THE GROUP SK_2 OF A BIQUATERNION ALGEBRA

BAPTISTE CALMÈS

Abstract. M. Rost has proved the existence of an exact sequence relating the group SK_1 - kernel of the reduced norm for K_1 - of a biquaternion algebra D whose center is a field F with Galois cohomology groups of F. In this paper, we relate the group SK_2 - kernel of the reduced norm for K_2 - of D with Galois cohomology of F through an exact sequence.

Introduction

To understand the K-theory of central simple algebras, one of the most useful tools is the reduced norm. It is defined for K_0, K_1 and K_2. It has been proved (see [13], proposition 4) that it cannot be defined for K_3 and satisfy reasonable properties. The definition of the reduced norm is trivial for K_0, elementary for K_1, but much less elementary for K_2. A definition for K_2 was given by Suslin in [23] (corollary 5.7), which uses the highly non trivial result that the K-cohomology group $H^0(X, \mathcal{K}_2)$ is isomorphic to K_2F when X is a complete smooth rational variety over the field F (see [23], corollary 5.6). The kernel of the reduced norm for K_i, $i = 0, 1, 2$, is denoted SK_i and is difficult to compute for $i = 1, 2$ (it is always zero for $i = 0$). The first result on SK_1 was obtained by Wang in 1949. He proved in [30] that SK_1A is zero when the index of A is a product of different prime numbers. Whether SK_1 was always zero or not was then known as the Tannaka-Artin problem. No example of an algebra with nonzero SK_1 was found until 1975, when Platonov gave the first such example (see [17]). In the eighties, a new approach has been initiated by Suslin, which is to relate SK_1 with Galois cohomology of the base field. Quite a few theorems were obtained in this direction (see [24], [12], [13] and [14]). The most explicit of these results is a theorem of Rost who proves the existence of an exact sequence $0 \to SK_1D \to H^4(F, \mu_2) \to H^4(F(q), \mu_2)$ when q is an Albert form with associated biquaternion algebra D. Since D is of index 4, it is the simplest case not covered by the theorem of Wang.

About the group SK_2, much less is known. Merkurjev has shown in [11] that SK_2 of a quaternion algebra is always trivial, but no would-be analogue of the Wang theorem is known. Once again, the simplest case when this group can be non-zero is the case of a biquaternion algebra. It is worth noting that an explicit biquaternion algebra for which SK_2 is non zero can be obtained in the following way. First, use Rost's theorem to obtain a biquaternion algebra with a non zero element x in SK_1D (see for example [13]). Then, the cup-product of x by t in $K_2D(t)$ is non zero by residue. Nevertheless, I believe it is of interest to continue Suslin's approach, that is to relate SK_2 with Galois cohomology. This is the subject of the present work. The main result in this paper is the following, which is an analogue of Rost's theorem for SK_2.

Main Theorem. *Let F be a field of characteristic zero, containing an algebraically closed subfield ($\bar{\mathbf{Q}} \subseteq F$). Let D be the biquaternion algebra $\binom{a\ b}{F} \otimes \binom{c\ d}{F}$. Let*

q be the Albert quadratic form $< a, b, -ab, -c, -d, cd >$ and q' a codimension-one subform of q. Let $\mathrm{N}_{q'} : H^3(X_{q'}, \mathcal{K}_5) \rightarrow K_2F$ be the usual norm map in K-cohomology (see [19]). There is an exact sequence

$$\ker \mathrm{N}_{q'} \longrightarrow SK_2D \longrightarrow H^5(F, \mathbf{Z}/2) \longrightarrow H^5(F(q), \mathbf{Z}/2)$$

The proof of this result is divided into four parts. In the first part, computations are made using spectral sequences in motivic cohomology, in order to identify specific K-cohomology groups with Galois cohomology groups (see section 1, equality (4)). In the second part, we first exhibit an isomorphism between the projective quadric X_q of an Albert form and the generalized Severi-Brauer variety $SB(2, D)$ of the associated biquaternion algebra D (see theorem 2.13). This isomorphism ultimately comes from the exceptionnal isomorphism between SL_4 and $Spin 3\mathbf{H}$, where $3\mathbf{H}$ is an orthogonal sum of three times the hyperbolic form $< 1, -1 >$. We then use Panin's decomposition of the K-theory of projective homogeneous varieties to decompose the K-theory of these two varieties and to pass from one decomposition to another using the previously described isomorphism. We also handle the functoriality of the decomposition along the natural morphism $X_{q'} \rightarrow X_q$ where q' is a codimension one subform of q. In the third part, we partially compute the topological filtration of X_q and $X_{q'}$. To fulfill this task, we use the isomorphism described in part two. Indeed, part of the topological filtration is easy to compute on $SB(2, D)$, which is a twisted Grassmannian, because we can use the theory of Schubert calculus (see section 3.1), but another part of the filtration is easier to compute on the quadric, because we can then use some results of Chernousov and Merkurjev on R-equivalence (see section 3.2). Finally, in the last part, we use the results of parts one, two and three to obtain the main theorem.

Notations

We now introduce some notations that are used throughout the article. Let F be a field and F_{sep} a separable closure of F. We usually use F as the base field, whenever a base field is needed.

Varieties. By a variety over F, we mean a separated integral scheme of finite type over Spec$F$. The field of functions of an integral scheme X over Spec$F$ is denoted $F(X)$. Let K be an extension of F, X_K denotes the variety $X \times_{\mathrm{Spec}F} \mathrm{Spec}K$ over K.

Quadratic forms. By a quadratic form, we mean a non degenerate (regular) quadratic form. Let φ be a quadratic form over a field. We denote X_φ the corresponding projective variety (defined by the equation $\varphi = 0$). The field $F(X_\varphi)$ is abbreviated in $F(\varphi)$. If K is an extension of F, q_K denotes the quadratic form obtained by extension of scalars from F to K.

The letter q always denotes an Albert quadratic form whose coefficients on an orthogonal basis is $q =< a, b, -ab, -c, -d, cd >$. The letter q' denotes a codimension 1 subform of q.

Cohomology. For a variety X, we denote $H^p(X, \mathcal{K}_q)$ (resp. $H^p(X, \mathcal{K}_q^M)$) the K-cohomology groups of X, that is, the cohomology of the Gersten complex of X for Quillen K-theory (resp. for Milnor K-theory).

The notation $H^i(K, \mathbf{Z}/m)$ is used for Galois cohomology of the field K with coefficients in $\mathbf{Z}/m$. Let $\mathbf{Z}/m(1)$ be the Galois module of the roots of unity μ_m. A twist (by j) shall mean that $\mathbf{Z}/m(1)$ has been tensored by itself over $\mathbf{Z}$ j times, as in $H^i(K, \mathbf{Z}/m(j))$.

Motivic cohomology groups of a scheme for the étale topology whith coefficients in the ring A over $\mathbf{Z}$, as they are defined by Voevodsky in [28], shall be denoted $H^i_{ét}(X, A(n))$. The group $H^i_{ét}(\mathrm{Spec}F, \mathbf{Z}/m(j))$ can be identified with the classical étale cohomology group (and therefore Galois cohomology) $H^i(F, \mathbf{Z}/m(j))$ (see [28]).

Central simple algebras. We say that a central simple algebra over F is split when its class in the Brauer group of F is trivial (i.e. when it is isomorphic to a matrix algebra over F).

The letter D always denotes the biquaternion algebra $\left(\frac{a\ b}{F}\right) \otimes_F \left(\frac{c\ d}{F}\right)$. It is easy to show that the Clifford algebra of the Albert form q is isomorphic to $M_2(D)$. They therefore define the same class in the Brauer group of F. Moreover, one can prove (see [10]) that D is a division algebra if and only if q is anisotropic, and that it is split if and only if q is hyperbolic.

Severi-Brauer varieties. A detailed account of Severi-Brauer varieties and their properties can be found in [1].

Let A be a degree n central simple algebra over F. The variety parametrizing the ideals of rank mn of A is called the generalized Severi-Brauer variety of A, and is denoted $SB(m, A)$ (or simply $SB(A)$ when $m = 1$). It is therefore equivalent for A not to be a division algebra and for $SB(A)$ to have a rationnal point. When A is split, $SB(m, A)$ is isomorphic to the Grassmann variety $Gr(m, n)$.

1. Motivic cohomology

In this section, we shall relate some Galois cohomology groups of the base field F and its extensions with K-cohomology groups of certain quadrics, using ideas originally described in [6]. These are mainly computations in spectral sequences involving motivic cohomology groups. For this reason, we require that the characteristic of F is zero. To be a bit more precise, we shall prove the following.

Theorem 1.1. *After localizing at 2, setting $X = X_q$ or $X = X_{q'}$ and $Y = SB(D)$, there are exact sequences*

$$0 \longrightarrow H^5_{ét}(F, \mathbf{Q}/\mathbf{Z}(4)) \longrightarrow H^6_{ét}(X_q, \mathbf{Z}(4)) \longrightarrow K_2(F) \oplus K_2(F) \tag{1}$$

$$0 \longrightarrow H^5_{ét}(F, \mathbf{Q}/\mathbf{Z}(4)) \longrightarrow H^6_{ét}(X_{q'}, \mathbf{Z}(4)) \longrightarrow K_2(F) \tag{2}$$

$$0 \longrightarrow H^2(X, \mathcal{K}_4) \longrightarrow H^6_{ét}(X, \mathbf{Z}(4)) \longrightarrow H^5_{ét}(F(X), \mathbf{Q}/\mathbf{Z}(4)) \tag{3}$$

and they induce an isomorphism

$$\ker(H^2(X, \mathcal{K}_4) \to H^2((X)_{F(Y)}, \mathcal{K}_4)) \simeq \ker(H^5(F, \mathbf{Z}/2) \to H^5(F(X), \mathbf{Z}/2)). \tag{4}$$

We shall now obtain the exact sequence (1) from the spectral sequence defined in [6], theorem 4.4. This spectral sequence is associated to a geometrically cellular variety X over a field F of characteristic 0 (i.e. a variety that is cellular over a separable closure of F) and a weight n. We will use it only for the quadrics X_q and $X_{q'}$, in weight $n = 4$. The E_2 terms of this spectral sequence are motivic cohomology groups of an étale algebra over F (see [6], §5.1), which will always be F or $F \times F$ in our case:

$$E_2^{p,q} = H^p(E_q, \mathbf{Z}(n - q)).$$

It converges, for the antidiagonals $p + q \leq 2n$, to the étale motivic cohomology group $H^{p+q}_{ét}(X, \mathbf{Z}(n))$.

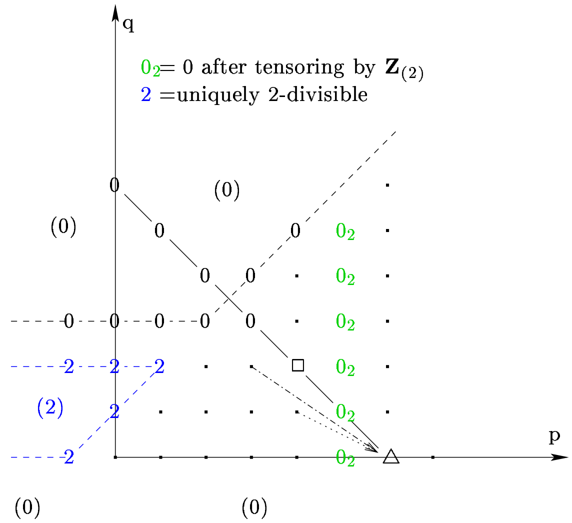


FIGURE 1. Kahn's spectral sequence in weight 4

In weight $n = 4$, the E_2 terms have the following properties:

(1) for $q < 0$, $E_2^{p,q} = 0$
(2) for $q > p$, $E_2^{p,q}$ is uniquely 2-divisible
(3) for $q > p$ and $q > 2$, $E_2^{p,q} = 0$
(4) for all q, $E_2^{5,q} \otimes \mathbf{Z}_{(2)} = 0$ ("Hilbert 90")
(5) $E_2^{3,3} = 0$

These properties are summed-up in figure 1.

Proof: 1. This follows from the definition of the spectral sequence. 2. This follows from the long exact sequence in cohomology associated to the short exact sequence $0 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z}/2 \longrightarrow 0$, using the fact that classical étale cohomology $H^i_{ét}(F, \mathbf{Z}/m(j)) = 0$ for $i < 0$. 3. In this case, the groups identify with sheaf cohomology in negative degree. 4. See [27] 5. The complex of sheaves $\mathbf{Z}(1)$ is just $\mathbf{G}_m$ in degree 1, so its cohomology in degree zero $H^0_{ét}(F, \mathbf{Z}(1))$ is zero. □

After localising at the prime 2, we are left with at most two non-zero terms on the $p + q = 6$ anti-diagonal (□ and △ in figure 1). This induces an exact sequence (all the groups are localized at 2)

$$0 \longrightarrow E_\infty^{6,0} \longrightarrow H^6_{ét}(X, \mathbf{Z}(4)) \longrightarrow E_\infty^{4,2}.$$

Let us now compute some of the differentials to relate the E_∞ terms with the E_2 terms. All the differentials that map to $E_2^{4,2}$ are zero, therefore $E_\infty^{4,2}$ is a subgroup of $E_2^{4,2}$. The differential d_i, $i > 3$ that map to $E_i^{6,0}$ are zero, as well as all the differentials coming from $E_i^{6,0}$, $i \geq 2$. The differential $d_2^{4,1}$ is zero (see [6], Corollary 8.6, a). If $d_3^{3,2}$ is zero, we will therefore have an exact sequence

$$0 \longrightarrow E_2^{6,0} \longrightarrow H^6_{ét}(X, \mathbf{Z}(4)) \longrightarrow E_2^{4,2}. \quad (5)$$

In fact, $d_3^{3,2}$ is zero if F contains an algebraically closed subfield F_0. This follows from

Lemma 1.2. Let $K_3(F)_{nd}$ be the cokernel of the natural map from the Milnor K-theory group $K_3^M F$ to the Quillen K-theory group $K_3 F$. If F contains an algebraically closed subfield F_0, then $K_3(F)_{nd}$ is divisible.

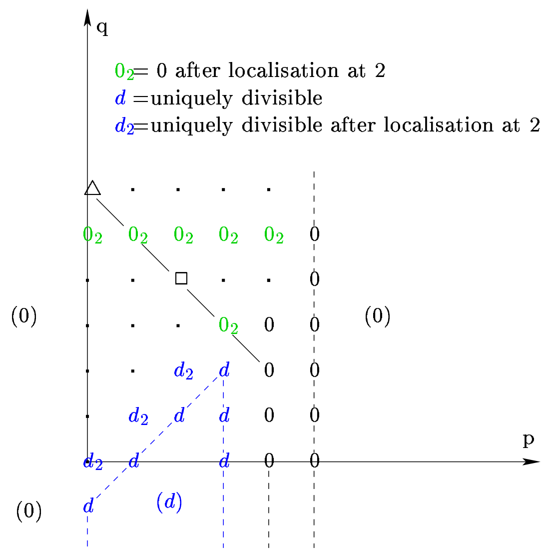


FIGURE 2. Coniveau spectral sequence in weight 4

Proof: Proposition 11.6 in [15] shows that the cokernel of the morphism $K_3(F_0)_{nd} \to K_3(F)_{nd}$ is uniquely divisible. Since F_0 is algebraically closed, the group $K_3(F_0)_{nd}$ is divisible and so is $K_3(F)_{nd}$. □

The differential $d_2^{1,3}$ is zero. All the differentials are killed by 4 by a transfer argument (the variety becomes cellular after a degree 4 extension). The differential $d_2^{3,2}$ is therefore also zero because $E_2^{5,1}$ is zero after localizing at 2. It follows that $E_3^{3,2} = E_2^{3,2}$. The latter can be idenntified with $H^3_{ét}(F \times F, \mathbf{Z}(1)) \simeq K_3(F)_{nd} \times K_3(F)_{nd}$ if $X = X_q$ and with $H^3_{ét}(F, \mathbf{Z}(1)) \simeq K_3(F)_{nd}$ if $X = X_{q'}$ (see [6], Lemma 8.2 for the computation of the étale algebra F or $F \times F$ involved). Lemma 1.2 implies $d_3^{3,2} = 0$ since it is torsion and comes from a divisible group. Hence, sequence (5) is exact. Identification of the E_2 terms yields $E_2^{4,2} = K_2F \times K_2F$ for $X = X_q$, $E_2^{4,2} = K_2F$ for $X = X_{q'}$ and $E_2^{6,0} = H^6_{ét}(F, \mathbf{Z}(4))$ in both cases. The long exact sequence in cohomology associated to the exact triangle $\mathbf{Z}(j) \to \mathbf{Q}(j) \to \mathbf{Q}/\mathbf{Z}(j) \to \mathbf{Z}(j)[1]$ and the fact that, for $i > j$, $H^i_{ét}(F, \mathbf{Q}(j)) = 0$ shows that $H^6_{ét}(F, \mathbf{Z}(4)) \simeq H^5_{ét}(F, \mathbf{Q}/\mathbf{Z}(4))$. Sequence (5) therefore becomes sequences (1) and (2) when specializing X to X_q or $X_{q'}$.

Let us now obtain the exact sequence (3). We will use the coniveau spectral sequence for étale motivic cohomology (see [6], lemma 5.1) once again in weight 4 and for the varieties $X = X_q$ or $X = X_{q'}$.

This spectral sequence has the following properties (see [6], §5.1):

(1) $E_1^{p,q} = 0$ for p such that $p \geq q$ and $p > n$, as well as for $p > q$ and $p = n$,
(2) $E_1^{p,q}$ is uniquely divisible for $q < p \leq n$,
(3) after localisation at 2, $E_1^{p,q}$ is uniquely divisible for $p = q < n$,
(4) after localisation at 2, $E_1^{n-1,n-1} = 0$,
(5) after localisation at 2, $E_1^{p,n+1} = 0$,
(6) $E_1^{p,q} = 0$ for $p > dimX$ or for $p < 0$.

These properties are summed-up on figure 2

As in the preceding sequence, we get, after localization at 2, the exact sequence

$$0 \longrightarrow E_\infty^{2,4} \longrightarrow H^6_{ét}(X, \mathbf{Z}(4)) \longrightarrow E_\infty^{0,6}. \tag{6}$$

The group $E_\infty^{2,4}$ can be identified with $E_2^{2,4}$ and $E_\infty^{0,6} \subset E_1^{0,6}$, since the needed differentials are evidently zero. The group $E_1^{0,6}$ can be identified with $H^6_{ét}(F(X), \mathbf{Z}(4)) \simeq H^5_{ét}(F(X), \mathbf{Q}/\mathbf{Z}(4))$ and $E_2^{2,4}$ with $H^2(X, \mathcal{K}_4^M)$. When F contains an algebraically closed subfield, the latter can be identified with $H^2(X, \mathcal{K}_4)$. I reproduce here a proof of this result by Kahn: it is obvious, on the Gersten complex, that the natural map $\varphi : H^2(X, \mathcal{K}_4^M) \to H^2(X, \mathcal{K}_4)$ is surjective. Using the Adams operations on algebraic K-theory, one can show that the exact sequence

$$0 \longrightarrow K_3^M(F) \longrightarrow K_3(F) \longrightarrow K_3(F)_{nd} \longrightarrow 0$$

is split up to 2-torsion. It follows that $\ker \varphi$ is killed by 2. We have an exact sequence

$$\bigoplus_{x \in X^{(1)}} K_3(F(x))_{nd} \longrightarrow H^2(X, \mathcal{K}_4^M) \xrightarrow{\varphi} H^2(X, \mathcal{K}_4).$$

Each $K_3(F(x))_{nd}$ is divisible (see lemma 1.2). Since their images in $H^2(X, \mathcal{K}_4)$ are killed by 2, they are zero.

With these identifications, we get sequence (3) from sequence (6).

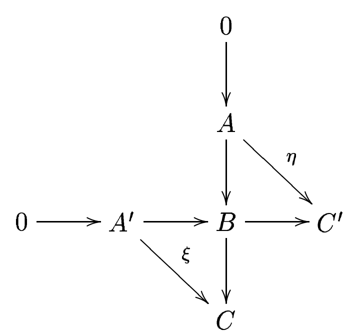
The following lemmas are well known.

Lemma 1.3. When X has a rational point,

$$\ker(H^5_{ét}(F, \mathbf{Q}/\mathbf{Z}(4)) \longrightarrow H^5_{ét}(F(X), \mathbf{Q}/\mathbf{Z}(4)))$$

is zero.

Lemma 1.4. Two forked exact sequences



give rise to a canonical isomorphism $\ker \eta \simeq \ker \xi$.

Now, lemma 1.4 applied to sequences (1) (resp. (2)) and (3) gives the isomorphisms

$$\ker(H^5_{ét}(F, \mathbf{Q}/\mathbf{Z}(4)) \xrightarrow{\eta} H^5_{ét}(F(q), \mathbf{Q}/\mathbf{Z}(4))) \simeq \ker(H^2(X_q, \mathcal{K}_4) \xrightarrow{\xi_q} K_2(F)^2 \tag{7}$$

$$\ker(H^5_{ét}(F, \mathbf{Q}/\mathbf{Z}(4)) \xrightarrow{\eta} H^5_{ét}(F(q'), \mathbf{Q}/\mathbf{Z}(4))) \simeq \ker(H^2(X_{q'}, \mathcal{K}_4) \xrightarrow{\xi_{q'}} K_2(F). \tag{8}$$

It is not difficult to show from the spectral sequences that η coincides with the extension of scalars.

Lemma 1.5. The quadric $X_{q'}$ has a rational point over $F(Y)$.

Proof: Since $D_{F(Y)}$ is split, $q_{F(Y)}$ is hyperbolic (see introduction on central simple algebras). The quadratic form q' is of codimension 1 in a 6-dimensional form q, so by the Witt index theorem, it is isotropic. □

This implies that η and therefore ξ_q and $\xi_{q'}$ are injective over $F(Y)$. A diagram chase using the fact that $K_2(F) \longrightarrow K_2(F(Y))$ is injective (see [23]) easily shows that $\ker \xi$ is isomorphic to $\ker(H^2(X, \mathcal{K}_4) \longrightarrow H^2(X_{F(Y)}, \mathcal{K}_4))$. Thus, the isomorphisms (7) and (8) become

(9)

$$\ker(H^5_{\acute{e}t}(F, \mathbf{Q}/\mathbf{Z}(4)) \xrightarrow{\eta} H^5_{\acute{e}t}(F(X), \mathbf{Q}/\mathbf{Z}(4))) \simeq \ker(H^2(X, \mathcal{K}_4) \xrightarrow{\xi} H^2(X_{F(Y)}, \mathcal{K}_4))$$

Lemma 1.6. The group $\ker \xi$ is killed by 2.

Proof: When X has a rational point, this group is zero (see lemma 1.3), so the result follows from a transfer argument using a quadratic extension over which q' (and therefore q) is isotropic. □

It is worth noting that the following result uses the Milnor conjecture.

Lemma 1.7. The 2-torsion part of $H^5_{\acute{e}t}(F, \mathbf{Q}/\mathbf{Z}(4))$ is $H^5_{\acute{e}t}(F, \mathbf{Z}/2(4))$.

Proof: The following diagram is commutative.

$$\begin{array}{ccccccc}
H^n_{\acute{e}t}(F, \mathbf{Q}_2/\mathbf{Z}_2(n)) & \xrightarrow{\times 2} & H^n_{\acute{e}t}(F, \mathbf{Q}_2/\mathbf{Z}_2/l(n)) & \xrightarrow{0} & H^{n+1}_{\acute{e}t}(F, \mathbf{Z}/2(n)) & \hookrightarrow & H^{n+1}_{\acute{e}t}(F, \mathbf{Q}_2/\mathbf{Z}_2(n)) \\
\uparrow \wr & & \uparrow \wr & & & \searrow & \downarrow \\
K^M_n(F) \otimes \mathbf{Q}_2/\mathbf{Z}_2 & \xrightarrow{\times 2} & K^M_n(F) \otimes \mathbf{Q}_2/\mathbf{Z}_2 & & & & H^{n+1}_{\acute{e}t}(F, \mathbf{Q}/\mathbf{Z}(n))
\end{array}$$

The top line comes from the long exact sequence in cohomology associated to the short exact sequence

$$0 \longrightarrow \mathbf{Z}/2 \longrightarrow \mathbf{Q}_2/\mathbf{Z}_2 \longrightarrow \mathbf{Q}_2/\mathbf{Z}_2 \longrightarrow 0.$$

The left vertical maps come from the Milnor conjecture ([27]) and the right vertical inclusion comes from the fact that the canonical map has a section. This shows the other properties of the diagram. The result is then implied by the fact that the next map in the top sequence is the multiplication by 2. □

Finally, the isomorphism (4) is just the 2-torsion part of the isomorphism (9).

Remark 1.8. (see [6], cor. 6.7 a) Using the same spectral sequences in weight 3 yields the isomorphism (for $X = X_q$ or $X = X_{q'}$)

(10) $$\ker(H^2(X, \mathcal{K}_3) \to H^2((X)_{F(Y)}, \mathcal{K}_3)) \simeq \ker(H^4(F, \mathbf{Z}/2) \to H^4(F(X), \mathbf{Z}/2))$$

without the hypothesis that F contains an algebraically closed subfield. This isomorphism was used by Rost to show his theorem, but is obtained by him in a more elementary way and in any characteristic different from 2.

2. Projective homogeneous varieties

2.1. **Panin's decomposition.** The K-theory of projective homogeneous varieties has been completely computed in terms of K-groups of algebras (Tits algebras) naturally associated to these varieties. Historically, Quillen, in [18] (1973) first computed the K-theory of projective spaces and their twisted forms (Severi-Brauer varieties) using resolutions. Then Swan, in [25] (1985) adapted Quillen's computations to quadrics. In 1989, Levine, Srinivas and Weyman computed in [9] the K-theory of twisted Grassmannians (generalised Severi-Brauer varieties) by descent methods. Panin had similar results by that time, using representation theory.

Finally, in 1994, he gave a general computation of the K-theory of projective homogeneous varieties using representations of algebraic groups (see [16]). We shall use this last computation for many reasons. First, it is easier to follow the functorial properties of these decompositions using Panin's viewpoint; morphisms coming from algebraic groups induce morphisms on the decomposition. Second, cup-products in K-theory are quite easy to understand on Panin's decomposition, and they are important to us because they respect the topological filtration. In this section, we shall therefore show some functorial properties of Panin's decomposition which can easily be deduced from [16] as well as the way to compute cup-products.

Let us first recall the settings. Let $\tilde{G}$ be an F-split simply connected semisimple algebraic group. Let $\tilde{Z}$ be the center of $\tilde{G}$ and $\tilde{Y}$ a subgroup of $\tilde{Z}$. Let $\tilde{T}$ be a maximal split torus in $\tilde{G}$, and $\tilde{P}$ a parabolic subgroup of $\tilde{G}$ containing $\tilde{T}$. We shall set $G = \tilde{G}/\tilde{Y}$ and $P = \tilde{P}/\tilde{Y}$. Let $\mathcal{F} = \tilde{G}/\tilde{P}$ be the quotient variety and ${}_\gamma\mathcal{F}$ the twist of $\mathcal{F}$ by a 1-cocycle $\gamma : Gal(F_{sep}/F) \to G(F_{sep})$.

For any affine algebraic group H, let $Rep_F(H)$ denote the exact category of finite dimensional F-rational linear representations of H and $R(H)$ the associated Grothendieck group. The tensor product of representations makes it a commutative ring. The forgetful functor from $Rep_F(H)$ to the category of F vector spaces induces on their Grothendieck groups the morphism $\dim : R(H) \to \mathbf{Z}$. Let χ be a character of $\tilde{Z}$ and denote $Rep_F^\chi(\tilde{P})$ (resp. $Rep_F^\chi(\tilde{G})$) the full subcategory of $Rep_F(\tilde{P})$ (resp. $Rep_F(\tilde{G})$) whose objects are the representations on which $\tilde{Z}$ acts via χ. Let $R^\chi(\tilde{P})$ (resp. $R^\chi(\tilde{G})$) be the associated Grothendieck group. The product on $R(\tilde{P})$ respects characters, that is

$$R^\chi(\tilde{P}) \otimes_{\mathbf{Z}} R^{\chi'}(\tilde{P}) \longrightarrow R^{\chi\chi'}(\tilde{P})$$

Furthermore, the characters induce the decompositions (see [16], lemma 2.8)

$$\bigoplus_\chi R^\chi(\tilde{P}) \simeq R(\tilde{P})$$

and

$$\bigoplus_\chi R^\chi(\tilde{G}) \simeq R(\tilde{G}).$$

An element in $R(\tilde{P})$ is said to be Ch-homogeneous if it lies in $R^\chi(\tilde{P})$ for a certain χ. Let W denote the Weyl group of $\tilde{G}$, and W_P the subgroup of W whose elements w verify $w\tilde{P}w^{-1} = \tilde{P}$. Let $\tilde{X} = \mathrm{Hom}(\tilde{T}, \mathbf{G}_m)$ and $\tilde{X}_\chi$ the subset of $\tilde{X}$ of those elements who induce the character χ on $\tilde{Z}$. Then

$$\begin{array}{rcl} R(\tilde{T}) & \simeq & \mathbf{Z}[\tilde{X}] \\ R(\tilde{P}) & \simeq & \mathbf{Z}[\tilde{X}]^{W_P} \\ R(\tilde{G}) & \simeq & \mathbf{Z}[\tilde{X}]^{W} \end{array}$$

and

$$\begin{array}{rcl} R^\chi(\tilde{T}) & \simeq & \mathbf{Z}[\tilde{X}_\chi] \\ R^\chi(\tilde{P}) & \simeq & \mathbf{Z}[\tilde{X}_\chi]^{W_P} \\ R^\chi(\tilde{G}) & \simeq & \mathbf{Z}[\tilde{X}_\chi]^{W}. \end{array}$$

Moreover, $R(\tilde{G})$ is a polynomial ring - in the classes of fundamental representations - and $R(\tilde{P})$ is a free $R(\tilde{G})$ module (see [16], th. 2.10).

Let $Vect^{\tilde{G}}(\mathcal{F})$ denote the category of vector $\tilde{G}$ equivariant vector bundles over $\mathcal{F}$. There are well known functors (see [16], § 1)

$$\mathrm{Ind} : Rep_F(\tilde{P}) \longrightarrow Vect^{\tilde{G}}(\mathcal{F})$$

and

$$\mathrm{Res} : Vect^{\tilde{G}}(\mathcal{F}) \longrightarrow Rep_F(\tilde{P})$$

which are equivalences of categories and induce in K-theory isomorphisms inverse to each other

$$\mathrm{Ind} : R(\tilde{P}) \longrightarrow K_0^{\tilde{G}}(\tilde{G}/\tilde{P})$$

and

$$\mathrm{Res} : K_0^{\tilde{G}}(\tilde{G}/\tilde{P}) \longrightarrow R(\tilde{P}).$$

Central simple algebras can be associated to every character χ and cocycle γ. These algebras are called Tits algebras and were first introduced by Tits (see [26]). Take V_χ in $Rep_F^\chi(\tilde{G})$ and let $A_\chi = \mathrm{End}_F(V_\chi)$, then twist A_χ in $A_{\chi,\gamma}$ by the cocycle obtained by pushing γ to $PGL(F_{sep}) = Aut_{F_{sep}}(A_\chi \otimes_F F_{sep})$ using V_χ.

Lemma 2.1. (see [16], § 3 and lemma 3.4)

(1) The class of $A_{\chi,\gamma}$ in the Brauer group of F is independent of the representation chosen in $Rep_F^\chi(\tilde{G})$.
(2) If we choose $V_{\chi\chi'} = V_\chi \otimes_F V_{\chi'}$, then $A_{\chi,\gamma} \otimes A_{\chi',\gamma} \simeq A_{\chi\chi',\gamma}$. In the general case, we only have $A_{\chi,\gamma} \otimes A_{\chi',\gamma} \sim A_{\chi\chi',\gamma}$.
(3) $A_{\chi^{-1},\gamma} \sim A_{\chi,\gamma}^{op}$
(4) If $a \in R^\chi(\tilde{G})$, then $\mathrm{ind}(A_{\chi,\gamma})$ divides $\dim(a)$.

Proof: 1, 2 and 3 are proved in [16]. To prove 4, we just need to show that for every representation $V'_\chi \in Rep_F^\chi(\tilde{G})$, $\mathrm{ind}(A_{\chi,\gamma})$ divides $\dim V'_\chi$. But since the degree of $A'_{\chi,\gamma} =_\gamma \mathrm{End}(V'_\chi)$ is $\dim V'_\chi$, it is divisible by $\mathrm{ind}(A'_{\chi,\gamma}) = \mathrm{ind}(A_{\chi,\gamma})$ (by 1). □

Let $U' \in Vect^G(\mathcal{F})$ be a vector bundle on which A_χ acts on the right. The twisted form ${}_\gamma U'$ of U' is naturally equipped with a right action of $A_{\chi,\gamma}$. The biexact functor

$$\begin{array}{ccc} Rep_F^\chi(\tilde{P}) \times (A_{\chi,\gamma} - mod) & \longrightarrow & Vect({}_\gamma\mathcal{F}) \\ (U, M) & \longmapsto & {}_\gamma(Ind(U) \otimes_F V_\chi^*) \otimes_{A_{\chi,\gamma}} M \end{array}$$

induces a pairing

$$\mu_{\chi,\gamma} : R^\chi(\tilde{P}) \otimes_{\mathbf{Z}} K_*(A_{\chi,\gamma}) \longrightarrow K_*({}_\gamma\mathcal{F}).$$

For a Ch-homogeneous element $a \in R(\tilde{P})$, define $\varphi_{a,\gamma}$ as

$$\begin{array}{cccc} \varphi_{a,\gamma} : & K_*(A_{\chi_a}) & \longrightarrow & K_*({}_\gamma\mathcal{F}) \\ & x & \longmapsto & \mu_{\chi_a,\gamma}(a \otimes x). \end{array}$$

The main theorem in [16] is the following.

Theorem 2.2. *(see [16], theorem 4.2) For any Ch-homogeneous basis $\{a_i | i = 1 \ldots n\} \in R(\tilde{P})$ of the free $R(\tilde{G})$-module $R(\tilde{P})$, the morphism*

$$\sum_{i=1}^n \varphi_{a_i,\gamma} : \bigoplus_{i=1}^n K_*(A_{\chi_{a_i},\gamma}) \longrightarrow K_*({}_\gamma\mathcal{F})$$

is an isomorphism.

Remark 2.3. It is clear on this decomposition that $K_0({}_\gamma\mathcal{F})$ injects in $K_0({}_\gamma\mathcal{F}_E)$ for any field extension E of F.

Let us now show a few properties of this decomposition.

Lemma 2.4. For a and b two Ch-homogeneous elements, $x \in K_*(A_{\chi_a})$ and $y \in K_*(A_{\chi_b})$,

$$\varphi_{ab,\gamma}(xy) = \varphi_{a,\gamma}(x).\varphi_{b,\gamma}(y).$$

Proof: This follows from the commutativity of the diagram

$$\begin{array}{ccc}
R^{\chi}(\tilde{P}) \otimes R^{\chi'}(\tilde{P}) \otimes K_i(A_{\chi,\gamma}) \otimes K_j(A_{\chi',\gamma}) & \xrightarrow{\sim} & R^{\chi}(\tilde{P}) \otimes K_i(A_{\chi,\gamma}) \otimes R^{\chi'}(\tilde{P}) \otimes K_j(A_{\chi',\gamma}) \\
\downarrow {\scriptstyle .\otimes.} & & \downarrow {\scriptstyle \mu_{\chi,\gamma}\otimes\mu_{\chi',\gamma}} \\
R^{\chi\chi'}(\tilde{P}) \otimes K_{i+j}(A_{\chi\chi',\gamma}) & & K_i({}_\gamma\mathcal{F}) \otimes K_j({}_\gamma\mathcal{F}) \\
\downarrow {\scriptstyle \mu_{\chi\chi',\gamma}} & & \downarrow {\scriptstyle \cdot} \\
K_{i+j}({}_\gamma\mathcal{F}) & \xrightarrow{\sim} & K_{i+j}({}_\gamma\mathcal{F})
\end{array}$$

which amounts to the identification of tensor products in the underlying categories
□

Lemma 2.5. The morphism $\varphi_{a,\gamma}$ commutes with extension of scalars and with the norm for a finite extension of the base field.

Proof: In the definition of $\mu_{\chi,\gamma}$, all the terms commute with the extension of scalars and the norm (which is just a restriction).
□

As mentionned above (lemma 2.1), $A_{\chi\chi',\gamma} \simeq A_{\chi,\gamma} \otimes A_{\chi',\gamma}$. Let $B_{\chi',\gamma}$ be the division algebra Brauer-equivalent to $A_{\chi',\gamma}$. Define $\overline{\mathrm{Res}} : K_*(A_{\chi\chi',\gamma}) \to K_*(A_{\chi,\gamma})$ as the composite of the Morita invariance morphism from $K_*(A_{\chi\chi',\gamma})$ to $K_*(A_{\chi,\gamma} \otimes B_{\chi',\gamma})$ with the restriction of the latter to $K_*(A_{\chi,\gamma})$.

Lemma 2.6. The following diagram is commutative.

$$\begin{array}{ccc}
R^{\chi}(\tilde{G}) \otimes R^{\chi'}(\tilde{P}) \otimes K_*(A_{\chi\chi',\gamma}) & \xrightarrow{(Id.Id)\otimes Id} & R^{\chi\chi'}(\tilde{P}) \otimes K_*(A_{\chi\chi',\gamma}) \\
{\scriptstyle (\frac{\dim}{\mathrm{ind}(A_{\chi,\gamma})}.Id)\otimes\overline{\mathrm{Res}}} \downarrow & & \downarrow {\scriptstyle \mu_{\chi\chi',\gamma}} \\
R^{\chi'}(\tilde{P}) \otimes K_*(A_{\chi,\gamma}) & \xrightarrow{\mu_{\chi',\gamma}} & K_*({}_\gamma\mathcal{F})
\end{array}$$

Proof: This amounts again to identifying tensor products in the underlying categories. □

Lemma 2.7. Let a be a Ch-homogeneous element of $R(\tilde{P})$ such that $a = \sum_k \lambda_k b_k$ where the b_k are also Ch-homogeneous and free (as a subset of the $R(\tilde{G})$-module $R(\tilde{P})$) and $\lambda_k \in R(\tilde{G})$ for all k, then

$$\varphi_{a,\gamma} = \sum_k \frac{\dim(\lambda_k)}{\mathrm{ind}(A_{\chi_{\lambda_k},\gamma})} \varphi_{b_k,\gamma} \circ \overline{\mathrm{Res}}_{A_{\chi_a},A_{\chi_{b_k}}}$$

Proof: Let us first prove that the λ_k are Ch-homogeneous. Let $\lambda_k = \sum_l \epsilon_{l,k}$ where the $\epsilon_{l,k} \in R(\tilde{G})$ are Ch-homogeneous of character $\chi_{l,k}$ ($\chi_{l,k} \neq \chi_{l',k}$ when $l \neq l'$). For every k, the product $\epsilon_{l,k} b_k$ is Ch-homogeneous.

$$\begin{array}{rcl}
a & = & \sum_k \sum_l \epsilon_{k,l} b_k \\
 & = & \sum_{\chi_{l,k}\chi_{b_k}=\chi_a} \epsilon_{k,l} b_k + \sum_{\chi' \neq \chi_a} \sum_{\chi_{l,k}\chi_{b_k}=\chi'} \epsilon_{k,l} b_k
\end{array}$$

Since a is homogeneous of character χ_a, each $\sum_{\chi_{l,k}\chi_{b_k}=\chi'} \epsilon_{k,l} b_k$ from the second sum is zero. Since the b_k are free, all the $\epsilon_{k,l}$ in this sum are zero.

$$a = \sum_{\chi_{l,k}\chi_{b_k}=\chi_a} \epsilon_{k,l} b_k$$

This implies that $\chi_{l,k} = \chi_a \chi_{b_k}^{-1}$ (independant of l). Thus, for each k, there is only one l such that $\epsilon_{k,l} \neq 0$ and λ_k is therefore Ch-homogène of character $\chi_a \chi_{b_k}^{-1}$. This fact, as well as lemma 2.6 proves the following equalities.

$$\begin{array}{rcl}\varphi_{a,\gamma}(x) & = & \mu_{\chi_a,\gamma}((\sum_k \lambda_k b_k)\otimes x)\\ & = & \sum_k \mu_{\chi_a,\gamma}(\lambda_k b_k \otimes x)\\ & = & \sum_k \mu_{\chi_{\lambda_k b_k},\gamma}(\lambda_k b_k \otimes x)\\ & = & \sum_k \frac{\dim(\lambda_k)}{\mathrm{ind}(A_{\chi_{\lambda_k},\gamma})}\mu_{\chi_{b_k},\gamma}(b_k\otimes \overline{\mathrm{Res}}_k(x))\\ & = & \sum_k \frac{\dim(\lambda_k)}{\mathrm{ind}(A_{\chi_{\lambda_k},\gamma})}\varphi_{b_k,\gamma}\circ \overline{\mathrm{Res}}_k(x)\end{array}$$

where $\overline{\mathrm{Res}}_k = \overline{\mathrm{Res}}_{A_{\chi_a},A_{\chi_{b_k}}}$. □

Lemmas 2.4 and 2.7 now enable us to compute cup-products. We shall now take care of the functoriality of the decomposition. For a detailed account of twisted forms, we refer the reader to [21], §5 and [20], §2.

We shall just need to investigate the simplest case of functorial behaviour, that is when all the particular subgroups used in the construction are preserved by a morphism between algebraic groups, as well as the cocycle used for twisting. A more general case would be for example when the center is not preserved, but we shall not need this. Let $\tilde{G}$ and $\tilde{G'}$ (resp. $\tilde{P}$ and $\tilde{P'}$, resp. $\tilde{Y}$ and $\tilde{Y'}$) two algebraic groups as above (resp. two parabolic subgroups, resp. two subgroups of the centers of $\tilde{G}$ and $\tilde{G'}$). Let $f:\tilde{G'}\to\tilde{G}$ be a morphism such that $\tilde{P'}$ (resp. $\tilde{Y'}$, $\tilde{Z'}$) is mapped to $\tilde{P}$ (resp. $\tilde{Y}$, $\tilde{Z}$). In such a case, a element γ' of $H^1(\mathrm{Gal}(F_{sep}/F),G')$ can be pushed to an element γ of $H^1(\mathrm{Gal}(F_{sep}/F),G)$ and we get a morphism ${}_{\gamma'}f : {}_{\gamma'}\mathcal{F}\to{}_{\gamma}\mathcal{F}$.

We must now explain how the algebras A_χ behave under this functoriality. Let $\gamma':\mathrm{Gal}(F_{sep}/F)\to G'$ be a cocycle. Let V_χ be a Ch-homogeneous representation of $\tilde{G}$ and $A_\chi=\mathrm{End}_F(V_\chi)$. Since Y' is mapped to Y by f, V_χ is pulled-backed to a Ch-homogeneous representation $V_{\chi'}$ of $\tilde{G'}$. Let $A_{\chi'}=\mathrm{End}_F(V_{\chi'})$. Evidently, we have $A_{\chi'}\simeq A_\chi$. Using $V_{\chi'}$, we can push γ' to $\overline{\gamma'}:\mathrm{Gal}(F_{sep})\to\mathrm{Aut}(A_{\chi'}\otimes_F F_{sep})$ by the composition

$$\mathrm{Gal}(F_{sep})\xrightarrow{\gamma} G'(F_{sep})\longrightarrow PGL_{F_{sep}}(V_{\chi'}\otimes_F F_{sep})\simeq\mathrm{Aut}(A_{\chi'}\otimes_F F_{sep})$$

in which the morphism from $G'(F_{sep})$ to $PGL_{F_{sep}}(V_{\chi'}\otimes_F F_{sep})$ is induced by the obvious one from $\tilde{G'}$ to $PGL_{F_{sep}}(V_{\chi'}\otimes_F F_{sep})$. It is well defined since $\tilde{Y'}$ is central in $\tilde{G'}$ and the representation is Ch-homogeneous. This defines a map from the characters to $Br(F)$ called Tit's map. From the following diagram, it is clear that $A_{\chi',\gamma'}\simeq A_{\chi,\gamma}$.

$$\begin{array}{ccccccc}\mathrm{Gal}(F_{sep}) & \xrightarrow{\gamma'} & G'(F_{sep}) & \longrightarrow & PGL_{F_{sep}}(V_{\chi'}\otimes_F F_{sep}) & \xrightarrow{\sim} & \mathrm{Aut}(A_{\chi'}\otimes_F F_{sep})\\ & \searrow^{\gamma} & \downarrow f & & \downarrow\wr & & \downarrow\wr\\ & & G(F_{sep}) & \longrightarrow & PGL_{F_{sep}}(V_{\chi}\otimes_F F_{sep}) & \xrightarrow{\sim} & \mathrm{Aut}(A_{\chi}\otimes_F F_{sep})\end{array}$$

Let $id_{\chi,\chi'}:K_*(A_{\chi',\gamma'})\to K_*(A_{\chi,\gamma})$ denote the isomorphism induced in K-theory. It is now easy to deduce the following lemma and proposition.

Lemma 2.8. The following diagram is commutative.

$$\begin{array}{ccc}R^{\chi}(\tilde{P})\otimes_{\mathbf{Z}}K_*(A_{\chi,\gamma}) & \xrightarrow{\mu_{\chi,\gamma}} & K_*({}_{\gamma}\mathcal{F})\\ \downarrow{\scriptstyle f^*\otimes id_{\chi,\chi'}} & & \downarrow{\scriptstyle f^*_{\gamma'}}\\ R^{\chi'}(\tilde{P'})\otimes_{\mathbf{Z}}K_*(A_{\chi',\gamma'}) & \xrightarrow{\mu_{\chi',\gamma'}} & K_*({}_{\gamma'}\mathcal{F})\end{array}$$

Proposition 2.9. *Let $\tilde{G}$, $\tilde{G}'$, $\tilde{P}$, $\tilde{P}'$, $\tilde{Y}$, $\tilde{Y}'$, f, γ' and γ be as above, let a be a Ch-homogeneous element of $R(\tilde{P})$, then $f^*(a) \in R(\tilde{P}')$ is Ch-homogeneous and we have the equality ${}_{\gamma'}f^* \circ \varphi_{a,\gamma} = \varphi_{f^*(a),\gamma'} \circ id_{\chi,\chi'}$.*

Proof: This follows from lemma 2.8 and the definition of $\varphi_{a,\gamma}$. □

In the following, we shall omit the morphism $id_{\chi,\chi'}$.

Lemma 2.10. Let $\tilde{G}_1$ and $\tilde{G}_2$ be algebraic groups equipped with subgroups as above. Let $\tilde{P}_1 \times \tilde{P}_2$ be equipped with the product subgroups. Let i_1 (resp. p_1) be the inclusion $\tilde{G}_1 \to \tilde{G}_1 \times \tilde{G}_2$ (resp. the projection $\tilde{G}_1 \times \tilde{G}_2 \to \tilde{G}_1$). Let γ be a cocycle on $G_1 \times G_2$. Then

$$_{p_1(\gamma)}i_1^* \circ \varphi_{p_1^*(a),\gamma} = \varphi_{a,p_1(\gamma)}.$$

Proof: From lemma 2.9 applied to p_1 and γ, we deduce ${}_{\gamma}p_1^* \circ \varphi_{a,p_1(\gamma)} = \varphi_{p_1^*(a),\gamma}$. The twisting respects the products, so that ${}_{\gamma}p_1 \circ_{p_1(\gamma)} i_1 = id$ (see [21], ch 1, § 5.3). Applying ${}_{p_1(\gamma)}i_1^*$ on the left-hand side proves the lemma. □

2.2. **Quadrics.** We now explain what this construction yields in the case of a quadric. This is done in [16] and is just repeated here for the sake of completeness and because we shall slightly modify the notations used in [16] to be coherent with the rest of our text. We only use the cases of a quadratic form of dimension $n = 4m + 2$ or $n' = 2m' + 1$.

Let $\mathbf{H}$ denote the hyperbolic form xy. Let $\tilde{G} = \mathrm{Spin}(h)$, where h is the hyperbolic form $[n/2]\mathbf{H}$, and $\tilde{G}' = \mathrm{Spin}(h')$, where h' is the hyperbolic form $[n/2]\mathbf{H} \perp < 1 >$. The centers $\tilde{Z}$ and $\tilde{Z}'$ are μ_4 and μ_2. We shall take $\tilde{Y}$ and $\tilde{Y}'$ equal to μ_2. This yields $G = SO(h)$ and $G' = SO(h')$. The tori T and T' are diagonal, and $\tilde{T}$ and $\tilde{T}'$ are their preimages in $\tilde{G}$ and $\tilde{G}'$. The group G (resp. G') acts on the projective space $\mathbf{P}^{n-1}$ (resp. $\mathbf{P}^{n'-1}$). Let P (resp. P') be the stabilizer of the projective point $(1 : 0 : \ldots : 0)$ and $\tilde{P}$ (resp. $\tilde{P}'$) the preimage of P (resp. P') in $\tilde{G}$ (resp. $\tilde{G}'$). The variety $\mathcal{F} = G/P$ (resp. $\mathcal{F}' = G'/P'$) is then the projective quadric defined by the equation $h = 0$ (resp. $h' = 0$).

Let r_i (resp. r_i') be the character of $\tilde{T}$ (resp. $\tilde{T}'$) induced by the character of T (resp. T') such that $r_i(a) = a_{2i-1,2i-1}$. Let δ' be the character of the spin representation of G' and δ_+ and δ_- the characters of the two spin representations of G. We have

$$\begin{array}{rcl} (\delta')^2 & = & r_1' \ldots r_{[n'/2]}' \\ \delta_+^2 & = & r_1 \ldots r_{[n/2]-1} r_{[n/2]}^{-1} \\ \delta_-^2 & = & r_1 \ldots r_{[n/2]-1} r_{[n/2]} \end{array}$$

The character group of h' is

$$\tilde{X}' = \mathbf{Z}.r_1' \oplus \cdots \oplus \mathbf{Z}.r_{[n'/2]-1}' \oplus \mathbf{Z}\delta'$$

and the character group of h is

$$\tilde{X} = \mathbf{Z}.r_1 \oplus \cdots \oplus \mathbf{Z}.r_{[n/2]-1} \oplus \mathbf{Z}\delta_+$$

The Weyl group W' of $\tilde{G}'$ is $\mathfrak{S}_{[n/2]} \times Sign_{[n/2]}'$ where $Sign_{[n/2]}'$ est le groupe $\mathbf{Z}/2^{[n/2]}$, and the Weyl group W of $\tilde{G}$ is $\mathfrak{S}_{[n/2]} \times Sign_{[n/2]}$, where $Sign_{[n/2]}$ is $\ker(\mathbf{Z}/2^{[n/2]} \longrightarrow \mathbf{Z}/2)$ (the morphism is the sum). The Weyl group acts by permuting the r_i (resp. r_i') for the factor $\mathfrak{S}_{[n/2]}$ and changing r_i in r_i^{-1}(resp. r_i' in $(r_i')^{-1}$)

for the factor $Sign_{[n/2]}$ (resp. $Sign'_{[n/2]}$). The group W_P (resp. W'_P) (see the beginning of section 2.1) is the stabilizer of r_1 (resp. r'_1). We get

$$R(\tilde{T}') = \mathbf{Z}[\tilde{X}'] = \mathbf{Z}[(r'_1)^{\pm 1}, \ldots, (r'_{[n/2]-1})^{\pm 1}, \delta']$$

and

$$R(\tilde{T}) = \mathbf{Z}[\tilde{X}] = \mathbf{Z}[r_1^{\pm 1}, \ldots, r_{[n/2]-1}^{\pm 1}, \delta_+, \delta_-].$$

We define

$$\eta' = \sum_{w \in Sign_{[n/2]} \cap W_P} (\delta')^w, \ \eta_+ = \sum_{w \in Sign_{[n/2]} \cap W_P} \delta_+^w, \ \eta_- = \sum_{w \in Sign_{[n/2]} \cap W_P} \delta_-^w.$$

They are fixed by W_P. We also define

$$\beta' = \sum_{w \in Sign_{[n/2]}} (\delta')^w, \ \beta_+ = \sum_{w \in Sign_{[n/2]}} \delta_+^w, \ \beta_- = \sum_{w \in Sign_{[n/2]}} \delta_-^w.$$

They are fixed by W. We denote θ_i (resp. θ_i^1) the i-th elementary symmetric polynomial in $y_1, \ldots, y_{[n/2]}$ (resp. $y_2, \ldots, y_{[n/2]}$) where $y_i = r_i + r_i^{-1}$. We define the same polynomials with the r'_i. We get

$$\begin{array}{rcccl} R(\tilde{P}') & = & \mathbf{Z}[\tilde{X}']^{W'_P} & = & \mathbf{Z}[(r'_1)^{\pm 1}, (\theta')_1^1, \ldots, (\theta')_{[n/2]-1}^1, \eta'] \\ R(\tilde{G}') & = & \mathbf{Z}[\tilde{X}']^{W'} & = & \mathbf{Z}[\theta'_1, \ldots, \theta'_{[n/2]-1}, \beta'] \end{array}$$

and

$$\begin{array}{rcccl} R(\tilde{P}) & = & \mathbf{Z}[\tilde{X}]^{W_P} & = & \mathbf{Z}[r_1^{\pm 1}, \theta_1^1, \ldots, \theta_{[n/2]-1}^1, \eta_-, \eta_+] \\ R(\tilde{G}) & = & \mathbf{Z}[\tilde{X}]^{W} & = & \mathbf{Z}[\theta_1, \ldots, \theta_{[n/2]-1}, \beta_-, \beta_+] \end{array}$$

The dimension of an element of $R(\tilde{G})$, $R(\tilde{P})$, or $R(\tilde{G})$ can be obtained by replacing the r_i, δ, δ_+ and δ_- by 1.

We get the decompositions

$$R(\tilde{P}) = R(\tilde{G}).1 \oplus R(\tilde{G}).r_1 \oplus \ldots \oplus R(\tilde{G}).r_1^{n-3} \oplus R(\tilde{G}).\eta_- \oplus R(\tilde{G}).\eta_+$$

and

$$R(\tilde{P}') = R(\tilde{G}').1 \oplus R(\tilde{G}').r'_1 \oplus \ldots \oplus R(\tilde{G}').(r'_1)^{n-3} \oplus R(\tilde{G}').\eta'$$

The algebras $A_{\chi,\gamma}$ are all F for the powers of r_1 (or r'_1) and $A_{\chi_{\eta'},\gamma} = C_0({}_\gamma h')$. We have $C_0({}_\gamma h) = C_0^+({}_\gamma h) \oplus C_0^-({}_\gamma h)$ which yields $A_{\chi_{\eta_+},\gamma} = C_0^+({}_\gamma h)$ et $A_{\chi_{\eta_-},\gamma} = C_0^-({}_\gamma h)$ (see [16], § 5.1). Any quadratic form with trivial discriminant and with the same dimension as h can be obtained as a twisted form of h. Any quadratic form with the same dimension as h' can be obtained as a twisted form of h'. The variety ${}_\gamma\mathcal{F}$ (resp. ${}_\gamma\mathcal{F}'$ is then the projective quadric $X_{{}_\gamma h}$ (resp. $X_{{}_\gamma h'}$).

We then get the decompositions

$$\sum_{i=1}^{n} \varphi_\gamma^i : K_*(F) \oplus \ldots \oplus K_*(F) \oplus K_*(C_0^-({}_\gamma h)) \oplus K_*(C_0^+({}_\gamma h)) \xrightarrow{\sim} K_*(X_{{}_\gamma h})$$

and

$$\sum_{i=1}^{n'} \varphi_\gamma^i : K_*(F) \oplus \ldots \oplus K_*(F) \oplus K_*(C_0({}_\gamma h')) \xrightarrow{\sim} K_*(X_{{}_\gamma h'})$$

2.3. **Generalized Severi-Brauer varieties.** In this section, we shall do the same thing as in the preceeding one, but for generalized Severi-Brauer varieties.

Let $\tilde{G} = SL_n$. Its center $\tilde{Z}$ is μ_n. We take $\tilde{Y} = \tilde{Z}$. We then get $G = PGL_n$. The torus T is the image in PGL_n of the diagonal subgroup of GL_n and $\tilde{T}$ is the diagonal subgroup in SL_n. We then take

$$\tilde{P} = \{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \text{ avec } \det(a)\det(b) = 1\} \subset SL_n$$

in which a (resp. b) is a square matrix with k (resp. $n-k$) rows. Let t_i be the character of $\tilde{T}$ induced by the character $t_i(a) = a_{i,i}$ on T. The Weyl group W is $\mathfrak{S}_n$, and W_P is the subgroup $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$. We get

$$\tilde{X} = (\mathbf{Z}.t_1 \oplus \cdots \oplus \mathbf{Z}.t_n)/\mathbf{Z}(t_1 + \cdots + t_n)$$

and, if we denote σ_i (resp. σ'_i, resp. σ''_i) the i-th elementary symmetric polynomial in the variables $t_1 \dots t_n$ (resp. $t_1 \dots t_k$, resp. $t_{k+1} \dots t_n$),

$$\begin{array}{lllll} R(\tilde{T}) & = & \mathbf{Z}[\tilde{X}] & = & \mathbf{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]/(t_1 \dots t_n - 1) \\ R(\tilde{P}) & = & \mathbf{Z}[\tilde{X}]^{W_P} & = & \mathbf{Z}[\sigma'_1, \dots, \sigma'_k, \sigma''_1, \dots, \sigma''_{n-k}]/(\sigma'_k \sigma''_{n-k} - 1) \\ R(\tilde{G}) & = & \mathbf{Z}[\tilde{X}]^{W} & = & \mathbf{Z}[\sigma_1, \dots, \sigma_n]/(\sigma_n - 1) \end{array}$$

The dimension of an element of $R(\tilde{T})$, $R(\tilde{G})$ or $R(\tilde{G})$ can be obtained by replacing t_i by 1.

We get the decomposition

$$R(\tilde{P}) = \bigoplus_{\alpha} R(\tilde{G}).\sigma_\alpha$$

where σ_α is the *Schur* polynomial (see for example [4], p. 49) whose multi-index α spans the sequences $\alpha_1, \dots, \alpha_k$ such that $n - k \geq \alpha_1 \geq \dots \geq \alpha_k \geq 0$.

The algebra $A_{\chi_\alpha,\gamma}$ is $A_\gamma^{\otimes d(\alpha)}$ where $d(\alpha) = \alpha_1 + \cdots + \alpha_k$ and $A_\gamma \simeq {}_\gamma\mathrm{End}(V)$. The vector space V is the n dimensional one whose subspaces are the points of the Grassmann variety. We get the following isomorphism.

$$\sum_\alpha \varphi_\gamma^\alpha : \bigoplus_\alpha K_*(A_\gamma^{\otimes d(\alpha)}) \xrightarrow{\sim} K_*({}_\gamma Gr(k,n))$$

Generalized Severi-Brauer varieties $SB(k, A)$ (see [1]) are twisted Grassmann varieties and therefore part of this framework.

2.4. **The special case of SL_4 and Spin_6.** Recall that q is an Albert quadratic form and D a biquaternion algebra (they are related to each other by the fact that the image of D in the Brauer group of F is the clifford invariant of q). In this section, we shall explain how the classical isomorphism between SL_4 and Spin_6 induces an isomorphism between the quadric X_q and the generalized Severi-Brauer variety $SB(2, D)$. In the split case, q is isomorphic to three times the hyperbolic form $\mathbf{H} =< 1, -1 >$, D is a matrix algebra and $SB(2, D)$ is the grassmannian variety $Gr(2, 4)$. It is well known that the quadric $X_{3\mathbf{H}}$ and $Gr(2, 4)$ are isomorphic. We shall see that such an isomorphism can be obtained from an isomorphism between SL_4 and $\mathrm{Spin}(3\mathbf{H})$ of which we shall give an explicit construction. This will permit us to relate their representation rings and compute the induced morphisms on Panin's decomposition of the K-theory.

Let us now briefly recall the classical isomorphism between SL_4 and $\mathrm{Spin}(3\mathbf{H})$. Let V be an F-vector space of dimension 4 with a basis $v_1, \dots, v_4$. Let $W = \Lambda^2 V$.

It is naturally equipped with a symmetric bilinear form

$$\begin{array}{ccc} \Lambda^2 V \times \Lambda^2 V & \longrightarrow & \Lambda^4 V \simeq F \\ (u_1 \wedge u_2, u_3 \wedge u_4) & \longmapsto & u_1 \wedge u_2 \wedge u_3 \wedge u_4. \end{array}$$

The quadratic form associated to this bilinear form is hyperbolic; it is given by the formula $x_1y_1 + x_2y_2 + x_3y_3$ on the basis $w_1 = v_1 \wedge v_2$, $w_2 = v_3 \wedge v_4$, $w_3 = v_2 \wedge v_3$, $w_4 = v_1 \wedge v_4$, $w_5 = v_1 \wedge v_3$ and $w_6 = v_4 \wedge v_2$. Let us denote this form h. An element g of $SL(V)$ acts on W by $u_1 \wedge u_2 \longmapsto g(u_1) \wedge g(u_2)$. This defines a morphism g_1 from $SL(V)$ to $GL(W)$. By definition of the determinant, $g(u_1) \wedge g(u_2) \wedge g(u_3) \wedge g(u_4) = det(g) u_1 \wedge u_2 \wedge u_3 \wedge u_4$, therefore h is conserved by the action of $SL(V)$ and g_1 actually maps to $SO(h)$. Since $\mathrm{Spin}(h)$ and $SO(h)$ are both simple and simply connected groups, g_1 lifts to a unique morphism g from $SL(V)$ to $\mathrm{Spin}(h)$. In fact, g is an isomorphism for $SL(V)$ and $\mathrm{Spin}(h)$ have the same dimension. Let f be the inverse of g.

Lemma 2.11. The following diagram has exact lines - as complexes of algebraic groups - and is commutative.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & Spin(h) & \longrightarrow & SO(h) & \longrightarrow & 1 \\ & & \| & & \uparrow g & & \| & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & SL(V) & \xrightarrow{g_1} & SO(h) & \longrightarrow & 1 \end{array}$$

Proof: The right square is commutative by definition of g and the left square has to be commutative, otherwise g would not be an isomorphism. □

From now on, let us set $\tilde{G}_1 = SL(V)$ and $\tilde{G}_2 = \mathrm{Spin}(h)$. As in sections 2.2 and 2.3, we will denote $\tilde{T}_1$ and $\tilde{T}_2$ the two maximal tori and $\tilde{P}_1$ and $\tilde{P}_2$ the two parabolic subgroups. Let us recall that $\tilde{T}_2$ is the preimage in $\mathrm{Spin}(h)$ of the diagonal torus T_2 of $SO(h)$. The morphism g_1 maps a matrix of $\tilde{T}_2$ as

$$\begin{pmatrix} t_1 & & & (0) \\ & t_2 & & \\ & & t_3 & \\ (0) & & & t_4 \end{pmatrix} \longmapsto \begin{pmatrix} t_1t_2 & & & & & \\ & t_3t_4 & & & (0) & \\ & & t_2t_3 & & & \\ & & & t_1t_4 & & \\ & (0) & & & t_1t_3 & \\ & & & & & t_4t_2 \end{pmatrix}$$

where $t_1t_2t_3t_4 = 1$. Thus, g induces an isomorphism between $\tilde{T}_1$ and $\tilde{T}_2$. The parabolic subgroup $\tilde{P}_1$ is the subgroup of $SL(V)$ which stabilises the plane $< v_1, v_2 >$, whereas $\tilde{P}_2$ is the preimage in $\mathrm{Spin}(h)$ of the subgroup in $SO(h)$ which fixes the projective point $(1 : 0 \ldots : 0)$. An element s in $SL(V)$ verifies $s(v_1 \wedge v_2) = \lambda v_1 \wedge v_2$ if and only if s stabilises the plane $< v_1, v_2 >$, so $\tilde{P}_1$ and $\tilde{P}_2$ are also isomorphic through g. This gives the classical isomorphism f from $Gr(2, V) = \tilde{G}_1/\tilde{P}_1$ to $X_h = \tilde{G}_2/\tilde{P}_2$.

Lemma 2.12. Let Pl_h denote the natural embedding of X_h in $\mathbf{P}^5$ and Plk the Plücker embedding of $Gr(2, V)$ in $\mathbf{P}^5$. The diagram

$$\begin{array}{ccc} X_h & \xrightarrow{Pl_h} & \mathbf{P}^5 \\ \downarrow f & \nearrow Plk & \\ Gr(2, V) & & \end{array}$$

is commutative.

Proof: The projective space $\mathbf{P}^5$ is the quotient $\tilde{G}_3/\tilde{P}_3$ where $\tilde{G}_3 = SL(W)$ and $\tilde{P}_3$ is the subgroup of $SL(W)$ that fixes the projective point $(1:0\ldots:0)$. By definition, the Plücker embedding is induced by the morphims g_1 (as described above). The embedding of X_h in $\mathbf{P}^5$ is induced by the natural embedding of $SO(h)$ in $SL(W)$. Since $\tilde{P}_3$ is the preimage of P_2 by definition, we are done. □

It is completely straight forward to check that the natural inclusion $SO(h') \to SO(h)$ maps the parabolic $\tilde{P}'_2$ to $\tilde{P}_2$ (not surjectively) and induces the inclusion $X_{h'} \to X_h$.

Let us now see how the representation rings $R(\tilde{P}_1)$, $R(\tilde{P}_2)$ and $R(\tilde{P}'_2)$ map to each other. From the mapping from $\tilde{T}_1$ to $\tilde{T}_2$ described above, we get

$$\begin{array}{rcl} (f^{-1})^*(r_1) & = & t_1t_2 \\ (f^{-1})^*(r_2) & = & t_2t_3 \\ (f^{-1})^*(r_3) & = & t_1t_3 \end{array}$$

To find the image of δ_+, we can use the fact that the spinorial representation whose highest weight is δ_+ is precisely the standard representation of $SL(V)$, $t_1+t_2+t_3$, so δ_+ maps to t_1, t_2 or t_3, according to the choice of the basis. In our case, $\delta_+^2 = r_1r_2r_3^{-1}$, $(f^{-1})^*(\delta_+^2) = (f^{-1})^*(r_1).(f^{-1})^*(r_2).(f^{-1})^*(r_3^{-1}) = t_2^2$, so $(f^{-1})^*(\delta_+) = t_2$ and $(f^{-1})^*(\delta_-) = (f^{-1})^*(\delta_+).(f^{-1})^*(r_3) = t_1t_2t_3$. Hence,

$$\begin{array}{c} (f^{-1})^*(1) = 1, (f^{-1})^*(r_1) = t_1t_2, (f^{-1})^*(r_2) = t_2t_3, (f^{-1})^*(r_3) = t_1t_3, \\ (f^{-1})^*(\delta_+) = t_2, (f^{-1})^*(\delta_-) = t_1t_2t_3 \end{array} \tag{11}$$

and

$$f^*(1) = 1, f^*(t_1) = \delta_- r_2^{-1}, f^*(t_2) = \delta_+, f^*(t_3) = \delta_- r_1^{-1}, f^*(t_4) = \delta_+ r_1^{-1} r_2^{-1}. \tag{12}$$

Let i denote the inclusion $SO(h') \to SO(h)$. Clearly, $i^*(r_1) = r'_1$, $i^*(r_2) = r'_2$ and $i^*(r_3) = 1$ since i maps $\tilde{T}'_2$ to $\tilde{T}_2$ as

$$\begin{pmatrix} r_1 & & & & \\ & r_1^{-1} & & (0) & \\ & & r_2 & & \\ & (0) & & r_2^{-1} & \\ & & & & 1 \end{pmatrix} \longmapsto \begin{pmatrix} r_1 & & & & & \\ & r_1^{-1} & & & (0) & \\ & & r_2 & & & \\ & & & r_2^{-1} & & \\ & (0) & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

From this, we can deduce that $i^*(\delta_+^2) = i^*(\delta_-^2) = (\delta')^2$, and since the caracter group is a free $\mathbf{Z}$-module, we must have $i^*(\delta_+) = i^*(\delta_-) = \delta'$.

We must now explain what happens in the non-split case, that is when we twist these varieties and morphisms by cocycles. It will turn out that the isomorphism $f : Gr(2,V) \to X_h$ will be twisted in an isomorphism ${}_\gamma f : SB(2,D) \simeq X_q$, and the morphism $i : X_{h'} \to X_h$ will be twisted in a morphism ${}_\gamma i : X_{q'} \to X_q$.

Lemma 2.11 induces the commutative diagram with exact lines

$$\begin{array}{ccccc} H^1(F, SO(h)) & \longrightarrow & H^2(F,\mu_2) & \xrightarrow{\sim} & {}_2Br(F) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(F, PGL(V)) & \longrightarrow & H^2(F,\mu_4) & \xrightarrow{\sim} & {}_4Br(F) \end{array}$$

These short exact sequences induce the following exact sequences in cohomology, since μ_2, resp. μ_4 is central in Spin(h), resp. $SL(V)$ (see [21], chapter I, § 5.7). The boundary morphisms between degree 1 and 2 terms induce the commutative diagram

$$\begin{array}{ccccc} H^1(F,SO(h)) & \longrightarrow & H^2(F,\mu_2) & \xrightarrow{\sim} & {}_2Br(F) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(F,PGL(V)) & \longrightarrow & H^2(F,\mu_4) & \xrightarrow{\sim} & {}_4Br(F) \end{array}$$

Let γ be an element of $H^1(F,SO(h))$ such that ${}_\gamma h = q$. Its image in ${}_2Br(F)$ is $w_2(q) - w_2(h)$, where w_2 is the Stiefel-Whitney invariant (see [21], chapter III, § 3.2, b.). It is given by the formula $w_2(q) = \sum_{i<j}(a_i, a_j)$, where the (a_i) are the coefficients of q on an orthogonal basis and (a_i, a_j) is the class of the quaternion algebra $\left(\frac{a_i\ a_j}{F}\right)$ in $Br(F)$. Using the relations

$$(a,b) = (b,a), (a^2,b) = 0, (a, 1-a) = 0, (a,-a) = 0, (a, bc) = (a,b) + (a,c),$$

we get $w_2(q) - w_2(h) = (a,b) + (c,d) = [D]$. Now, the image of a cocycle $\gamma \in H^1(F,PGL(V)) = H^1(F,Aut(End(V)))$ in ${}_4Br(F)$ is the class of the twisted form ${}_\gamma End(V)$ of the algebra $End(V)$ (see [20], chapter X, §4 et §5). Furthermore, the twisting of Grassmann varieties is compatible with the twisting of algebras, meaning that the twisted form ${}_\gamma Gr(k,V)$, $\gamma \in H^1(F,PGL(V))$ is the generalized Severi-Brauer variety $SB(k, {}_\gamma End(V))$ (see [1], after theorem 1). We have therefore proved the following result

Theorem 2.13. *The isomorphism f from $SL(V)$ to* Spin(h) *induces an isomorphism from $SB(2,D)$ to X_q.*

The commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & Spin(h) & \longrightarrow & SO(h) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle f} & & \downarrow & & \\ 1 & \longrightarrow & \mu_6 & \longrightarrow & SL(V) & \xrightarrow{g_1} & SO(h) & \longrightarrow & 1 \end{array}$$

induces the following commutative diagram in cohomology.

$$\begin{array}{ccccc} H^1(F,SO(h)) & \longrightarrow & H^2(F,\mu_2) & \xrightarrow{\sim} & {}_2Br(F) \\ \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\ H^1(F,PGL(W)) & \longrightarrow & H^2(F,\mu_6) & \xrightarrow{\sim} & {}_6Br(F) \end{array}$$

This shows that the cocycle used to twist X_h is sent by the morphism that induces the inclusion of X_h in $\mathbf{P}^5$ to a cocycle whose image is trivial in $Br(F)$. The only forms of the projective space are the Severi-Brauer varieties, and a cocycle that produces a non-split Severi-Brauer variety has a non-zero image in $Br(F)$ (since the corresponding algebra cannot be split). So we have proved the following lemma

Lemma 2.14. The commutative diagram of lemma 2.12 twists to a commutative diagram

$$\begin{array}{ccc} X_q & \xrightarrow{{}_\gamma Pl_h} & \mathbf{P}^5 \\ {\scriptstyle {}_\gamma f}\downarrow & \nearrow{\scriptstyle {}_\gamma Plk} & \\ SB(2,D) & & \end{array}$$

Let us now handle the twisting of the morphism between $X_{h'}$ and X_h. We want to understand how the decomposition $h \simeq h' \perp < 1 >$ can be twisted in $q \simeq q' \perp < d_\pm q' >$ (since $d_\pm q = 1$). The decomposition in the split case yields a morphism $O(h') \to SO(h)$. But since $\dim h'$ is odd, we have $O(h') \simeq SO(h') \times \mu_2$ (by $M \mapsto$

$(\det(M)M, \det(M)))$. This induces a morphism $SO(h') \times \mu_2 \to SO(h)$, where $-1 \in \mu_2$ is sent to $-Id \in SO(h)$. The element $\gamma'' \in H^1(\mathrm{Gal}(F_(sep)/F), O(h'))$ twisting h' in q' will therefore yield by push-forward an element $\gamma \in H^1(\mathrm{Gal}(F_{sep}/F), SO(h))$ twisting $h \simeq h' \perp< 1 >$ in $q \simeq q' \perp< d_\pm q' >$. To explain what happens on Panin's decomposition, we shall use the groups and subgroups defined in table 1 below.

	'	"	"'	
$\tilde{G_2}$	$\mathrm{Spin}(h')$	$\mathrm{Spin}(h') \times \mu_2$	$\mathrm{Spin}(h) \times \mu_2$	$\mathrm{Spin}(h)$
$\tilde{P_2}$	$\mathrm{Fix}(1:0:\ldots:0)$	$\mathrm{Fix}(1:0:\ldots:0) \times \mu_2$	$\mathrm{Fix}(1:0:\ldots:0) \times \mu_2$	$\mathrm{Fix}(1:0:\ldots:0)$
$\tilde{Z_2}$	μ_2	$\mu_2 \times \mu_2$	$\mu_4 \times \mu_2$	μ_4
$\tilde{Y_2}$	μ_2	$\mu_2 \times \{1\}$	μ_4	μ_2
G_2	$SO(h')$	$SO(h') \times \mu_2$	$SO(h)$	$SO(h)$
P_2	$\mathrm{Fix}(1:0:\ldots:0)$	$\mathrm{Fix}(1:0:\ldots:0) \times \mu_2$	$\mathrm{Fix}(1:0:\ldots:0)$	$\mathrm{Fix}(1:0:\ldots:0)$
Z_2	$\{1\}$	$\{1\} \times \mu_2$	μ_2	μ_2
$\mathcal{F} = G/P$	X_h'	X_h'	X_h	X_h

TABLE 1. Notations

Thus, $\tilde{G_2''} = \mathrm{Spin}(h') \times \mu_2$ for example. Note that the inclusion $\tilde{Y_2'''} = \mu_4 \hookrightarrow \mu_4 \times \mu_2 = \tilde{Z_2'''}$ is the identity onto μ_4 and the quotient map onto μ_2. We shall define $\tilde{i}_1 : \tilde{G'}_2 \longrightarrow \tilde{G''}_2$ as the natural inclusion, $\tilde{i}_2 : \tilde{G''}_2 \longrightarrow \tilde{G'''}_2$ as the identity on μ_2 and the inclusion $\mathrm{Spin}(h') \to \mathrm{Spin}(h)$ on $\mathrm{Spin}(h')$, $\tilde{i}_3 : \tilde{G}_2 \longrightarrow \tilde{G'''}_2$ as the natural inclusion and $\tilde{i} : \tilde{G'}_2 \to \tilde{G}_2$ as the inclusion $\mathrm{Spin}(h') \to \mathrm{Spin}(h)$. All these morphisms respect the subgroups mentioned in table 1, therefore they induce morphisms $i_1 : G_2' \to G_2''$, $i_2 : G_2'' \to G_2'''$, $i_3 : G_2 \to G_2'''$ and $i : G_2' \to G_2$. Notice that $\tilde{i_3} \circ \tilde{i} = \tilde{i_2} \circ \tilde{i_1}$, and that i_3 is the identity morphism of $S0(h)$. The morphism i_1 induces the identity on $X_{h'}$, i_2 and i the inclusion $X_{h'} \hookrightarrow X_h$ and i_3 the identity on X_h. We shall denote these induced morphisms by the same names. Now, let γ'' and $\gamma = i_2(\gamma'')$ be as above, and let γ' be the projection of γ'' on $H^1(\mathrm{Gal}(F_{sep}/F), SO(h'))$. The twisted morphism ${}_{\gamma'}i_1 : X_{q'} \to X_{q'}$ is also the identity, ${}_{\gamma''}i_2 : X_{q'} \to X_q$ is the inclusion induced by the decomposition $q' \perp< d_\pm q' > \simeq q$ and $i_3 : X_q \to X_q$ is the identity. The only reason why we introduce these new groups and morphisms is to avoid explaining how to twist i to get the morphism $X_{q'} \to X_q$ because it cannot be obtained by the simple functorial behaviour explained above (clearly, $i_*(\gamma') \neq \gamma$). Instead, we shall consider $({}_\gamma i_3)^{-1} \circ_{\gamma''} i_2 \circ_{\gamma'} i_1$, which we shall denote (improperly) ${}_\gamma i$.

Lemma 2.15. The commutative diagram

$$\begin{array}{ccc} G_2'' = SO(h') \times \mu_2 & \xrightarrow{i} & SO(h) = G_2''' \\ \downarrow & & \downarrow \\ SL(W') & \longrightarrow & SL(W) \end{array}$$

induces the - cartesian - commutative diagram

$$\begin{array}{ccc} X_{q'} & \xrightarrow{{}_\gamma i} & X_q \\ {\scriptstyle {}_{\gamma''}Plh' = Plq'} \downarrow & & \downarrow {\scriptstyle {}_\gamma Plh = Plq} \\ \mathbf{P}^4 & \longrightarrow & \mathbf{P}^5 \end{array}$$

Proof: We just have to prove that the cocycle $\gamma'' \in H^1(\mathrm{Gal}(F_{sep}/F), SO(h') \times \mu_2)$ (resp. $\gamma \in H^1(\mathrm{Gal}(F_{sep}/F), SO(h))$) is pushed forward to the trivial cocycle in $H^1(\mathrm{Gal}(F_{sep}/F), SL(W'))$ (resp. $H^1(\mathrm{Gal}(F_{sep}/F), SL(W))$). Actually, we have already done so for γ in the proof of lemma 2.14. If we decompose γ'' as (γ', e) (coming from $H^1(\mathrm{Gal}(F_{sep}/F), SO(h')) \oplus H^1(\mathrm{Gal}(F_{sep}/F), \mu_2)$, the same proof applies for γ' as for γ. For the cocycle e, we just have to notice that to find the twisted form of the projective space that we might obtain, we have to push the cocycle to $H^2(\mathrm{Gal}(F_{sep}/F), \mu_2)$ using the exact sequence

$$1 \to \mu_6 \to SL(W) \to PSL(W) \to 1$$

and since the map $\mu_2 \to SL(W)$ factors through μ_6, this push-forward has to be zero. □

2.5. **K-theory and morphisms.** We shall now use the results of section 2.1 to follow the morphisms introduced in the last section on Panin's decompositions of the K-theory of quadrics and generalized Severi-Brauer varieties.

In section 2.3 and 2.2, we have seen that $(1, r_1, r_1^2, r_1^3, \eta_-, \eta_+)$ is a basis of the $R(\tilde{G}_2)$-module $R(\tilde{P}_2)$, $(1, r_1', (r_1')^2, \eta')$ is a basis of the $R(\tilde{G}_2')$-module $R(\tilde{P}_2')$ and $(\sigma_{0,0}, \sigma_{1,0}, \sigma_{1,1}, \sigma_{2,0}, \sigma_{2,1}, \sigma_{2,2})$ is a basis of the $R(\tilde{G}_1)$-module $R(\tilde{P}_1)$. Let us recall that $\sigma_{i,j}$ is the Schur polynomial in t_1 and t_2, and σ_i (resp. σ_i', σ_i'') is the elementary symmetric polynomial of degree i in $t_1, \ldots, t_4$ (resp. t_1, t_2, t_3, t_4). Furthermore, $t_1t_2t_3t_4 = \sigma_2'\sigma_2'' = 1$. From the set of equations (11), we get

$$\begin{array}{rcl}
(f^{-1})^*(r_1) & = & t_1t_2 = \sigma_2' = \sigma_{1,1} \\
(f^{-1})^*(r_1^2) & = & (\sigma_2')^2 = \sigma_{1,1}^2 = \sigma_{2,2} \\
(f^{-1})^*(r_1^3) & = & (\sigma_2')^3 = \sigma_{1,1}^3 = \sigma_4\sigma_{2,0} - \sigma_3\sigma_{2,1} + \sigma_2\sigma_{2,2} \\
(f^{-1})^*(\eta_+) & = & t_2 + t_1t_2t_3(t_2t_3)^{-1} = \sigma_1' = \sigma_{1,0} \\
(f^{-1})^*(\eta_-) & = & t_1t_2t_3 + t_2(t_2t_3)^{-1} = \sigma_2'\sigma_1'' = \sigma_1\sigma_{1,1} - \sigma_{2,1}
\end{array}$$

Note that the last equalities on the right can easily be checked since they are equalities between polynomials. However, there is a way to find such equalities systematically (see for example [3]). We now choose the cocycles as in section 2.4, and we do not mention them anymore in the notations. The algebras corresponding to each character are only define up to their class in the Brauer group of F (see section 2.1, lemma 2.1). According to sections 2.2 and 2.3, we can choose $A_{\sigma_\alpha} = D^{\otimes|\alpha|}$ for the $\tilde{P}_1$ characters, $A_{r_1^i} = D^{\otimes 2i}$, $A_{\eta_+} = D$ and $A_{\eta_-} = D^{\otimes 3}$ for the $\tilde{P}_2$ characters and $A_{(r_1')^i} = D^{\otimes 2i}$, $A_{\eta'} = D$ for the $\tilde{P}_2'$ characters. Using proposition 2.9, we get

$$(f^{-1})^*\varphi_1(x) = \varphi_{\sigma_{0,0}}(x),\ (f^{-1})^*\varphi_{r_1}(x) = \varphi_{\sigma_{1,1}}(x),\ (f^{-1})^*\varphi_{(r_1)^2}(x) = \varphi_{\sigma_{2,2}}(x),$$

$$(f^{-1})^*\varphi_{\eta_+}(x) = \varphi_{\sigma_{1,0}}(x).$$

On the other hand,

$$(f^{-1})^*\varphi_{(r_1)^3}(x) = \varphi_{\sigma_4\sigma_{2,0} - \sigma_3\sigma_{2,1} + \sigma_2\sigma_{2,2}}(x)$$

and

$$(f^{-1})^*\varphi_{\eta_-}(x) = \varphi_{\sigma_1\sigma_{1,1} - \sigma_{2,1}}(x)$$

Let $\mathrm{M}_{A,A'}$ be the Morita morphism between the K-theory of two Brauer-equivalent algebras A and A'. Let A and B be algebras with a non-zero morphism from A to B. Denote $\mathrm{Res}_{B,A}$ the restriction in K-theory and $\mathrm{I}_{A,B}$ the morphism induced by the functoriality of the K-theory of algebras. For central simple algebras, these morphisms do not depend on the non-zero morphism. From lemma 2.7, we deduce, when D is a field ($\mathrm{ind}(D) = \deg(D) = 4$),

$$\begin{array}{l}
\varphi_{\sigma_4\sigma_{2,0} - \sigma_3\sigma_{2,1} + \sigma_2\sigma_{2,2}}(x) \\
= \varphi_{\sigma_{2,0}}(\overline{\mathrm{Res}}_{D^{\otimes 6}, D^{\otimes 2}}(x)) - \varphi_{\sigma_{2,1}}(\overline{\mathrm{Res}}_{D^{\otimes 6}, D^{\otimes 3}}(x)) + 6\varphi_{\sigma_{2,2}}(\overline{\mathrm{Res}}_{D^{\otimes 6}, D^{\otimes 4}}(x)) \\
= \varphi_{\sigma_{2,0}}(\mathrm{M}_{D^{\otimes 6}, D^{\otimes 2}}(x)) - \varphi_{\sigma_{2,1}}(\mathrm{Res}_{D^{\otimes 4}, D^{\otimes 3}} \circ \mathrm{M}_{D^{\otimes 6}, D^{\otimes 4}}(x)) \\
+6\varphi_{\sigma_{2,2}}(\mathrm{M}_{D^{\otimes 6}, D^{\otimes 4}}(x))
\end{array}$$

and

$$\varphi_{\sigma_1\sigma_{1,1}-\underline{\sigma_{2,1}}}(x)$$
$$= \varphi_{\sigma_{1,1}}(\overline{\mathrm{Res}}_{D^{\otimes 3},D^{\otimes 2}}(x)) - \varphi_{\sigma_{2,1}}(x) = \varphi_{\sigma_{1,1}}(\mathrm{Res}_{D^{\otimes 3},D^{\otimes 2}}(x)) - \varphi_{\sigma_{2,1}}(x)$$

This sums up as

$$\begin{array}{lcl}(f^{-1})^*\varphi_1 & = & \varphi_{\sigma_{0,0}} \\ (f^{-1})^*\varphi_{r_1} & = & \varphi_{\sigma_{1,1}} \\ (f^{-1})^*\varphi_{(r_1)^2} & = & \varphi_{\sigma_{2,2}} \\ (f^{-1})^*\varphi_{(r_1)^3} & = & \varphi_{\sigma_{2,0}} \circ \mathrm{M}_{D^{\otimes 6},D^{\otimes 2}} \\ & & -\varphi_{\sigma_{2,1}} \circ \mathrm{Res}_{D^{\otimes 4},D^{\otimes 3}} \circ \mathrm{M}_{D^{\otimes 6},D^{\otimes 4}} \\ & & +6\varphi_{\sigma_{2,2}} \circ \mathrm{M}_{D^{\otimes 6},D^{\otimes 4}} \\ (f^{-1})^*\varphi_{\eta_+} & = & \varphi_{\sigma_{1,0}} \\ (f^{-1})^*\varphi_{\eta_-} & = & \varphi_{\sigma_{1,1}} \circ \mathrm{Res}_{D^{\otimes 3},D^{\otimes 2}} - \varphi_{\sigma_{2,1}}\end{array} \tag{13}$$

from which we can easily deduce the inverse morphisms

$$\begin{array}{lcl}f^*\varphi_{\sigma_{0,0}} & = & \varphi_1 \\ f^*\varphi_{\sigma_{1,0}} & = & \varphi_{\eta_+} \\ f^*\varphi_{\sigma_{1,1}} & = & \varphi_{r_1} \\ f^*\varphi_{\sigma_{2,0}} & = & 16\varphi_{r_1} - 6\varphi_{(r_1)^2} \circ \mathrm{M}_{D^{\otimes 2},D^{\otimes 4}} + \varphi_{(r_1)^3} \circ \mathrm{M}_{D^{\otimes 2},D^{\otimes 6}} \\ & & -\varphi_{\eta_-} \circ \mathrm{I}_{D^{\otimes 2},D^{\otimes 3}} \\ f^*\varphi_{\sigma_{2,1}} & = & \varphi_{r_1} \circ \mathrm{Res}_{D^{\otimes 3},D^{\otimes 2}} - \varphi_{\eta_-} \\ f^*\varphi_{\sigma_{2,2}} & = & \varphi_{(r_1)^2}\end{array} \tag{14}$$

These equalities stay true when D is not a field (which wasn't the case of the formulas containing the $\overline{\mathrm{Res}}$ morphisms).

Let us now compute the functoriality along ${}_\gamma i : X_{q'} \to X_q$. From the definition of ${}_\gamma i$, we get ${}_\gamma i^* \circ \varphi_{a,\gamma} =_{\gamma'} i_1^* \circ_{\gamma''} i_2^* \circ ({}_\gamma i_3^*)^{-1} \circ \varphi_{a,\gamma}$. Let $\tilde{p}_3$ (resp. $\tilde{p}_1$) be the projection $\mathrm{Spin}(h) \times \mu_2 \to \mathrm{Spin}(h)$ (resp. $\mathrm{Spin}(h') \times \mu_2 \to \mathrm{Spin}(h')$). Since $p_3 \circ i_3 = id$, the formula $({}_\gamma i_3^*)^{-1} \circ \varphi_{\tilde{i}_3^*(a),\gamma} = \varphi_{a,i_3(\gamma)}$ (lemma 2.9) yields $({}_\gamma i_3^*)^{-1} \circ \varphi_{a,\gamma} = \varphi_{\tilde{p}_3^*(a),i_3(\gamma)}$. The morphism i_3 is in fact the identity of $SO(h)$, so we have $i_3(\gamma) = \gamma$. Thus, ${}_\gamma i^* \circ \varphi_{a,\gamma} =_{\gamma'} i_1^* \circ_{\gamma''} i_2^* \circ \varphi_{\tilde{p}_3^*(a),\gamma}$. From lemma 2.9 and $i_2(\gamma'') = \gamma$, we have ${}_{\gamma''} i_2^* \circ \varphi_{\tilde{p}_3^*(a),\gamma} = \varphi_{\tilde{i}_2^* \circ \tilde{p}_3^*(a),\gamma''}$. Actually, $\tilde{p}_3 \circ \tilde{i}_2 = \tilde{i} \circ \tilde{p}_1$, so we now juste have to compute ${}_{\gamma'} i_1^* \circ \varphi_{\tilde{p}_1^* \circ \tilde{i}^*(a),\gamma''}$. Since $\gamma' = p_1(\gamma)$, lemma 2.10 yields ${}_\gamma i^* \varphi_{a,\gamma} = \varphi_{i^*(a),\gamma'}$.

Therefore, dropping the cocycles on the notations, we have

$$\begin{array}{lcl}i^*\varphi_1 & = & \varphi_1 \\ i^*\varphi_{r_1} & = & \varphi_{r_1'} \\ i^*\varphi_{(r_1)^2}(x) & = & \varphi_{(r_1')^2} \\ i^*\varphi_{(r_1)^3}(x) & = & \varphi_{(r_1')^3} \\ & = & \varphi_{-1-((\beta')^2-\theta_1'-1)r_1'+(\theta_1'+1)(r_1')^2+\beta'\eta'} \\ & = & -\varphi_1 - 11\varphi_{r_1'} + 5\varphi_{(r_1')^2} + \varphi_{\eta'} \circ \mathrm{I}_{F,D} \\ i^*\varphi_{\eta_+} & = & \varphi_{\eta'} \\ i^*\varphi_{\eta_-} & = & \varphi_{\eta'}\end{array} \tag{15}$$

in which we have used lemma 2.7 to compute $\varphi_{(r_1')^3}$.

2.6. **Cup-products.** We now give the results of some cup-products that we use in the next section. These computations are straight forward applications of lemmas 2.4 and 2.7. For the K-theory of $SB(2, D)$, to use lemma 2.7, we need to decompose products of Schur polynomials as given in table 2. The computation of the cup-products on X_q can then be deduced from the ones on $SB(2, D)$. Of course, they could also be computed directly. To avoid lengthy formulas, these cup-products will be given in tables. It should be understood that the intersection between a line and a column gives the cup-product between the morphism at the top of the column and the morphism at the beginning of the line. Moreover, the Morita morphisms $\mathrm{M}_{D^{\otimes i},D^{\otimes j}}$ (resp. the restriction morphisms $\mathrm{Res}_{D^{\otimes i},D^{\otimes j}}$ will be abreviated as $\mathrm{M}_{i,j}$ (resp. $\mathrm{R}_{i,j}$). Thus, for example, in table 3, we can read

$$\begin{aligned}\varphi_{\sigma_{1,0}}(x).\varphi_{\sigma_{2,2}}(y) = \ & \varphi_{\sigma_{1,0}} \circ \mathrm{M}_{D^{\otimes 5},D}(x.y) - \varphi_{\sigma_{1,1}} \circ \mathrm{Res}_{D^{\otimes 3},D^{\otimes 2}} \circ \mathrm{M}_{D^{\otimes 5},D^{\otimes 3}}(x.y) \\ & +\varphi_{\sigma_{2,2}} \circ \mathrm{Res}_{D^{\otimes 5},D^{\otimes 4}}(x.y)\end{aligned}$$

α	$1,0$	$1,1$	$2,0$	$2,1$	$2,2$
$1,0$	$\sigma_{2,0}$ $+\sigma_{1,1}$	$\sigma_{2,1}$	$\sigma_3\sigma_{0,0} - \sigma_2\sigma_{1,0}$ $+\sigma_1\sigma_{2,0} + \sigma_{2,1}$	$\sigma_4\sigma_{0,0} - \sigma_2\sigma_{1,1}$ $+\sigma_1\sigma_{2,1} + \sigma_{2,2}$	$\sigma_4\sigma_{1,0} - \sigma_3\sigma_{1,1}$ $+\sigma_1\sigma_{2,2}$
$1,1$		$\sigma_{2,2}$	$\sigma_4\sigma_{0,0} - \sigma_2\sigma_{1,1}$ $+\sigma_1\sigma_{2,1}$	$\sigma_4\sigma_{1,0} - \sigma_3\sigma_{1,1}$ $+\sigma_1\sigma_{2,2}$	$\sigma_4\sigma_{2,0} - \sigma_3\sigma_{2,1}$ $+\sigma_2\sigma_{2,2}$
$2,0$			$\sigma_1\sigma_3\sigma_{0,0}$ $+(\sigma_3 - \sigma_1\sigma_2)\sigma_{1,0}$ $-\sigma_2\sigma_{1,1}$ $+(\sigma_1^2 - \sigma_2)\sigma_{2,0}$ $+\sigma_1\sigma_{2,1} + \sigma_{2,2}$	$\sigma_1\sigma_4\sigma_{0,0} + \sigma_4\sigma_{1,0}$ $-\sigma_1\sigma_2\sigma_{1,1}$ $+(\sigma_1^2 - \sigma_2)\sigma_{2,1}$ $+\sigma_1\sigma_{2,2}$	$\sigma_1\sigma_4\sigma_{1,0}$ $+(-\sigma_1\sigma_3 + \sigma_4)\sigma_{1,1}$ $+(\sigma_1^2 - \sigma_2)\sigma_{2,2}$
$2,1$				$\sigma_1\sigma_4\sigma_{1,0}$ $+(-\sigma_1\sigma_3 + \sigma_4)\sigma_{1,1}$ $+\sigma_4\sigma_{2,0} - \sigma_3\sigma_{2,1}$ $+\sigma_1^2\sigma_{2,2}$	$\sigma_1\sigma_4\sigma_{2,0}$ $+(-\sigma_1\sigma_3 + \sigma_4)\sigma_{2,1}$ $+(\sigma_1\sigma_2 - \sigma_3)\sigma_{2,2}$
$2,2$					$\sigma_4^2\sigma_{0,0} - \sigma_3\sigma_4\sigma_{1,0}$ $+(-\sigma_2\sigma_4 + \sigma_3^2)\sigma_{1,1}$ $+\sigma_2\sigma_4\sigma_{2,0}$ $+(-\sigma_2\sigma_3 + \sigma_1\sigma_4)\sigma_{2,1}$ $+(\sigma_2^2 - \sigma_1\sigma_3)\sigma_{2,2}$

TABLE 2. Decomposition of the products of Schur polynomials

	$\varphi_{0,0}$	$\varphi_{1,0}$	$\varphi_{1,1}$	$\varphi_{2,0}$	$\varphi_{2,1}$	$\varphi_{2,2}$
$\varphi_{0,0}$	$\varphi_{0,0}$	$\varphi_{1,0}$	$\varphi_{1,1}$	$\varphi_{2,0}$	$\varphi_{2,1}$	$\varphi_{2,2}$
$\varphi_{1,0}$		$\varphi_{1,1}$ $+\varphi_{2,0}$	$\varphi_{2,1}$	$\varphi_{0,0}\mathrm{R}_{1,0}\mathrm{M}_{3,1}$ $-6\varphi_{1,0}\mathrm{M}_{3,1}$ $+\varphi_{2,0}\mathrm{R}_{3,2}$ $+\varphi_{2,1}$	$\varphi_{0,0}\mathrm{M}_{4,0}$ $-6\varphi_{1,1}\mathrm{M}_{4,2}$ $+\varphi_{2,1}\mathrm{R}_{4,3}$ $+\varphi_{2,2}$	$\varphi_{1,0}\mathrm{M}_{5,1}$ $-\varphi_{1,1}\mathrm{R}_{3,2}\mathrm{M}_{5,3}$ $+\varphi_{2,2}\mathrm{R}_{5,4}$
$\varphi_{1,1}$			$\varphi_{2,2}$	$\varphi_{0,0}\mathrm{M}_{4,0}$ $-6\varphi_{1,1}\mathrm{M}_{4,2}$ $+\varphi_{2,1}\mathrm{R}_{4,3}$	$\varphi_{1,0}\mathrm{M}_{5,1}$ $-\varphi_{1,1}\mathrm{R}_{3,2}\mathrm{M}_{5,3}$ $+\varphi_{2,2}\mathrm{R}_{5,4}$	$\varphi_{2,0}\mathrm{M}_{6,2}$ $-\varphi_{2,1}\mathrm{R}_{4,3}\mathrm{M}_{6,4}$ $+6\varphi_{2,2}\mathrm{M}_{6,4}$
$\varphi_{2,0}$				$16\varphi_{0,0}\mathrm{M}_{4,0}$ $-5\varphi_{1,0}\mathrm{R}_{2,1}\mathrm{M}_{4,2}$ $-6\varphi_{1,1}\mathrm{M}_{4,2}$ $+10\varphi_{2,0}\mathrm{M}_{4,2}$ $+\varphi_{2,1}\mathrm{R}_{4,3}$ $+\varphi_{2,2}$	$\varphi_{0,0}\mathrm{R}_{1,0}\mathrm{M}_{5,1}$ $+\varphi_{1,0}\mathrm{M}_{5,1}$ $-6\varphi_{1,1}\mathrm{R}_{3,2}\mathrm{M}_{5,3}$ $+10\varphi_{2,1}\mathrm{M}_{5,3}$ $+\varphi_{2,2}\mathrm{R}_{5,4}$	$\varphi_{1,0}\mathrm{R}_{2,1}\mathrm{M}_{6,2}$ $-15\varphi_{1,1}\mathrm{M}_{6,2}$ $+10\varphi_{2,2}\mathrm{M}_{6,4}$
$\varphi_{2,1}$					$\varphi_{1,0}\mathrm{R}_{2,1}\mathrm{M}_{6,2}$ $-15\varphi_{1,1}\mathrm{M}_{6,2}$ $+\varphi_{2,0}\mathrm{M}_{6,2}$ $-\varphi_{2,1}\mathrm{R}_{4,3}\mathrm{M}_{6,4}$ $+16\varphi_{2,2}\mathrm{M}_{6,4}$	$\varphi_{2,0}\mathrm{R}_{3,2}\mathrm{M}_{7,3}$ $-15\varphi_{2,1}\mathrm{M}_{7,3}$ $+5\varphi_{2,2}\mathrm{R}_{5,4}\mathrm{M}_{7,5}$
$\varphi_{2,2}$						$\varphi_{0,0}\mathrm{M}_{8,0}$ $-\varphi_{1,0}\mathrm{R}_{2,1}\mathrm{M}_{8,2}$ $+10\varphi_{1,1}\mathrm{M}_{8,2}$ $+6\varphi_{2,0}\mathrm{M}_{8,2}$ $-5\varphi_{2,1}\mathrm{R}_{4,3}\mathrm{M}_{8,4}$ $+20\varphi_{2,2}\mathrm{M}_{8,4}$

TABLE 3. Cup-products for the K-theory of $SB(2,D)$

3. TOPOLOGICAL FILTRATION

In this section, we shall compute part of the topological filtration of the quadric X_q. For this task, Panin's decomposition cannot be used directly, since it does not respect the topological filtration. We shall therefore introduce new morphisms, which map to the different levels of the filtration. The definition of those morphims uses the reduced norm. Since it is only defined for K_0, K_1 and K_2, those morphisms will only be defined for those K-theory levels.

	φ_1	φ_{r_1}	$\varphi_{r_1^2}$	$\varphi_{r_1^3}$	φ_{η_-}	φ_{η_+}
φ_1	φ_1	φ_{r_1}	$\varphi_{r_1^2}$	$\varphi_{r_1^3}$	φ_{η_-}	φ_{η_+}
φ_{r_1}		$\varphi_{r_1^2}$	$\varphi_{r_1^3}$	$\varphi_1 \mathrm{M}_{8,0}$ $+26\varphi_{r_1}\mathrm{M}_{8,2}$ $-16\varphi_{r_1^2}\mathrm{M}_{8,4}$ $+6\varphi_{r_1^3}\mathrm{M}_{8,6}$ $-\varphi_{\eta_-}\mathrm{R}_{4,3}\mathrm{M}_{8,4}$ $-\varphi_{\eta_+}\mathrm{R}_{2,1}\mathrm{M}_{8,2}$	$\varphi_{r_1}\mathrm{R}_{3,2}\mathrm{M}_{5,3}$ $-\varphi_{\eta_+}\mathrm{M}_{5,1}$	$\varphi_{r_1}\mathrm{R}_{3,2}$ $-\varphi_{\eta_-}$
$\varphi_{r_1^2}$			$\varphi_1 \mathrm{M}_{8,0}$ $+26\varphi_{r_1}\mathrm{M}_{8,2}$ $-16\varphi_{r_1^2}\mathrm{M}_{8,4}$ $+6\varphi_{r_1^3}\mathrm{M}_{8,6}$ $-\varphi_{\eta_-}\mathrm{R}_{4,3}\mathrm{M}_{8,4}$ $-\varphi_{\eta_+}\mathrm{R}_{2,1}\mathrm{M}_{8,2}$	$6\varphi_1\mathrm{M}_{10,0}$ $+125\varphi_{r_1}\mathrm{M}_{10,2}$ $-70\varphi_{r_1^2}\mathrm{M}_{10,4}$ $+20\varphi_{r_1^3}\mathrm{M}_{10,6}$ $-5\varphi_{\eta_-}\mathrm{R}_{4,3}\mathrm{M}_{10,4}$ $-5\varphi_{\eta_+}\mathrm{R}_{2,1}\mathrm{M}_{10,2}$	$-\varphi_{r_1}\mathrm{R}_{3,2}\mathrm{M}_{7,3}$ $+\varphi_{r_1^2}\mathrm{R}_{5,4}\mathrm{M}_{7,5}$ $+\varphi_{\eta_-}\mathrm{M}_{7,1}$	$-\varphi_{r_1}\mathrm{R}_{3,2}\mathrm{M}_{5,3}$ $+\varphi_{r_1^2}\mathrm{R}_{5,4}$ $+\varphi_{\eta_+}\mathrm{M}_{5,1}$
$\varphi_{r_1^3}$				$20\varphi_1\mathrm{M}_{12,0}$ $+366\varphi_{r_1}\mathrm{M}_{12,2}$ $-195\varphi_{r_1^2}\mathrm{M}_{12,4}$ $+50\varphi_{r_1^3}\mathrm{M}_{12,6}$ $-15\varphi_{\eta_-}\mathrm{R}_{4,3}\mathrm{M}_{12,4}$ $-15\varphi_{\eta_+}\mathrm{R}_{2,1}\mathrm{M}_{12,2}$	$\varphi_{r_1}\mathrm{R}_{3,2}\mathrm{M}_{9,3}$ $-\varphi_{r_1^2}\mathrm{R}_{5,4}\mathrm{M}_{9,5}$ $+\varphi_{r_1^3}\mathrm{R}_{7,6}\mathrm{M}_{9,7}$ $-\varphi_{\eta_+}\mathrm{M}_{9,3}$	$\varphi_{r_1}\mathrm{R}_{3,2}\mathrm{M}_{7,3}$ $-\varphi_{r_1^2}\mathrm{R}_{5,4}\mathrm{M}_{7,5}$ $+\varphi_{r_1^3}\mathrm{R}_{7,6}$ $-\varphi_{\eta_-}\mathrm{M}_{7,3}$
φ_{η_-}					$17\varphi_{r_1}\mathrm{M}_{6,2}$ $-6\varphi_{r_1^2}\mathrm{M}_{6,4}$ $+\varphi_{r_1^3}$ $-\varphi_{\eta_+}\mathrm{R}_{2,1}\mathrm{M}_{6,2}$	$-\varphi_1\mathrm{M}_{4,0}$ $+6\varphi_{r_1}\mathrm{M}_{4,2}$ $-\varphi_{r_1^2}$
φ_{η_+}						$17\varphi_{r_1}$ $-6\varphi_{r_1^2}\mathrm{M}_{2,4}$ $+\varphi_{r_1^3}\mathrm{M}_{6,2}$ $-\varphi_{\eta_-}\mathrm{I}_{2,3}$

TABLE 4. Cup-products for the K-theory of X_q

Let $K_iX^{(j)}$ be the group of level j in the topological filtration of K_iX and let $K_iX^{(j/j+1)} = K_iX^{(j)}/K_iX^{(j+1)}$.

3.1. Computation of $K_iX_q^{(1)}$ and $K_iX_q^{(2)}$ $(i = 0, 1, 2)$.

Definition 3.1. For $i = 0, 1$ and 2, we shall define

$$\begin{array}{ll} \Psi_0, \Psi_1, \Psi_2, \Psi_3 : & K_iF \longrightarrow K_iX_q \\ \Psi'_0, \Psi'_1, \Psi'_2 : & K_iF \longrightarrow K_iX_{q'} \end{array}$$

by

$$\begin{aligned} \Psi_0 &= \varphi_1 \\ \Psi_1 &= \varphi_1 - \varphi_{r_1} \circ \mathrm{M}_{F,D^{\otimes 2}} \\ \Psi_2 &= \varphi_1 - 2\varphi_{r_1} \circ \mathrm{M}_{F,D^{\otimes 2}} + \varphi_{(r_1)^2} \circ \mathrm{M}_{F,D^{\otimes 4}} \\ \Psi_3 &= \varphi_1 - 3\varphi_{r_1} \circ \mathrm{M}_{F,D^{\otimes 2}} + 3\varphi_{(r_1)^2} \circ \mathrm{M}_{F,D^{\otimes 4}} - \varphi_{(r_1)^3} \circ \mathrm{M}_{F,D^{\otimes 6}} \\ \Psi'_0 &= \varphi_{1'} \\ \Psi'_1 &= \varphi_{1'} - \varphi_{r'_1} \circ \mathrm{M}_{F,D^{\otimes 2}} \\ \Psi'_2 &= \varphi_{1'} - 2\varphi_{r'_1} \circ \mathrm{M}_{F,D^{\otimes 2}} + \varphi_{(r'_1)^2} \circ \mathrm{M}_{F,D^{\otimes 4}} \end{aligned}$$

and

$$\begin{array}{ll} \Psi_{2'}, \Psi_{2''}, \Psi_{3'} : & K_iD \longrightarrow K_iX_q \\ \Psi'_{2'} : & K_iD \longrightarrow K_iX_{q'} \end{array}$$

by

$$\begin{aligned} \Psi_{2'} &= \varphi_1 \circ \mathrm{Nrd} + \varphi_{r_1} \circ \mathrm{M}_{F,D^{\otimes 2}} \circ \mathrm{Nrd} - \varphi_{\eta_+} \\ \Psi_{2''} &= \varphi_1 \circ \mathrm{Nrd} + \varphi_{r_1} \circ \mathrm{M}_{F,D^{\otimes 2}} \circ \mathrm{Nrd} - \varphi_{\eta_-} \circ \mathrm{M}_{D,D^{\otimes 3}} \\ \Psi_{3'} &= \varphi_1 \circ \mathrm{Nrd} + 4\varphi_{r_1} \circ \mathrm{M}_{F,D^{\otimes 2}} \circ \mathrm{Nrd} - \varphi_{(r'_1)^2} \circ \mathrm{M}_{F,D^{\otimes 4}} - \varphi_{\eta_+} - \varphi_{\eta_-} \circ \mathrm{M}_{D,D^{\otimes 3}} \\ \Psi'_{2'} &= \varphi_{1'} \circ \mathrm{Nrd} + \varphi_{r'_1} \circ \mathrm{M}_{F,D^{\otimes 2}} \circ \mathrm{Nrd} - \varphi_{\eta'} \end{aligned}$$

Remark 3.2. Note that $\Psi_{3'} = \Psi_{2'} + \Psi_{2''} - \Psi_2 \circ \mathrm{Nrd}$.

These morphisms are related in the following way.

Lemma 3.3. Recall that i denotes the inclusion $X_{q'} \hookrightarrow X_q$. For $j = 0, 1, 2$ and $2'$, $i^*\Psi_j = \Psi'_j$. Moreover, $i^*\Psi_3 = -2\Psi'_2 + \Psi'_{2'} \circ \mathrm{I}_{F,D}$ and $i^*\Psi_{3'} = 2\Psi'_{2'} - \Psi'_2 \circ \mathrm{Nrd}$.

Proof: This follows from definition 3.1 and equalities (15). □

Lemma 3.4. Let $k, k' \in K_iF$ and $d \in K_iD$.
For $j = 0, 1, 2, 3$, $\Psi_0(k).\Psi_j(k') = \Psi_j(k.k')$
For $j = 0, 1, 2$, $\Psi'_0(k).\Psi'_j(k') = \Psi'_j(k.k')$
Moreover, $\Psi_1(k).\Psi_1(k') = \Psi_2(k.k')$
$\Psi'_1(k).\Psi'_1(k') = \Psi'_2(k.k')$
$\Psi_1(k).\Psi_2(k') = \Psi_3(k.k')$
$\Psi_1(k).\Psi_{2'}(d) = \Psi_1(k).\Psi_{2''}(d) = \Psi_{3'}(k.d)$

Proof: This follows from the definition of these morphisms and table 4. □

Theorem 3.5. *For $i = 0$, 1 and 2, the morphisms*

$$\begin{aligned}
&\Psi_0 \oplus \Psi_1 \oplus \Psi_2 \oplus \Psi_3 \oplus \Psi_{2'} \oplus \Psi_{2''} : && K_iF^{\oplus 4} \oplus K_iD^{\oplus 2} \longrightarrow K_iX_q \\
&\Psi_0 \oplus \Psi_1 \oplus \Psi_2 \oplus \Psi_3 \oplus \Psi_{2'} \oplus \Psi_{3'} : && K_iF^{\oplus 4} \oplus K_iD^{\oplus 2} \longrightarrow K_iX_q \\
&\Psi'_0 \oplus \Psi'_1 \oplus \Psi'_2 \oplus \Psi'_{2'} : && K_iF^{\oplus 3} \oplus K_iD \longrightarrow K_iX_{q'}
\end{aligned}$$

are isomorphisms.

Proof: The first morphism is the composition of the morphism

$$\varphi_1 \oplus \varphi_{r_1} \oplus \varphi_{r_1^2} \oplus \varphi_{r_1^3} \oplus \varphi_{\eta_+} \oplus \varphi_{\eta_-} :$$

$$K_iF \oplus K_iD^{\otimes 2} \oplus K_iD^{\otimes 4} \oplus K_iD^{\otimes 6} \oplus K_iD^{\otimes 1} \oplus K_iD^{\otimes 3} \longrightarrow K_iX_q$$

which is an isomorphism and the isomorphism

$$K_iF^{\oplus 4} \oplus K_iD^{\oplus 2} \longrightarrow K_iF \oplus K_iD^{\otimes 2} \oplus K_iD^{\otimes 4} \oplus K_iD^{\otimes 6} \oplus K_iD^{\otimes 1} \oplus K_iD^{\otimes 3}$$

given by the matrix

$$\begin{pmatrix}
\mathrm{Id} & \mathrm{Id} & \mathrm{Id} & \mathrm{Id} & \mathrm{Nrd} & \mathrm{Nrd} \\
0 & -\mathrm{M}_{F,D^{\otimes 2}} & -2\mathrm{M}_{F,D^{\otimes 2}} & -3\mathrm{M}_{F,D^{\otimes 2}} & \mathrm{M}_{F,D^{\otimes 2}} \circ \mathrm{Nrd} & \mathrm{M}_{F,D^{\otimes 2}} \circ \mathrm{Nrd} \\
0 & 0 & \mathrm{M}_{F,D^{\otimes 4}} & 3\mathrm{M}_{F,D^{\otimes 4}} & 0 & 0 \\
0 & 0 & 0 & -\mathrm{M}_{F,D^{\otimes 6}} & 0 & 0 \\
0 & 0 & 0 & 0 & -\mathrm{Id} & 0 \\
0 & 0 & 0 & 0 & 0 & -\mathrm{M}_{D,D^{\otimes 3}}
\end{pmatrix}.$$

This matrix is invertible because it is upper-triangular, with invertible morphisms on the diagonal. The result for the second morphism is then a simple consequence of remark 3.2. For the last morphism, the same kind of proof as for the first one applies. □

Theorem 3.6. *For $i = 0, 1, 2$ and for $j = 0, 1, 2, 3$, Ψ_j maps to $K_iX_q^{(j)}$.*

Before proving this theorem, we shall obtain a simple corollary.

Corollary 3.7. *For $i = 0, 1, 2$ and for $j = 0, 1, 2$, Ψ'_j maps to $K_iX_{q'}^{(j)}$.*

Proof: This is a consequence of lemma 3.3, since i^* preserves the topological filtration. □

Let us now prove theorem 3.6. First of all, the theorem reduces to the fact that $\Psi_1([F])$ lies in $K_0X_q^{(1)}$ ($[F]$ is the class in K_0F of F itself). Indeed, the cup-product by $[F]$ is the identity on K_iF or K_iD, thus the formulas of lemma 3.4 imply the other cases since the cup-product respects the filtration. In order to prove that $\Psi_1([F])$ is in $K_0X_q^{(1)}$, we shall make a few computations in the split case

α	$(0,0)$	$(1,1)$	$(2,0)$	$(2,2)$	$(1,0)$	$(2,1)$
$\lvert\alpha\rvert$	0	2	2	4	1	3
$S^\alpha\mathcal{J}$	1	$\Lambda^2\mathcal{J}$	$S^2\mathcal{J}$	$\Lambda^2\mathcal{J}\otimes\Lambda^2\mathcal{J}$	$\mathcal{J}$	$\mathcal{J}\otimes\Lambda^2\mathcal{J}$
$\dim S^\alpha\mathcal{J}$	1	1	3	1	2	2

TABLE 5.

$(X_q = X_h)$. We make use of elements of K_0X_h, whose codimensions are known. These elements come from the embedding of X_h in $\mathbf{P}^5$, and they generate K_0X_h (see [8], § 3.2). Let $\mathcal{Q}$ be the class in K_0X_h of a rationnal point, $\mathcal{H}$ the class of a hyperplane section $((\mathbf{P}^4\cap X_h)\subset\mathbf{P}^5)$, $\mathcal{D}$ the class of a line $((\mathbf{P}^1\cap X_h)\subset\mathbf{P}^5)$. These classes are independant of the choice of the embeddings of the projective spaces in $\mathbf{P}^5$. Let $\mathcal{P}_1$ (resp. $\mathcal{P}_2$ be the class of the intersection of X_h and the projective plane $w_2 = w_4 = w_6 = 0$ (resp. $w_2 = w_4 = w_5 = 0$) in the basis chosen in section 2.4. These two classes are different in K_0X_h. We will also denote $\mathcal{I}$ the class of the structural sheaf of X_h. By construction, the codimensions of $\mathcal{I}$, $\mathcal{H}$, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{D}$ and $\mathcal{Q}$ are respectively 0, 1, 2, 2, 3 and 4. To keep the notations simple, we will denote identically the images of these elements in $K_0Gr(2,V)$ by the isomorphism $(f^{-1})^*$. The cup-products between these elements are given by the formulas (see [8])

(16)
$$\mathcal{H}^2 = \mathcal{P}_1+\mathcal{P}_2-\mathcal{D},\ \mathcal{H}.\mathcal{P}_1 = \mathcal{H}.\mathcal{P}_2 = \mathcal{D},\ {\mathcal{P}_1}^2 = {\mathcal{P}_2}^2 = \mathcal{Q},\ \mathcal{P}_1.\mathcal{P}_2 = 0,\ \mathcal{H}.\mathcal{D} = \mathcal{Q}.$$

All the other ones are zero for codimension reasons. The subquadric $X_{h'}$ of equation $x_1'y_1'+x_2'y_2'+(z')^2 = 0$ includes in the quadric X_h of equation $x_1y_1+x_2y_2+x_3y_3 = 0$ by $x_1 = x_1'$, $y_1 = y_1'$, $x_2 = x_2'$, $y_2 = y_2'$ and $x_3 = y_3 = z'$, so $\mathcal{H}' = i^*\mathcal{H}$ is the class of $X_{h'}\cap\mathbf{P}^3$, $\mathcal{D}' = i^*\mathcal{P}_1 = i^*\mathcal{P}_2$ is the class of $X_{h'}\cap\mathbf{P}^1$, $\mathcal{Q}' = i^*\mathcal{D}$ is the class of a rationnal point and $i^*\mathcal{Q} = 0$. The non trivial cup-products between these elements are

$$(\mathcal{H}')^2 = 2\mathcal{D}'-\mathcal{Q}',\ \mathcal{H}'.\mathcal{D}' = \mathcal{Q}'. \tag{17}$$

We also introduce elements of $K_0Gr(2,V)$ which are classes of vector bundles. Let $\mathcal{J}$ be the canonical bundle of $Gr(2,V)$ - the fiber above a point is the subspace of V that this point represents. Let S^α be the Schur functor of multi-index α. We shall use the vector bundles $S^\alpha\mathcal{J}$, where $\alpha = (0,0)$, $(1,0)$, $(1,1)$, $(2,0)$, $(2,1)$ and $(2,2)$. Table 5 shows their values in terms of symmetric and exterior powers of $\mathcal{J}$. By definition, the morphism $\varphi_{\sigma_{0,0}}$ (resp. $\varphi_1 = \Psi_0$, $\varphi_1' = \Psi_0'$) is equal to the pull-back along the structural morphism of $Gr(2,V)$ (resp. X_h, $X_{h'}$). Of course, this is also true in the non-split case. In the following, we will simply replace $\varphi_{\sigma_{0,0}}(k)$ (resp. $\varphi_1(k)$, $\Psi_0(k)$, $\varphi_1'(k)$, $\Psi_0'(k)$) by k, to shorten the formulas.

Lemma 3.8. In the split case, $\varphi_{\sigma_\alpha}\circ \mathrm{M}_{F,D^{\otimes|\alpha|}}(k) = k.S^\alpha\mathcal{J}$. In particular, $\varphi_{\sigma_\alpha}\circ \mathrm{M}_{F,D^{\otimes|\alpha|}}([F]) = S^\alpha\mathcal{J}$.

Proof: From the cup-products in table 3, we get $\varphi_{\sigma_\alpha}\circ\mathrm{M}_{F,D^{\otimes|\alpha|}}(k) = k.\varphi_{\sigma_\alpha}\circ \mathrm{M}_{F,D^{\otimes|\alpha|}}([F])$. The identification of $\varphi_{\sigma_\alpha}(\mathrm{M}_{F,D^{\otimes|\alpha|}}([F]))$ and $S^\alpha\mathcal{J}$ easily follows from the definition of φ_{σ_α} (see section 2.3). □

The Plücker embedding Plk of $Gr(2,V)$ in $\mathbf{P}^5$ sends a subspace U of V to Λ^2U in Λ^2V, so $Plk^*(\mathcal{O}_{\mathbf{P}^5}(-1)) = \Lambda^2\mathcal{J}$. Since $Plk\circ f = Pl_h$ (see lemma 2.12), the classical equality $\mathcal{O}_{\mathbf{P}^5}(-1) = \mathcal{O}_{\mathbf{P}^5}-\mathcal{H}$ pulls back to K_0X_h as

$$\Lambda^2\mathcal{J} = \mathcal{I}-\mathcal{H} \tag{18}$$

Since $\mathcal{H} = \mathcal{I}-\mathcal{O}_{X_h}(-1)$, it can also be defined in the non-split case by the same formula. Its codimension is 1 - even in the non-split case - since it is in the kernel of the rank application (on vector bundles). From the definition of Ψ_1, section 2.5 and lemma 3.8, we get

$$\Psi_1([F]) = \varphi_1([F]) - \varphi_{r_1}([F]) = f^*\varphi_{\sigma_{0,0}}([F]) - f^*\varphi_{\sigma_{1,1}}([F]) = \mathcal{I} - \Lambda^2\mathcal{J} = \mathcal{H}.$$

The equality $\Psi_1([F]) = \mathcal{H}$ has to be true in the non-split case since the extension of scalars is injective on K_0 (see remark 2.3), so we have proved $\Psi_1([F]) \in K_0X_q^{(1)}$ and theorem 3.6. We shall establish the following (which is a little more difficult):

Theorem 3.9. *For $i = 0, 1, 2$ and for $j = 2, 3$, $\Psi_{j'}$ maps to $K_iX_q^{(j)}$. $\Psi_{2''}$ maps to $K_iX_q^{(2)}$.*

As for theorem 3.6, we have a simple corollary.

Corollary 3.10. *For $i = 0, 1, 2$, $\Psi'_{2'}$ maps to $K_iX_{q'}^{(2)}$.*

Let us now prove the theorem 3.9. It reduces to the case of $\Psi_{2'}$ since we have $\Psi_1([F]).\Psi_{2'}(d) = \Psi_{3'}(d)$ (see lemma 3.4) and $\Psi_{2''}(d) = \Psi_{3'}(d) - \Psi_{2'}(d) + \Psi_2 \circ \mathrm{Nrd}(d)$ (see remark 3.2).

By definition, the bundle $\mathcal{J}$ fits into an exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow V \otimes \mathcal{O}_{Gr(2,V)} \longrightarrow \mathcal{J}' \longrightarrow 0$$

and the dual sequence is

$$0 \longrightarrow \mathcal{J}'^* \longrightarrow V \otimes \mathcal{O}_{Gr(2,V)} \longrightarrow \mathcal{J}^* \longrightarrow 0.$$

Let ϕ be an element of V^* whose kernel is $< v_1, v_2, v_3 >$ (these are the elements of the basis of V chosen at the beginning of section 2.4). Such an element gives rise to a section s of $\mathcal{J}^*$ through the composition

$$\mathcal{O}_{Gr(2,V)} \xrightarrow{\phi \otimes \cdot} V^* \otimes \mathcal{O}_{Gr(2,V)} \longrightarrow \mathcal{J}^*$$

The zero locus of s is the set of points x such that

$$\mathcal{J}_x \longrightarrow V \otimes \mathcal{O}_{Gr(2,V),x} \xrightarrow{\phi.\mathrm{Id}} \mathcal{O}_{Gr(2,V),x}$$

is zero, that is if $\mathcal{J}_x$ in V is included in $\ker\phi$. Through the Plücker embedding, this condition becomes $\Lambda^2\mathcal{J}_x \subset \Lambda^2 \ker\phi$. Since $\Lambda^2 \ker\phi = < v_1 \wedge v_2, v_1 \wedge v_2, v_1 \wedge v_3 > = < w_1, w_3, w_5 >$, we obtain the subvariety $w_2 = w_4 = w_6 = 0$ whose class is $\mathcal{P}_1$ in $K_0Gr(2, V)$.

The Koszul exact sequence (see [5], IV, §2) for the bundle $\mathcal{J}$ (of rank 2) and the section s is

$$0 \longrightarrow \Lambda^2\mathcal{J} \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_{Gr(2,V)} \longrightarrow \mathcal{O}_s \longrightarrow 0$$

where $\mathcal{O}_s$ is the structural sheaf of the zero locus of s. Thus $\mathcal{J} = S^{1,1}\mathcal{J} + \mathcal{I} - \mathcal{P}_1$ in $K_0Gr(2, V)$. The cup-products (16), the table 5 and the equality $S^{1,1}\mathcal{J} + S^{2,0}\mathcal{J} = (S^{1,0})^2$ give

$$\begin{aligned}
S^{0,0}\mathcal{J} &= \mathcal{I}\\
S^{1,0}\mathcal{J} &= 2\mathcal{I} - \mathcal{H} - \mathcal{P}_1\\
S^{1,1}\mathcal{J} &= \mathcal{I} - \mathcal{H}\\
S^{2,0}\mathcal{J} &= 3\mathcal{I} - 3\mathcal{H} - 3\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{D} + \mathcal{Q}\\
S^{2,1}\mathcal{J} &= 2\mathcal{I} - 3\mathcal{H} + \mathcal{P}_2\\
S^{2,2}\mathcal{J} &= (\mathcal{I} - \mathcal{H})^2 = \mathcal{I} - 2\mathcal{H} + \mathcal{P}_1 + \mathcal{P}_2 - \mathcal{D}
\end{aligned}$$

From (12), we get

$$\begin{aligned}
\varphi_{(r_1)^i} \circ \mathrm{M}_{F,D^{\otimes 2i}}(k) &= f^*(\varphi_{(\sigma_{1,1})^i})(k)\\
&= f^*(k).f^*((\mathcal{I} - \mathcal{H})^i)\\
&= k.(\mathcal{I} - \mathcal{H})^i
\end{aligned}$$

$$\begin{aligned}\varphi_{\eta_+}\circ \mathrm{M}_{F,D}(k) &= f^*(\varphi_{\sigma_{1,0}}(k))\\ &= f^*(k.(2\mathcal{I}-\mathcal{H}-\mathcal{P}_1))\\ &= k.(2\mathcal{I}-\mathcal{H}-\mathcal{P}_1)\end{aligned}$$

and

$$\begin{aligned}\varphi_{\eta_-}\circ \mathrm{M}_{F,D^{\otimes 3}}(k) &= f^*(\varphi_{\sigma_{1,1}}\circ \mathrm{Res}_{D^{\otimes 3},D^{\otimes 2}}\circ \mathrm{M}_{F,D^{\otimes 3}}(k) - \varphi_{\sigma_{2,1}}\circ \mathrm{M}_{F,D^{\otimes 3}}(k))\\ &= f^*(\varphi_{\sigma_{1,1}}\circ 4\mathrm{M}_{F,D^{\otimes 2}}(k) - \varphi_{\sigma_{2,1}}\circ \mathrm{M}_{F,D^{\otimes 3}}(k))\\ &= f^*(k.4S^{1,1}\mathcal{J} - k.S^{2,1}\mathcal{J})\\ &= f^*(k.4(\mathcal{I}-\mathcal{H}) - k.(2\mathcal{I}-3\mathcal{H}+\mathcal{P}_2))\\ &= k.(2\mathcal{I}-\mathcal{H}-\mathcal{P}_2)\end{aligned}$$

Thus, in the split case,

$$\begin{gathered}\Psi_0(k)=k.\mathcal{I},\ \Psi_1(k)=k.\mathcal{H},\ \Psi_2(k)=k.\mathcal{H}^2,\ \Psi_3(k)=k.\mathcal{H}^3,\\ \Psi_{2'}\circ \mathrm{M}_{F,D}(k)=k.\mathcal{P}_1,\ \Psi_{2''}\circ \mathrm{M}_{F,D}(k)=k.\mathcal{P}_2,\ \Psi_{3'}(k)=k.\mathcal{D}.\end{gathered}\tag{19}$$

and applying i^*

$$\Psi'_0(k)=k.\mathcal{I}',\ \Psi'_1(k)=k.\mathcal{H}',\ \Psi'_2(k)=k.(\mathcal{H}')^2,\ \Psi'_{2'}\circ \mathrm{M}_{F,D}(k)=k.\mathcal{D}'.\tag{20}$$

In particular, this proves that when X_h is split, $\Psi_{2'}$ maps to $K_0X_q^{(2)}$, so theorem 3.9 is proved in this case. We shall now establish the result in the non-split case. Let K be the function fields of the Severi-Brauer variety of D. It has two important properties. First, it splits D (and equivalently X_q), second, K_2F injects in K_2K (see [23], § 5). Instead of K, we could use any other field that has these two properties.

Definition 3.11. For $i=0,1,2$, using the Brown-Gersten-Quillen spectral sequence (see [18] or [22]), we define ξ_0, ξ_1, ξ'_0 and ξ'_1 as the compositions

$$\begin{aligned}\xi_0 &: K_iF\longrightarrow K_iX_q\longrightarrow K_iX_q^{(0/1)}\hookrightarrow H^0(X_q,\mathcal{K}_i)\\ \xi'_0 &: K_iF\longrightarrow K_iX_{q'}\longrightarrow K_iX_{q'}^{(0/1)}\hookrightarrow H^0(X_{q'},\mathcal{K}_i)\\ \xi_1 &: K_iF\xrightarrow{.\mathcal{H}} K_iX_q^{(1)}\longrightarrow K_iX_q^{(1/2)}\hookrightarrow H^1(X_q,\mathcal{K}_{i+1})\\ \xi'_1 &: K_iF\xrightarrow{.\mathcal{H}'} K_iX_{q'}^{(1)}\longrightarrow K_iX_{q'}^{(1/2)}\hookrightarrow H^1(X_{q'},\mathcal{K}_{i+1})\end{aligned}$$

Proposition 3.12. *The morphisms ξ_0, ξ_1, ξ'_0 and ξ'_1 are isomorphisms.*

Proof: We shall only handle the case of ξ_0 and ξ_1, since the same proof can be applied to ξ'_0 and ξ'_1. In the split case, $K_iX_h^{(j)}$ is generated by the cup-products of K_iF with the elements of $(\mathcal{I},\mathcal{H},\mathcal{P}_1,\mathcal{P}_2,\mathcal{D},\mathcal{Q})$ whose codimension is greater than j (see [8], § 3.2). Thus

$$K_iF\longrightarrow K_iX_q\longrightarrow K_iX_q^{(0/1)}$$

and

$$K_iF\xrightarrow{.\mathcal{H}} K_iX_q^{(1)}\longrightarrow K_iX_q^{(1/2)}$$

are isomorphisms. Furthermore, in the split case, the B.G.Q. spectral sequence degenerates, so the inclusions

$$K_iX_q^{(0/1)}\hookrightarrow H^1(X_q,\mathcal{K}_i)$$

and

$$K_iX_q^{(1/2)}\hookrightarrow H^1(X_q,\mathcal{K}_{i+1})$$

are isomorphisms. Hence, ξ_0 and ξ_1 are isomorphisms.

In the non-split case, (see [6], § 5.3, § 5.4 and corollary 8.6), ξ_0 et ξ_1 are isomorphisms after localisation at 2. Their kernels and cokernels are therefore 2-torsion free so by a transfer argument to a degree 4 extension that splits X_q, they are zero. □

Corollary 3.13. *For $X = X_q$ and $X = X_{q'}$, for $i = 0,1,2$ and for $j = 0,1$, the morphism*

$$K_iX^{(j/j+1)} \hookrightarrow H^1(X, \mathcal{K}_{i+j})$$

is an isomorphism, as well as the composition

$$K_iF \xrightarrow{\Psi_j} K_iX^{(j)} \longrightarrow K_iX^{(j/j+1)}.$$

Corollary 3.14. *For $X = X_q$ and $X = X_{q'}$, for $i = 0,1,2$ and for $j = 0,1$, $K_iX^{(j/j+1)}$ injects in $K_i(X)_K^{(j/j+1)}$.*

Since $\Psi_{2'}$ maps to $K_iX_q^{(2)}$ in the split case, its image in $K_iX_q^{(0/1)}$ (and then $K_iX_q^{(1/2)}$) has to be zero. This is also true in the non-split case by extension of scalars to K and corollary 3.14. Thus theorem 3.9 is proved.

Corollary 3.15. *For $i = 0,1,2$,*

(1) *the morphism $\Psi_1 \oplus \Psi_2 \oplus \Psi_{2'} \oplus \Psi_3 \oplus \Psi_{3'}$ induces an isomorphism between $K_iF \oplus K_iF \oplus K_iD \oplus K_iF \oplus K_iD$ and $K_iX_q^{(1)}$,*
(2) *the morphism $\Psi_2 \oplus \Psi_{2'} \oplus \Psi_3 \oplus \Psi_{3'}$ induces an isomorphism between $K_iF \oplus K_iD \oplus K_iF \oplus K_iD$ and $K_iX_q^{(2)}$,*
(3) *the morphism $\Psi'_1 \oplus \Psi'_2 \oplus \Psi'_{2'}$ induces an isomorphism between $K_iF \oplus K_iF \oplus K_iD$ and $K_iX_{q'}^{(1)}$,*
(4) *the morphism $\Psi'_2 \oplus \Psi'_{2'}$ induces an isomorphism between $K_iF \oplus K_iD$ and $K_iX_{q'}^{(2)}$.*
(5) *in the split case, $\Psi_3 \oplus \Psi_{3'}$ induces an isomorphism between $K_iF \oplus K_iD$ and $K_iX_q^{(3)}$.*

Proof: Points 1, 2, 3, 4 and 5 are true in the split case because of (19) and points 1, 2, 3 and 4 directly follow in the general case from corollary 3.14 and the fact that K_iF injects in K_iK. □

3.2. **The group $K_1X_q^{(4)}$.** Let X be a smooth projective variety of dimension d over F. We shall now use the norm map $\mathrm{N}_X^i : H^d(X, \mathcal{K}_{i+d}) \longrightarrow K_iF$. It commutes with the extension of scalars for a field extension and with the norm for finite field extensions.

Proposition 3.16. *The morphism N has the following properties.*

(1) *Let π be the structural morphism of X and $p : H^d(X, \mathcal{K}_{d+i}) \to K_iX$ the morphism given by the B.G.Q. spectral sequence, then $\mathrm{N}_X^i = \pi_* \circ p$.*
(2) *Let ϕ be a quadratic form and L an extension of F such that X_ϕ has an L-rationnal point, then the morphism*

$$\mathrm{N}^1_{X_\phi} : H^d((X_\phi)_L, \mathcal{K}_{d+1}) \longrightarrow K_1L$$

is an isomorphism.
(3) *Let $X(L)$ be the set of L-rationnal points of X. The morphism*

$$\sum \mathrm{N}^1_{L/F} : \bigoplus_{X(L)\neq\varnothing} H^d(X_L, \mathcal{K}_{d+1}) \longrightarrow H^d(X, \mathcal{K}_{d+1})$$

is surjective.

Proof: 1. This is a consequence of the functoriality of the B.G.Q. spectral sequence with respect to proper morphisms. 2. See [2], example 2.3. 3. This can be seen easily on the Gersten complex. □

Let $\mathrm{S}\Gamma(q)$ be the special Clifford group of q and $\mathrm{Spin}(q)$ the Spin group, kernel of the spinorial norm $sn : \mathrm{S}\Gamma(q) \longrightarrow F^*$.

Theorem 3.17. *(see [7], prop. 4.2 and cor. 4.3) The following diagram is commutative and has exact rows and columns.*

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Spin}(q) & \longrightarrow & D^* & \xrightarrow{\mathrm{Nrd}} & F^* \\
 & & \downarrow & & \downarrow & & \| \\
1 & \longrightarrow & \mathrm{S}\Gamma(q) & \longrightarrow & D^* \times F^* & \xrightarrow{\omega} & F^* \\
 & & {\scriptstyle sn}\downarrow & & \downarrow & & \\
 & & F^* & = & F^* & & \\
 & & & & \downarrow & & \\
 & & & & 1 & &
\end{array}$$

where $\omega(d, f) = \mathrm{Nrd}(d)/f^2$.

This diagram is functorial with respect to the extension of scalars, thus the similar diagram of algebraic groups has the same properties.

In [2], Chernousov and Merkurjev define a morphism $\alpha : \mathrm{S}\Gamma(\phi) \longrightarrow A_0(X_\phi, \mathcal{K}_1)$ for any quadratic form ϕ. In our case, q is of dimension 6 so $A_0(X_q, \mathcal{K}_1)$ coincides with $H^4(X_q, \mathcal{K}_5)$. This morphism commutes with the extension of scalars for any field extension and with the norm for finite field extensions.

Proposition 3.18. *([2], prop. 3.5) The morphism α has the property that $\mathrm{N}^1_{X_q} \circ \alpha = sn$.*

For an algebraic group G, let RG denote its subgroup of R-equivalence (see [2], § 1.1).

Theorem 3.19. *([2], prop. 3.5) The morphism α induces isomorphisms (also denoted α)*

$$\mathrm{S}\Gamma(q)/R\mathrm{Spin}(q) \simeq H^4(X_q, \mathcal{K}_5)$$

and therefore

$$\mathrm{Spin}(q)/R\mathrm{Spin}(q) \simeq \ker \mathrm{N}_{X_q}.$$

Theorem 3.20. *(see [29] or [2], théorème 6.1) The subgroup of R-equivalence of the group $\mathrm{SL}_1(D)$ is $R\mathrm{SL}_1(D) = [D^*, D^*]$.*

The commutative diagram of theorem 3.17 therefore induce an injective morphism

$$\beta : \mathrm{S}\Gamma(q)/R\mathrm{Spin}(q) \longrightarrow K_1 D \oplus K_1 F$$

such that $p_2 \circ \beta = sn$, where $p_2 : K_1 D \oplus K_1 F \to K_1 F$ is the projection on the second factor.

Corollary 3.21. *This gives rise to isomorphisms*

$$\ker(sn : \mathrm{S}\Gamma(q)/R\mathrm{Spin}(q) \longrightarrow K_1 F) \simeq SK_1 D$$

and

$$\ker(\mathrm{N}^1_{X_q} : H^4(X_q, \mathcal{K}_5) \longrightarrow K_1 F) \simeq SK_1 D$$

Let us now use these tools to compute $K_1 X_q^{(4)}$.

Definition 3.22. For $i = 0, 1, 2$, we define the morphism

$$\Theta : K_iD \oplus K_iF \longrightarrow K_iX_q$$

by

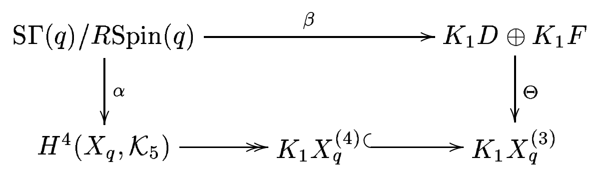
$$\Theta(d, f) = \Psi_{3'}(d) - \Psi_3(f)$$

Remark 3.23. The morphism Θ is injective.

Definition 3.24. For $i = 0, 1, 2$, let VK_iD be the kernel of the morphism

$$\begin{array}{ccc} K_iD \oplus K_iF & \longrightarrow & K_iF \\ (d, f) & \longmapsto & \mathrm{Nrd}(d) - 2f. \end{array}$$

Proposition 3.25. *The diagram*

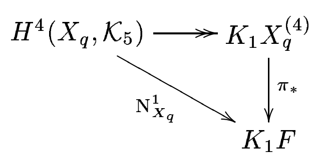


is commutative.

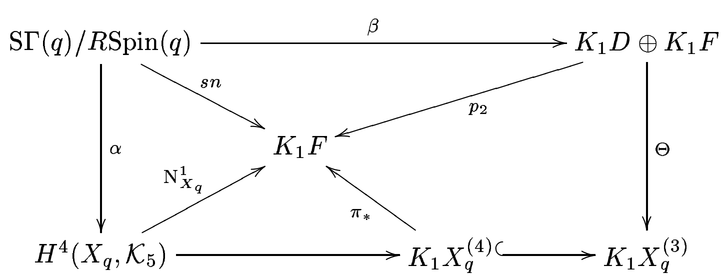
Proof: Proposition 3.16, point 3. and isomorphism $\alpha : \mathrm{S}\Gamma(q)/R\mathrm{Spin}(q) \to H^4(X_q, \mathcal{K}_5)$ - commuting with $\mathrm{N}_{L/F}$ - prove that the morphism

$$\sum \mathrm{N}_{L/F} : \bigoplus_{X_q(L) \neq \varnothing} \mathrm{S}\Gamma(q)/R\mathrm{Spin}(q)_L \longrightarrow \mathrm{S}\Gamma(q)/R\mathrm{Spin}(q)$$

is surjective. Since all the morphisms in the diagram commute to the norm, the theorem can be proved in the case where the quadric is isotropic. We shall suppose so from now on. In the isotropic case, $\mathrm{N}^1_{X_q} : H^4(X_q, \mathcal{K}_5) \longrightarrow K_1F$ is isomorphism. The commutative diagram



therefore shows that π_* induces an isomorphism between $K_1X_q^{(4)}$ and K_1F. Its inverse π_*^{-1} is the morphism $K_1F \overset{\cdot \mathcal{Q}}{\longrightarrow} K_{X_q}$ (recall that $\mathcal{Q}$ is the class of a rationnal point of X_q). In the diagram



all the triangles are commutative, but we still have to understand what happens with the right quadrangle. The norms $\mathrm{N}^1_{X_q}$, sn and the morphism π_* restricted to $K_1X_q^{(4)}$ are isomorphisms, so we have to show that $\pi_*^{-1} \circ p_2 \circ \beta = \Theta \circ \beta$. The image of β in $K_1D \oplus K_1F$ is VK_1D by definition of β and the morphism p_2 restricted to VK_1D is an isomorphism. Let us compute its inverse $((p_2)_{|\mathrm{VK}_1D})^{-1} : K_1F \longrightarrow \mathrm{VK}_1D$.

Lemma 3.26. When D is not a division algebra, the composition $\mathrm{VK}_1D \hookrightarrow K_1D \oplus K_1F \xrightarrow{p_2} K_1F$ has a section $s : K_1F \longrightarrow \mathrm{VK}_1D$.

Proof: Since D - whose degree is 4 - is not a division algebra, it is similar to a quaternion algebra Q. We define $t = \mathrm{M}_{Q,D} \circ \mathrm{I}_{F,Q}$. Thus

$$\begin{aligned}\mathrm{Nrd}_D \circ t &= \mathrm{Nrd}_D \circ \mathrm{M}_{Q,D} \circ \mathrm{I}_{F,Q} \\ &= \mathrm{Nrd}_Q \circ \mathrm{I}_{F,Q} \\ &= \deg(Q)\mathrm{Id}_{K_1F} \\ &= 2\mathrm{Id}_{K_1F}\end{aligned}$$

The morphism $s = (t, \mathrm{Id}) : K_1F \longrightarrow K_1D \oplus K_1F$ factors through VK_1D and is therefore the desired section. □

This section has to be $((p_2)_{|\mathrm{VK}_1D})^{-1}$ (since it is an isomorphism).

To check that $\pi_*^{-1} \circ p_2 \circ \beta = \Theta \circ \beta$, it is then sufficient to prove that $\pi_*^{-1} = \Theta \circ ((p_2)_{|\mathrm{VK}_1D})^{-1}$. We have the following equalities:

$$\begin{aligned}\Theta \circ ((p_2)_{|\mathrm{VK}_1D})^{-1}(k) &= \Psi_{3'} \circ \mathrm{M}_{Q,D} \circ \mathrm{I}_{F,Q}(k) - \Psi_3(k) \\ &= \Psi_{3'} \circ \mathrm{M}_{Q,D} \circ \mathrm{I}_{F,Q}(k.[F]) - \Psi_3(k.[F]) \\ &= \Psi_{3'}(k.\mathrm{M}_{Q,D} \circ \mathrm{I}_{F,Q}([F])) - \Psi_3(k.[F]) \\ &= k.(\Psi_{3'} \circ \mathrm{M}_{Q,D} \circ \mathrm{I}_{F,Q}([F]) - \Psi_3([F]))\end{aligned}$$

We shall now show that $\Psi_{3'} \circ \mathrm{M}_{Q,D} \circ \mathrm{I}_{F,Q}([F]) - \Psi_3([F])$ is the class in K_0X_q of a rationnal point. Since the extension of scalars is injective on K_0X_q, we can extend the scalars to an extension E of F such that X_q is split - and therefore so is D. Then

$$\begin{aligned}&\mathrm{Ext}_{E/F}(\Psi_{3'} \circ \mathrm{M}_{Q,D} \circ \mathrm{I}_{F,Q}([F]) - \Psi_3([F])) \\ &= \Psi_{3'} \circ \mathrm{M}_{Q_E,D_E} \circ \mathrm{I}_{E,Q_E} \circ \mathrm{Ext}_{E/F}([F]) - \Psi_3 \circ \mathrm{Ext}_{E/F}([F]) \\ &= \Psi_{3'} \circ \mathrm{M}_{Q_E,D_E} \circ \mathrm{I}_{E,Q_E}([E]) - \Psi_3([E]) \\ &= \Psi_{3'} \circ 2\mathrm{M}_{E,D_E}([E]) - \Psi_3([E]) \\ &= 2\Psi_{3'} \circ \mathrm{M}_{E,D_E}([E]) - \Psi_3([E])\end{aligned}$$

and in the split case, we already now that $\Psi_{3'} \circ \mathrm{M}_{F,D}$ is the cup-product by $\mathcal{D}$ and that Ψ_3 is the cup-product by $\mathcal{H}^3$ (see (19)). As $2\mathcal{D} - \mathcal{H}^3 = \mathcal{Q}$, we get

$$\begin{aligned}\Theta \circ ((p_2)_{|\mathrm{VK}_1D})^{-1}(k) &= k.\mathcal{Q} \\ &= \pi_*^{-1}(k)\end{aligned}$$

This ends the proof of proposition 3.25. □

Corollary 3.27. *The morphism* Θ *induces an isomorphism* $\mathrm{VK}_1D \to K_1X_q^{(4)}$.

Proof: This follows from the fact that α is an isomorphism and that β and Θ are injective. □

Corollary 3.28. *The morphism* $H^4(X_q, \mathcal{K}_5) \to K_1X_q^{(4)}$ *is an isomorphism and the differential* $d_2^{2,-4}$ *is zero in the B.G.Q. spectral sequence.*

We shall now prove the result for which we needed corollary 3.28.

Proposition 3.29. *Let* $X = X_q$ *or* $X = X_{q'}$.

(1) *In the B.G.Q. spectral sequence for* X, *the differential* $d_2^{0,-3}$ *is zero.*
(2) $K_2X^{2/3} \simeq H^2(X, \mathcal{K}_4)$.

Proof: Point 2 is a consequence of point 1 and, for X_q, corollary 3.28 ($d_2^{2,-4}$ is trivialy zero for $X_{q'}$). Let us therefore prove point 1. Since all the differentials are killed by 4 by a transfer argument, we can and will assume that all the groups are localized at the prime 2. The coniveau spectral sequence in étale motivic cohomology in weight 3 gives the surjection

$$H^3_{ét}(X, \mathbf{Z}(3)) \twoheadrightarrow H^0(X, \mathcal{K}_3^M)$$

The spectral sequence defined in [6] theorem 4.4 - that we have already used in section 1 - yields the exact sequence - here is the place where we use the localization at 2 -

$$0 \longrightarrow H^3_{ét}(F, \mathbf{Z}(3)) \longrightarrow H^3_{ét}(X, \mathbf{Z}(3)) \longrightarrow H^1_{ét}(F, \mathbf{Z}(2)) \longrightarrow 0$$

in which $H^3_{ét}(F, \mathbf{Z}(3)) \simeq K_3^M(F)$ and $H^1_{ét}(F, \mathbf{Z}(2)) \simeq K_3(F)_{ind}$. This exact sequence is split by a section given by the multiplication by a hyperplane section h. The diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^3_{ét}(F, \mathbf{Z}(3)) & \longrightarrow & H^3_{ét}(X, \mathbf{Z}(3)) & \overset{s}{\rightleftarrows} & H^1_{ét}(F, \mathbf{Z}(2)) & \longrightarrow & 0 \\
 & & \wr\uparrow & & \downarrow p & & & & \\
 & & K_3^M(F) & \xrightarrow{f} & H^0(X, \mathcal{K}_3^M) & & & &
\end{array}$$

is commutative. The morphism p is a localisation, hence $p \circ s = 0$ for h is zero at the generic point. The top row is exact, so by diagram chase, the morphism $K_3^M(F) \longrightarrow H^0(X_q, \mathcal{K}_3^M)$ is surjective. Again, a diagram chase in the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_3^M(F) & \longrightarrow & K_3(F) & \longrightarrow & K_3(F)_{nd} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow g & & \downarrow\wr & & \\
0 & \longrightarrow & H^0(X, \mathcal{K}_3^M) & \longrightarrow & H^0(X, \mathcal{K}_3) & \longrightarrow & H^0(X, (\mathcal{K}_3)_{nd}) & &
\end{array}$$

yields that the morphism $K_3(F) \longrightarrow H^0(X, \mathcal{K}_3)$ is surjective. Since it factors through $K_3(X)$, its composition with $d_2^{0,-3}$ has to be zero, therefore $d_2^{0,-3}$ is zero. □

4. The group SK_2D

We shall now collect the results obtained in the preceding sections to prove the main result of this article (see introduction). As in section 3, let K be the function field of the Severi-Brauer variety of D.

From corollary 3.15, points 3 and 4, theorem 3.9 and corollary 3.10, we get the commutative diagram

$$\begin{array}{ccc}
K_2F \oplus K_2D & \xrightarrow{\overline{\Psi_2} \oplus \overline{\Psi_{2'}}} & K_2X_q^{(2/3)} \\
\Vert & & \downarrow i^* \\
K_2F \oplus K_2D & \xrightarrow{\overline{\Psi'_2} \oplus \overline{\Psi'_{2'}}} & K_2X_{q'}^{(2/3)}
\end{array}$$

where $\overline{\Psi_2}$, $\overline{\Psi_{2'}}$, $\overline{\Psi'_2}$ and $\overline{\Psi'_{2'}}$ are just the the morphisms Ψ_2, $\Psi_{2'}$, Ψ'_2 and $\Psi'_{2'}$ followed by the projection to the quotient. Since the reduced norm commutes to the extension of scalars and K_2F injects in K_2K, $SK_2D \simeq \ker(K_2D \to K_2D_K)$. We therefore get a commutative diagram

$$\begin{array}{ccc} SK_2D & \xrightarrow{m} & \ker(K_2X_q^{(2/3)} \to K_2(X_q)_K^{(2/3)}) \\ & \searrow^{m'} & \downarrow \\ & & \ker(K_2X_{q'}^{(2/3)} \to K_2(X_{q'})_K^{(2/3)}). \end{array}$$

Furthermore, the top horizontal arrow is surjective because $\overline{\Psi_2} \oplus \overline{\Psi_{2'}}$ is an isomorphism in the split case (see corollary 3.15, point 5). Proposition 3.29 yields that $\ker(K_2X_q^{(2/3)} \to K_2(X_q)_K^{(2/3)}) \simeq \ker(H^2(X_q, \mathcal{K}_4) \to H^2((X_q)_K, \mathcal{K}_4))$ and (4) is an isomorphism between the latter and $\ker(H^5(F, \mathbf{Z}/2) \to H^5(F(q), \mathbf{Z}/2))$ when F contains an algebraically closed subfield. So we already have the exact sequence

$$SK_2D \longrightarrow H^5(F, \mathbf{Z}/2) \longrightarrow H^5(F(q), \mathbf{Z}/2). \tag{21}$$

The following lemma is well known.

Lemma 4.1. Let ϕ and ϕ' be quadratic forms such that ϕ becomes isotropic over $F(\phi')$, then there is an inclusion (inside $H^n(F, \mathbf{Z}/2)$)

$$\ker(H^n(F, \mathbf{Z}/2) \to H^n(F(\phi), \mathbf{Z}/2)) \subset \ker(H^n(F, \mathbf{Z}/2) \to H^n(F(\phi'), \mathbf{Z}/2))$$

This yields

$$\ker(H^5(F, \mathbf{Z}/2) \to H^5(F(q), \mathbf{Z}/2)) \subset \ker(H^5(F, \mathbf{Z}/2) \to H^5(F(q'), \mathbf{Z}/2))$$

and therefore

$$\ker(K_2X_q^{(2/3)} \to K_2(X_q^{(2/3)})_K) \hookrightarrow \ker(K_2X_{q'}^{(2/3)} \to K_2(X_{q'})_K^{(2/3)}).$$

Thus, the morphisms m and m' have the same kernel.

Lemma 4.2. For $i = 0, 1, 2$, let $p_1 : K_iF \oplus K_iD \to K_iF$ be the projection on the first factor. Then the composition

$$H^3(X_{q'}, \mathcal{K}_{3+i}) \twoheadrightarrow K_iX_{q'}^{(3)} \hookrightarrow K_iX_{q'}^{(2)} \xrightarrow{(\Psi'_2 \oplus \Psi'_{2'})^{-1}} K_iF \oplus K_iD \xrightarrow{p_1} K_iF$$

is minus the norm map $\mathrm{N}_{X_{q'}}$. It becomes an isomorphism in the split case.

Proof: This can be checked after extension the scalars to K. Using proposition 3.16, point 1, the result follows from equalities (20), $(\mathcal{H}')^2 = 2\mathcal{D}' - \mathcal{Q}'$, $\pi_*(\mathcal{D}') = 0$ and $\pi_*(\mathcal{Q}) = [F]$. The norm becomes an isomorphism in the split case because the B.G.Q. spectral sequence degenerates, thus $H^{3+i}(X_{q'}, \mathcal{K}_i) \simeq K_iX_{q'}^{(3)}$, and $K_iX_{q'}^{(3)} \simeq K_iF$ by the map given above (see the proof of proposition 3.12). □

Since the norm $N_{X_{q'}}$ is an isomorphism in the split case and since K_2F injects in K_2K, we can identify $\ker \mathrm{N}_{X_{q'}}$ with $\ker(H^3(X_{q'}, \mathcal{K}_5) \to H^3((X_{q'})_K, \mathcal{K}_5))$, and we get the diagram with exact rows

$$\begin{array}{ccccccccc} & & H^3(X_{q'}, \mathcal{K}_5) & \longrightarrow & K_2X_{q'}^{(2)} & \longrightarrow & K_2X_{q'}^{(2/3)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^3((X_{q'})_K, \mathcal{K}_5) & \longrightarrow & K_2(X_{q'})_K^{(2)} & \longrightarrow & K_2(X_{q'})_K^{(2/3)} & \longrightarrow & 0. \end{array}$$

The kernels of the vertical maps are therefore related through an exact sequence

$$\ker \mathrm{N}_{X_{q'}} \longrightarrow SK_2D \xrightarrow{m'} \ker(K_2X_{q'}^{(2/3)} \to K_2(X_{q'})_K^{(2/3)})$$

which, pasted to the sequence (21), gives rise to the desired exact sequence.

Remark 4.3. By the same method applied to K_1, Rost's theorem can be established (in this case, $\ker \mathrm{N}_{X_{q'}}$ is zero, by another theorem of Rost).

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