overset{\gamma_Y}{\searrow} & \cong\downarrow{\scriptstyle \alpha_Y} & & \cong\downarrow{\scriptstyle \eta_Y} & & \cong\downarrow{\scriptstyle \beta_Y} \\ (fY)^*[-1] & \xrightarrow{\delta\cdot(kY)^*[-1]} & (gY)^* & \xrightarrow{(jY)^*} & (hY)^* & \xrightarrow{(iY)^*} & (fY)^* \end{array}$$

be a system of isomorphisms of exact triangles in $\boldsymbol{D}$ which are functorial in $Y \in \boldsymbol{A}$, and let $\gamma_Y : g(Y^\sharp)[-1] \to (gY)^*$ be the composition marked in the diagram. Suppose that the diagrams

$$
(10)\qquad
\begin{array}{ccc}
fY & \xrightarrow{f\varpi_Y} & f(Y^{\sharp\sharp}) \\
{\scriptstyle \bar{\varpi}_{fY}}\downarrow & & \downarrow{\scriptstyle \alpha_{Y^\sharp}} \\
(fY)^{**} & \xrightarrow{\beta_Y^*} & g(Y^\sharp)^*
\end{array}
\qquad\qquad
\begin{array}{ccc}
hY & \xrightarrow{h\varpi_Y} & h(Y^{\sharp\sharp}) \\
{\scriptstyle \bar{\varpi}_{hY}}\downarrow & & \downarrow{\scriptstyle \eta_{Y^\sharp}} \\
h(Y^{**}) & \xrightarrow{\eta_Y^*} & h(Y^\sharp)^*
\end{array}
$$

commute. Recall that we write $\underline{\boldsymbol{D}}[-1] = (\boldsymbol{D}, {}^*[-1], -\delta, \delta \cdot \bar{\varpi})$ for the triangulated category with shifted duality.

Lemma 5.3. *In this situation if there is a thick invariant subcategory $\underline{\boldsymbol{U}} = (\boldsymbol{U}, {}^*, \delta, \bar{\varpi}) \subset \underline{\boldsymbol{D}}$ such that for all $Y \in \boldsymbol{A}$ one has $hY \in \boldsymbol{U}$, then $(g[-1], \gamma) : \underline{\boldsymbol{A}} \to \underline{\boldsymbol{D}}/\underline{\boldsymbol{U}}[-1]$ and $(h, \eta) : \underline{\boldsymbol{A}} \to \underline{\boldsymbol{U}}$ are duality-preserving functors, and there are commutative diagrams*

$$
(11)\qquad
\begin{array}{ccccc}
GW^n(\underline{\boldsymbol{A}}) & \xrightarrow{(g[-1],\gamma)} & GW^{n-1}(\underline{\boldsymbol{D}}/\underline{\boldsymbol{U}}) & \xrightarrow{\partial} & W^n(\underline{\boldsymbol{U}}) \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \| \\
W^n(\underline{\boldsymbol{A}}) & \xrightarrow{(g[-1],\gamma)} & W^{n-1}(\underline{\boldsymbol{D}}/\underline{\boldsymbol{U}}) & \xrightarrow{\partial} & W^n(\underline{\boldsymbol{U}})
\end{array}
\qquad W^n(\underline{\boldsymbol{A}}) \xrightarrow{(h,\eta)} W^n(\underline{\boldsymbol{U}})
$$

in which the maps marked $(g[-1], \gamma)$ and (h, η) are induced by the duality-preserving functors, the maps ∂ are the connecting maps of the exact sequences of the Localization Theorem 2.1 and the vertical maps $GW \twoheadrightarrow W$ are the natural quotient maps.

Proof. Because η is an isomorphism of exact functors making the righthand square of (10) commute, $(h, \eta) : \underline{\boldsymbol{A}} \to \underline{\boldsymbol{D}}$ is a duality-preserving functor by definition (cf. [7] §4 (15)).

For $(g[-1], \gamma)$ note first that the $\gamma_Y = \alpha_Y \circ k(Y^\sharp)$ become isomorphisms in $\boldsymbol{D}/\boldsymbol{U}$ because the α_Y are already isomorphisms in $\boldsymbol{D}$ while the mapping cone $h(Y^\sharp)$ of $k(Y^\sharp)$ is in $\boldsymbol{U}$. Moreover, the diagram

$$
\begin{array}{ccccc}
gY[-1] & & \xrightarrow{\quad g\varpi_Y[-1]\quad} & & g(Y^{\sharp\sharp})[-1] \\
 & \searrow^{k(Y)} & & & \downarrow{\scriptstyle k(Y^{\sharp\sharp})} \\
{\scriptstyle \delta\cdot\bar{\varpi}_{gY[-1]}}\downarrow & & fY & \xrightarrow{f\varpi_Y} & f(Y^{\sharp\sharp}) \\
 & & {\scriptstyle \delta\cdot\bar{\varpi}_{fY}}\downarrow & & \downarrow{\scriptstyle \delta\cdot\alpha_{Y^\sharp}} \\
(gY)^{**}[-1] & \xrightarrow{k(Y)^{**}} & (fY)^{**} & \xrightarrow{\beta_Y^*} & g(Y^\sharp)^*
\end{array}
$$

commutes, and the composition of the arrows along the righthand edge of the square gives $\delta \cdot \gamma_{Y^\sharp}$ while the composition along the bottom edge gives $\delta \cdot \gamma_Y^*[-1]$. It follows that $(g[-1], \gamma)$ is duality-preserving.

The two squares of the diagram in the statement of the lemma commute because of the compatibility of the localization exact sequences for Witt and Grothendieck-Witt groups. So it only remains to show that the two maps $W^n(\underline{\boldsymbol{A}}) \to W^n(\underline{\boldsymbol{U}})$ are the same.

So let $\varphi : Y[n] \cong Y^\sharp$ be an isomorphism in $\boldsymbol{A}$ which is symmetric with respect to the n-th shifted duality. Complete the diagram (9) by adding an extra row.

$$\begin{array}{ccccccc}
gY[n-1] & \xrightarrow{k(Y[n])} & fY[n] & \xrightarrow{i(Y[n])} & hY[n] & \xrightarrow{j(Y[n])} & gY[n] \\
{\scriptstyle g(\varphi)[-1]}\Big\downarrow\cong & & {\scriptstyle f(\varphi)}\Big\downarrow\cong & & {\scriptstyle h(\varphi)}\Big\downarrow\cong & & {\scriptstyle g(\varphi)}\Big\downarrow\cong \\
g(Y^\sharp)[-1] & \longrightarrow & f(Y^\sharp) & \longrightarrow & h(Y^\sharp) & \longrightarrow & g(Y^\sharp) \\
\Big\downarrow\cong & \overset{\gamma_Y}{\searrow} & \cong\Big\downarrow & & {\scriptstyle \eta_Y}\Big\downarrow\cong & & \cong\Big\downarrow \\
(fY)^*[-1] & \longrightarrow & (gY)^* & \xrightarrow{(jY)^*} & (hY)^* & \longrightarrow & (fY)^*
\end{array}$$

The morphism $W^n(\underline{\boldsymbol{A}}) \to W^n(\underline{\boldsymbol{U}})$ induced by (h, η) sends $\big[Y[n], \varphi\big] \mapsto \big[hY[n], \eta_Y \circ h(\varphi)\big]$. The composition $W^n(\underline{\boldsymbol{A}}) \to W^{n-1}(\underline{\boldsymbol{D}}/\underline{\boldsymbol{U}}) \to W^n(\underline{\boldsymbol{U}})$ sends

$$\big[Y[n], \varphi\big] \mapsto \big[\mathrm{Cone}\big(gY[n-1], \gamma_Y \circ g(\varphi)[-1]\big)\big].$$

But since

$$gY[n-1] \xrightarrow{\gamma_Y \circ g(\varphi)[-1]} (gY)^* \xrightarrow{(\eta_Y \circ h(\varphi))^{-1}(jY)^*} hY \xrightarrow{j(Y)[n]} gY[n]$$

is an exact triangle, we have $\mathrm{Cone}(gY[n-1], \gamma_Y \circ g(\varphi)[-1]) \cong (hY[n], \eta_Y \circ h(\varphi))$. □

We will use the following easy and well known exercise several times.

Exercise 5.4. Suppose one is given a morphism $Y \to Z$ and two exact triangles in a triangulated category $\boldsymbol{D}$, the rows of the following diagram,

$$\begin{array}{ccccccc}
C[-1] & \longrightarrow & A & \longrightarrow & Y & \longrightarrow & C \\
\vdots & & \exists!\,\vdots & & \Big\downarrow & & \exists!\,\vdots \\
B[-1] & \longrightarrow & D & \longrightarrow & Z & \longrightarrow & B.
\end{array}$$

such that $\mathrm{Hom}_{\boldsymbol{D}}(A, B) = 0$ and $\mathrm{Hom}_{\boldsymbol{D}}(A, B[-1]) = 0$. Then there is a unique morphism $A \to D$ making the middle square commute, and a unique $C \to B$ making the righthand square commute, and together they make the entire diagram into a morphism of exact triangles.

Our first application of Lemma 5.3 concerns mutations and duality.

Theorem 5.5. *Let $\underline{\boldsymbol{D}} = (\boldsymbol{D}, ^*, \delta, \varpi)$ be a triangulated category with a duality containing a thick invariant subcategory $\underline{\boldsymbol{B}} = (\boldsymbol{B}, ^*, \delta, \varpi) \subset \boldsymbol{D}$ such that $(^\perp\boldsymbol{B}, \boldsymbol{B})$, $(\boldsymbol{B}, \boldsymbol{B}^\perp)$, and $(\boldsymbol{B}^\perp, {}^\perp\boldsymbol{B})$ are admissible pairs of subcategories of $\boldsymbol{D}$. Then there is an equivalence of triangulated categories with duality $\underline{\boldsymbol{B}} \simeq \underline{\boldsymbol{D}}/\underline{\boldsymbol{B}}[-1]$ and natural isomorphisms*

$$GW^{n-1}(\underline{\boldsymbol{D}}/\underline{\boldsymbol{B}}) \cong GW^n\underline{\boldsymbol{B}}, \qquad W^{n-1}(\underline{\boldsymbol{D}}/\underline{\boldsymbol{B}}) \cong \boldsymbol{W}^n\underline{\boldsymbol{B}}. \tag{12}$$

For the categories of the theorem we have $GW^n\underline{\boldsymbol{D}} \cong K_0\boldsymbol{B}$ for all n by Theorem 2.5(b), so the localization exact sequences have the same form $GW^{n-1}\underline{\boldsymbol{B}} \to K_0\boldsymbol{B} \to GW^n\underline{\boldsymbol{B}} \to W^n\underline{\boldsymbol{B}} \to 0$ as in the Fundamental Theorem 2.2. This is not a coincidence. If $\boldsymbol{B} \simeq \boldsymbol{E}[\boldsymbol{w}^{-1}]$ is the derived category of a complicial exact category with weak equivalences, then one can construct a $\boldsymbol{D} = \boldsymbol{P}[\boldsymbol{v}^{-1}]$ with the properties of Theorem 5.5 and use it and the Localization Theorem 2.1 to prove the Fundamental Theorem. The construction of $\boldsymbol{D}$ comes from surgery theory (e.g. Ranicki [24] §2.1): $(\boldsymbol{P}, \boldsymbol{v})$ is the category of pairs in the algebraic cobordism category $(\boldsymbol{E}, \boldsymbol{w})$.

Proof of Theorem 5.5. Since $(\boldsymbol{B}^{\perp}, {}^{\perp}\boldsymbol{B})$ is an admissible pair of subcategories of $\boldsymbol{D}$, for any $Y \in \boldsymbol{B}$ we have a functorial exact triangle

$$\ell_{\perp\boldsymbol{B}}Y[-1] \xrightarrow{kY} r_{\boldsymbol{B}^{\perp}}Y \xrightarrow{iY} Y \xrightarrow{jY} \ell_{\perp\boldsymbol{B}}Y$$

Since the duality preserves $\boldsymbol{B}$, it exchanges $\boldsymbol{B}^{\perp}$ and ${}^{\perp}\boldsymbol{B}$, and so by Exercise 5.4 there exist unique morphisms α_Y and β_Y making

$$\begin{array}{ccccccc}
\ell_{\perp\boldsymbol{B}}(Y^*)[-1] & \xrightarrow{k(Y^*)} & r_{\boldsymbol{B}^{\perp}}(Y^*) & \xrightarrow{i(Y^*)} & Y^* & \xrightarrow{j(Y^*)} & \ell_{\perp\boldsymbol{B}}(Y^*) \\
{\scriptstyle \beta_Y[-1]}\downarrow{\scriptstyle \cong} & \searrow^{\lambda_Y} & {\scriptstyle \alpha_Y}\downarrow{\scriptstyle \cong} & & \| & & {\scriptstyle \beta_Y}\downarrow{\scriptstyle \cong} \\
(r_{\boldsymbol{B}^{\perp}}Y)^*[-1] & \xrightarrow{\delta\cdot(kY)^*[-1]} & (\ell_{\perp\boldsymbol{B}}Y)^* & \xrightarrow{(jY)^*} & Y^* & \xrightarrow{(iY)^*} & (r_{\boldsymbol{B}^{\perp}}Y)^*
\end{array}$$

commute. Standard arguments using Exercise 5.4 show that the α_Y and β_Y are isomorphisms and are functorial in Y. The same lemma also shows that $\alpha_{Y^*} \circ r_{\boldsymbol{B}^{\perp}}\varpi_Y = \beta_Y^* \circ \varpi_{r_{\boldsymbol{B}^{\perp}}Y}$ because both compositions both make the middle square in the diagram

$$\begin{array}{ccccccc}
\ell_{\perp\boldsymbol{B}}Y[-1] & \longrightarrow & r_{\boldsymbol{B}^{\perp}}Y & \xrightarrow{iY} & Y & \longrightarrow & \ell_{\perp\boldsymbol{B}}Y \\
 & & {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \varpi_Y} & & \\
r_{\boldsymbol{B}^{\perp}}(Y^*)^* & \longrightarrow & \ell_{\perp\boldsymbol{B}}(Y^*)^* & \xrightarrow{j(Y^*)^*} & Y^{**} & \longrightarrow & r_{\boldsymbol{B}^{\perp}}(Y^*)^*
\end{array}$$

commute. So the lefthand square of (10) commutes, while the righthand square commutes because $h = 1_{\boldsymbol{B}}$ and $\eta_Y = 1_{Y^*}$ for all Y. Thus all the hypotheses of Lemma 5.3 hold, and so $(\ell_{\perp\boldsymbol{B}}[-1], \lambda) : \underline{\boldsymbol{B}} \to \underline{\boldsymbol{D}}/\underline{\boldsymbol{B}}[-1]$ is a duality-preserving functor. Since $\ell_{\perp\boldsymbol{B}}[-1]$ is the composition of two equivalences $\mathbf{lmut}_{\boldsymbol{B}^{\perp}}[-1] : \boldsymbol{B} \simeq {}^{\perp}\boldsymbol{B}$ and ${}^{\perp}\boldsymbol{B} \simeq \boldsymbol{D}/\boldsymbol{B}$ by Corollary 5.2 and Proposition 5.1, $(\ell_{\perp\boldsymbol{B}}[-1], \lambda)$ is a duality-preserving equivalence $\underline{\boldsymbol{B}} \simeq \underline{\boldsymbol{D}}/\underline{\boldsymbol{B}}[-1]$. Since duality-preserving equivalences induce isomorphisms of Grothendieck-Witt groups and of Witt groups, this completes the proof of the theorem. □

Theorem 5.6. *Let* $\underline{\boldsymbol{D}}$ *be a small* (TR4+) *triangulated category with duality containing* $\frac{1}{2}$ *with thick subcategories* $\boldsymbol{A}$, $\boldsymbol{A}_\ell$, $\boldsymbol{A}_r$, *and* $\boldsymbol{C}$ *such that* $(\boldsymbol{A}_\ell, \boldsymbol{C}, \boldsymbol{A}_r)$, $(\boldsymbol{A}, \boldsymbol{A}_\ell, \boldsymbol{C})$, *and* $(\boldsymbol{C}, \boldsymbol{A}_r, \boldsymbol{A})$ *are admissible triples of subcategories of* $\boldsymbol{D}$, *and with the duality fixing* $\boldsymbol{A}$ *and* $\boldsymbol{C}$ *and exchanging* $\boldsymbol{A}_\ell$ *and* $\boldsymbol{A}_r$.

(a) *Then there are duality-preserving equivalences* $\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}} \rangle \simeq \underline{\boldsymbol{C}} \times \underline{\boldsymbol{A}}$ *and* $\underline{\boldsymbol{A}}[1] \simeq \underline{\boldsymbol{D}}/\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}} \rangle$.

(b) *There is a natural long exact sequence of Witt groups*

$$\cdots \to W^{n-1}\underline{\boldsymbol{C}} \to W^{n-1}(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \xrightarrow{p} W^n\underline{\boldsymbol{A}} \xrightarrow{\partial} W^n\underline{\boldsymbol{C}} \to W^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \to W^{n+1}\underline{\boldsymbol{A}} \to \cdots$$

which can be extended to Grothendieck-Witt groups

$$GW^{n-1}\underline{\boldsymbol{C}} \to GW^{n-1}(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \xrightarrow{p} GW^n\underline{\boldsymbol{A}} \xrightarrow{\partial} W^n\underline{\boldsymbol{C}} \to W^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \to W^{n+1}\underline{\boldsymbol{A}} \to \cdots.$$

(c) *The maps marked* p *come from a composition of duality-preserving functors* $\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}} \to \underline{\boldsymbol{D}}/\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}} \rangle \simeq \underline{\boldsymbol{A}}[1]$ *where the functor component of the first arrow is naturally equivalent to the right adjoint* $r_{\boldsymbol{A}_\ell} : \langle \boldsymbol{A}_\ell, \boldsymbol{C} \rangle \to \langle \boldsymbol{A}_\ell, \boldsymbol{C} \rangle/\boldsymbol{C} \simeq \boldsymbol{A}_\ell$ *of the inclusion* $\boldsymbol{A}_\ell \hookrightarrow \langle \boldsymbol{A}_\ell, \boldsymbol{C} \rangle$.

Proof. (a) We have assumed that $(\boldsymbol{A}, \boldsymbol{A}_\ell, \boldsymbol{C})$ and $(\boldsymbol{C}, \boldsymbol{A}_r, \boldsymbol{A})$ are admissible triples of subcategories of $\boldsymbol{D}$, so we have $\boldsymbol{A} \perp \boldsymbol{C}$ and $\boldsymbol{C} \perp \boldsymbol{A}$. It follows that there is a duality-preserving equivalence $\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}} \rangle \simeq \underline{\boldsymbol{C}} \times \underline{\boldsymbol{A}}$.

Next let $\boldsymbol{B}$, $\boldsymbol{B}_\ell$, and $\boldsymbol{B}_r$ be the essential images of $\boldsymbol{A}$, $\boldsymbol{A}_\ell$, and $\boldsymbol{A}_r$ in $\boldsymbol{D}/\boldsymbol{C}$. Since $\boldsymbol{C}$ and $\boldsymbol{A}$ are fixed by the duality on $\boldsymbol{D}$, the duality descends to $\boldsymbol{D}/\boldsymbol{C}$ where it fixes $\boldsymbol{B} = \langle \boldsymbol{C}, \boldsymbol{A}\rangle/\boldsymbol{C}$. Therefore $\underline{\boldsymbol{B}} \subset \underline{\boldsymbol{D}}/\underline{\boldsymbol{C}}$ is a thick invariant subcategory of a triangulated category with duality, while $(\boldsymbol{B}_\ell, \boldsymbol{B}_r)$, $(\boldsymbol{B}, \boldsymbol{B}_\ell)$, and $(\boldsymbol{B}_r, \boldsymbol{B})$ are admissible pairs of subcategories of $\boldsymbol{D}/\boldsymbol{C}$ by Proposition 2.4. So by Theorem 5.5 we have a duality-preserving equivalence $\underline{\boldsymbol{B}}[1] \simeq (\underline{\boldsymbol{D}}/\underline{\boldsymbol{C}})/\underline{\boldsymbol{B}}$. But since we have $\boldsymbol{B} = \langle \boldsymbol{C}, \boldsymbol{A}\rangle/\boldsymbol{C}$, there is an identification $(\underline{\boldsymbol{D}}/\underline{\boldsymbol{C}})/\underline{\boldsymbol{B}} \cong \underline{\boldsymbol{D}}/\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}}\rangle$, and on the other hand we have $\underline{\boldsymbol{A}} \simeq \underline{\boldsymbol{B}}$ by Proposition 2.4. Together this gives a duality-preserving equivalence $\underline{\boldsymbol{A}}[1] \simeq \underline{\boldsymbol{D}}/\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}}\rangle$.

(b) Apply the Localization Theorem 2.1 to $\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}}\rangle/\underline{\boldsymbol{A}} \hookrightarrow \underline{\boldsymbol{D}}/\underline{\boldsymbol{A}} \twoheadrightarrow \underline{\boldsymbol{D}}/\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}}\rangle$ while taking into account the duality-preserving equivalences $\underline{\boldsymbol{C}} \simeq \langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}}\rangle/\underline{\boldsymbol{A}}$ and $\underline{\boldsymbol{D}}/\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}}\rangle \simeq \underline{\boldsymbol{A}}[1]$.

(c) Since $(\boldsymbol{A}, \boldsymbol{A}_\ell, \boldsymbol{C})$ is an admissible triple of subcategories of $\boldsymbol{D}$, the quotient functor $\boldsymbol{D}/\boldsymbol{A} \to \boldsymbol{D}/\langle \boldsymbol{C}, \boldsymbol{A}\rangle$ is naturally equivalent to $\langle \boldsymbol{A}_\ell, \boldsymbol{C}\rangle \to \langle \boldsymbol{A}_\ell, \boldsymbol{C}\rangle/\boldsymbol{C} \simeq \boldsymbol{A}_\ell$, and this is the right adjoint of the inclusion $\boldsymbol{A}_\ell \hookrightarrow \langle \boldsymbol{A}_\ell, \boldsymbol{C}\rangle$ according to Propositions 5.1 and 2.3. □

We investigate further the exact sequences of Theorem 5.6.

Proposition 5.7. *Suppose that in the situation of Theorem 5.6 there exist exact functors $f, g, h : \boldsymbol{A} \to \boldsymbol{D}$ and morphisms of exact functors i, j, k, α, β, η, γ as in Lemma 5.3 such that for all $Y \in \boldsymbol{A}$ one has $fY \in \langle \boldsymbol{A}_\ell, \boldsymbol{C}\rangle$ and $gY \in \langle \boldsymbol{C}, \boldsymbol{A}_r\rangle$, and that there are a duality-preserving functor $(c, \zeta) : \underline{\boldsymbol{A}} \to \underline{\boldsymbol{C}}$ and functorial isomorphisms $hY \cong Y \oplus cY$ in $\boldsymbol{D}$ which identify the morphisms $\eta_Y : h(Y^*) \cong (hY)^*$ with $\begin{pmatrix} 1_{Y^*} & 0 \\ 0 & \zeta_Y \end{pmatrix} : Y^* \oplus c(Y^*) \cong Y^* \oplus (cY)^*$.*

(a) *Then the connecting map $\partial : W^n\underline{\boldsymbol{A}} \to W^n\underline{\boldsymbol{C}}$ of the long exact sequence of Theorem 5.6 is the same as the map induced by (c, ζ), while the map $\partial : GW^n\underline{\boldsymbol{A}} \to W^n\underline{\boldsymbol{C}}$ is the composition of the former map with the natural map $GW^n\underline{\boldsymbol{A}} \twoheadrightarrow W^n\underline{\boldsymbol{A}}$.*

(b) *If we have $(c, \zeta) \cong 0$, then the long exact sequences of Theorem 5.6 decompose into short exact sequences $0 \to W^n\underline{\boldsymbol{C}} \to W^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \leftrightarrows W^{n+1}\underline{\boldsymbol{A}} \to 0$ and $GW^n\underline{\boldsymbol{C}} \to GW^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \leftrightarrows GW^{n+1}\underline{\boldsymbol{A}} \to 0$ in which the surjections are split by sections induced by the duality-preserving functor $(g[-1], \gamma) : \underline{\boldsymbol{A}}[1] \to \underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}$ of Lemma 5.3.*

Proof. The long exact sequences of Theorem 5.6 are localization sequences in which one has made certain identifications using isomorphisms $GW^{n+1}\underline{\boldsymbol{A}} \cong GW^n(\underline{\boldsymbol{D}}/\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}}\rangle)$ and $W^{n+1}\underline{\boldsymbol{A}} \cong W^n(\underline{\boldsymbol{D}}/\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}}\rangle)$ coming from duality-preserving equivalences which are the compositions of the natural equivalences $\underline{\boldsymbol{A}} \simeq \underline{\boldsymbol{B}}$ with equivalences $(\ell_{\perp\boldsymbol{B}}[-1], \lambda) : \underline{\boldsymbol{B}} \simeq \underline{\boldsymbol{D}}/\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}}\rangle[-1]$ coming from Theorem 5.5. The latter equivalences make use of a left adjoint of the inclusion ${}^\perp\boldsymbol{B} \hookrightarrow \boldsymbol{D}/\boldsymbol{C}$, and we are free to specify this left adjoint as we wish because any two left adjoints of the same functor are canonically isomorphic, and the duality-preserving equivalences they determine are then canonically isomorphic and so induce the same maps of triangulated Grothendieck-Witt and Witt groups (cf. [7] Lemma 4.1(b), [27] Proposition 2.1). Thus the identifications of Grothendieck-Witt and of Witt groups used in Theorem 5.6 do not depend on the choice of a particular left adjoint $\ell_{\perp\boldsymbol{B}}$.

Now $\ell_{\perp\boldsymbol{B}} : \boldsymbol{D}/\boldsymbol{C} \to {}^\perp\boldsymbol{B}$ is characterized by there being exact triangles in $\boldsymbol{D}/\boldsymbol{C}$

$$\ell_{\perp\boldsymbol{B}}Y[-1] \longrightarrow r_{\boldsymbol{B}^\perp}Y \longrightarrow Y \longrightarrow \ell_{\perp\boldsymbol{B}}Y$$

which are functorial in $Y \in \boldsymbol{B}$ and such that $r_{\boldsymbol{B}^\perp}Y \in \boldsymbol{B}^\perp$ and $\ell_{\perp\boldsymbol{B}}Y \in {}^\perp\boldsymbol{B}$ for all Y. The exact triangles in $\boldsymbol{D}$

$$gY[-1] \xrightarrow{kY} fY \xrightarrow{iY} hY \xrightarrow{jY} gY$$

are functorial in $Y \in \boldsymbol{A} \simeq \boldsymbol{B}$, and when one passes to the quotient category $\boldsymbol{D}/\boldsymbol{C}$ one has natural isomorphisms $hY \cong Y$ as well as having $fY \in \langle \boldsymbol{A}_\ell, \boldsymbol{C}\rangle/\boldsymbol{C} = \boldsymbol{B}^\perp$ and $gY \in$

$\langle \boldsymbol{C}, \boldsymbol{A}_r \rangle / \boldsymbol{C} = {}^{\perp}\boldsymbol{B}$. Hence we may let the left adjoint $\ell_{\perp \boldsymbol{B}}$ be the composition

$$\boldsymbol{B} \simeq \boldsymbol{A} \xrightarrow{g} \langle \boldsymbol{C}, \boldsymbol{A}_r \rangle \twoheadrightarrow \langle \boldsymbol{C}, \boldsymbol{A}_r \rangle / \boldsymbol{C} = {}^{\perp}\boldsymbol{B}.$$

So the isomorphisms $GW^n \underline{\boldsymbol{A}} \cong GW^{n-1}(\underline{\boldsymbol{D}}/\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}} \rangle)$ and $W^n \underline{\boldsymbol{A}} \cong W^{n-1}(\underline{\boldsymbol{D}}/\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}} \rangle)$ used in constructing the long exact sequences of Theorem 5.6 are the same as the isomorphisms induced by the duality-preserving equivalences $(g[-1], \gamma) : \underline{\boldsymbol{A}} \simeq \underline{\boldsymbol{D}}/\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}} \rangle [-1]$ where we set $\gamma_Y = \alpha_Y \circ k(Y^*)$ as in Lemma 5.3.

(a) The connecting map $\partial : W^n(\underline{\boldsymbol{A}}) \to W^n(\underline{\boldsymbol{C}})$ of Theorem 5.6(b) is really a composition

$$W^n \underline{\boldsymbol{A}} \xrightarrow{(g[-1],\gamma)\, \cong} W^{n-1}(\underline{\boldsymbol{D}}/\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}} \rangle) \xrightarrow{d} W^n(\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}} \rangle / \underline{\boldsymbol{A}}) \xrightarrow{\cong} W^n \underline{\boldsymbol{C}}$$

with d the connecting map of a localization sequence. By Lemma 5.3 the composition of the first two maps is the map induced by the duality-preserving functor $(h, \eta) : \underline{\boldsymbol{A}} \to \langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}} \rangle / \underline{\boldsymbol{A}}$. The third map is induced by the duality-preserving equivalence $\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}} \rangle / \underline{\boldsymbol{A}} \simeq \underline{\boldsymbol{C}}$, so the composition of all the maps is the same as the map induced by the summand $(c, \zeta) : \underline{\boldsymbol{A}} \to \underline{\boldsymbol{C}}$ of (h, η). The assertion for Grothendieck-Witt groups also follows from Lemma 5.3.

(b) According to the construction of Lemma 5.3 each γ_Y is an arrow in $\boldsymbol{D}$ whose mapping cone is isomorphic to hY. Under the hypotheses of part (b) we have $hY \in \boldsymbol{A}$ for all Y, so the γ_Y are isomorphisms already in $\boldsymbol{D}/\boldsymbol{A}$ and not only in $\boldsymbol{D}/\langle \boldsymbol{C}, \boldsymbol{A} \rangle$. It follows that our duality-preserving equivalence has a factorization by duality-preserving functors

$$\underline{\boldsymbol{A}} \xrightarrow{(g[-1],\gamma)} \underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}\,[-1] \twoheadrightarrow \underline{\boldsymbol{D}}/\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}} \rangle\,[-1]$$

as asserted by the proposition. This then implies that the duality-preserving functor $(g[-1], \gamma)$ is a section up to isomorphism of the composition $\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}\,[-1] \twoheadrightarrow \underline{\boldsymbol{D}}/\langle \underline{\boldsymbol{C}}, \underline{\boldsymbol{A}} \rangle\,[-1] \xleftarrow{\cong} \underline{\boldsymbol{A}}$ which induces the maps $GW^{n-1}(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \to GW^n \underline{\boldsymbol{A}}$ and $W^{n-1}(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \to W^n \underline{\boldsymbol{A}}$ in the long exact sequences of Theorem 5.6. So the long exact sequences decompose into short exact sequences whose surjections are split by maps induced by $(g[-1], \gamma)$. □

In this paper we do not extend the long exact sequence to the left beyond Grothendieck-Witt groups into higher quadratic K-theory, so we cannot use the categorical splitting of Proposition 5.7(b) to deduce that the maps $GW^n \underline{\boldsymbol{C}} \to GW^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}})$ are injective. But we have an ad-hoc method to show the injectivity.

Proposition 5.8. *Suppose that the hypotheses of Theorem 5.6 and Proposition 5.7(b) hold, and that $\boldsymbol{C}$ has an admissible triple of subcategories $(\boldsymbol{E}_\ell, \boldsymbol{F}, \boldsymbol{E}_r)$ with $\boldsymbol{E}_\ell$ and $\boldsymbol{E}_r$ exchanged by the duality such that the natural duality-preserving functor $\underline{\boldsymbol{F}} \to \underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}$ induces an injection $GW^n \underline{\boldsymbol{F}} \hookrightarrow GW^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}})$. Then the split short exact sequence of Grothendieck-Witt groups of Proposition 5.7(b) is $0 \to GW^n \underline{\boldsymbol{C}} \to GW^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \leftrightarrows GW^{n+1} \underline{\boldsymbol{A}} \to 0$.*

Proof. The forgetful maps $GW^n \to K_0$ are functorial, so we have a commutative diagram

(13)
$$\begin{array}{ccc} GW^n \underline{\boldsymbol{C}} & \xrightarrow{a} & GW^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \\ {\scriptstyle b}\downarrow & & \downarrow \\ K_0 \boldsymbol{C} & \xrightarrow{c} & K_0(\boldsymbol{D}/\boldsymbol{A}), \end{array}$$

in which the map c is injective by Theorem 2.6 because $(\boldsymbol{C}, \boldsymbol{A}_r, \boldsymbol{A})$ is an admissible triple of subcategories of $\boldsymbol{D}$, and so $(\boldsymbol{C}, \boldsymbol{A}_r)$ is an admissible pair of subcategories of $\langle \boldsymbol{C}, \boldsymbol{A}_r \rangle \simeq \boldsymbol{D}/\boldsymbol{A}$. The kernel of $a : GW^n \underline{\boldsymbol{C}} \to GW^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}})$ is therefore contained in the kernel of the forgetful map $b : GW^n \underline{\boldsymbol{C}} \to K_0 \boldsymbol{C}$. By Theorem 2.5(c) the kernel of b is the same as the kernel of the forgetful map $GW^n\underline{\boldsymbol{F}} \to K_0\boldsymbol{F}$. So the kernel of $GW^n\underline{\boldsymbol{C}} \to GW^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}})$ is contained in $GW^n\underline{\boldsymbol{F}}$ and thus in the kernel of $GW^n\underline{\boldsymbol{F}} \to GW^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}})$. By hypothesis this last map is injective. So $GW^n\underline{\boldsymbol{C}} \to GW^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}})$ is also injective, which is what we had to show. □

When $\boldsymbol{F} = \boldsymbol{0}$ the previous proposition can be simplified because the hypotheses of Proposition 5.7(b) hold automatically.

Corollary 5.9. *If in the situation of Theorem 5.6 the category $\boldsymbol{C}$ has an admissible pair of subcategories $(\boldsymbol{E}_\ell, \boldsymbol{E}_r)$ which are exchanged by the duality, then the long exact sequences of Theorem 5.6 decompose into isomorphisms $W^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \cong W^{n+1}\underline{\boldsymbol{A}}$ and split short exact sequences $0 \to GW^n\underline{\boldsymbol{C}} \to GW^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \leftrightarrows GW^{n+1}\underline{\boldsymbol{A}} \to 0$. The maps $W^{n+1}\underline{\boldsymbol{A}} \to W^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}})$ and $GW^{n+1}\underline{\boldsymbol{A}} \to GW^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}})$ come from a duality-preserving functor $(\ell_{\langle \boldsymbol{E}_r, \boldsymbol{A}_r\rangle}[-1], \gamma)$.*

Proof. Because $W^n\underline{\boldsymbol{C}} = 0$ by the Additivity Theorem 2.5(b), the long exact sequence of Witt groups of Theorem 5.6 yields $W^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \cong W^{n+1}\underline{\boldsymbol{A}}$. Since $\boldsymbol{D}$ has an admissible pair of subcategories $(\langle \boldsymbol{A}_\ell, \boldsymbol{E}_\ell\rangle, \langle \boldsymbol{E}_r, \boldsymbol{A}_r\rangle)$ which are exchanged by the duality, one is in the situation of Lemma 5.3 with $g[-1] \to f \to h \to g$ given by the triangle $\ell_{\langle \boldsymbol{E}_r, \boldsymbol{A}_r\rangle}[-1] \to r_{\langle \boldsymbol{A}_\ell, \boldsymbol{E}_\ell\rangle} \to 1_{\boldsymbol{D}} \to \ell_{\langle \boldsymbol{E}_r, \boldsymbol{A}_r\rangle}$ of Proposition 2.3(b), with α and β created by Exercise 5.4, and with $\eta = 1$. Since $h = 1_{\boldsymbol{D}}$ we are in the situation of Proposition 5.7(b), and the admissible triple of subcategories $(\boldsymbol{E}_\ell, \boldsymbol{0}, \boldsymbol{E}_r)$ of $\boldsymbol{C}$ puts us in the situation of Proposition 5.8. So we have split exact sequences $0 \to GW^n\underline{\boldsymbol{C}} \to GW^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \leftrightarrows GW^{n+1}\underline{\boldsymbol{A}} \to 0$, with the surjection split by $(\ell_{\langle \boldsymbol{E}_r, \boldsymbol{A}_r\rangle}[-1], \gamma)$. An explicit splitting of the injection may be read off of diagram (13) because the maps b and c of that diagram are split injective by the Additivity Theorems 2.5(b) and 2.6. □

A (TR4+) triangulated category with duality satisfying the conditions of the next proposition satisfies the hypotheses of Theorem 5.6.

Proposition 5.10. *Let $(\boldsymbol{A}_\ell, \boldsymbol{C}, \boldsymbol{A}_r)$ be an admissible triple of subcategories of a triangulated category $\boldsymbol{D}$ with a duality which exchanges $\boldsymbol{A}_\ell$ and $\boldsymbol{A}_r$. Suppose that*

(i) *the quotient functor $Q_r : \boldsymbol{D} \twoheadrightarrow \boldsymbol{D}/\langle \boldsymbol{A}_\ell, \boldsymbol{C}\rangle \simeq \boldsymbol{A}_r$ has a left adjoint $F : \boldsymbol{A}_r \to \boldsymbol{D}$ such that the unit $1_{\boldsymbol{A}_r} \to Q_rF$ of the adjunction is an isomorphism,*

(ii) *the quotient functor $Q_\ell : \boldsymbol{D} \twoheadrightarrow \boldsymbol{D}/\langle \boldsymbol{C}, \boldsymbol{A}_r\rangle \simeq \boldsymbol{A}_\ell$ has a right adjoint $G : \boldsymbol{A}_\ell \to \boldsymbol{D}$ such that the counit $Q_\ell G \to 1_{\boldsymbol{A}_\ell}$ of the adjunction is an isomorphism, and*

(iii) *the essential images of F and of G are the same subcategory $\boldsymbol{A} \subset \boldsymbol{D}$.*

Then $\boldsymbol{A}$ and $\boldsymbol{C}$ are fixed by the duality, and $(\boldsymbol{A}, \boldsymbol{A}_\ell, \boldsymbol{C})$ and $(\boldsymbol{C}, \boldsymbol{A}_r, \boldsymbol{A})$ are admissible triples of subcategories of $\boldsymbol{D}$.

Proof. The unit $1_{\boldsymbol{A}_r} \to Q_rF$ of the adjunction is an isomorphism, so F is fully faithful (MacLane [22] Chap. IV, §3, Theorem 1), and therefore it factors as an equivalence followed by the inclusion of its essential image $\boldsymbol{A}_r \simeq \boldsymbol{A} \hookrightarrow \boldsymbol{D}$. Since F has a right adjoint Q_r, the inclusion $\boldsymbol{A} \hookrightarrow \boldsymbol{D}$ has a right adjoint, and so $(\boldsymbol{A}, \boldsymbol{A}^\perp)$ is an admissible pair of subcategories $\boldsymbol{D}$ by Proposition 2.3(c). Now $Y \in \boldsymbol{A}^\perp = F(\boldsymbol{A}_r)^\perp$ is equivalent to $Q_rY \cong 0$. Since Q_r is the quotient by a thick subcategory, this is equivalent to $Y \in \langle \boldsymbol{A}_\ell, \boldsymbol{C}\rangle$. Thus we have $\boldsymbol{A}^\perp = \langle \boldsymbol{A}_\ell, \boldsymbol{C}\rangle$, and so $(\boldsymbol{A}, \langle \boldsymbol{A}_\ell, \boldsymbol{C}\rangle)$ is an admissible pair of subcategories of $\boldsymbol{D}$. In particular, we have $\langle \boldsymbol{A}, \boldsymbol{A}_\ell, \boldsymbol{C}\rangle = \boldsymbol{D}$, and $\boldsymbol{A} \perp \boldsymbol{A}_\ell$ and $\boldsymbol{A} \perp \boldsymbol{C}$. Since $(\boldsymbol{A}_\ell, \boldsymbol{C}, \boldsymbol{A}_r)$ is an admissible triple of subcategories, we also have $\boldsymbol{A}_\ell \perp \boldsymbol{C}$. So $(\boldsymbol{A}, \boldsymbol{A}_\ell, \boldsymbol{C})$ is an admissible triple of subcategories of $\boldsymbol{D}$.

A dual argument shows that $(\boldsymbol{C}, \boldsymbol{A}_r, \boldsymbol{A})$ is also an admissible triple of subcategories of $\boldsymbol{D}$.

By (4) we have $\boldsymbol{C} = \boldsymbol{A}_\ell^\perp \cap {}^\perp\boldsymbol{A}_r$ and $\boldsymbol{A} = {}^\perp\langle \boldsymbol{A}_\ell, \boldsymbol{C}\rangle = \langle \boldsymbol{C}, \boldsymbol{A}_r\rangle^\perp$. So a duality which exchanges $\boldsymbol{A}_\ell$ and $\boldsymbol{A}_r$ fixes $\boldsymbol{C}$ and $\boldsymbol{A}$. □

Finally we give a method for applying this theory to categories which may not be constructed as quotient categories.

Proposition 5.11. *Let $\underline{\boldsymbol{D}}$, $(\boldsymbol{A}_\ell, \boldsymbol{C}, \boldsymbol{A}_r)$, and $\boldsymbol{A}$ be as in Theorem 5.6. Let $\underline{\boldsymbol{D}} \to \underline{\boldsymbol{D}}'$ be a duality-preserving functor of triangulated categories with duality such that the composition $\langle \boldsymbol{A}_\ell, \boldsymbol{C}\rangle \hookrightarrow \boldsymbol{D} \to \boldsymbol{D}'$ is an equivalence of categories and the composition $\boldsymbol{A} \hookrightarrow \boldsymbol{D} \to \boldsymbol{D}'$ vanishes. Then the induced $\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}} \to \underline{\boldsymbol{D}}'$ is a duality-preserving equivalence.*

Proof. Since $\underline{\boldsymbol{A}} \subset \underline{\boldsymbol{D}}$ is a thick invariant subcategory such that the composition $\underline{\boldsymbol{A}} \hookrightarrow \underline{\boldsymbol{D}} \to \underline{\boldsymbol{D}}'$ vanishes, the last duality-preserving functor factors uniquely through the quotient map $\underline{\boldsymbol{D}} \twoheadrightarrow \underline{\boldsymbol{D}}/\underline{\boldsymbol{A}} \to \underline{\boldsymbol{D}}'$. Moreover, the composition of the three functors $\langle \boldsymbol{A}_\ell, \boldsymbol{C}\rangle \hookrightarrow \boldsymbol{D} \twoheadrightarrow \boldsymbol{D}/\boldsymbol{A} \to \boldsymbol{D}'$ is an equivalence of categories by hypothesis, and the composition of the first two functors is an equivalence of categories by Proposition 5.1 because $(\boldsymbol{A}, \langle \boldsymbol{A}_\ell, \boldsymbol{C}\rangle)$ is an admissible pair of subcategories of $\boldsymbol{D}$. So $\boldsymbol{D}/\boldsymbol{A} \to \boldsymbol{D}'$ is an equivalence of categories, and $\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}} \to \underline{\boldsymbol{D}}'$ is a duality-preserving equivalence. □

6. Sheaves of modules over the exterior algebra

Before applying the methods of the previous section to derived categories of linear complexes, we need additional analysis of categories of linear complexes. In particular we need to calculate certain adjoint functors explicitly. A convenient way of doint this is to use the identification of linear complexes with modules over the exterior algebra. This identification goes back at least to Bernstein-Gelfand-Gelfand [9].

Let $s = r + 1 = \operatorname{rk} F$, and let $\Lambda = \bigoplus_{i=0}^s \Lambda^i F^\vee$ be the sheaf of exterior algebras over $\mathcal{O}_X$ generated by $F^\vee$. It is a sheaf of graded algebras; the summand $\Lambda^i F^\vee$ is of degree i. We will write Λ-$\mathbf{VB}$ for the category of sheaves of graded left Λ-modules $\mathcal{M} = \bigoplus_{i\in\mathbb{Z}} \mathcal{M}_i$ such that the graded pieces $\mathcal{M}_i$ are all vector bundles on X. The full subcategory of graded modules such that $\mathcal{M}_i = 0$ for $i \notin [a,b]$ will be denoted Λ-$\mathbf{VB}^{[a,b]}$. There are projection functors $\rho^{[a,b]} : \Lambda\text{-}\mathbf{VB} \to \Lambda\text{-}\mathbf{VB}^{[a,b]}$ given by $\rho^{[a,b]}\mathcal{M} = \bigoplus_{i=a}^b \mathcal{M}_i$. Given a sheaf of graded Λ-modules $\mathcal{M} = \bigoplus_{i\in\mathbb{Z}} \mathcal{M}_i$, the same module with a shifted grading will be written $\mathcal{M}\{n\} = \bigoplus_i \mathcal{M}_{i+n}$. The following result is well known.

Theorem 6.1. *There are isomorphisms of categories Λ-$\mathbf{VB} \cong \mathbf{LC}$ and Λ-$\mathbf{VB}^{[a,b]} \cong \mathbf{LC}^{[a,b]}$.*

Proof. To a linear complex

$$\mathcal{A}^\bullet : \quad \cdots \longrightarrow \pi^*\mathcal{A}_{-1} \otimes \mathcal{O}_P(-1) \xrightarrow{d} \pi^*\mathcal{A}_0 \otimes \mathcal{O}_P \xrightarrow{d} \pi^*\mathcal{A}_1 \otimes \mathcal{O}_P(1) \longrightarrow \cdots \tag{14}$$

of vector bundles on P one associates the sheaf $\mathcal{A} := \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_i$ of graded $\mathcal{O}_X$-modules. The coboundaries $\pi^*\mathcal{A}_i \otimes \mathcal{O}_P(i) \to \pi^*\mathcal{A}_{i+1} \otimes \mathcal{O}_P(i+1)$ correspond via twisting, the π^*-π_* adjunction, and the projection formula $\pi_*(\pi^*\mathcal{A}_{i+1} \otimes \mathcal{O}_P(1)) \cong \mathcal{A}_{i+1} \otimes F$ to morphisms $\mathcal{A}_i \to \mathcal{A}_{i+1} \otimes F$ of sheaves of $\mathcal{O}_X$-modules. One has $d \circ d = 0$ if and only if the composition of $\mathcal{A}_i \to \mathcal{A}_{i+1} \otimes F \to \mathcal{A}_{i+2} \otimes F \otimes F$ with the map induced by the projection $F \otimes F \twoheadrightarrow S^2F$ vanishes. This condition is equivalent to having the image of $\mathcal{A}_i$ lie in $\mathcal{A}_{i+2} \otimes \Lambda^2 F \subset \mathcal{A}_{i+2} \otimes F \otimes F$, and thus to having the maps $\mathcal{A}_i \to \mathcal{A}_{i+1} \otimes F$ make $\mathcal{A}$ into a sheaf of graded comodules over the exterior coalgebra $\bigoplus \Lambda^i F$. Since F is locally free of finite rank, this is equivalent to having the dual maps $F^\vee \otimes \mathcal{A}_i \to \mathcal{A}_{i+1}$ make $\mathcal{A}$ into a sheaf of graded modules over the exterior algebra

$\Lambda = \bigoplus \Lambda^i F^\vee$. This correspondence is functorial and invertible, so we have an isomorphism of categories $\mathbf{LC} \cong \Lambda\text{-}\mathbf{VB}$. It induces an isomorphism of subcategories $\mathbf{LC}^{[a,b]} \cong \Lambda\text{-}\mathbf{VB}^{[a,b]}$ □

We now begin a long series of lemmas needed for the proofs of Theorems 1.2, 1.4, and 1.5.

Lemma 6.2. (a) *The functor* $\mathrm{gr}_n : \Lambda\text{-}\mathbf{VB} \to \mathbf{VB}_X$ *defined by* $\mathrm{gr}_n(\bigoplus_i \mathcal{A}_i) := \mathcal{A}_n$ *has a left adjoint* free_n *and a right adjoint* cofree_n.

(b) *For* $n \in [a, b]$ *the functor* $\mathrm{gr}_n : \Lambda\text{-}\mathbf{VB}^{[a,b]} \to \mathbf{VB}_X$ *has a left adjoint* $\rho^{[a,b]} \circ \mathrm{free}_n$ *and a right adjoint* $\rho^{[a,b]} \circ \mathrm{cofree}_n$.

(c) *There are natural isomorphisms* $\mathrm{free}_n(\mathcal{G}) \cong \mathrm{cofree}_{n+s}(\mathcal{G} \otimes \det F^\vee)$.

Proof. (a) For $R \to S$ a morphism of rings the restriction-of-scalars functor from left S-modules to left R-modules has a left adjoint $M \mapsto S \otimes_R M$ and a right adjoint $M \mapsto \mathrm{Hom}_R(S, M)$. An analogous calculation shows that $\mathrm{gr}_n : \Lambda\text{-}\mathbf{VB} \to \mathbf{VB}_X$ has left and right adjoints $\mathrm{free}_n \mathcal{G} := \Lambda \otimes_{\mathcal{O}_X} \mathcal{G}\{-n\}$ and $\mathrm{cofree}_n \mathcal{G} := \mathcal{H}om_{\mathcal{O}_X}(\Lambda, \mathcal{G})\{-n\}$ respectively. Part (b) is proven similarly

(c) The multiplication maps $\Lambda^{s-i}F^\vee \times \Lambda^i F^\vee \to \Lambda^s F^\vee = \det F^\vee$ are perfect pairings and the resulting isomorphisms $\Lambda^i F^\vee \cong \mathcal{H}om_{\mathcal{O}_X}(\Lambda^{s-i}F^\vee, \det F^\vee)$ can be organized into an isomorphism $\Lambda \cong \mathcal{H}om_{\mathcal{O}_X}(\Lambda, \det F^\vee)\{-s\}$ of left Λ-modules. Tensoring with $\mathcal{G}$ and shifting by n gives the isomorphism $\mathrm{free}_n \mathcal{G} \cong \mathrm{cofree}_{n+s}(\mathcal{G} \otimes \det F^\vee)$. □

We are interested in the unsigned duality on $\mathbf{LC}^{[a,b]}$ sending the linear complex

$$0 \to \pi^*\mathcal{A}_a \otimes \mathcal{O}_P(a) \xrightarrow{d} \cdots \xrightarrow{d} \pi^*\mathcal{A}_b \otimes \mathcal{O}_P(b) \to 0$$

to the linear complex

$$0 \to \pi^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}_b, \det F^\vee) \otimes \mathcal{O}_P(a) \xrightarrow{d^\vee} \cdots \xrightarrow{d^\vee} \pi^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}_a, \det F^\vee) \otimes \mathcal{O}_P(b) \to 0$$

with the natural unsigned biduality maps. We are also interested in dualities obtained from this one by twisting by a line bundle L on X and by shifting.

The functor on $\Lambda\text{-}\mathbf{VB}$ corresponding to the above duality functor should send $\mathcal{A}$ to $\mathcal{A}^\flat := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \det F^\vee)\{-a-b\}$, except that this last object is naturally a right Λ-module rather than a left Λ-module. But in the proof of Theorem 6.1 the coboundaries in the linear complex were encoded as the action of the linear part $F^\vee \subset \Lambda$ on the sheaf of graded modules. So we want to give $\mathcal{A}^\flat$ a left Λ-module structure such that the action of the linear part $F^\vee \subset \Lambda$ is the same as the action of $F^\vee \subset \Lambda^{\mathrm{op}}$ in the natural right module structure. This can be done by factoring the action of Λ through the unique graded involution $\iota : \Lambda \to \Lambda^{\mathrm{op}}$ whose linear part is $1 : F^\vee \to F^\vee$. In degree d this involution is $(-1)^{\lfloor d/2 \rfloor} : \Lambda^d F^\vee \to \Lambda^d F^\vee$. Thus we equip $\mathcal{A}^\flat = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \det F^\vee)\{-a-b\}$ with the left module structure such that for an affine open subset $U \subset X$ and for $x \in \Lambda^d F^\vee(U)$ and $\phi \in \mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{A}_i(U), \det F^\vee(U))$ the product $x\phi \in \mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{A}_{i-d}(U), \det F^\vee(U))$ sends $a \in \mathcal{A}_{i-d}(U)$ to $x\phi(a) = \phi(\iota(x)a) = (-1)^{\lfloor d/2 \rfloor}\phi(xa)$.

The biduality maps $\varpi_{\Lambda,\mathcal{A}} : \mathcal{A} \cong \mathcal{A}^{\flat\flat}$ are the standard evaluation maps.

We now let $b = a + s = a + r + 1$, and we equip $\mathbf{VB}_X$ with the standard duality twisted by a line bundle L on X.

Lemma 6.3. *For any* $a \in \mathbb{Z}$ *and any line bundle* L *on* X *there are natural isomorphisms* $\eta_{\mathcal{E}} : \mathrm{free}_a(\mathcal{E}^\vee \otimes L) \cong \mathrm{free}_a(\mathcal{E})^\flat \otimes L$ *such that*

$$(\mathrm{free}_a, \eta) : (\mathbf{VB}_X, {}^\vee \otimes L, (-1)^{\lfloor s/2 \rfloor}\varpi_X) \to (\Lambda\text{-}\mathbf{VB}^{[a,a+s]}, {}^\flat \otimes L, \varpi_\Lambda) \tag{15}$$

is a duality-preserving functor.

Proof. We treat only the case $a = 0$ and $L = \mathcal{O}_X$, the general case being essentially the same except for notation. Let $\mathrm{tr} : \Lambda \to \Lambda^s F^\vee\{-s\}$ denote the projection onto the graded piece of degree s. Since the multiplication maps $\Lambda^i F^\vee \times \Lambda^{s-i} F^\vee \to \Lambda^s F^\vee = \det F^\vee$ are perfect pairings, the map $\eta : \Lambda \to \Lambda^\flat = \mathcal{H}om_{\mathcal{O}_X}(\Lambda, \det F^\vee\{-s\})$ given by $x \mapsto (y \mapsto \mathrm{tr}(\iota(x)y))$ is an isomorphism. Tensoring with $\mathcal{E}^\vee$ gives the asserted natural isomorphisms $\eta_\mathcal{E} : \mathrm{free}_0(\mathcal{E}^\vee) \cong \mathrm{free}_0(\mathcal{E})^\flat$.

We now need to show that the diagram (cf. [7] §4 (15)) characterizing duality-preserving functors

$$\begin{array}{ccc} \mathrm{free}_0\,\mathcal{E} & \xrightarrow{\varpi_\Lambda} & (\mathrm{free}_0\,\mathcal{E})^{\flat\flat} \\ {\scriptstyle (-1)^{\lfloor s/2 \rfloor}\varpi_X}\downarrow & & \downarrow{\scriptstyle \eta^\flat_\mathcal{E}} \\ \mathrm{free}_0\,\mathcal{E}^{\vee\vee} & \xrightarrow{\eta_{\mathcal{E}^\vee}} & (\mathrm{free}_0\,\mathcal{E}^\vee)^\flat \end{array}$$

commutes. But this diagram is the componentwise tensor product of the two squares

$$\begin{array}{ccc} \Lambda & \xrightarrow{\varpi_\Lambda} & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\Lambda, \det F^\vee\{-s\}), \det F^\vee\{-s\}) \\ {\scriptstyle (-1)^{\lfloor s/2 \rfloor}}\downarrow & & \downarrow{\scriptstyle \eta^\flat} \\ \Lambda & \xrightarrow{\eta} & \mathcal{H}om_{\mathcal{O}_X}(\Lambda, \det F^\vee\{-s\}) \end{array} \qquad \begin{array}{ccc} \mathcal{E} & \xrightarrow{\varpi_X} & \mathcal{E}^{\vee\vee} \\ {\scriptstyle \varpi_X}\downarrow & & \downarrow{\scriptstyle 1} \\ \mathcal{E}^{\vee\vee} & \xrightarrow{1} & \mathcal{E}^{\vee\vee} \end{array}$$

and the righthand square obviously commutes. The top map of the lefthand square sends $x \mapsto (\psi \mapsto \psi(x))$, while the right map is the dual of $y \mapsto (z \mapsto \mathrm{tr}(\iota(y)z)) = \psi_y$. Their composition is $x \mapsto (y \mapsto \psi_y(x)) = (y \mapsto \mathrm{tr}(\iota(y)x))$. The left and bottom maps of the lefthand square send $x \mapsto (y \mapsto (-1)^{\lfloor s/2 \rfloor}\,\mathrm{tr}(\iota(x)y))$. Since ι is an anti-involution which acts by $(-1)^{\lfloor s/2 \rfloor}$ in degree s, we have $(-1)^{\lfloor s/2 \rfloor}\iota(x)y = \iota(\iota(x)y) = \iota(y)x$. This completes the proof. Note that the sign $(-1)^{\lfloor s/2 \rfloor}$ comes from the action of the involution on the "socle" of Λ. □

Lemma 6.3 has the following consequence.

Lemma 6.4. *Suppose that $r + 1 = s = 2\rho$ is even, and let $\phi : \Lambda^\rho F^\vee \times \Lambda^\rho F^\vee \to \det F^\vee$ be the $(-1)^\rho$-symmetric bilinear form defined by the product. Then the composition of the duality-preserving functor (free_a, η) of Lemma 6.3 with $\mathrm{gr}_{a+\rho} : (\Lambda\text{-}\mathbf{VB}^{[a,a+2\rho]}, {}^\flat \otimes L, \varpi_\Lambda) \to (\mathbf{VB}_X, {}^\vee \otimes L \otimes \det F^\vee, \varpi_X)$ is the duality-preserving functor denoted*

$$(\mathbf{VB}_X, {}^\vee \otimes L, (-1)^\rho \varpi_X) \xrightarrow{-\otimes(\Lambda^\rho F^\vee, (-1)^{\lfloor \rho/2 \rfloor}\phi)} (\mathbf{VB}_X, {}^\vee \otimes L \otimes \det F^\vee, \varpi_X)$$

and consisting of the exact functor $- \otimes \Lambda^\rho F^\vee$ on $\mathbf{VB}_X$ plus the isomorphisms

$$\mathcal{E}^\vee \otimes L \otimes \Lambda^\rho F^\vee \xrightarrow{1 \otimes (-1)^{\lfloor \rho/2 \rfloor}\widetilde{\phi}} \mathcal{E}^\vee \otimes L \otimes \mathcal{H}om_{\mathcal{O}_X}(\Lambda^\rho F^\vee, \det F^\vee) \cong (\mathcal{E} \otimes \Lambda^\rho F^\vee)^\vee \otimes L \otimes \det F^\vee$$

where $\widetilde{\phi}(x)(y) = \phi(x, y) = xy$.

This duality-preserving functor induces the operation $-\otimes(\Lambda^\rho F^\vee, (-1)^{\lfloor \rho/2 \rfloor}\phi)$ on symmetric objects, whence the notation.

Proof. As in the previous lemma is is enough if we treat the case $a = 0$, $L = \mathcal{O}_X$ and $\mathcal{E} = \mathcal{O}_X$. Then the composite functor $\mathrm{gr}_\rho\,\mathrm{free}_0$ sends $\mathcal{O}_X \mapsto \Lambda \mapsto \Lambda^\rho F^\vee$, while the composite isomorphism $\mathrm{gr}_\rho\,\mathrm{free}_0(\mathcal{O}_X^\vee) \cong \mathrm{gr}_\rho(\mathrm{free}_0(\mathcal{O}_X)^\flat) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathrm{gr}_\rho\,\mathrm{free}_0(\mathcal{O}_X), \det F^\vee)$ is the map $\Lambda^\rho F^\vee \cong \mathcal{H}om_{\mathcal{O}_X}(\Lambda^\rho F^\vee, \det F^\vee)$ given by $x \mapsto (y \mapsto \iota(x)y)$. But x is of degree ρ in the exterior algebra, so we have $\iota(x)y = (-1)^{\lfloor \rho/2 \rfloor}xy$. □

Let $T_{\mathrm{un}}^{a+s} : \mathrm{D^b}(\Lambda\text{-}\mathbf{VB}^{[a,a+s]}) \to \mathrm{D^b}(\Lambda\text{-}\mathbf{VB}^{[a,a+s]})$ be the unsigned translation which translates a chain complex $a+s$ places to the left without changing the sign of the coboundary. Since the usual signed translation of chain complexes $X \mapsto X[-s]$ translates a complex $-s$ places to the left while changing the sign of the coboundary by $(-1)^{-s}$, there are natural (unsigned) identifications $T_{\mathrm{un}}^{a+s}(X^\flat[-s]) = (T_{\mathrm{un}}^{a+s}X)^\flat[2a+s]$ which combine with the functor to give an isomorphism of triangulated categories with duality

$$(16) \qquad \big(\mathrm{D^b}(\Lambda\text{-}\mathbf{VB}^{[a,a+s]}), {}^\flat \otimes L[-s], \varpi_\Lambda\big) \cong \big(\mathrm{D^b}(\Lambda\text{-}\mathbf{VB}^{[a,a+s]}), {}^\flat \otimes L[2a+s], \varpi_\Lambda\big).$$

The biduality maps are not multiplied by $(-1)^{a+s}$ as they would be for the usual signed translation.

Lemma 6.5. *There is a natural isomorphism of triangulated categories with duality*

$$(17) \qquad \big(\mathrm{D^b}(\Lambda\text{-}\mathbf{VB}^{[a,a+s]}), {}^\flat \otimes L[2a+s], \varpi_\Lambda\big) \cong \big(\mathrm{D^b}(\mathbf{LC}^{[a,a+s]}), {}^\natural \otimes \pi^*(L \otimes \det F^\vee), \varpi_{\mathbf{LC}}\big).$$

Proof. The isomorphism of exact categories of Theorem 6.1 extends to the derived categories and induces an isomorphism of triangulated categories with duality

$$\big(\mathrm{D^b}(\Lambda\text{-}\mathbf{VB}^{[a,a+s]}), {}^\flat \otimes L[2a+s], \varpi_\Lambda\big) \cong \big(\mathrm{D^b}(\mathbf{LC}^{[a,a+s]}), {}^\flat \otimes \pi^*L[2a+s], \varpi_{\mathbf{LC}}\big)$$

where we use the duality on the second category which corresponds to the duality on the first category under the isomorphism. This duality functor on the second category sends a double complex $E \in \mathrm{Ch^b}(\mathbf{LC}^{[a,a+s]})$ as in (7) to a double complex $E^\flat \otimes \pi^*L[2a+s]$ which is essentially a twisted version $E^\natural \otimes \pi^*(L \otimes \det F^\vee)$ of (8) except that the vertical and horizontal coboundaries are $d_v^\vee$ and $(-1)^{2a+s}d_h^\vee$ instead of $(-1)^{a+b}d_v^\vee = (-1)^{2a+s}d_v^\vee$ and $d_h^\vee$. There are functorial isomorphisms $\mu_E : E^\flat \otimes L[2a+s] \cong E^\natural \otimes \pi^*(L \otimes \det F^\vee)$ which are given by $(-1)^{(2a+s)(i+j)}$ on the bigraded piece of the double complex of bidegree (i,j). The signs are invariant under the action $(i,j) \mapsto (2a+s-i, -2a-s-j)$ of the duality on the double grading, and so we have $\mu_{E^\flat \otimes \pi^*L[2a+s]} = \mu_E^\natural \otimes \pi^*(L \otimes \det F^\vee)$. It now follows easily that

$$\big(\mathrm{D^b}(\mathbf{LC}^{[a,a+s]}), {}^\flat \otimes \pi^*L[2a+s], \varpi_{\mathbf{LC}}\big) \xrightarrow{(1,\mu)\ \cong} \big(\mathrm{D^b}(\mathbf{LC}^{[a,a+s]}), {}^\natural \otimes \pi^*(L \otimes \det F^\vee), \varpi_{\mathbf{LC}}\big)$$

is a duality-preserving functor because the required diagram commutes (cf. [7] §4 (15)), and $(1,\mu)$ is an isomorphism because it has a strict inverse $(1,\mu^{-1})$. This completes the proof. □

Lemma 6.6. *For any line bundle L on X and any $a \in \mathbb{Z}$ there is a duality-preserving functor*

$$\big(\mathrm{D^b}(\mathbf{VB}_X), {}^\vee \otimes L \otimes \det F, \varpi_X\big)[-s] \longrightarrow \big(\mathrm{D^b}(\mathbf{LC}^{[a,a+s]}), {}^\natural \otimes \pi^*L, \varpi_{\mathbf{LC}}\big)$$

whose functor component is $T_{\mathrm{un}}^{a+s} \circ \mathrm{free}_a : \mathrm{D^b}(\mathbf{VB}_X) \to \mathrm{D^b}(\mathbf{LC}^{[a,a+s]})$.

Proof. From the definition of shifted dualities in §2 above we have

$$\big(\mathrm{D^b}(\mathbf{VB}_X), {}^\vee \otimes L \otimes \det F, 1, \varpi_X\big)[-s] = \big(\mathrm{D^b}(\mathbf{VB}_X), {}^\vee \otimes L \otimes \det F[-s], (-1)^{\lceil (-s)/2 \rceil}\varpi_X\big).$$

Since $(-1)^{\lceil (-s)/2 \rceil} = (-1)^{\lfloor s/2 \rfloor}$, the composition of the duality-preserving functors (15)(16)(17) of Lemmas 6.3 and 6.5 is a duality-preserving functor from this triangulated category with duality to $\big(\mathrm{D^b}(\mathbf{LC}^{[a,a+s]}), {}^\natural \otimes \pi^*L, \varpi_{\mathbf{LC}}\big)$ whose functor component is $1 \circ T_{\mathrm{un}}^{a+s} \circ \mathrm{free}_a$. □

Lemma 6.7. *The following composition vanishes*

$$\mathrm{D^b}(\mathbf{VB}_X) \xrightarrow{T_{\mathrm{un}}^{a+s} \circ \mathrm{free}_a} \mathrm{D^b}(\mathbf{LC}^{[a,a+s]}) \xrightarrow{\mathrm{Tot}} \mathrm{D^b}(\mathbf{VB}_P).$$

Proof. The composition of the functors is given by $\mathcal{G} \mapsto T_{\mathrm{un}}^{a+s}\pi^*\mathcal{G} \otimes_{\mathcal{O}_P} \operatorname{Tot}\operatorname{free}_a(\mathcal{O}_X)$, so it is enough to show that $\operatorname{Tot}\operatorname{free}_a(\mathcal{O}_X) \cong 0$ in the derived category. But that is equivalent to $\operatorname{Tot}\operatorname{free}_a(\mathcal{O}_X)$ being acyclic, which can be checked locally. So choose a $y \in P$, let $x = \pi(y) \in X$, replace X by the spectrum of the local ring $\mathcal{O}_{X,x}$, and choose a basis of the free module $F_x \cong \mathcal{O}_{X,x}^{r+1}$. This basis may be identified with a basis $t_0, t_1, \ldots, t_r$ of the linear forms on the relative projective space $\mathbb{P}(F_x) \to \operatorname{Spec}\mathcal{O}_{X,x}$. the complex $\operatorname{Tot}\operatorname{free}_a(\mathcal{O}_X)$ is obtained from the globalized Koszul cohomology complex

$$0 \to \mathcal{O}_P \xrightarrow{(t_0,\ldots,t_r)} \pi^*F^\vee \otimes \mathcal{O}_P(1) \to \cdots \to \Lambda^{r+1}F^\vee \otimes \mathcal{O}_P(r+1) \to 0$$

by twisting by $\mathcal{O}_P(a)$ and shifting. At any point of $\pi^{-1}(x)$ at least one of the t_i locally invertible, and the Koszul complex of a sequence containing an invertible element is acyclic. Therefore $\operatorname{Tot}\operatorname{free}_a(\mathcal{O}_X)$ is acyclic everywhere, and the lemma is proven. □

7. Proofs of Theorems 1.2, 1.4, and 1.5

In this section we prove Theorems 1.2, 1.4, and 1.5.

Proof of Theorem 1.2. Let $\underline{\boldsymbol{D}} = (\mathrm{D^b}(\mathbf{LC}^{[-r-1,0]}), \natural \otimes \pi^*L, 1, \varpi_{\mathbf{LC}})$. It is a (TR4+) triangulated category with a twist of the duality of (8). By Corollary 3.5 it has a triple of admissible subcategories

$$\boldsymbol{A}_\ell = \mathrm{D^b}(\mathbf{LC}^{[0,0]}), \qquad \boldsymbol{C} = \mathrm{D^b}(\mathbf{LC}^{[-r,-1]}), \qquad \boldsymbol{A}_r = \mathrm{D^b}(\mathbf{LC}^{[-r-1,-r-1]}),$$

with $\boldsymbol{A}_\ell$ and $\boldsymbol{A}_r$ exchanged by the duality. The projection $\mathrm{gr}_{-r-1} : \boldsymbol{D} \to \boldsymbol{C}/\langle \boldsymbol{A}_\ell, \boldsymbol{C}\rangle \simeq \boldsymbol{A}_r$ has a left adjoint free_{-r-1} according to Lemma 6.2(b), and the unit of the adjunction is an isomorphism $1_{\boldsymbol{A}_r} \cong \mathrm{gr}_{-r-1}\mathrm{free}_{-r-1}$. Similarly, $\mathrm{gr}_0 : \boldsymbol{D} \to \boldsymbol{C}/\langle \boldsymbol{A}_r, \boldsymbol{D}\rangle \simeq \boldsymbol{A}_\ell$ has a right adjoint cofree_0, and the counit of the adjunction is an isomorphism $\mathrm{gr}_0\,\mathrm{cofree}_0 \cong 1_{\boldsymbol{A}_\ell}$. The essential images of free_{-r-1} and cofree_0 are the same subcategory $\boldsymbol{A} \subset \boldsymbol{D}$ by Lemma 6.2(c). So Proposition 5.10 applies, and therefore the hypotheses of Theorem 5.6 hold: $\boldsymbol{A}$ and $\boldsymbol{C}$ are fixed by the duality, and $(\boldsymbol{A}_\ell, \boldsymbol{C}, \boldsymbol{A}_r)$, $(\boldsymbol{A}, \boldsymbol{A}_\ell, \boldsymbol{C})$, and $(\boldsymbol{C}, \boldsymbol{A}_r, \boldsymbol{A})$ are admissible triples of subcategories of $\boldsymbol{D}$. Moreover, $\boldsymbol{C}$ has an admissible pair of subcategories $\boldsymbol{E}_\ell = \mathrm{D^b}(\mathbf{LC}^{[-r/2,-1]})$ and $\boldsymbol{E}_r = \mathrm{D^b}(\mathbf{LC}^{[-r,-(r/2)-1]})$ which are exchanged by the duality. So by Corollary 5.9 we have isomorphisms $W^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \cong W^{n+1}\underline{\boldsymbol{A}}$, and split short exact sequences $0 \to GW^n\underline{\boldsymbol{C}} \to GW^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \leftrightarrows GW^{n+1}\underline{\boldsymbol{A}} \to 0$.

The triangulated subcategory with duality $\underline{\boldsymbol{A}}$ is the essential image of the duality-preserving functor of Lemma 6.6 (with $a = -s = -r-1$) whose functor component free_{-r-1} is fully faithful because it has a right adjoint gr_{-r-1} such that the unit of the adjunction is an isomorphism $1 \cong \mathrm{gr}_{-r-1}\mathrm{free}_{-r-1}$ (MacLane [22] Chap. IV, §3, Theorem 1). This gives us a duality-preserving equivalence $(\mathrm{D^b}(\mathbf{VB}_X), {}^\vee \otimes L \otimes \det F, \varpi_X)[-r-1] \simeq \underline{\boldsymbol{A}}$. So Lemma 6.6 gives us natural isomorphisms $GW^{n+1}(\underline{\boldsymbol{A}}) \cong GW^{n-r}(X, L \otimes \det F)$ and similarly for Witt groups. By periodicity modulo 2 in the line bundle argument this is naturally isomorphic to $GW^{n-r}(X, L \otimes \det F^\vee)$ (and similarly for Witt groups). Indeed one would end up directly with the latter group if one replaced free_{-r-1} by cofree_0 in the above argument.

Now let $\underline{\boldsymbol{D}}' = (\mathrm{D^b}(\mathbf{VB}_P), {}^\vee \otimes \pi^*L \otimes \mathcal{O}_P(-r-1), \varpi_P)$. By Proposition 3.3 we have a duality-preserving functor $\underline{\boldsymbol{D}} \to \underline{\boldsymbol{D}}'$. The subcategory $\boldsymbol{A}$ is in the kernel of the functor by Lemma 6.7, and the composition $\langle \boldsymbol{A}_\ell, \boldsymbol{C}\rangle = \mathrm{D^b}(\mathbf{LC}^{[-r,0]}) \hookrightarrow \boldsymbol{D} \to \mathrm{D^b}(\mathbf{VB}_P)$ is an equivalence by Theorem 3.2. So by Proposition 5.11 the induced functor $\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}} \to \underline{\boldsymbol{D}}'$ is a duality-preserving equivalence. So we have natural isomorphisms of Grothendieck-Witt groups $GW^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \cong GW^n(P, \pi^*L \otimes \mathcal{O}_P(-r-1))$ and similar isomorphisms for Witt groups.

By Corollary 3.5 $\boldsymbol{C}$ has an admissible r-tuple $(\mathrm{D^b}(\mathbf{LC}^{[-1,-1]}),\dots,\mathrm{D^b}(\mathbf{LC}^{[-r,-r]}))$ of subcategories permuted by the duality and each equivalent to $\mathrm{D^b}(\mathbf{VB}_X)$. So the Additivity Theorem 2.5(b) gives isomorphisms $GW^n\underline{\boldsymbol{C}} \cong K_0(X)^{r/2}$.

Therefore the split short exact sequences of Corollary 5.9 may be written as

$$0 \longrightarrow K_0(X)^{r/2} \xrightarrow{H} GW^n(P,\pi^*L\otimes\mathcal{O}_P(-r-1)) \underset{\pi_*}{\overset{\Omega}{\rightleftarrows}} GW^{n-r}(X,L\otimes\det F^\vee) \longrightarrow 0$$

The map H sends $([\mathcal{E}_1],\dots,[\mathcal{E}_{r/2}]) \mapsto \sum_{i=1}^{r/2}[\![\mathbf{H}(\pi^*\mathcal{E}_i\otimes\mathcal{O}_P(-i))]\!]$.

By Theorem 5.6(c) the map π_* comes from a duality-preserving functor whose functor component is essentially the right adjoint of the inclusion $\boldsymbol{A}_\ell \hookrightarrow \langle \boldsymbol{A}_\ell, \boldsymbol{C}\rangle$. This inclusion can be identified with $\pi^* : \mathrm{D^b}(\mathbf{VB}_X) \to \mathrm{D^b}(\mathbf{VB}_P)$, whose right adjoint is the derived direct image $\mathbf{R}\pi_* : \mathrm{D^b}(\mathbf{VB}_P) \to \mathrm{D^b}(\mathbf{VB}_X)$. So the map π_* is a sort of direct image map. (Before calling it *the* direct image one should check that the natural transformation part of the duality-preserving functor inducing π_* corresponds to the isomorphisms of Grothendieck duality.)

The map Ω is induced by the composition of duality-preserving functors

$$\mathrm{D^b}(\mathbf{VB}_X, {}^\vee\otimes L\otimes\det F^\vee)[-r] \xrightarrow{\mathrm{cofree}_0} \underline{\boldsymbol{A}}[1] \xrightarrow{(\ell_{\langle \boldsymbol{E}_r,\boldsymbol{A}_r\rangle}[-1],\gamma)} \underline{\boldsymbol{D}}/\underline{\boldsymbol{A}} \simeq \mathrm{D^b}(\mathbf{VB}_P, {}^\vee\otimes\pi^*L)$$

Now as is well known for the Koszul complex $\mathrm{cofree}_0(\mathcal{O}_X)$

$$0 \to \Lambda^{r+1}(\pi^*F)\otimes\mathcal{O}_P(-r-1) \xrightarrow{d_{r+1}} \cdots \xrightarrow{d_2} \pi^*F\otimes\mathcal{O}_P(-1) \xrightarrow{d_1} \mathcal{O}_P \to 0$$

the kernel of d_1 is the relative cotangent bundle $\Omega_{P/X}$, and the kernel of d_i is its i-th exterior power $\Omega^i_{P/X}$. Since $\ell_{\langle \boldsymbol{E}_r,\boldsymbol{A}_r\rangle}[-1]\circ\mathrm{cofree}_0(\mathcal{O}_X)$ is the truncated complex

$$0 \to \Lambda^{r+1}(\pi^*F)\otimes\mathcal{O}_P(-r-1) \xrightarrow{d_{r+1}} \cdots \xrightarrow{d_{r/2+2}} \Lambda^{r/2+1}(\pi^*F)\otimes\mathcal{O}_P(-r/2-1) \to 0$$

shifted into cochain degrees between $-r$ and $-r/2$, the composition sends $\mathcal{O}_X \mapsto \Omega^{r/2}_{P/X}[r/2]$. The exterior product defines a nonsingular $(-1)^{r/2}$-symmetric form $\Omega^{r/2}_{P/X}\times\Omega^{r/2}_{P/X} \to \omega_{P/X}$ giving a natural class $\boldsymbol{\Omega} = [\![\Omega^{r/2}_{P/X}[r/2], w]\!] \in GW^r(P,\omega_{P/X})$. The map labeled Ω above is thus $[\![\mathcal{E},\phi]\!] \mapsto [\![\Omega^{r/2}_{P/X}[r/2]\otimes\pi^*\mathcal{E}, w\otimes\pi^*\phi]\!]$, which is $e \mapsto \boldsymbol{\Omega}\cdot\pi^*e$ using the product of Gille-Nenashev [17] adapted to Grothendieck-Witt groups.

The inverse isomorphisms π_* and Ω between Witt groups are defined similarly. □

Proof of Theorem 1.4. Write $\rho = (r+1)/2$, let $\boldsymbol{D} = \mathrm{D^b}(\mathbf{LC}^{[-\rho,\rho]}, \natural\otimes\pi^*L, 1, \varpi_{\mathbf{LC}})$ using the duality of Proposition 3.3, and let

$$\boldsymbol{A}_\ell = \mathrm{D^b}(\mathbf{LC}^{[\rho,\rho]}), \qquad \boldsymbol{C} = \mathrm{D^b}(\mathbf{LC}^{[-(\rho-1),\rho-1]}), \qquad \boldsymbol{A}_r = \mathrm{D^b}(\mathbf{LC}^{[-\rho,-\rho]}).$$

As in the proof of Theorem 1.2 just above the essential images of cofree_ρ and $\mathrm{free}_{-\rho}$ are the same subcategory $\boldsymbol{A}\subset\boldsymbol{D}$, and these categories satisfy the hypotheses of Theorem 5.6. So Theorem 5.6 gives us a long exact sequence

$$GW^n\underline{\boldsymbol{C}} \xrightarrow{i} GW^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \xrightarrow{p} GW^{n+1}\underline{\boldsymbol{A}} \xrightarrow{\partial} W^{n+1}\underline{\boldsymbol{C}} \to W^{n+1}(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \to W^{n+2}\underline{\boldsymbol{A}} \to \cdots$$

and a similar sequence for Witt groups. As in the proof of Theorem 1.2 we may identify $GW^{n+1}\underline{\boldsymbol{A}} \cong GW^{n-r}(X, L\otimes\det F^\vee)$ and $GW^n(\underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}) \cong GW^n(P,\pi^*L)$ and $p = \pi_*$. Similar identifications hold for Witt groups. As in Theorem 1.1 we may identify

$$GW^n\underline{\boldsymbol{C}} \cong GW^n\big(\mathrm{D^b}(\mathbf{LC}^{[0,0]}), \natural \otimes \pi^*L\big) \times \prod_{i=1}^{\rho-1} K_0(\mathrm{D^b}(\mathbf{LC}^{[i,i]})) \cong GW^n(X,L) \times K_0(X)^{\rho-1}$$

and $W^n\underline{\boldsymbol{C}} \cong W^n(X,L)$, and the map i has the same formula as in Theorem 1.1. Substituting these identifications the long exact sequence gives everything except the formula for $\boldsymbol{\theta}$.

Earlier we equipped $\mathcal{M} := \bigoplus_{i=-\rho}^{\rho} \Lambda^{i+\rho}F^\vee$ with the nonsingular $(-1)^\rho$-symmetric bilinear form $\psi : \mathcal{M} \times \mathcal{M} \to \det F^\vee$ given by $\psi(x,y) = \mathrm{tr}(\iota(x)y)$. The restriction to degree 0 is $(-1)^{\lfloor \rho/2 \rfloor}\phi : \Lambda^\rho F^\vee \times \Lambda^\rho F^\vee \to \det F^\vee$ where $\phi(x,y) = xy$. The submodule $\mathcal{S}_1 := \bigoplus_{i=0}^{\rho} \Lambda^{i+\rho}F^\vee \subset \mathcal{M}$ has a natural epimorphism onto the degree 0 piece $\Lambda^\rho F^\vee$. Combining the inclusion and the epimorphism gives an embedding $\mathcal{S}_1 \rightarrowtail (\mathcal{M},\psi) \perp \big(\Lambda^\rho F^\vee, (-1)^{\lfloor \rho/2 \rfloor -1}\phi\big)$ as a degreewise split lagrangian submodule. The cokernel is isomorphic to $\mathcal{Q}_1 := \bigoplus_{i=-\rho}^{0} \Lambda^{i+\rho}F^\vee$. These generate a commutative diagram with exact rows in $\mathrm{D^b}(\Lambda\text{-}\mathbf{VB}^{[-\rho,\rho]})$

$$\begin{array}{ccccccc}
\mathcal{Q}_1[-1] & \overset{k}{\dashrightarrow} & \mathcal{S}_1 & \overset{i}{\rightarrowtail} & \mathcal{M} \oplus \Lambda^\rho F^\vee & \overset{j}{\twoheadrightarrow} & \mathcal{Q}_1 \\
{\scriptstyle \beta[-1]}\downarrow\cong & & {\scriptstyle \alpha}\downarrow\cong & & {\scriptstyle \begin{pmatrix} \psi & 0 \\ 0 & (-1)^{\lfloor \rho/2 \rfloor -1}\phi \end{pmatrix}}\downarrow\cong & & {\scriptstyle \beta}\downarrow\cong \\
\mathcal{S}_1^\flat[-1] & \underset{k^\flat[-1]}{\dashrightarrow} & \mathcal{Q}_1^\flat & \underset{j^\flat}{\rightarrowtail} & \mathcal{M}^\flat \oplus (\Lambda^\rho F^\vee)^\flat & \underset{i^\flat}{\twoheadrightarrow} & \mathcal{S}_1^\flat
\end{array}$$

where $^\flat$ is as in Lemma 6.3.

Now $\mathrm{free}_{-\rho}$ and $\mathrm{gr}_{-\rho}$ give adjoint equivalences $\mathrm{D^b}(\mathbf{VB}_X) \simeq \boldsymbol{A}$, and thus for any $\mathcal{G} \in \boldsymbol{A}$ there are natural isomorphisms $\mathcal{G} \cong \mathrm{free}_{-\rho}\,\mathrm{gr}_{-\rho}\mathcal{G} \cong (\mathrm{gr}_{-\rho}\mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{M}$. Hence if we set

$$f\mathcal{G} := (\mathrm{gr}_{-\rho}\mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{S}_1, \qquad h\mathcal{G} := (\mathrm{gr}_{-\rho}\mathcal{G}) \otimes_{\mathcal{O}_X} (\mathcal{M} \oplus \Lambda^\rho F^\vee),$$
$$g\mathcal{G} := (\mathrm{gr}_{-\rho}\mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{Q}_1, \qquad c\mathcal{G} := (\mathrm{gr}_{-\rho}\mathcal{G}) \otimes_{\mathcal{O}_X} \Lambda^\rho F^\vee,$$

then tensoring the first line of the above diagram by $\mathrm{gr}_{-\rho}\mathcal{G}$ gives us an exact triangle of exact functors $g[-1] \to f \to h \to g$ as in Lemma 5.3 with $h\mathcal{G} \cong \mathcal{G} \oplus c\mathcal{G}$ and $c\mathcal{G} \in \boldsymbol{C}$ as in Proposition 5.7. Tensoring the entire diagram above by $(\mathrm{gr}_{-\rho}\mathcal{G})^\vee \otimes L$ gives the diagram (9) of the same lemma with in addition $(h,\eta) \cong 1 \oplus (c,\zeta)$ as in Proposition 5.7. So by Proposition 5.7(a) the coboundary $\partial : W^n\underline{\boldsymbol{A}} \to W^n\underline{\boldsymbol{C}}$ in the long exact sequence of Witt groups is the morphism induced by the duality-preserving functor (c,ζ) which is multiplication by $[\Lambda^\rho F^\vee, (-1)^{\lfloor \rho/2 \rfloor -1}\phi] \in W^{2\rho}(X, \det F^\vee)$. A calculation using dual local bases of F and $F^\vee$ shows that this corresponds under the periodicity isomorphism $W^{2\rho}(X,\det F^\vee) \cong W^{2\rho}(X, \det F)$ to multiplication by the class $\Theta = [\Lambda^\rho F, (-1)^{\lfloor \rho/2 \rfloor -1}u]$ where $u : \Lambda^\rho F \times \Lambda^\rho F \to \det F$ is the multiplication map. The map $GW^n(\underline{\boldsymbol{A}}) \to W^n(\underline{\boldsymbol{C}})$ is the composition of the natural surjection $GW^n(\underline{\boldsymbol{A}}) \twoheadrightarrow W^n(\underline{\boldsymbol{A}})$ with the map on Witt groups according to Proposition 5.7(a). □

Proof of Theorem 1.5. By hypothesis we have $\Theta = 0$, so we also have $[\Lambda^\rho F^\vee, (-1)^{\lfloor \rho/2 \rfloor}\phi] = 0$ in $W^{2\rho}(X, \det F^\vee)$. Therefore there exists a vector bundle with a nonsingular $(\det F)$-twisted $(-1)^\rho$-symmetric bilinear form with a lagrangian subbundle $\mathcal{S}_2 \subset (\mathcal{N},\sigma)$ such that there is also a Lagrangian subbundle $\mathcal{S}_3 \subset (\Lambda^\rho F^\vee, (-1)^{\lfloor \rho/2 \rfloor}\phi) \perp (\mathcal{N},\sigma)$. It follows the acyclic complex $\mathcal{C} = (\cdots \to 0 \to \mathcal{S}_2 \to \mathcal{N} \to \mathcal{N}/\mathcal{S}_2 \to 0 \to \cdots)$ of vector bundles on X with a nondegenerate $(-1)^\rho$-symmetric bilinear form $\tau : \mathcal{C} \cong \mathcal{C}^\flat$ induced by σ which we view as a complex of graded Λ-modules concentrated in degree 0 and on which the positive-degree part of Λ acts trivially, and such that if we equip $\mathcal{P} := \mathcal{S}_3 \oplus \bigoplus_{i=1}^{\rho} \Lambda^{i+\rho} F^\vee$ with its natural structure as a graded Λ-submodule of $\mathcal{M} \oplus \mathcal{N} = \bigoplus_{i=-\rho}^{\rho} \Lambda^{i+\rho} F^\vee \oplus \bigoplus_{i=0}^{0} \mathcal{N}$, then we obtain a lagrangian subcomplex $\mathcal{K} \subset (\mathcal{M}, \psi) \perp (\mathcal{C}, \tau)$

$$\begin{array}{ccccccccccccc}
\mathcal{K} & & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{N}/\mathcal{S}_2 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{M} \oplus \mathcal{C} & & \cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{S}_2 & \longrightarrow & \mathcal{M} \oplus \mathcal{N} & \longrightarrow & \mathcal{N}/\mathcal{S}_2 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

If we now set

$$f\mathcal{G} := (\mathrm{gr}_{-\rho}\mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{K}, \qquad h\mathcal{G} := (\mathrm{gr}_{-\rho}\mathcal{G}) \otimes_{\mathcal{O}_X} (\mathcal{M} \oplus \mathcal{C}),$$
$$g\mathcal{G} := (\mathrm{gr}_{-\rho}\mathcal{G}) \otimes_{\mathcal{O}_X} (\mathcal{M} \oplus \mathcal{C})/\mathcal{K}, \qquad c\mathcal{G} := (\mathrm{gr}_{-\rho}\mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{C},$$

we find that we are in the situation of Proposition 5.7(b) with $(c, \zeta) \cong 0$ and $(h, \eta) \cong 1$ because $\mathcal{C}$ is acyclic and $\mathrm{gr}_{-\rho}\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{M} \cong \mathcal{G}$ for $\mathcal{G} \in \boldsymbol{A}$. Hence the long exact sequences decompose into short exact sequences as in Proposition 5.7(b) with the splittings induced by the duality-preserving functor $(g[-1], \gamma) : \underline{\boldsymbol{A}}[1] \to \underline{\boldsymbol{D}}/\underline{\boldsymbol{A}}$. Translating this into Grothendieck-Witt groups of schemes, we have short exact sequences

$$0 \longrightarrow W^n(X, L) \xrightarrow{\pi^*} W^n(P, \pi^* L) \underset{\pi_*}{\overset{\Psi}{\rightleftarrows}} W^{n-r}(X, L \otimes \det F^\vee) \longrightarrow 0,$$

$$GW^n(X, L) \times K_0(X)^{\rho-1} \xrightarrow{(\pi^*, H)} GW^n(P, \pi^* L) \underset{\pi_*}{\overset{\Psi}{\rightleftarrows}} GW^{n-r}(X, L \otimes \det F^\vee) \longrightarrow 0,$$

with the splittings Ψ of the surjections π_* of the form $e \mapsto \pi^* e \cdot \boldsymbol{\Psi}$ with $\boldsymbol{\Psi} \in GW^r(P, \pi^* \det F)$ of the form $[\![((\mathcal{M} \oplus \mathcal{C})/\mathcal{P}) \otimes \det F, \psi]\!]$ or the corresponding Witt class. If $\mathcal{S}_3$ is a split Lagrangian with Lagrangian complement $\mathcal{S}_3^\vee \otimes \det F^\vee$ (which can be arranged), then $\boldsymbol{\Psi}$ is the class of a $(-1)^{\rho-1}$-symmetric short complex

$$\begin{array}{ccccccc}
0 \longrightarrow & \Omega^{\rho}_{P/X}(\rho) \oplus \pi^*(\mathcal{S}_2 \otimes \det F) & \longrightarrow & \pi^* \mathcal{S}_3^\vee & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \\
0 \longrightarrow & \pi^*(\mathcal{S}_3 \otimes \det F) & \longrightarrow & \Omega^{\rho-1}_{P/X}(\rho) \oplus \pi^*((\mathcal{N}/\mathcal{S}_2) \otimes \det F) & \longrightarrow & 0
\end{array}$$

in the Grothendieck-Witt group or Witt group of short complexes. (See [27] §7 for an exposition of such groups in this framework.)

Finally according to Proposition 5.8, the map $GW^n(X, L) \times K_0(X)^{\rho-1} \to GW^n(P, \pi^* L)$ is injective if the component $\pi^* : GW^n(X, L) \to GW^n(P, \pi^* L)$ is. But we have seen that $\pi_*(\pi^* e \cdot \boldsymbol{\Psi}) = e$ for any $e \in GW^n(X, L)$ for any n and L. So π^* is injective. □

8. The ring structure of $W^{\mathrm{tot}}(\mathbb{P}^r_X)$

In this section we investigate the ring structure of $W^{\mathrm{tot}}(\mathbb{P}^r_X)$. The first thing we need is a simple, usable sufficient condition for the exact sequence (1) to split. The exact sequence occurs when F is of even rank 2ρ over X, and it splits when $\Theta = 0$ where $\Theta \in W^{2\rho}(X, \det F)$ is the Witt class of the nonsingular $(-1)^\rho$-symmetric bilinear form $u : \Lambda^\rho F \times \Lambda^\rho F \to \Lambda^{2\rho} F = \det F$ defined by the exterior product.

Proposition 8.1. *If F has a quotient bundle of odd rank, then we have $\Theta = 0$.*

Proof. By hypothesis there is an exact sequence $0 \to G \to F \to H \to 0$ of vector bundles with G and H of odd rank. Let $\operatorname{rk} G = 2s-1$. Then we have a descending filtration

$$\Lambda^\rho F = U_0 \supset U_1 \supset \cdots \supset U_{2s-1} \supset U_{2s} = 0$$

where each U_i is the image of $\Lambda^i G \otimes \Lambda^{\rho-i} F \to \Lambda^\rho F$ under the exterior product map. Local calculations show that each U_i is locally free and that $U_i^\perp = U_{2s-i}$. So $U_s \subset (\Lambda^\rho F, u)$ is a Lagrangian subbundle, and we have $\Theta = 0$. □

Geometrically $\mathbb{P}(H) \subset \mathbb{P}(F)$ is a linear subbundle of even relative dimension over X. Since quotient bundles of rank 1 correspond to global sections ([18] Proposition II.7.12), we get the following corollary.

Corollary 8.2. *If $\pi : P \to X$ has a global section, then we have $\Theta = 0$.*

With direct images for Grothendieck-Witt and Witt groups, one can give an alternate proof of Proposition 8.1. The composition $\mathbb{P}(H) \hookrightarrow \mathbb{P}(F) \to X$ is a projective bundle with fibers of even dimension $2t$, so by Theorem 1.2 the compositions of direct image maps

$$GW^{n+2t}(\mathbb{P}(H), i^*\pi^*L \otimes \omega_{\mathbb{P}(H)/X}) \xrightarrow{i_*} GW^{n+r}(\mathbb{P}(F), \pi^*L \otimes \omega_{\mathbb{P}(F)/X}) \xrightarrow{\pi_*} GW^n(X, L)$$

are split surjections. So the maps π_* are split surjections, and $\Theta = 0$.

Now if either r is even or $\Theta = 0$, then we have an isomorphism of graded $W^{\mathrm{tot}}(X)$-modules

$$W^{\mathrm{tot}}(P) \cong W^{\mathrm{tot}}(X) \oplus W^{\mathrm{tot}}(X) \cdot \xi$$

where ξ is the image in the Witt group of either the $\mathbf{\Omega}$ of Theorem 1.2 or the $\mathbf{\Psi}$ of Theorem 1.5. It is natural to ask whether $\xi \cdot \xi = 0$ (Balmer [6] Proposition 5.1).

Theorem 8.3. *If $\pi : P \to X$ has two disjoint sections, then we have $\xi \cdot \xi = 0$ in the total derived Witt ring.*

This holds in particular if P is a trivial projective bundle $\mathbb{P}^r_X$, or if $P = \mathbb{P}(F_0 \oplus L' \oplus L'')$ with L' and L'' line bundles on X, or if X is quasi-projective over an infinite field and $r > \dim X$. (In the last case there is a line bundle L on X such that $E^\vee \otimes L$ is generated by global sections, so that by [19] Remark 6 a general morphism $E \to L^{\oplus 2}$ is surjective outside a locus of codimension $\operatorname{rk}(E) - \operatorname{rk}(L^{\oplus 2}) + 1 = r > \dim X$, i.e. is surjective everywhere.) The theorem follows from the next proposition because the tensor product of two complexes with disjoint homological support is acyclic.

Proposition 8.4. *If s is a section of $\pi : P \to X$, then the derived Witt class ξ has a representative whose homological support is $s(X)$.*

Proof. The section corresponds to an exact sequence $0 \to G \to F \to L \to 0$ of vector bundles on X with L a line bundle ([18] Proposition II.7.12). We first consider the special case where the exact sequence is split and $F = G \oplus L$. Then the globalized Koszul complex on $\pi^*F \otimes \mathcal{O}_P(-1) \to \mathcal{O}_P \to 0$ splits degreewise and is the mapping cone on a chain map

$$\begin{array}{ccccccccccc}
0 \longrightarrow & \Lambda^r(\pi^*G) \otimes \pi^*L \otimes \mathcal{O}_P(-r-1) & \longrightarrow \cdots \longrightarrow & \pi^*G \otimes \pi^*L \otimes \mathcal{O}_P(-2) & \longrightarrow & \pi^*L \otimes \mathcal{O}_P(-1) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Lambda^r(\pi^*G) \otimes \mathcal{O}_P(-r) & \longrightarrow \cdots \longrightarrow & \pi^*G \otimes \mathcal{O}_P(-1) & \longrightarrow & \mathcal{O}_P & \longrightarrow 0
\end{array}$$

The mapping cone is acyclic, so the chain map is a quasi-isomorphism. We will call it $x : \mathcal{P} \to \mathcal{P} \otimes \pi^*L^\vee \otimes \mathcal{O}_P(1)$. The bottom line is the globalized Koszul complex on $\pi^*G \otimes \mathcal{O}_P(-1) \to \mathcal{O}_P$.

Since F is the vector bundle of "linear forms" on $P = \mathbb{P}(F)$, and since $G \subset F$ is the subbundle of "linear forms" vanishing on $s(X)$, the bottom line is a locally free resolution of $\mathcal{O}_{s(X)}$. Since we have a quasi-isomorphism, the top line is also a locally free resolution of $\mathcal{O}_{s(X)}$. The chain map x is multiplication by a "linear form" which does not vanish on $s(X)$.

The bottom line is naturally symmetrically isomorphic to its shifted twisted dual. Combining this isomorphism with the quasi-isomorphism above gives a symmetric quasi-isomorphism $\psi : \mathcal{P} \xrightarrow{\sim} \mathcal{P}^\sharp = \mathcal{P}^\vee \otimes \pi^* \det F \otimes \mathcal{O}_P(-r-1)[r]$ between chain complexes with homological support $s(X)$.

We claim that this symmetric quasi-isomorphism is a representative of the Witt class of ξ. If r is even, this is done by algebraic surgery (see e.g. Ranicki [24] §1.5, or the exposition in Walter [27] §5 (15)) along the inclusion $\sigma\mathcal{P} \hookrightarrow (\mathcal{P}, \psi)$ of the brutally truncated subcomplex which is the same as $\mathcal{P}$ in chain degrees 0 to $r/2 - 1$ and zero in all other degrees. The righthand portion of the complexes obtained from the surgery breaks off by a homotopy equivalence, and what remains is isomorphic to the truncation

$$0 \to \Lambda^{r+1}(\pi^*F) \otimes \mathcal{O}_P(-r-1) \to \cdots \to \Lambda^{r/2+1}(\pi^*F) \otimes \mathcal{O}_P(-r/2-1) \to 0$$

shifted into chain degrees r to $r/2$. This is a locally free resolution of $\Omega^{r/2}_{P/X}[r/2]$. Thus our original symmetric quasi-isomorphism may be transformed by algebraic surgery and quasi-isomorphisms into the $(\Omega^{r/2}_{P/X}[r/2], w)$ of the proof of Theorem 1.2. Since this does not change the Witt class, $\Omega \in W^r(P, \omega_{P/X})$ has a representative with homological support $s(X)$.

A similar surgery is done when r is odd.

If the exact sequence $0 \to G \to F \to L \to 0$ does not split, then the globalized Koszul complex $\mathcal{K}$ on $\pi^*F \otimes \mathcal{O}_P(-1) \to \mathcal{O}_P \to 0$ fits into an exact sequence $0 \to \mathcal{P} \otimes \pi^* L^\vee \otimes \mathcal{O}_P(1) \xrightarrow{i} \mathcal{K} \to \mathcal{P}[1] \to 0$, but there is no degreewise splitting, and the maps x and ψ above become morphisms in the derived category rather than chain maps. However, the morphism ψ can be defined as a fraction of chain maps $\mathcal{P} \xleftarrow{\sim} C(i)[-1] \xrightarrow{\sim} C(i)^\sharp[1] \xleftarrow{\sim} \mathcal{P}^\sharp$, and one can modify the argument above replacing $\mathcal{P}$ by the shifted mapping cone $C(i)[-1]$. We omit the details. $\square$

9. $\mathbb{P}^1$ bundles over the projective plane

In this section we compute the Grothendieck-Witt groups of a $\mathbb{P}^1$ bundle over the projective plane $\mathbb{P}^2_k$.

The projective plane itself has eight derived Witt groups indexed by $\mathbb{Z}/4 \times \operatorname{Pic}(\mathbb{P}^r) = \mathbb{Z}/4 \times \mathbb{Z}/2$. Of these the "trivial" and "canonical" Witt groups are nonzero

$$W^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}) \xleftarrow[\cong]{\text{pullback}} W(k), \qquad W^r(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(-r-1)) \xrightarrow[\cong]{\text{direct image}} W(k) \cdot \xi, \tag{18}$$

while the six $W^i(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(t))$ with $(i,t) \not\equiv (0,0), (r, -r-1)$ in $\mathbb{Z}/4 \times \mathbb{Z}/2$ all vanish. The ring structure is given by the product in $W(k)$ plus the equation $\xi \cdot \xi = 0$ of Theorem 8.3.

Let F be a rank 2 vector bundle on $\mathbb{P}^2$, and let $\pi : P = \mathbb{P}(F) \to \mathbb{P}^2$ for the corresponding $\mathbb{P}^1$ bundle. Then $\operatorname{Pic}(P) \cong \mathbb{Z}^2$, and we will use the notation $\mathcal{O}_P(a,b) = \pi^*\mathcal{O}_{\mathbb{P}^2}(a) \otimes \mathcal{O}_P(b)$. The scheme P has sixteen derived Witt groups, which we wish to compute.

We will need to make reference to the Chern classes c_1 and c_2 of F, which are integers satisfying $\mathcal{O}_{\mathbb{P}^2}(c_1) \cong \det F$ and $\chi(\mathbb{P}^2, F) = (c_1^2 + 3c_1)/2 + 2 - c_2$ (Riemann-Roch, e.g. [20] p. 154).

Theorem 9.1. (a) *If c_1 is even, then the four derived Witt groups $W^0(P, \mathcal{O}_P)$, $W^1(P, \mathcal{O}_P)$, $W^2(P, \mathcal{O}_P(-3,0))$, and $W^3(P, \mathcal{O}_P(-3,0))$ are isomorphic to $W(k)$, while the other twelve vanish.*

(b) *If c_1 is odd, then two of the derived Witt groups of P are given by isomorphisms $W^0(P, \mathcal{O}_P) \cong W(k)$ and $W^3(P, \mathcal{O}_P) \cong W(k)$ and two others by the exact sequence*

$$0 \to W^1(P, \mathcal{O}_P(c_1, 0)) \to W(k) \xrightarrow{\Theta} W(k) \to W^2(P, \mathcal{O}_P(c_1, 0)) \to 0.$$

The other twelve derived Witt groups vanish. Moreover, Θ belongs to the ideal $I \subset W(k)$ of even-dimensional quadratic forms if and only if c_2 is even.

Proof. The class Θ is the class of the skew-symmetric bilinear form $F \times F \to \Lambda^2 F \cong \mathcal{O}_{\mathbb{P}^2}(c_1)$ in $W^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(c_1))$. If c_1 is even, this derived Witt group vanishes, so $\Theta = 0$. We may then use Theorems 1.3 and 1.5 and (18) to calculate the derived Witt groups of P, leading to (a). If c_1 is odd, then replacing F by $F(-(c_1+3)/2)$ if necessary, which does not change $\mathbb{P}(F)$ ([18] Lemma II.7.9), we may assume that $c_1 = -3$. This modification does not change the parity of c_1 and c_2 because for rank 2 vector bundles we have $c_1(F(t)) = c_1(F) + 2t$ and $c_2(F(t)) = c_2(F) + tc_1(F) + t^2$ (cf. [14] Example 3.2.2). Theorems 1.3 and 1.5 and (18) show that there are exact sequences

$$0 \to W^3(P, \mathcal{O}_P) \to W(k) \cdot \xi \xrightarrow{\Theta} W(k) \to W^0(P, \mathcal{O}_P) \to 0,$$

$$0 \to W^1(P, \mathcal{O}_P(c_1, 0)) \to W(k) \xrightarrow{\Theta} W(k) \cdot \xi \to W^2(P, \mathcal{O}_P(c_1, 0)) \to 0,$$

and that the other twelve derived Witt groups of P vanish. The map Θ is multiplication by $\Theta \in W(k) \cdot \xi$. Since $\xi \cdot \xi = 0$ by Theorem 8.3, the first exact sequence reduces to a zero map and two isomorphisms. As for the second exact sequence, the isomorphism $W^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) \cong W(k) \cdot \xi$ is constructed using a duality-preserving functor whose functor component is the total direct image $\mathbf{R}\Gamma$. Consequently $\Theta \in W^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3))$ corresponds to a class in $W(k)$ which has a representative of dimension $\chi(\mathbb{P}^2, F) = 2 - c_2$. This gives (b). □

Examples 9.2. An example of Theorem 9.1(a) is $F = \mathcal{O}_{\mathbb{P}^2}^2$, in which case $P = \mathbb{P}^1 \times \mathbb{P}^2$. An example of (b) with $\Theta = 0$ is $F = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$, in which case P is isomorphic to the blowup of $\mathbb{P}^3$ at a rational point. An example of (b) with $\Theta = \langle 1 \rangle$ invertible is $F = \Omega_{\mathbb{P}^2}$, in which case P is isomorphic to the flag variety of $\mathrm{GL}_3(k)$. Note that in this case only two of the derived Witt groups are nonzero.

We now give examples of (b) with Θ the class of an arbitrary two-dimensional nondegenerate quadratic form. If $k = \mathbb{R}$ and $\Theta = \langle 1, 1 \rangle$, then the real algebraic variety P has three nonzero derived Witt groups $W^0(P) \cong \mathbb{Z} \cong W^3(P)$ and $W^2(P, \mathcal{O}_P(-3,0)) \cong \mathbb{Z}/2$, but its complexification has four nonzero derived Witt groups.

Let $a, b \in k^\times$, let $R = k[X,Y,Z]$, and let F be the kernel

$$0 \to F \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2} \xrightarrow{(XY, aX^2 - bY^2, Z)} \mathcal{O}_{\mathbb{P}^2}(1) \to 0. \tag{19}$$

Then $c_1 = -3$ and $c_2 = 4$, and we have $H^0(\mathbb{P}^2, F) = 0$ and $H^2(\mathbb{P}^2, F) = 0$, while $\mathbf{R}\Gamma$ identifies Θ with the Witt class of a symmetric bilinear form on the two-dimensional space $H^1(\mathbb{P}^2, F)$. We will not show it here, but one may use the derived functor $\mathbf{R}\Gamma_*$ of the graded global sections $\Gamma_*(\mathcal{G}) = \bigoplus_{i \in \mathbb{Z}} \Gamma(\mathbb{P}^2, \mathcal{G}(i))$ to give a factorization of $\mathbf{R}\Gamma$ through the derived category of graded R-modules which is compatible with duality. Hence the symmetric bilinear form on the space $H^1(\mathbb{P}^2, F)$ comes from a symmetric bilinear form on the graded R-module $H^1_*(\mathbb{P}^2, F)$. The long exact sequence of graded cohomology modules associated to (19) shows that this graded module is $R/(XY, aX^2 - bY^2, Z)(1)$ because $H^0_*(\mathcal{O}_{\mathbb{P}^2}(i)) = R(i)$ and $H^1_*(\mathcal{O}_{\mathbb{P}^2}(i)) = 0$ by Hartshorne [18] Theorem III.5.1. It has a basis consisting of $\overline{1}$ (in degree -1), $\overline{X}$ and $\overline{Y}$ (in degree 0), and $a\overline{X}^2 = b\overline{Y}^2$ (in degree 1). The only nondegenerate symmetric bilinear forms on this graded module which are compatible with the module structure are given by multiplication in the Gorenstein ring $R/(XY, aX^2 - bY^2, Z)$ followed by projection onto the socle generated by $a\overline{X}^2 = b\overline{Y}^2$, and then identification of the socle with k. The quadratic form on $H^1(\mathbb{P}^2, F)$ is thus $u \cdot a\overline{X} + v \cdot b\overline{Y} \mapsto au^2 + bv^2$ up to a multiplicative constant, and we have $\Theta = \langle a, b\rangle$.

10. Symmetric powers of morphisms

We review some notions of multilinear algebra which we will need in the next section.

Let $\phi : G \to H$ be a morphism of vector bundles on a scheme, and let n be a positive integer. The morphism has a symmetric power $S^n(\phi)$ which is the complex

$$0 \to \Lambda^n G \to \Lambda^{n-1} G \otimes H \to \cdots \to G \otimes S^{n-1} H \to S^n H \to 0$$

whose differentials are the compositions

$$\begin{array}{ccc} \Lambda^i G \otimes S^{n-i} H & \xrightarrow{\text{comult}\otimes 1} & \Lambda^{i-1} G \otimes G \otimes S^{n-i} H & & \\ & & \downarrow{\scriptstyle 1\otimes\phi\otimes 1} & & \\ & & \Lambda^{i-1} G \otimes H \otimes S^{n-i} H & \xrightarrow{1\otimes\text{mult}} & \Lambda^{i-1} G \otimes S^{n-i+1} H. \end{array}$$

The linear complex we have called $\text{cofree}_0(\mathcal{O}_X)$ is $S^{r+1}(\pi^* F \otimes \mathcal{O}_P(-1) \to \mathcal{O}_P)$.

These symmetric powers are functorial, commute with base change, and have a direct sum formula $S^n(\phi \oplus \psi) \cong \bigoplus_{i=0}^n S^i(\phi) \otimes S^{n-i}(\psi)$. We will use the following two facts

Proposition 10.1. (a) *If ϕ is an isomorphism, then $S^n(\phi)$ is acyclic for $n \geq 1$.*

(b) *Given an morphism of exact sequences of vector bundles of the form*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \|{\scriptstyle 1} & & \\ 0 & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

the induced chain map $S^n(\phi_1) \to S^n(\phi_2)$ is a quasi-isomorphism.

Locally, we can use the direct sum formula to reduce (a) to the case of an isomorphism of rank one bundles, which is easy to check. Locally the exact sequences of (b) are split, and then (b) follows from (a) and the direct sum formulas.

11. The derived equivalence

The main theorem of this section is a version of the descriptions of Beilinson [8], Bernstein-Gelfand-Gelfand [9], Bondal [11], etc., of the bounded derived category of coherent sheaves on $\mathbb{P}^n$ over an algebraically closed field. We have taken pains to give a proof which is valid for the relative case and which uses adjoint functors acting on the underlying complicial categories with weak equivalences (a "Quillen equivalence" in the language of closed model categories). This is useful for higher algebraic K-theory. The proof hides the notion of exceptional sheaf (due to Drézet and Le Potier) which was prominent in Bondal's proof, but it uses in a major way Beilinson's idea of the resolution of the diagonal.

Recall that **LC** is the category of linear complexes of vector bundles on $P = \mathbb{P}(F)$ of the form (5)

$$\cdots \to \pi^*\mathcal{E}_{-1} \otimes \mathcal{O}_P(-1) \to \pi^*\mathcal{E}_0 \otimes \mathcal{O}_P \to \pi^*\mathcal{E}_1 \otimes \mathcal{O}_P(1) \to \cdots$$

with each $\mathcal{E}_i$ a vector bundle on X, and the term $\pi^*\mathcal{E}_i \otimes \mathcal{O}_P(i)$ occurring in cochain degree i. The corresponding category of linear complexes of quasi-coherent sheaves will be denoted by **Qcoh-LC**.

Theorem 11.1. *Let X be a noetherian scheme, let F be a vector bundle on X of rank $r+1$, and let $\pi : P = \mathbb{P}(F) \to X$ be the projective bundle. For any integer a the functor* Tot *induces an equivalence* $\mathrm{D}(\mathbf{Qcoh}\text{-}\mathbf{LC}^{[a,a+r]}) \to \mathrm{D}(\mathbf{Qcoh}_P)$ *of triangulated categories.*

Proof. Different values of a give different functors Tot which differ only by self-equivalences of $\mathrm{D}(\mathbf{Qcoh}_P)$ (twisting and shifting). So it is enough to consider the single value $a = -r$.

Let $\mathfrak{A} \subset \mathbf{Qcoh}_P$ be the full exact subcategory of sheaves $\mathcal{G}$ such that $R^p\pi_*(\mathcal{G} \otimes \mathcal{O}_P(i)) = 0$ for $0 \le i \le r$ and $p > 0$. The proof of the theorem depends on the following five claims.

(a) For any object $\mathcal{A}$ in $\mathrm{Ch}(\mathbf{Qcoh}_P)$ there is a quasi-isomorphism $\mathcal{A} \xrightarrow{\sim} \mathcal{B}$ with $\mathcal{B} \in \mathrm{Ch}(\mathfrak{A})$.

(b) The functor $\mathrm{Tot} : \mathrm{Ch}(\mathbf{Qcoh}\text{-}\mathbf{LC}^{[-r,0]}) \to \mathrm{Ch}(\mathbf{Qcoh}_P)$ factors through the inclusion $\mathrm{Ch}(\mathfrak{A}) \subset \mathrm{Ch}(\mathbf{Qcoh}_P)$.

(c) The functor Tot has a right adjoint Kr such that $\mathrm{Kr}|_{\mathrm{Ch}(\mathfrak{A})}$ is exact.

(d) The unit $\mathcal{E} \to \mathrm{Kr}\,\mathrm{Tot}\,\mathcal{E}$ of the adjunction is a quasi-isomorphism for all $\mathcal{E}$.

(e) The counit $\mathrm{Tot}\,\mathrm{Kr}\,\mathcal{G} \to \mathcal{G}$ of the adjunction is a quasi-isomorphism for all $\mathcal{G} \in \mathrm{Ch}(\mathfrak{A})$.

These claims imply that

$$\mathrm{Tot} : \mathrm{D}(\mathbf{Qcoh}\text{-}\mathbf{LC}^{[a,a+r]}) \rightleftarrows \mathrm{D}(\mathbf{Qcoh}_P) : \mathbf{R}\mathrm{Kr}$$

are inverse equivalences, where $\mathbf{R}\mathrm{Kr}$ is computed by replacing objects of $\mathrm{Ch}(\mathbf{Qcoh}_P)$ by quasi-isomorphic objects of $\mathrm{Ch}(\mathfrak{A})$ and applying Kr. We now prove the claims.

(a) The functors $\mathcal{G} \to \pi_*(\mathcal{G} \otimes \mathcal{O}_P(i))$ are of finite cohomological dimension, so on may let $\mathcal{B}$ be an appropriate truncation of a Cartan-Eilenberg injective resolution of $\mathcal{A}$.

(b) Any sheaf $\pi^*E \otimes \mathcal{O}_P(j)$ with $j \ge -r$ is in $\mathfrak{A}$.

(c) When a chain-complex-valued functor F is the totalization of a bounded complex of functors $0 \to F_n \to \cdots \to F_1 \to F_0 \to 0$, its right adjoint G is the totalization of the complex $0 \to G^0 \to G^1 \to \cdots \to G^n \to 0$ of right adjoint functors with each morphism $G^i \to G^{i+1}$ conjugate to the morphism $F_{i+1} \to F_i$ of left adjoints. The unit $1 \to GF$ of the adjunction is a morphism from 1 into a complex of functors $\cdots \to \bigoplus G^iF_i \to \bigoplus G^{i+1}F_i \to \cdots$ whose components are the units $1 \to G^iF_i$ of the individual adjunctions (up to sign). The counit of the adjunction is similar.

In our case Tot is the totalization of the complex of functors

$$0 \to \mathbf{tw}_{\mathcal{O}(-r)} \circ \pi^* \circ \mathrm{gr}_{-r} \to \cdots \to \mathbf{tw}_{\mathcal{O}(-1)} \circ \pi^* \circ \mathrm{gr}_{-1} \to \cdots \to \mathbf{tw}_{\mathcal{O}} \circ \pi^* \circ \mathrm{gr}_0 \to 0 \tag{20}$$

where $\mathbf{tw}_{\mathcal{O}(a)}$ is $\mathcal{G} \mapsto \mathcal{G} \otimes \mathcal{O}_P(a)$. Its right adjoint Kr is the totalization of the complex of right adjoint functors

$$0 \to \rho^{[-r,0]}\,\mathrm{cofree}_0 \circ \pi_* \circ \mathbf{tw}_{\mathcal{O}} \to \cdots \to \rho^{[-r,-r]}\,\mathrm{cofree}_{-r} \circ \pi_* \circ \mathbf{tw}_{\mathcal{O}(r)} \to 0, \tag{21}$$

the right adjoints of the gr_i coming from Lemma 6.2(b). All the functors involved in this complex are exact except π_* which is of finite cohomological dimension, so Kr is of finite cohomological dimension. The sheaves in $\mathfrak{A}$ are Kr-acyclic, so $\mathrm{Kr}|_{\mathrm{Ch}(\mathfrak{A})}$ is exact.

(d) The problem of showing that the unit maps $\mathcal{E} \to \operatorname{Kr}\operatorname{Tot}\mathcal{E}$ of the adjunction are quasi-isomorphisms may be reduced by standard truncation arguments and the compatibility of the functors and morphisms with tensor products with vector bundles to the cases where $\mathcal{E}$ is a linear complex of the form

$$0 \to \cdots \to 0 \to \mathcal{O}_P(-i) \to 0 \to \cdots \to 0. \tag{22}$$

Applying Kr Tot gives a complex of linear complexes which, when pruned of many zeros, is

$$\begin{array}{ccccccc}
\pi^*(\Lambda^{r-i}F)(-r) & \longrightarrow & \pi^*(\Lambda^{r-i-1}F)(-r+1) & \longrightarrow \cdots \longrightarrow & \pi^*F(-i-1) & \longrightarrow & \mathcal{O}_P(-i) \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi^*(\Lambda^{r-i-1}F\otimes F)(-r) & \longrightarrow & \pi^*(\Lambda^{r-i-2}F\otimes F)(-r+1) & \longrightarrow \cdots \longrightarrow & \pi^*F(-i-1) & & \\
\downarrow & & \downarrow & & & & \\
\vdots & & \vdots & & & & \\
\downarrow & & \downarrow & & & & \\
\pi^*(F\otimes S^{r-i-1}F)(-r) & \longrightarrow & \pi^*(S^{r-i-1}F)(-r+1) & & & & \\
\downarrow & & & & & & \\
\pi^*(S^{r-i}F)(-r) & & & & & &
\end{array}$$

The vertical maps in the diagram are the maps between cofree complexes in (21). Since the maps in the left adjoint complex (20) were compositions of maps induced by the comultiplication $\mathbf{tw}_{\mathcal{O}(-i-1)} \to \pi^*F^\vee \otimes \mathbf{tw}_{\mathcal{O}(-i)}$, isomorphisms passing $\pi^*F^\vee$ inside π^*, and the multiplication $F^\vee \otimes \mathrm{gr}_{-i-1} \to \mathrm{gr}_{-i}$, the maps in the right adjoint complex (21) are the compositions of the conjugate morphisms between adjoint functors induced by the comultiplication $\mathrm{cofree}_{-i} \to \pi^*F \otimes \mathrm{cofree}_{-i-1}$, isomorphisms passing π^*F inside π_*, and the multiplication $\pi^*F \otimes \mathbf{tw}_{\mathcal{O}(i)} \to \mathbf{tw}_{\mathcal{O}(i+1)}$. This identifies the vertical maps in the large diagram above as the maps in the symmetric powers $S^n(1_F)$ of §10. So by Proposition 10.1(a), all the columns of the large diagram above are exact save the rightmost. Since the unit of the adjunction is the map which sends the $\mathcal{O}_P(-i)$ of (22) onto the $\mathcal{O}_P(-i)$ on the top right of the big diagram above, the unit is a quasi-isomorphism.

(e) Consider the diagram

$$\begin{array}{ccccc}
P = \Delta & \xhookrightarrow{\;i\;} & P \times_X P & \xrightarrow{\;q\;} & P \\
 & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\
 & & P & \xrightarrow{\;\pi\;} & X
\end{array}$$

where the square is a pullback diagram and i is the inclusion of the diagonal. Write $f = \pi p = \pi q$. The "multiplication" map $\pi^*F \to \mathcal{O}_P(1)$ pulls back along p and q to two multiplication maps, which, after twisting, are the components of a morphism $\mu : (f^*F)(-1,0) \to \mathcal{O}_{P\times P} \oplus$

$\mathcal{O}_{P\times P}(-1,1)$. This μ fits into a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_P(-1)\boxtimes\Omega_{P/X}(1) & \longrightarrow & (f^*F)(-1,0) & \longrightarrow & \mathcal{O}_{P\times P}(-1,1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\mu_1} & & \downarrow{\scriptstyle\mu} & & \|{\scriptstyle 1} & & \\ 0 & \longrightarrow & \mathcal{O}_{P\times P} & \longrightarrow & \mathcal{O}_{P\times P}\oplus\mathcal{O}_{P\times P}(-1,1) & \longrightarrow & \mathcal{O}_{P\times P}(-1,1) & \longrightarrow & 0 \end{array}$$

In bihomogeneous coordinates μ is given locally by a matrix

$$\begin{pmatrix} X_0 & X_1 & X_2 & \cdots & X_r \\ Y_0 & Y_1 & Y_2 & \cdots & Y_r \end{pmatrix}.$$

Thus μ is surjective off the diagonal $\Delta\subset P\times_X P$, while on the diagonal around a point where (for example) $X_0\neq 0$ and $Y_0\neq 0$ the map μ_1 locally has matrix

$$\begin{pmatrix} \frac{X_1}{X_0}-\frac{Y_1}{Y_0} & \frac{X_2}{X_0}-\frac{Y_2}{Y_0} & \cdots & \frac{X_r}{X_0}-\frac{Y_r}{Y_0} \end{pmatrix}.$$

Thus locally the components of μ_1 form a regular sequence of r equations defining the diagonal $\Delta\subset P\times_X P$. Hence $\operatorname{coker}(\mu_1)=\mathcal{O}_\Delta$, and $S^r(\mu_1)$ is a globalized Koszul complex which is a locally free resolution of $\mathcal{O}_\Delta$. Because of Proposition 10.1(b) $S^r(\mu)$ is also a locally free resolution of $\mathcal{O}_\Delta$. The augmentation is given by restriction to the diagonal and subtracting

$$\mathcal{O}_{P\times P}\oplus\mathcal{O}_{P\times P}(-1,1)\xrightarrow{\mathrm{rest}|_\Delta}\mathcal{O}_\Delta\oplus\mathcal{O}_\Delta(-1,1)\cong\mathcal{O}_\Delta\oplus\mathcal{O}_\Delta\xrightarrow{(1\ -1)}\mathcal{O}_\Delta.$$

Thus $S^r(\mu)\to\mathcal{O}_\Delta$ is a quasi-isomorphism, and induced maps of the form $q^*\mathcal{G}\otimes S^r(\mu)\to q^*\mathcal{G}\otimes\mathcal{O}_\Delta\cong i_*\mathcal{G}$ are also quasi-isomorphisms. We claim that if $\mathcal{G}\in\mathfrak{A}$, then all the sheaves in the complexes $q^*\mathcal{G}\otimes S^r(\mu)$ and $i_*\mathcal{G}$ are p_*-acyclic with p_* of finite cohomological dimension, and hence that $p_*(q^*\mathcal{G}\otimes S^r(\mu))\to p_*i_*\mathcal{G}\cong\mathcal{G}$ is also a quasi-isomorphism. On the one hand it is clear that $i_*\mathcal{G}$ is p_*-acylic because i and $pi=1_P$ are finite morphisms. On the other hand $q^*\mathcal{G}\otimes S^r(\mu)$ is a complex whose terms are direct sums $\bigoplus_{i=0}^{r-n}p^*(\pi^*(\Lambda^nF)(-n-i))\otimes q^*(\mathcal{G}(i))$. These summands are p_*-acyclic if the $q^*(\mathcal{G}(i))$ are, but because π is flat we have natural isomorphisms $\pi^*R^t\pi_*\cong R^tp_*q^*$, so it is enough if the $\mathcal{G}(i)$ are π_*-acyclic for $0\le i\le r$. If $\mathcal{G}\in\mathfrak{A}$, then we have the required acyclicities, and $p_*(q^*\mathcal{G}\otimes S^r(\mu))\xrightarrow{\sim}\mathcal{G}$ is a quasi-isomorphism. The same statement holds for chain complexes $\mathcal{G}\in\mathrm{Ch}(\mathfrak{A})$.

If one now takes the complex $p_*(q^*\mathcal{G}\otimes S^r(\mu))$, expands the definition of $S^r(\mu)$ and uses the natural isomorphisms $\pi^*\pi_*\cong p_*q^*$, then the complex becomes the totalization of

$$\begin{array}{ccccccc}
\pi^*(\pi_*(\mathcal{G})\otimes\Lambda^rF)(-r) & \to & \pi^*(\pi_*(\mathcal{G})\otimes\Lambda^{r-1}F)(-r+1) & \to\cdots\to & \pi^*(\pi_*(\mathcal{G})\otimes F)(-1) & \to & \pi^*\pi_*(\mathcal{G}) \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi^*(\pi_*(\mathcal{G}(1))\otimes\Lambda^{r-1}F)(-r) & \to & \pi^*(\pi_*(\mathcal{G}(1))\otimes\Lambda^{r-2}F)(-r+1) & \to\cdots\to & \pi^*\pi_*(\mathcal{G}(1))(-1) & & \\
\downarrow & & \downarrow & & & & \\
\vdots & & \vdots & & & & \\
\downarrow & & \downarrow & & & & \\
\pi^*(\pi_*(\mathcal{G}(r-1))\otimes F)(-r) & \to & \pi^*(\pi_*(\mathcal{G}(r-1)))(-r+1) & & & & \\
\downarrow & & & & & & \\
\pi^*(\pi_*(\mathcal{G}(r)))(-r) & & & & & &
\end{array}$$

The i-th line is $\rho^{[-r,-i]} \operatorname{cofree}_{-i} \pi_*(\mathcal{G}(i))$, and the totalization of the diagram is $\operatorname{Tot}\operatorname{Kr}\mathcal{G}$. The quasi-isomorphism $p_*(q^*\mathcal{G} \otimes S^r(\mu)) \xrightarrow{\sim} \mathcal{G}$ discussed above is constructed out of an augmentation map which turns into a map $\bigoplus_{i=0}^r \pi^*(\pi_*(\mathcal{G}(i)))(-i) \to \mathcal{G}$ whose components are essentially the counits of the π^*-π_* adjunction. So the quasi-isomorphism above corresponds to the counit of the adjunction $\operatorname{Tot}\operatorname{Kr}\mathcal{G} \to \mathcal{G}$. The counit is thus a quasi-isomorphism for all $\mathcal{G} \in \operatorname{Ch}(\mathfrak{A})$, as claimed. This completes the proof of the theorem. □

The functors Tot and Kr respect bounded, coherent cohomology. So we have a corollary.

Corollary 11.2. *Let X be a noetherian scheme, let F be a vector bundle on X of rank $r+1$, and let $\pi : P = \mathbb{P}(F) \to X$ be the projective bundle. For any integer a the functor* Tot *induces an equivalence $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbf{Qcoh}\text{-}\mathbf{LC}^{[a,a+r]}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbf{Qcoh}_P)$ of triangulated categories.*

Another version of this derived equivalence is Theorem 3.2, which we now restate and prove.

Theorem 11.3. *Let X be a noetherian scheme, let F be a vector bundle on X of rank $r+1$, and let $\pi : P = \mathbb{P}(F) \to X$ be the projective bundle. For any integer a the functor* Tot *induces an equivalence $\mathrm{D}^{\mathrm{b}}(\mathbf{LC}^{[a,a+r]}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{VB}_P)$ of triangulated categories.*

Proof. As in the proof of Theorem 11.1, it is enough to consider the single value $a = -r$. We again let $\mathfrak{A} \subset \mathbf{Qcoh}_P$ be the full exact subcategory of sheaves $\mathcal{G}$ such that $R^p\pi_*(\mathcal{G} \otimes \mathcal{O}_P(i)) = 0$ for $0 \le i \le r$ and $p > 0$, and let we $\mathfrak{A}_{\mathbf{VB}} = \mathfrak{A} \cap \mathbf{VB}_P$. We need to establish five claims analogous to those of the proof of Theorem 11.1.

(a) For any $\mathcal{A}$ in $\operatorname{Ch}^{\mathrm{b}}(\mathbf{VB}_P)$ there is a quasi-isomorphism $\mathcal{A} \xrightarrow{\sim} \mathcal{B}$ with $\mathcal{B} \in \operatorname{Ch}^{\mathrm{b}}(\mathfrak{A}_{\mathbf{VB}})$.
(b) The functor $\operatorname{Tot} : \operatorname{Ch}^{\mathrm{b}}(\mathbf{LC}^{[-r,0]}) \to \operatorname{Ch}^{\mathrm{b}}(\mathbf{VB}_P)$ factors through the subcategory $\operatorname{Ch}^{\mathrm{b}}(\mathfrak{A}_{\mathbf{VB}}) \subset \operatorname{Ch}^{\mathrm{b}}(\mathbf{VB}_P)$.
(c) The functor Tot has a right adjoint Kr such that $\operatorname{Kr}|_{\operatorname{Ch}^{\mathrm{b}}(\mathfrak{A}_{\mathbf{VB}})}$ is exact.
(d) The unit $\mathcal{E} \to \operatorname{Kr}\operatorname{Tot}\mathcal{E}$ of the adjunction is a quasi-isomorphism for all $\mathcal{E}$.
(e) The counit $\operatorname{Tot}\operatorname{Kr}\mathcal{G} \to \mathcal{G}$ of the adjunction is a quasi-isomorphism for all $\mathcal{G} \in \operatorname{Ch}^{\mathrm{b}}(\mathfrak{A}_{\mathbf{VB}})$.

We prove these claims.

(a) Since $\mathcal{A}$ is a bounded complex of coherent sheaves on P which is projective over X which is noetherian, there exists an N such that the $\mathcal{A}^i \otimes \mathcal{O}_P(n)$ are π_*-acyclic for all i and all $n \ge N$. The counit of the adjunction of Theorem 11.1 gives us a quasi-isomorphism $G \xrightarrow{\sim} \mathcal{O}_P(N)$ with $G := \operatorname{Tot}\operatorname{Kr}(\mathcal{O}_P(N))$ a bounded complex of vector bundles of the form $\bigoplus_{i=0}^r \pi^* G_{ij} \otimes \mathcal{O}_P(-i)$. Dualizing and twisting gives a quasi-isomorphism $\mathcal{O}_P \xrightarrow{\sim} G^\vee \otimes \mathcal{O}_P(N)$ with the target a bounded complex of vector bundles of the form $\bigoplus_{i=0}^r \pi^* G_{ij}^\vee \otimes \mathcal{O}_P(N+i)$. So $\mathcal{A} \xrightarrow{\sim} \mathcal{A} \otimes G^\vee \otimes \mathcal{O}_P(N)$ is a quasi-isomorphism from $\mathcal{A}$ into $\operatorname{Ch}^{\mathrm{b}}(\mathfrak{A}_{\mathbf{VB}})$.

(b) If $\mathcal{G}$ is a vector bundle on X, and if $i \ge -r$, then $\pi^*\mathcal{G} \otimes \mathcal{O}_P(i)$ is in $\mathfrak{A}_{\mathbf{VB}}$.

(c) One uses the same formulas as in Theorem 11.1. Grauert's theorems on cohomology and base change (Hartshorne [18] Theorem III.12.11) say that if $\mathcal{E}$ is a coherent sheaf on P which is π_*-acyclic and flat over X, then $\pi_*\mathcal{E}$ is locally free on X. In particular, if $\mathcal{E}$ is in $\mathfrak{A}_{\mathbf{VB}}$, then the $\pi_*(\mathcal{E}(i))$ are in $\mathbf{VB}_X$ for $0 \le i \le r$, and so $\operatorname{Kr}(\mathcal{E}) \in \operatorname{Ch}^{\mathrm{b}}(\mathbf{LC}^{[-r,0]})$.

(d)(e). These follow from the corresponding claims in the proof of Theorem 11.1 given that the vector-bundle versions of the functors Tot and Kr are the restrictions of the quasicoherent-sheaf versions. □

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