Traces in oriented homology theories of algebraic varieties

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1 Introduction

The present preprint is the first one in a short series of preprints. The main aim of this series is to give a proof of "the only non-trivial result on algebraic cobordism" stated by V.Voevodsky in [V, Th. 3.22]. In this preprint we introduce the notion of oriented homology theory on algebraic varieties following the outline of [PS], [P1] and [P2].

So in this preprint we consider a field k and the category of pairs (X, U) with a smooth variety X over k and its open subset U. By a homology theory we mean a covariant functor A from this category to the category of abelian groups endowed with a functor transformation $\partial : A(X, U) \to A(U)$ and satisfying the localization, Nisnevich excision and homotopy invariance properties (2.0.1).

We consider three structures a homology theory A can be equipped with: *an orientation on A*, *a Thom structure on A* and *a Chern structure on A*. We describe relations of these structures to each other. These should be considered as a preliminary to construct trace structure on an oriented homology theory. The proper construction of the trace structure is postponed to the next preprint.

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1.1 Terminology and notation

Let k be a field. The term "variety" is used in this text to mean a reduced quasi-projective scheme over k. If X is a variety and $U \subset X$ is a Zariski open then $Z := X - U$ is considered as a closed subscheme with a unique structure of the reduced scheme, so Z is considered as a closed subvariety of X. We fix the following notation:

- $\mathcal{A}b$ - the category of abelian groups;
- $\mathcal{S}m$ - the category of smooth varieties;

  $\mathcal{S}m\mathcal{O}p$ - the category of pairs (X, U) with smooth X and open U in X. Morphisms are morphisms of pairs.

  We identify the category $\mathcal{S}m$ with a full subcategory of $\mathcal{S}m\mathcal{O}p$ assigning to a variety X the pair $(X, \emptyset)$;

- $pt = \mathrm{Spec}(k)$;

  For a smooth X and an effective divisor $D \subset X$ we write $L(D)$ for a line bundle over X whose sheaf of sections is the sheaf $\mathcal{L}_X(D)$ (see [Har, Ch.II,§6, 6.13]).

  $\mathbf{P}(V) = Proj(S^*(V^\vee))$ - the space of lines in a finite dimensional k-vector space V;

  $L_V = \mathcal{O}_V(-1)$ - the tautological line bundle over $\mathbf{P}(V)$;

  $\mathbf{1}_X$ - the trivial rank one bundle over X, often we will write $\mathbf{1}$ for $\mathbf{1}_X$;

- $\mathbf{P}(E)$ - the space of lines in a vector bundle E;

  $L_E = \mathcal{O}_E(-1)$ - the tautological line bundle on $\mathbf{P}(E)$;

  E^0 - the complement to the zero section of E;

  $E^\vee$ - the vector bundle dual to E;

  $z : X \to E$ - the zero section of a vector bundle E;

2 Homology theories

2.0.1 Definition. *A homology theory is a covariant functor $\mathcal{S}m\mathcal{O}p \xrightarrow{A_*} \mathcal{A}b$ together with a functor morphism $\partial : A_*(X,U) \to A_*(U)$ satisfying the following properties*

1. *Localization: the sequence $A_*(U) \xrightarrow{j_*} A(X) \xrightarrow{i_*} A_*(X,U) \xrightarrow{\partial_P} A_*(U) \xrightarrow{j_*} A(X)$ is exact for each pair $P = (X,U) \in \mathcal{S}m\mathcal{O}p$, where $j : U \hookrightarrow X$ and $i : (X,\emptyset) \hookrightarrow (X,U)$ are the natural inclusions;*

2. *Excision: the operator $A_*(X',U') \to A_*(X,U)$ induced by a morphism $e : (X',U') \to (X,U)$ is an isomorphism, if the morphism e is etale and for $Z = X - U$, $Z' = X' - U'$ one has $e^{-1}(Z) = Z'$ and $e : Z' \to Z$ is an isomorphism;*

3. *Homotopy invariance: the operator $A_*(X \times \mathbf{A}^1) \to A_*(X)$ induced by the projection $X \times \mathbf{A}^1 \to X$ is an isomorphism.*

The operator ∂_P is called the boundary operator and is written usually as ∂. Let us stress that the functor takes values in the category of abelian groups rather than in the category of graded abelian groups. The subscript $*$ in A_* is used only to stress that we work with a covariant functor.

A morphism of homology theories $\varphi : (A_*, \partial_{A_*}) \to (B_*, \partial_{B_*})$ is a functor transformation $\varphi : A_* \to B_*$ commuting with the boundary morphisms in the sense that for every pair $P = (X,U) \in \mathcal{S}m\mathcal{O}p$ one has $\partial^P_{B_*} \circ \varphi_P = \varphi_U \circ \partial^P_{A_*}$.

We write also $A^Z_*(X)$ for $A_*(X,U)$, where $Z = X - U$, and call the group $A^Z_*(X)$ homology of X with the support on Z. The operator

$$A_*(X) \xrightarrow{i_*} A^Z_*(X) \tag{1}$$

is called the support restriction operator for the pair (X,U).

Below in the text we will often denote a support restriction operaror appearing in various contexts by β.

2.1 General properties of homology theories

We specify here certain properties of an arbitrary homology theory A_* which are useful below in the text.

2.1.1. The localization property implies that $A_*^{\emptyset}(X) = A_*(X,X) = 0$. Therefore $A_*(\emptyset) = A_*^{\emptyset}(\emptyset) = 0$.

2.1.2. If any two of morphisms $(X,U) \to (Y,V)$, $X \to Y$, $U \to V$, defined by a morphism $f : (X,U) \to (Y,V)$, induce isomorphisms on the level of A_* then the third of these morphisms induces an isomorphism on the level of A_*.

2.1.3. Localization sequence for a triple. Let $T \subset Y \subset X$ be closed subsets of a smooth variety X. Let $\partial : A_*^T(X) \to A_*(X-T)$ be the boundary map for the pair $(X, X-T)$. Consider the support restriction map $e_* : A_*(X-T) \to A_*^{Y-T}(X-T)$ and set $\partial_{Y,T} = e_* \circ \partial : A_*^T(X) \to A_*^{Y-T}(X-T)$.

We claim that the sequence

$$\ldots \to A_*^Y(X) \xrightarrow{\alpha} A_*^T(X) \xrightarrow{\partial} A_*^{Y-T}(X-T) \xrightarrow{\beta} A_*^Y(X) \xrightarrow{\alpha} A_*^T(X) \to \ldots$$

with the obvious mappings α and β is a complex and moreover it is exact. We call this sequence the localization sequence for the triple $(X, X-T, X-Y)$. If $Y = X$, then this sequence coincides with the localization sequence for the pair $(X, X-T)$.

2.1.4. Mayer-Vietoris sequence. If $X = U_1 \cup U_2$ is a union of two open subsets U_1 and U_2 and if Y is a closed subset in X, then set $T_i = Y - U_i$, $Y_i = Y \cap U_i$, $U_{12} = U_1 \cap U_2$ $Y_{12} = U_{12} \cap Y$. Consider the morphism of the localization sequences for the triples $X \supset Y \supset T_1$ and $U_2 \supset Y_2 \supset T_1$ induced by the inclusion of the triples $(U_2, U_2 - T_1, U_2 - Y_2) \subset (X, X - T_1, X - Y)$

$$\begin{array}{ccccccc}
A_*^{Y_{12}}(U_2 - T_2) & \xrightarrow{\beta_2} & A_*^{Y_2}(U_2) & \longrightarrow & A_*^{T_1}(U_2) & \xrightarrow{\partial} & A_*^{Y_{12}}(U_2 - T_2) \\
\downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \gamma} & & \downarrow \\
A_*^{Y_1}(X - T_1) & \xrightarrow{\alpha_1} & A_*^{Y}(X) & \xrightarrow{e_*} & A_*^{T_1}(X) & \xrightarrow{\partial} & A_*^{Y_1}(X - T_1).
\end{array}$$

The map γ is an isomorphism by the excision property. Also by excision property we may identify $A_*^{Y_{12}}(U_2 - T_2)$ with $A_*^{Y_{12}}(U_{12})$ and $A_*^{Y_1}(X - T_1)$ with $A_*^{Y_1}(U_1)$.

Set $d = \partial \circ \gamma^{-1} \circ e_* : A_*^Y(X) \to A_*^{Y_{12}}(U_{12})$.

We claim that the sequence

$$\ldots \to A_*^Y(X) \xrightarrow{d} A_*^{Y_{12}}(U_{12}) \xrightarrow{(\beta_1, -\beta_2)} A_*^{Y_1}(U_1) \oplus A_*^{Y_2}(U_2) \xrightarrow{\alpha_1 + \alpha_2} A_*^Y(X) \to \ldots$$

is exact and call this sequence *the Mayer-Vietoris sequence* of the open covering $X = U_1 \cap U_2$. The proof of the exactness is straightforward and we skip it.

The Mayer-Vietoris sequence is natural in the following sence.If $f : X' \to X$ is a morphism and $X' = U_1' \cup U_2'$ is a Zariski covering of X' such that $f(U_i') \subset U_i$ and if Y' is a closed subset in X' containing $f^{-1}(Y)$, then the mappings $f_* : A_*^{Y'}(X') \to A_*^Y(X)$, $f_* : A_*^{Y_i'}(U_i') \to A_*^{Y_i}(U_i)$, $f_* : A_*^{Y_{12}'}(U_{12}') \to A_*^{Y_{12}}(U_{12})$ form a morphism of the corresponding Mayer-Vietoris sequences.

2.1.5. Let $i_r : X_r \hookrightarrow X_1 \amalg X_2$ be the natural inclusion ($r = 1, 2$). Let $Y_r \subset X_r$ be a closed subset for ($r = 1, 2$) Then the induced map $A_*^{Y_1}(X_1) \oplus A_*^{Y_2}(X_2) \to A_*^{Y_1 \amalg Y_2}(X_1 \amalg X_2)$ is an isomorphism.

Proof. This follows from the Mayer-Vietoris property and the fact that $A_*^{\emptyset}(\emptyset) = 0$.

2.1.6. Strong homotopy invarance. Let $p : T \to X$ be an affine bundle (i.e., a torsor under a vector bundle). Let $Z \subset X$ be a closed subset and let $S = p^{-1}(Z)$. Then the natural map $p_* : A_*^S(T) \to A_*^Z(X)$ is an isomorphism. If $s : X \to T$ is a section then the induced operator $s_* : A_*^Z(X) \to A_*^S(T)$ is an isomorphism as well.

Proof. First consider the case $Z = X$. Then $S = T$ and we have to check that the pull-back map $p_* : A_*(T) \to A_*(X)$ is an isomorphism. Choose a finite Zariski open covering $X = \cup U_i$ such that $T_i = p^{-1}(U_i)$ is isomorphic to the trivial vector bundle over each U_i and then use the morphism of the Mayer-Vietoris sequences and the homotopy invariance property of the homology theory A_*.

To prove the general case consider the localization sequences for the pairs $(X, X - Z)$ and $(T, T - S)$. The natural mappings form a morphism of these two long exact sequences. The 5-Lemma completes the proof.

2.1.7. Deformation to the normal cone. The deformation to the normal cone is a well-known construction (for example, see [Fu]). Since the construction and its property (3) play an important role in what follows we give here some details.

Let $i : Y \hookrightarrow X$ be a closed imbedding of smooth varieties with the normal bundle N. There exists a smooth variety X_t together with a smooth morphism $p_t : X_t \to \mathbf{A}^1$ and a closed imbedding $i_t : Y \times \mathbf{A}^1 \hookrightarrow X_t$ such that the map $p_t \circ i_t$ coincides with the projection $Y \times \mathbf{A}^1 \to \mathbf{A}^1$ and

- the fiber of p_t over $1 \in \mathbf{A}^1$ is canonically isomorphic to X and the base change of i_t by means of the imbedding $1 \hookrightarrow \mathbf{A}^1$ coincides with the imbedding $i : Y \hookrightarrow X$;
- the fiber of p_t over $0 \in \mathbf{A}^1$ is canonically isomorphic to N and the base change of i_t by means of the imbedding $0 \hookrightarrow \mathbf{A}^1$ coincides with the zero section $Y \hookrightarrow N$.

Thus we have the diagram

$$(N, N - Y) \xrightarrow{i_0} (X_t, X_t - Y \times \mathbf{A}^1) \xleftarrow{i_1} (X, X - Y) \tag{2}$$

Here and further we identify a variety with its image under the zero section of any vector bundle over this variety.

Let us recall a construction of X_t, p_t and i_t. For that take X_t' to be the blow-up of $X \times \mathbf{A}^1$ with the center $Y \times \{0\}$. Set $X_t = X_t' - \tilde{X}$ where $\tilde{X}$ is the the proper preimage of $X \times \{0\}$ under the blow-up map. Let $\sigma : X_t \to X \times \mathbf{A}^1$ be the restriction of the blow-up map $\sigma' : X_t' \to X \times \mathbf{A}^1$ to X_t and set p_t to be the composition of σ and the projection $X \times \mathbf{A}^1 \to \mathbf{A}^1$.

The proper preimage of $Y \times \mathbf{A}^1$ under the blow-up map is mapped isomorphically to $Y \times \mathbf{A}^1$ under the blow-up map. Thus the inverse isomorphism gives the desired imbedding $i_t : Y \times \mathbf{A}^1 \hookrightarrow X_t$ (observe that $i_t(Y \times \mathbf{A}^1)$ does not cross $\tilde{X}$).

It's not difficult to check that the imbedding i_t satisfies the mentioned two properties (the preimage of $X \times 0$ under the map σ' consists of two irreducible components: the proper preimage of X and the exceptional divizor $\mathbb{P}(N \oplus 1)$. Their intersection is $\mathbb{P}(N)$ and $i_t(Y \times \mathbb{A}^1)$ crosses $\mathbb{P}(N \oplus 1)$ along $\mathbb{P}(1)$ = the zero section of the normal bundle N).

We claim that the diagram (2) consists of isomorphisms on the level of A_*. We will not here give the proof of this theorem because the proof is straightforward analogous to one given in cohomological context in [P2] (c.f. Theorem **2.2.8**).

2.1.8 Theorem. *The following diagram consists of isomorphisms*

$$A_*^Y(N) \xrightarrow{i_*^0} A_*^{Y \times \mathbf{A}^1}(X_t) \xleftarrow{i_*^1} A_*^Y(X). \tag{3}$$

Moreover for each closed subset $Z \subset Y$ the following diagram consists of isomorphisms as well

$$A_*^Z(N) \xrightarrow{i_*^0} A_*^{Z \times \mathbf{A}^1}(X_t) \xleftarrow{i_*^1} A_*^Z(X). \tag{4}$$

2.1.9 Corollary. *Let $j_0 : \mathbf{P}(1 \oplus N) \hookrightarrow X_t'$ be the imbedding of the exceptional divisor into X_t' and let $j_1 = e_t \circ i_1 : X \hookrightarrow X_t'$, where $e_t : X_t \hookrightarrow X_t'$ is the open inclusion. Then the mapping*

$$A_*^{\mathbf{P}(1)}(\mathbf{P}(1 \oplus N)) \xrightarrow{j_*^0} A_*^{Y \times \mathbf{A}^1}(X_t') \tag{5}$$

is an isomorphism.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} A_*^Y(N) & \xrightarrow{i_*^0} & A_*^{Y \times \mathbf{A}^1}(X_t) \\ {\scriptstyle e_*}\downarrow & & \downarrow{\scriptstyle e_*^t} \\ A_*^{\mathbf{P}(1)}(\mathbf{P}(1 \oplus N)) & \xrightarrow{j_*^0} & A_*^{Y \times \mathbf{A}^1}(X_t') \end{array}$$

where the vertical arrows are the natural mappings. These vertical arrows are isomorphisms by the excision property. The operator i_*^0 is an ismomorphism by the first item of Theorem 2.1.8. Thus the operator j_*^0 is an isomorphism. □

2.1.10. Let X be a smooth variety and let L be a line bundle over X. Let $E = 1 \oplus L$ and let $\bar{i}_L : X = \mathbf{P}(L) \hookrightarrow \mathbf{P}(E)$ be the closed imbedding induced by the direct summand L of E. Let $A_*(\mathbf{P}(E)) \xrightarrow{\beta} A_*^{\mathbf{P(1)}}(\mathbf{P}(E))$ be the support restriction operator and let $\bar{i}_*^L : A_*(\mathbf{P}(L)) \to A_*(\mathbf{P}(E))$ be the natural mapping. We claim that the following sequence

$$0 \to A_*(\mathbf{P}(L)) \xrightarrow{\bar{i}_*^L} A_*(\mathbf{P}(E)) \xrightarrow{\beta} A_*^{\mathbf{P(1)}}(\mathbf{P}(E)) \to 0. \tag{6}$$

is exact.

To prove this consider $U = \mathbf{P}(E) - \mathbf{P}(1)$ with the open inclusion $j : U \hookrightarrow \mathbf{P}(E)$ and observe that U becomes a line bundle over X by means of the linear projection $q : U \to \mathbf{P}(L) = X$ (the line bundle is isomorphic to $L^\vee$) The obvious inclusion i_L :

$\mathbf{P}(L) \hookrightarrow U$ is just the zero section of this line bundle, $\bar{i}_L = j \circ i_L$ and the natural operator $i^L_* : A(U) \to A(\mathbf{P}(L))$ is an isomorphism (the inverse to the one q_*).

Now consider the pair $(\mathbf{P}(E), U)$. By the localization property 2.0.1 the following sequence

$$\ldots \to A_*(U) \xrightarrow{j_*} A_*(\mathbf{P}(E)) \xrightarrow{\beta} A_*^{\mathbf{P}(\mathbf{1})}(\mathbf{P}(E)) \to \ldots$$

is exact. If $\mathbf{P}(E) \xrightarrow{p} X$ is the natural projection then the operator $i^L_* \circ p_* : A_*(\mathbf{P}(E)) \to A_*(U)$ splits j_*. This implies the surjectivity of j_* and the injectivity of β in the long sequence above. To proof that the sequence (6) is short exact it remains to recall that the operator i^L_* is an isomorpism and $\bar{i}_L = j \circ i_L$.

2.1.11. We use here the notation from 2.1.7. Let $e_t : X_t \hookrightarrow X'_t$ be the open inclusion and let $p : \mathbf{P}(1 \oplus N) \to Y$ be the projection and let $s : Y \to \mathbf{P}(1 \oplus N)$ be the section of the projection identifying Y with the subvariety $\mathbf{P}(1)$ in $\mathbf{P}(1 \oplus N)$. The following commutative diagram will be repeatedly used below in the text

$$\begin{array}{ccccc}
\mathbf{P}(1 \oplus N) & \xrightarrow{j_0} & X'_t & \xleftarrow{j_1} & X \\
{\scriptstyle s}\uparrow & & {\scriptstyle I_t}\uparrow & & \uparrow{\scriptstyle i} \\
Y & \xrightarrow{k_0} & Y \times \mathbf{A}^1 & \xleftarrow{k_1} & Y,
\end{array}$$

where $I_t = e_t \circ i_t$ and j_0 is the inclusion of the exceptional divisor and $j_1 = e_t \circ i_1$ and k_0, k_1 are the closed imbedding given by $y \mapsto (y, 0)$ and $y \mapsto (y, 1)$ respectively.

2.1.12 Lemma (Useful lemma). *Under the notation from 2.1.7 let $j_t : V_t = X'_t - Y \times \mathbf{A}^1 \to X'$ be the inclusion. If the support restriction operator $A_*(\mathbf{P}(1 \oplus N)) \to A_*^{\mathbf{P}(1)}(\mathbf{P}(1 \oplus N))$ is surjective then*

$$Im(j^0_*) + Im(j^t_*) = A_*(X'_t),$$

in the other words the operator

$$j^0_* \oplus j^t_* : A_*(\mathbf{P}(1 \oplus N)) \oplus A_*(V_t) \to A_*(X'_t)$$

is an epimorphism. In particular this holds if Y is a divisor on X.

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
A_*(\mathbf{P}(1 \oplus N)) & \xrightarrow{j^0_*} & A_*(X'_t) \\
{\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \beta_t} \\
A_*^{\mathbf{P}(1)}(\mathbf{P}(1 \oplus N)) & \xrightarrow{j^0_*} & A_*^{Y \times \mathbf{A}^1}(X'_t).
\end{array}$$

where β and β_t are the support restriction operators. The bottom operator j^0_* is an isomorphism by Corollary 2.1.9. The map β is surective by the very assumption (if Y is a divisor in X then α is surjective by 2.1.10). Since the composition $\beta_t \circ j^0_*$ coincides with the one $j^0_* \circ \beta$ it is surjective as well.

The localization sequence for the pair (X'_t, V_t) shows that $Im(j^t_*) = Ker(\beta_t)$. The Lemma follows. $\square$

2.1.13. Let $i : \mathbf{P}(V) \hookrightarrow \mathbf{P}(W)$ and $j : \mathbf{P}(V) \hookrightarrow \mathbf{P}(W)$ be two linear imbeddings (imbeddings induced by linear imbeddings V into W). If the dimension of V is strictly less than the dimension of W, then $i_* = j_* : A_*(\mathbf{P}(V)) \to A_*(\mathbf{P}(W))$.

In fact, in this case there exists a linear automorphism ϕ of W which has the determinant 1 and such that $j = \phi \circ i$. Since ϕ is a composite of elementary matrices and each elementary matrix induces the identity automorphism $A(\mathbf{P}(W))$ (by the homotopy invariance of A) one gets the relation $\phi_* = id$. Therefore $j_* = \phi_* \circ i_* = i_*$.

2.2 Chern and Thom structures on A

In this section A_* is a homology theory. If X is a smooth variety we write $\mathbf{1}_X$ for the trivial rank one bundle over X. Often we will just write $\mathbf{1}$ for $\mathbf{1}_X$ if it is clear from a context what the variety X is.

2.2.1 Definition. *A Chern structure on A_* is an assignment which associate to each smooth X, closed subset $Z \subset X$ and each line bundle L/X a homomorphism $e^Z(L) : A_*^Z(X) \to A_*^Z(X)$ satisfying the following properties*

1. *functoriality:*

   - *Let $\varphi : (X', U') \to (X, U)$ be a morphism of varieties, $Z = X - U$, $Z' = X' - U'$ and L be a linear bundle on X. Then the following diagram commutes*

$$\begin{array}{ccc} A_*^{Z'}(X') & \xrightarrow{e^{Z'}(\varphi^* L)} & A_*^{Z'}(X') \\ {\scriptstyle\varphi_*}\downarrow & & {\scriptstyle\varphi_*}\downarrow \\ A_*^{Z}(X) & \xrightarrow{e^{Z}(L)} & A_*^{Z}(X). \end{array}$$

   - *Chern homomorphisms commutes with ∂ from a long localization sequence.*
   - *$e^Z(L_1) = e^Z(L_2)$ for isomorphic line bundles L_1 and L_2;*

2. *nondegeneracy: the operator $(p_*, p_* \circ e) : A_*(X \times \mathbf{P}^1) \to A_*(X) \oplus A_*(X)$ is an isomorphism where $e = e(q^*\mathcal{O}(-1))$, $\mathcal{O}(-1)$ is the tautological line bundle on $\mathbf{P}^1$, $q : X \times \mathbf{P}^1 \to \mathbf{P}^1$ and $p : X \times \mathbf{P}^1 \to X$ are projections;*

3. *vanishing: $e^Z(\mathbf{1}_X)$ is a null homomorphism for any pair $Z \subset X$.*

The homomorphisms $e^Z(L)$ will be called Chern homomorphisms of the line bundle L. (It will be proved below in 2.3.9 that these homomorphisms are nilpotent).

if $Z = X$ the superscript in $e^Z(L)$ will be omitted.

2.2.2 Lemma. *Let L be a tautological line bundle $\mathcal{O}(-1)$ over $X \times \mathbf{P}^1$. Then one has the following relations:*

- *$e(L)^2 = e(L)e(L^\vee) = 0$;*

- $e(L^\vee) = -e(L) \in Hom(A_*(X \times \mathbf{P}^1), A_*(X \times \mathbf{P}^1))$.

Proof. To prove the first assertion we show that for any two line bundle L_1 and L_2 over $\mathbf{P}^1$ one has $e(q^*L_2)e(q^*L_1) = 0$ where $q : X \times \mathbf{P}^1 \to \mathbf{P}^1$ is a projection.

Fix two points $\{0\}, \{\infty\} \in \mathbf{P}^1$ and consider a commutative diagram

$$\begin{array}{ccccc} A_*(X \times \mathbf{P}^1) & \xrightarrow{\beta} & A_*^{X\times\{0\}}(X \times \mathbf{P}^1) & \xrightarrow{\gamma} & A_*^{X\times\{0\}}(X \times (\mathbf{P}^1 - \{\infty\})) \\ {\scriptstyle e(q^*L_1)}\downarrow & & {\scriptstyle e^{X\times\{0\}}(q^*L_1)}\downarrow & & {\scriptstyle e^{X\times\{0\}}(q^*L_1)}\downarrow \\ A_*(X \times \mathbf{P}^1) & \xrightarrow{\beta} & A_*^{X\times\{0\}}(X \times \mathbf{P}^1) & \xrightarrow{\gamma} & A_*^{X\times\{0\}}(X \times (\mathbf{P}^1 - \{\infty\})) \end{array}$$

where β is a support restriction and γ is an excision isomorphism. The right vertical arrow is null because $q^*L_1|_{X\times\mathbf{A}^1}$ is a trivial line bundle. Therefore the middle vertical arrow is null. We get $\beta \circ e(q^*L_1) = 0$. By 2.1.10) the sequence $0 \to A_*(X) \xrightarrow{s_*} A_*(X \times \mathbf{P}^1) \xrightarrow{\beta} A_*^{X\times\{0\}}(X \times \mathbf{P}^1) \to 0$ is exact.

It implies that $Im(e(q^*L_1)) \subset Im(s_*)$. Since the line bundle $s^*q^*L_2$ is trivial then $e(q^*L_2)e(q^*L_1)(A_*(X \times \mathbf{P}^1)) \subset (e(q^*L_2)s_*)(A_*(X)) = (s_* \circ e(s^*q^*L_2))(A_*(X)) = 0$ and the first part of the lemma is proved.

In the proof of the second part we assume for simplicity that $X = pt$. It can do because all line bundles to be considered below becames trivial being restricted to X.

Let $i_1, i_2 : \mathbf{P}^1 \hookrightarrow \mathbf{P}^1 \times \mathbf{P}^1$ be closed imbeddings, $p_1, p_2 : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ be corresponding projections and $\Delta : \mathbf{P}^1 \hookrightarrow \mathbf{P}^1 \times \mathbf{P}^1$ be a diagonal imbedding. Also let $s : pt \to \mathbf{P}^1 \times \mathbf{P}^1$ be any closed imbedding and $P : \mathbf{P}^1 \times \mathbf{P}^1 \to pt$ be a projection. The projection $\mathbf{P}^1 \to pt$ will be denoted by a small letter p.

We claim the following relation holds:

$$\Delta_* = i_*^1 + i_*^2 - s_*p_*. \tag{7}$$

By property 2 there is an isomorphism $A_*(\mathbf{P}^1 \times \mathbf{P}^1) \longrightarrow A_*(pt) \oplus A_*(pt) \oplus A_*(pt) \oplus A_*(pt)$ given by $(P_*, P_* \circ e(p_1^*L), P_* \circ e(p_2^*L), P_* \circ e(p_1^*L)e(p_2^*L))$.

In order to check the relation 7 we must check four equations:

$$\begin{aligned} P_*\Delta_* &= P_*(i_*^1 + i_*^2 - s_*p_*) \\ P_*e(p_1^*L)\Delta_* &= P_*e(p_1^*L)(i_*^1 + i_*^2 - s_*p_*) \\ P_*e(p_2^*L)\Delta_* &= P_*e(p_2^*L)(i_*^1 + i_*^2 - s_*p_*) \\ P_*e(p_1^*L)e(p_2^*L)\Delta_* &= P_*e(p_1^*L)e(p_2^*L)(i_*^1 + i_*^2 - s_*p_*). \end{aligned}$$

The first one is obvious. To prove the other three ones it is enough to observe that

$$e(p_r^*L)\Delta_* = \Delta_*e(L); \quad e(p_r^*L)i_*^r = i_*^r e(L); \quad e(p_r^*L)i_*^k = 0 \quad \text{for} \quad k \neq r.$$

Then both hands in the second and third equations are equal to $p_*e(L)$ and both hands in the fourth equation are zero.

Now we are ready to complete the proof of this lemma. Consider a line bundle $M = p_1^*L \otimes p_2^*L^\vee$ over $\mathbf{P}^1 \times \mathbf{P}^1$. Then $i_1^*M = L$; $\quad i_2^*M = L^\vee$; $\quad \Delta^*M = \mathbf{1}_{\mathbf{P}^1}$.

By equation 7 one has $P_* e(M)\Delta_* = P_* e(M)(i^1_* + i^2_* - s_* p_*)$. Computing left and right hand we get $0 = p_*(e(L) + e(L^\vee))$.

To prove an equality $u = v \in Hom(A_*(\mathbf{P}^1), A_*(\mathbf{P}^1))$ it is sufficient by property 2 in Definition 2.2.1 to prove that $p_* u = p_* v \quad \& \quad p_* e(L) u = p_* e(L) v$.

The first one in our case $(u = -e(L), v = e(L^\vee))$ is just have checked. In the second one both hands are zero because $e(L)^2 = 0$ and $e(L)e(L^\vee) = 0$ by the first assertion of this Lemma.

□

2.2.3 Definition. *If one has a Chern structure $(L, X, Z) \mapsto e^Z(L)$ on a homology theory A_* then the assignment $(L, X, Z) \mapsto (e')^Z(L) = e^Z(L^\vee)$ will be called a dual Chern structure with respect to $\{e\}$.*

We must check that all properties of Chern structure are valid. The only point to prove is the nondegeneracy. It is hold because by Lemma 2.2.2 $e'(\mathcal{O}(-1)) = e(\mathcal{O}(1)) = -e(\mathcal{O}(-1)$.

2.2.4 Definition. *One says that A_* is endowed with a Thom structure if for each smooth variety X, closed subset $Z \subset X$ and each line bundle L/X it is chosen and fixed a homomorphism $th^Z(L) : A^Z_*(L) \to A^Z_*(X)$ satisfying the following properties*

1. *functoriality:*

   Let $Z \subset X$ and $Z' \subset X'$ be closed subsets, L/X be a line bundle over X; $\varphi : X' \to X$ be a morphism such that $\varphi^{-1}(Z) \subset Z'$. Then the following diagram commutes

$$\begin{array}{ccc} A^{Z'}_*(L') & \xrightarrow{\varphi^L_*} & A^Z_*(L) \\ {\scriptstyle th^{Z'}(L')}\downarrow & & {\scriptstyle th^Z(L)}\downarrow \\ A^{Z'}_*(X') & \xrightarrow{\varphi_*} & A^Z_*(X), \end{array}$$

   where $L' = L \times_X X'$ is a line bundle over X' and $\varphi^L : L' \to L$ is a morphism of line bundles induced by φ.

   Homomorphisms $th^Z(L)$ commutes with ∂ from a long localization sequence.

2. *If $\tau : L_1\ to L_2$ is an isomorphism of line bundles over X then for any closed subset $Z \subset X$ one has $th^Z(L_2) \circ \tau_* = th^Z(L_1) \in Hom(A^Z_*(L_1), A^Z_*(X))$.*

3. *nondegeneracy: $th(\mathbf{1}) : A^X_*(X \times \mathbf{A}^1) \to A_*(X)$ is an isomorphism (here X is identified with $X \times \{0\}$).*

2.2.5 Remark. *Chern and Thom homomorphisms commutes with ∂ from the Mayer-Vietoris sequence.*

2.2.6 Lemma. *Any homomorphism $th^Z(L)$ is an isomorphism.*

Proof. The usual arguments using Mayer-Vietoris and long exact sequence for a triple. □

Now we are going to describe a one-to-one correspondence between Chern and Thom structures on A_*.

2.2.7 Lemma. *Suppose that a homology theory theory A_* is equipped with a Chern structure. Then the following assertions hold*

1. *Let $e = e(\mathcal{O}_{\mathbf{P}^n}(-1))$ and $\beta : A_*(\mathbf{P}^n) \to A_*^{\mathbf{P}^{n-1}}(\mathbf{P}^n)$ be a support restriction. Then there exists a unique homomorphism $\widehat{e} : A_*^{\mathbf{P}^{n-1}}(\mathbf{P}^n) \to A_*(\mathbf{P}^n)$ such that $\widehat{e} \circ \beta = e$.*

2. *Let L/X be a line bundle, $Z \subset X$ be a closed subset. Let $p : \mathbf{P}(1 \oplus L) \to X$ be a projection, $s : X \to \mathbf{P}(1 \oplus L)$ be a closed imbedding identifying X with $\mathbf{P}(1)$, $s_L : X \to \mathbf{P}(1 \oplus L)$ be a closed imbedding identifying X with $\mathbf{P}(L)$ and $\beta : A_*^{p^{-1}(Z)}(\mathbf{P}(1 \oplus L)) \to A_*^{s_*(Z)}(\mathbf{P}(1 \oplus L))$ be the support restriction. Consider the line bundle $M = \mathcal{O}_{1\oplus L}(1) \otimes p^*L$ over $\mathbf{P}(1 \oplus L)$. Then*

   - *$s^*M = L$, $s_L^*M = 1_X$;*
   - *there exists a unique homomorphism $\overline{\alpha^Z(L)} : A_*^{s_*(Z)}(\mathbf{P}(1 \oplus L)) \to A_*^Z(X)$ such that $\overline{\alpha^Z(L)} \circ \beta = p_* \circ e(M)$.*

Proof. To prove the first statement consider a long localization sequence for pair $(\mathbf{P}^n, \mathbf{P}^n - \mathbf{P}^{n-1})$. As $A_*(\mathbf{P}^n - \mathbf{P}^{n-1}) = A_*(\mathbf{A}^n) \cong A_*(pt)$ by homotopic invariance each third arrow in this sequence $j_* : A_*(\mathbf{P}^n - \mathbf{P}^{n-1}) \to A_*(\mathbf{P}^n)$ is mono. Therefore the long sequence splits into short exact sequences

$$0 \to A_*(\mathbf{A}^n) \xrightarrow{j_*} A_*(\mathbf{P}^n) \xrightarrow{\beta} A_*^{\mathbf{P}^{n-1}}(\mathbf{P}^n) \to 0. \tag{8}$$

As $j^*\mathcal{O}_{\mathbf{P}^n}(-1)$ is trivial then $e \circ j_* = j_* \circ e(j^*\mathcal{O}_{\mathbf{P}^n}(-1)) = 0$. Hence e factors through the factorgroup $A_*^{\mathbf{P}^{n-1}}(\mathbf{P}^n)$ as required.

The proof of the second statement is similar the first one. First of all we compute s^*M and s_L^*M. One has $s^*\mathcal{O}_{1\oplus L}(-1) = \mathbf{1_X}$ and $s_L^*\mathcal{O}_{1\oplus L}(-1) = L$. Therefore $s^*M = \mathbf{1_X}^\vee \otimes s^*p^*L = L$ and $s_L^*M = L^\vee \otimes L = \mathbf{1_X}$.

Consider a long localization sequence for triple $\mathbf{P}(1 \oplus L) \supset p^{-1}(Z) \supset s_*(Z)$:

$$\ldots \to A_*^{p^{-1}(Z)-s_*(Z)}(\mathbf{P}(1 \oplus L) - \mathbf{P}(1)) \xrightarrow{j_*} A_*^{p^{-1}(Z)}(\mathbf{P}(1 \oplus L)) \xrightarrow{\beta} A_*^{s_*(Z)}(\mathbf{P}(1 \oplus L)) \to \ldots$$

where $A_*^{p^{-1}(Z)-s_*(Z)}(\mathbf{P}(1 \oplus L) - s_*(Z)) = A_*^{p^{-1}(Z)-s_*(Z)}(\mathbf{P}(1 \oplus L) - \mathbf{P}(1))$ by the excision property.

Since $\mathbf{P}(1 \oplus L) - \mathbf{P}(1)$ is a line bundle over $\mathbf{P}(L)$ (isomorphic to $L^\vee$) then $s_*^L : A_*^Z(X) \to A_*^{p^{-1}(Z)-s_*(Z)}(\mathbf{P}(1 \oplus L) - \mathbf{P}(1))$ and $(p|_{\mathbf{P}(1\oplus L)-\mathbf{P}(1)})_*$ are inverse to each other isomorphisms. Hence j_* in the long sequences has splitting $(s_*^L)^{-1} \circ p_*$ and this sequence splits into short exact ones:

$$0 \to A^Z(X) \xrightarrow{s_*^L} A_*^{p^{-1}(Z)}(\mathbf{P}(1 \oplus L)) \xrightarrow{\beta} A_*^{s_*(Z)}(\mathbf{P}(1 \oplus L)) \to 0. \tag{9}$$

Since $s_L^* M$ is trivial then the homomorpfism $e^{p^{-1}(Z)}(M)$ factors through β: $e^{p^{-1}(Z)}(M) = \widehat{e^{p^{-1}(Z)}}(M) \circ \beta$. We can set

$$\overline{\alpha^Z(L)} = p_* \circ \widehat{e^{p^{-1}(Z)}}(M). \tag{10}$$

□

2.2.8 Remark. *Notation used in the previous lemma will be used below in this section without any further explanation.*

2.2.9 Lemma. *Let $\gamma : A_*^Z(L) = A_*^{s_*(Z)}(\mathbf{P}(1 \oplus L) - \mathbf{P}(L)) \to A_*^{s_*(Z)}(\mathbf{P}(1 \oplus L))$ be the excision isomorphism. Suppose that A_* is endowed with a Chern structure. Let $\alpha^Z(L) = \overline{\alpha^Z(L)} \circ \gamma$ where a homomorphism $\overline{\alpha^Z(L)}$ is constructed in Lemma 2.2.7. Then homomorphisms $\overline{\alpha^Z(L)}$ define a Thom structure on a homology theory A_*.*

Proof. The functiorial properties of such constructed $\overline{\alpha^Z(L)}$ are obvious consequences from the analogous properties of Chern homomorphisms.

The only item we need to check is nondegeneracy.

By the nondegeneracy property of Chern structure we have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_*(X \times \mathbf{A}^1) & \xrightarrow{j_*} & A_*(X \times \mathbf{P}^1) & \xrightarrow{\beta} & A_*^{X \times \{0\}}(\mathbf{P}^1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \overline{\alpha(\mathbf{1})}} & & \\ 0 & \longrightarrow & A_*(X) & \xrightarrow{i_*} & A_*(X \times \mathbf{P}^1) & \xrightarrow{p_* \circ \zeta} & A_*(X) & \longrightarrow & 0 \end{array}$$

where $\zeta = e(\mathcal{O}(1)) : A_*(X \times \mathbf{P}^1) \to A_*(X \times \mathbf{P}^1)$ is the Chern homomorphism. The top row is exact in the same way as sequence 8. The bottom row is exact by the nondegeneracy of the dual Chern structure (c.f. Definition 2.2.3).

As two of vertical arrows are isomorphisms the third one is an isomorphism as well. □

2.2.10 Remark. *The bar-notation introduced above will be used below without any further explanation.*

2.2.11 Lemma. *Let $\{th^Z(L)\}$ be a Thom structure on a homology theory A. Define a homomorphism $c^Z(L)$ as the composition*

$$\left[A_*^Z(X) \xrightarrow{s_*} A_*^{p^{-1}(Z)}(\mathbf{P}(1 \oplus L)) \xrightarrow{\beta} A_*^{s_*(Z)}(\mathbf{P}(1 \oplus L)) \xrightarrow{\overline{th^Z(L)}} A_*^Z(X) \right].$$

Then the assignment $(L, X, Z) \mapsto c^Z(L)$ is a Chern structure on A_*. Moreover, if we begin with a Chern structure $\{e^Z(L)\}$ and take a Thom structure to be the one corresponding to the Chern structure $\{e^Z(L)\}$ by the construction from lemma 2.2.9 then for any (L, X, Z) one has $c^Z(L) = e^Z(L)$.

Proof. Functorial properties are obvious.

To proof the nondegeneracy property for a just defined Chern structure it is sufficient to prove nondegeneracy of the dual Chern structure. The last will follow from the fact that the sequence

$$0 \to A_*(X) \to A_*(X \times \mathbf{P}^1) \xrightarrow{p_* \circ c(\mathcal{O}(1))} A_*(X) \to 0$$

is exact. For simplicity of notation we proceed only the case $X = pt$.

Consider $\mathbf{P}^1 \subset \mathbf{P}^2$ and a point $\{\infty\} \in \mathbf{P}^2 - \mathbf{P}^1$. Then $\mathbf{P}^2 - \{\infty\}$ is turned out to be a line bundle L over $\mathbf{P}^1$ which is isomorphic to $\mathcal{O}(1)$. Then $\mathbf{P}(1 \oplus L)$ is isomorphic to the blow up $\widehat{\mathbf{P}^2}_{\{\infty\}}$ of the $\mathbf{P}^2$ at the point $\{\infty\}$. The exceptional divisor D is identified with $\mathbf{P}(L)$ under this isomorphism. We will write $\widehat{\mathbf{P}^2}$ for $\widehat{\mathbf{P}^2}_{\{\infty\}}$ and $\sigma : \widehat{\mathbf{P}^2} \to \mathbf{P}^2$ for the blowing up. By the excision property one has $A_*^{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L) - D) = A_*^{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L))$ and $A_*^{\mathbf{P}^1}(\mathbf{P}^2 - \{\infty\}) = A_*^{\mathbf{P}^1}(\mathbf{P}^2)$.

Since $\sigma : \mathbf{P}(1 \oplus L) - D \to \mathbf{P}^2 - \{\infty\}$ is an isomorphism then the operator $A_*^{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L)) \xrightarrow{\sigma_*} A_*^{\mathbf{P}^1}(\mathbf{P}^2)$ is an isomorphism.

Take a projective line ℓ' on $\mathbf{P}^2$ crossing the point ∞. Let $\{0\} = \mathbf{P}^1 \cap \ell'$ and $\ell \subset \widehat{\mathbf{P}^2}$ be the strict transform of ℓ'. It is easy to see that ℓ is just a fibre of projective bundle $\mathbf{P}(1 \oplus L)$ over a point $\{0\}$.

Then closed imbedding $i : \ell \hookrightarrow \mathbf{P}(1 \oplus L)$ can be regarded as a morphism of the projective bundles $\ell/\{0\}$ and $\mathbf{P}(1 \oplus L)/\mathbf{P}^1$.

Let $i_0 : \mathbf{P}^1 \to \mathbf{P}^2$, $\tau : \mathbf{P}^1 \to \ell$ be arbitrary linear embeddings. By the functoriality of the Thom isomorphisms we have a commutative diagram

$$\begin{array}{ccccccccc}
A_*(\mathbf{P}^1) & \xrightarrow{\tau_*} & A_*(\ell) & \xrightarrow{\beta_0} & A_*^{\{0\}}(\ell) & \xrightarrow{id} & A_*^{\{0\}}(\ell) & \xrightarrow{\overline{th(\mathbf{1})}} & A_*(\{0\}) \\
{\scriptstyle id}\downarrow & & {\scriptstyle \sigma_* i_*}\downarrow & & {\scriptstyle \sigma_* i_*}\downarrow & & {\scriptstyle i_*}\downarrow & & {\scriptstyle i_*}\downarrow \\
A_*(\mathbf{P}^1) & \xrightarrow{i^0_*} & A_*(\mathbf{P}^2) & \xrightarrow{\beta'} & A_*^{\mathbf{P}^1}(\mathbf{P}^2) & \xrightarrow{\sigma_*^{-1}} & A_*^{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L)) & \xrightarrow{\overline{th(L)}} & A_*(\mathbf{P}^1).
\end{array}$$

The left hand side square commutes by 2.1.13. The bottom row composition map is the homomorphism $c(L)$ by the very definition of $c(L)$. Since the diagram commutes one has the relation $p_* \circ c(L) = \beta_0 \circ \overline{th(1)}$.

Replacing in the short exact sequence of type 8.

$$0 \to A_*(pt) \to A_*(\mathbf{P}^1) \xrightarrow{\beta} A_*^{\{0\}}(\mathbf{P}^1) \to 0$$

the term $A_*^{\{0\}}(\mathbf{P}^1)$ by the one $A_*(pt)$ we get the short exact sequence

$$0 \to A_*(pt) \to A_*(\mathbf{P}^1) \xrightarrow{\overline{th(1)} \circ \beta} A_*(pt) \to 0$$

Since $p_* \circ c(L) = \overline{th(1)} \circ \beta$ the nondegeneracy property of the $c(L)$ follows .

It remains to prove the relation $c(L) = e(L)$ for any line bundle L over X.

As in the proof of the second part of lemma 2.2.7 consider the commutative diagram

$$\begin{array}{ccccc}
A_*^Z(X) & \xrightarrow{s_*} & A_*^{p^{-1}(Z)}(\mathbf{P}(1 \oplus L)) & \xrightarrow{\beta} & A_*^{s_*(Z)}(\mathbf{P}(1 \oplus L)) \\
{\scriptstyle e^Z(L)}\downarrow & & {\scriptstyle e^{p^{-1}(Z)}(M)}\downarrow & & {\scriptstyle \overline{\alpha^Z(L)}}\downarrow \\
A_*^Z(X) & \xrightarrow{s_*} & A_*^{p^{-1}(Z)}(\mathbf{P}(1 \oplus L)) & \xrightarrow{p_*} & A_*^Z(X).
\end{array}$$

The right hand side square commutes by the very definition of the operator $\overline{\alpha^Z(L)}$ from the Lemma 2.2.7. The left hand side square commutes by the functoriality of the Chern classes and the relation $s^*M = L$.

Then $c^Z(L) = \overline{\alpha^Z(L)} \circ \beta \circ s_* = p_* \circ s_* \circ e^Z(L) = e^Z(L)$.

□

2.3 Projective bundle theorem.

The nearest aim is to compute the homology of a projective bundle.

2.3.1 Theorem (Projective Bundle Homology). *Let A_* be a homology theory equipped with a Chern structure $(L, X, Z) \mapsto e^Z(L)$ on A_*. Let X be a smooth variety and let E/X be a vector bundle with $rk(E) = n+1$. For the endomorphism $e = e(\mathcal{O}_E(-1)) : A_*(\mathbf{P}(E)) \to A_*(\mathbf{P}(E))$ we have an isomorphism*

$$(p_*, p_* \circ e, \dots, p_* \circ e^n) : A_*(\mathbf{P}(E)) \to A_*(X) \oplus A_*(X) \cdots \oplus A_*(X)$$

where $p_* : \mathbf{P}(E) \to X$ is a projection.

Moreover, for the trivial rank $n+1$ bundle E we have $e^{n+1} = 0$. In addition, all the assertions hold if the endomorphism e is replaced by the one $e(\mathcal{O}_E(1)) : A_*(\mathbf{P}(E)) \to A_*(\mathbf{P}(E))$.

Proof. This variant of the proof is based on an unpublished notes of I.Panin.

Let $\mathbf{P}^{n-1} \subset \mathbf{P}^n$ be a hyperplane and $0 \in \mathbf{P}^n - \mathbf{P}^{n-1}$ be a point. Recall that the projection $\mathbf{P}^n - \{0\} \to \mathbf{P}^{n-1}$ with the center 0 makes $\mathbf{P}^n - \{0\}$ into a line bundle over $\mathbf{P}^{n-1}$ which is isomorphic to $\mathcal{O}_{\mathbf{P}^{n-1}}(1)$. Let us denote this bundle by L below. Projective bundle $\mathbf{P}(1 \oplus L)$ is isomorphic to the blow up $\widehat{\mathbf{P}^n}_{\{0\}}$ of the $\mathbf{P}^n$ at the point $\{0\}$. We will write $\widehat{\mathbf{P}}^n$ for the $\widehat{\mathbf{P}^n}_{\{0\}}$ and $\sigma : \widehat{\mathbf{P}^n} \to \mathbf{P}^n$ for the blowing up. If $\mathbf{P}(1 \oplus L)$ is identified with $\widehat{\mathbf{P}}^n$ then $\sigma(\mathbf{P}(1)) = \mathbf{P}^{n-1} \subset \mathbf{P}^n$.

Let us regard our homology theory A_* to be equipped with Thom structure corresponding given Chern structure by Lemma 2.2.9. We will write below in the proof e_k for $e(\mathcal{O}_{\mathbf{P}^k}(-1))$.

2.3.2 Lemma. *Let L_1 and L_2 be two line bundle over $\mathbf{P}(1 \oplus L)$, and $i : \mathbf{P}^1 \hookrightarrow \mathbf{P}(1 \oplus L)$ be a closed imbedding of any fiber. Then the following three conditions are equivalent:*

1. *$L_1 \cong L_2$;*
2. *$s^*L_1 \cong s^*L_2$ and $i^*L_1 \cong i^*L_2$;*
3. *$s_L^*L_1 \cong s_L^*L_2$ and $i^*L_1 \cong i^*L_2$.*

Proof. This lemma is an easy consequence of a general projective bundle Picar group computation. Recall that the sequence $0 \to Pic(X) \xrightarrow{p^*} Pic(\mathbf{P}(E)) \xrightarrow{i^*} Pic(\mathbf{P}^n) = \mathbb{Z} \to 0$ is exact for any projective bundle $p : E \to X$. Here $i : \mathbf{P}^n \hookrightarrow X$ is any fibre of this bundle. If $E = \mathbf{P}(1 \oplus L)$ then s^* and s_L^* be various splitting of this sequence. The lemma follows from this remark directly.

□

2.3.3 Lemma. *Let $M = \mathcal{O}_{1\oplus L}(1)\otimes p^*L$ be a line bundle over $\mathbf{P}(1\oplus L)$. Then the diagram below commutes.*

$$\begin{array}{ccccc} A_*(\mathbf{P}(1\oplus L)) & \xrightarrow{\beta} & A_*^{\mathbf{P}(1)}(\mathbf{P}(1\oplus L)) & \xrightarrow{\widehat{e}(M^\vee)} & A_*(\mathbf{P}(1\oplus L)) \\ \downarrow{\sigma_*} & & \downarrow{\widehat{\sigma_*}} & & \downarrow{\sigma_*} \\ A_*(\mathbf{P}^n) & \xrightarrow{\beta'} & A_*^{\mathbf{P}^{n-1}}(\mathbf{P}^n) & \xrightarrow{\widehat{e}(\mathcal{O}_{\mathbf{P}^n}(-1))} & A_*(\mathbf{P}^n) \end{array}$$

Proof. The left square commutes by natural reason. Homomorphisms β' and β are taken from sequences 8 and 9. Therefore they both are onto. This implies that one has to check the relation $\sigma_*\widehat{e}(M)\circ\beta = \widehat{e}_n\circ\beta\sigma_*$ in order to prove that right square commutes.

Last relation is equivalent to $\sigma_* e(M^\vee) = e_n\sigma_*$. It follows from the functoriality of Chern homomorphism and the fact that $\sigma^*\mathcal{O}_{\mathbf{P}^n}(-1) = \mathcal{O}_{1\oplus L}(-1)\otimes p^*L^\vee$. Last equality is easy to prove using lemma 2.3.2: $s_L^*\sigma^*O_{\mathbf{P}^n}(-1) = \sigma^*O_{\mathbf{P}^n}(-1)|_{\{\infty\}}$ is trivial and s_L^*M is trivial by lemma 2.2.7; restricting both side of the desired relation onto a fibre we see that in both cases we get the line bundle $L^\vee = \mathcal{O}_{\mathbf{P}^{n-1}}(-1)$. So two bundles over $\mathbf{P}(1\oplus L))$ are isomorphic to each other and lemma follows. □

Before the general case we prove the theorem 2.3.1 for the trivial bundle $E = X\times\mathbf{A}^{n+1}$. For simplicity of notation we procceed only the case $X = pt$. In this case $\mathbf{P}(E) = \mathbf{P}^n$ and we proceed the proof by the induction on the integer n. By induction assumption there is an isomorphism

$$\gamma_{n-1} = ((p_{n-1})_*, (p_{n-1})_*\circ e_{n-1}, \ldots, (p_{n-1})_*\circ e_{n-1}^{n-1}) : A_*(\mathbf{P}^{n-1}) \to A_*(pt)\oplus\cdots\oplus A_*(pt)$$

and $e_{n-1}^n = 0$.

Let $u : A_*(\mathbf{P}^n)\to A_*(\mathbf{P}^{n-1})$ be a composition map

$$\left[A_*(\mathbf{P}^n) \overset{(p_*,p_*e_n,\ldots,p_*e_n^{n-1})}{\longrightarrow} A_*(pt)\oplus\cdots\oplus A_*(pt) \xrightarrow{\gamma_{n-1}^{-1}} A_*(\mathbf{P}^{n-1})\right].$$

Since $i_*\circ e_{n-1} = e_n\circ i_* \in Hom(A_*(\mathbf{P}^{n-1}), A_*(\mathbf{P}^n))$ where $i : \mathbf{P}^{n-1}\hookrightarrow\mathbf{P}^n$ is a linear embedding then $ui_* = id_{A_*(\mathbf{P}^{n-1})}$.

Under this notation if one prove that

$$(p_*, u\circ e_n) : A_*(\mathbf{P}^n)\to A_*(pt)\oplus A_*(\mathbf{P}^{n-1}) \tag{11}$$

is an isomorphism then the theorem follows.

2.3.4 Lemma. *Under induction hypothesis there is a following short exact sequence:*

$$0\to A_*(\mathbf{P}^{n-1}) \xrightarrow{i_*} A_*(\mathbf{P}^n) \xrightarrow{\beta} A_*^{\{\infty\}}(\mathbf{P}^n)\to 0.$$

Proof. Given a point $\{\infty\}$ not lying on $\mathbf{P}^{n-1}$ consider a long exact sequence for the pair $(\mathbf{P}^n, \mathbf{P}^n - \{\infty\})$:

$$\ldots\to A_*(\mathbf{P}^n - \{\infty\}) \xrightarrow{j_*} A_*(\mathbf{P}^n) \xrightarrow{\beta} A_*^{\{\infty\}}(\mathbf{P}^n)\to\ldots$$

Since $\mathbf{P}^n - \{\infty\}$ can be regarded as a linear bundle on $\mathbf{P}^{n-1}$ then zero section natural map $z_* : A_*(\mathbf{P}^{n-1}) \to A_*(\mathbf{P}^n - \{\infty\})$ is an isomorphism by strong homotopic invariance 2.1.6. Moreover, since $j_* z_* = i_*$ then the homomorphism $A_*(\mathbf{P}^n) \xrightarrow{z_* \circ u} A_*(\mathbf{P}^n - \{\infty\})$ is the splitting for j_*. Hence, j_* is mono and the long exact sequence splits into short exact sequences: $0 \to A_*(\mathbf{P}^n - \{\infty\}) \xrightarrow{j_*} A_*(\mathbf{P}^n) \xrightarrow{\beta} A_*^{\{\infty\}}(\mathbf{P}^n) \to 0$. We can write $A_*(\mathbf{P}^{n-1})$ for $A_*(\mathbf{P}^n - \{\infty\})$ and so we get a desired sequence. □

2.3.5 Lemma. *Under induction hypothesis one has:*

1. *The image of the operator* $e_n : A_*(\mathbf{P}^n) \to A_*(\mathbf{P}^n)$ *lies in the image of* $i_* : A_*(\mathbf{P}^{n-1}) \to A_*(\mathbf{P}^n)$ *and* $e_n^{n+1} = 0$.
2. *The image of an operator* $e(M) : A_*(\mathbf{P}(1 \oplus L)) \to A_*(\mathbf{P}(1 \oplus L))$ *lies in the image of* $s_* : A_*(\mathbf{P}^{n-1}) \hookrightarrow A_*(\mathbf{P}(1 \oplus L))$ *where* $s : \mathbf{P}^{n-1} \hookrightarrow \mathbf{P}(1 \oplus L)$ *is a closed imbedding identifying* $\mathbf{P}^{n-1}$ *with* $\mathbf{P}(1)$ *as in the lemma 2.2.7.*
3. *The following relations hold*

$$s_* P_* e(M) = e(M); \quad i_* u e_n = e_n \tag{12}$$

where $P : \mathbf{P}(1 \oplus L) \longrightarrow \mathbf{P}^{n-1}$ *is a projection.*

Proof. Consider a diagram

$$\begin{array}{ccccccc}
 & & A_*(\mathbf{P}^n) & \xrightarrow{\beta} & A_*^{\{\infty\}}(\mathbf{P}^n) & \xleftarrow{j_*} & A_*^{\{\infty\}}(\mathbf{A}^n) \\
 & & \downarrow{\scriptstyle e_n} & & \downarrow{\scriptstyle e_n^{\{\infty\}}} & & \downarrow{\scriptstyle e_n^{\{\infty\}}} \\
A_*(\mathbf{P}^{n-1}) & \xrightarrow{i_*} & A_*(\mathbf{P}^n) & \xrightarrow{\beta} & A_*^{\{\infty\}}(\mathbf{P}^n) & \xleftarrow{j_*} & A_*^{\{\infty\}}(\mathbf{A}^n)
\end{array}$$

It is commutative from functoriality of Chern homomorphism. Show that the right vertical arrow is null. Since the bundle $\mathcal{O}_{\mathbf{P}^n}(-1)$ becames trivial being restricted to $\mathbf{A}^n = \mathbf{P}^n - \mathbf{P}^{n-1} \xrightarrow{j} \mathbf{P}^n$. then the right vertical arrow is null by property 3 in definition 2.2.1. The middle vertical arrow is null as well because j_* is an excision isomorphism.

Therefore the diagram shows that $\beta \circ e_n = e_n^{\{\infty\}} \circ \beta = 0$. Since $\mathrm{Ker}(\beta) = \mathrm{Im}(i_*)$ by lemma 2.3.4 then $\mathrm{Im}(e_n) \subset \mathrm{Ker}(\beta) = \mathrm{Im}(i_*)$ as required.

This fact implies that $e_n^{n+1} = 0$. Indeed, let $e(a) = i_*(b)$ for some $b \in A_*(\mathbf{P}^{n-1})$. Since $e_{n-1}^n = 0$ then $e_n^{n+1}(a) = e_n^n \circ e(a) = e_n^n \circ i_*(b) = i_* \circ e_{n-1}^n(b) = 0$.

In order to proof the second part of the lemma we consider the commutative diagram:

$$\begin{array}{ccccccc}
 & & A_*(\mathbf{P}(1 \oplus L)) & \xrightarrow{\beta} & A_*^{\mathbf{P}(L)}(\mathbf{P}(1 \oplus L)) & \xleftarrow{j_*} & A_*^{\mathbf{P}(L)}(\mathbf{P}(1 \oplus L) - \mathbf{P}(1)) \\
 & & \downarrow{\scriptstyle e(M)} & & \downarrow{\scriptstyle e^{\mathbf{P}(L)}(M)} & & \downarrow{\scriptstyle e^{\mathbf{P}(L)}(M)} \\
A_*(\mathbf{P}(1)) & \xrightarrow{i_*} & A_*(\mathbf{P}(1 \oplus L)) & \xrightarrow{\beta} & A_*^{\mathbf{P}(L)}(\mathbf{P}(1 \oplus L)) & \xleftarrow{j_*} & A_*^{\mathbf{P}(L)}(\mathbf{P}(1 \oplus L) - \mathbf{P}(1))
\end{array}$$

.

As above j_* is an excision isomorphism. The projection $P : \mathbf{P}(1 \oplus L) - \mathbf{P}(1)X$ is a homotopic equivalence with inverse s_L. Since $s_L^* M = 1_X$ by lemma 2.2.7 then $j^* M = P^* s_L^* M$ is a trivial line bundle. Hence, $e^{\mathbf{P}(L)}(M) = 0$.

By relation 9 one has $\mathrm{Ker}(\beta) = \mathrm{Im}(s_*)$. Since $\beta \circ e(M) = e^{\mathbf{P}(L)}(M) \circ \beta = 0$ then $\mathrm{Im}(e(M)) \subset \mathrm{Ker}(\beta) = \mathrm{Im}(s_*)$. □

2.3.6 Remark. *One can write $M^\vee$ for M and write $\widehat{e}(M^\vee)$ for $e(M)$ in the second and the third assertion of the lemma because their images coincide.*

We are ready now to complete the proof of the projective bundle theorem 2.3.1.

Denote by $\overline{\alpha}$ the Thom isomorphism $\overline{th'(L)}$ in the dual Thom structure, i.e. $\overline{\alpha} = P_* \circ \widehat{e}(M^\vee) = P_* \circ \widehat{e}(\mathcal{O}_{1\oplus L}(-1) \otimes p^* L^\vee) \in Hom(A_*^{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L)), A_*(\mathbf{P}^{n-1}))$ (c.f. formula 10). Here hat-notation is borrowed from lemma 2.2.7.

By the short exact sequence 8

$$0 \to A_*(\mathbf{A}^n) \xrightarrow{j_*} A_*(\mathbf{P}^n) \xrightarrow{\beta'} A_*^{\mathbf{P}^{n-1}}(\mathbf{P}^n) \to 0$$

one has an isomorphism $A_*(\mathbf{P}^n) \xrightarrow{(p_*,\ \ \beta')} A_*(pt) \oplus A_*^{\mathbf{P}^{n-1}}(\mathbf{P}^n)$. The isomorphism of the second summand $A_*^{\mathbf{P}^{n-1}}(\mathbf{P}^n)$ with $A_*(\mathbf{P}^{n-1})$ is given by formula $\overline{\alpha} \circ \widehat{\sigma}_*^{-1}$ where $\widehat{\sigma}_* : A_*^{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L)) \to A_*^{\mathbf{P}^{n-1}}(\mathbf{P}^n)$ is an isomorphism. By equation 11 it is sufficient to prove that $\overline{\alpha} \circ \widehat{\sigma}_*^{-1} \circ \beta' = u \circ e_n$.

Applying i_* to both sides of desired equality one has to prove

$$i_* \overline{\alpha} \widehat{\sigma}_*^{-1} \beta' = i_* u \circ e_n. \tag{13}$$

Recall that $\overline{\alpha} = P_* \circ \widehat{e}(M^\vee)$. Therefore $i_* \overline{\alpha} = \sigma_* s_* P_* \widehat{e}(M^\vee) = \sigma_* \widehat{e}(M^\vee)$ by the lemma 2.3.5. Since $\sigma_* \widehat{e}(M^\vee) \widehat{\sigma}_*^{-1} = \widehat{e_n}$ by lemma 2.3.3 then the left hand of equality 13 is equal to $\widehat{e_n} \beta' = e_n$. But the right hand of this equality is equal to e_n by relation 12. Now we have finished the proof for the case of the trivial vector bundle E.

The general case can be proceeded in the usual way by the Mayer-Vietoris argument. □

2.3.7 Lemma. *Let Z be a closed subset in X, $L = q^* \mathcal{O}_{\mathbf{P}^n}(-1)$ be a line bundle over $X \times \mathbf{P}^n$.*

Then there is an isomorphism

$$\left(p_*, p_* \circ e^{Z \times \mathbf{P}^n}(L), ..., p_* \circ (e^{Z \times \mathbf{P}^n}(L))^n\right) : A_*^{Z \times \mathbf{P}^n}(X \times \mathbf{P}^n) \to A_*^Z(X) \oplus \cdots \oplus A_*^Z(X).$$

Proof. The proof is a straightforward consequence from theorem 2.3.1, the long localization sequence for the pair $(X \times \mathbf{P}^n, (X - Z) \times \mathbf{P}^n)$ and five lemma. □

2.3.8 Corollary. *Let $E = M \oplus N$ be vector bundles of constant rank. Then there is an exact sequence*

$$0 \to A_*(\mathbf{P}(N)) \xrightarrow{i_*} A_*(\mathbf{P}(E)) \xrightarrow{\beta} A_*^{\mathbf{P}(M)}(\mathbf{P}(E)) \to 0.$$

Proof. Consider a long exact sequence for the pair $(\mathbf{P}(E), \mathbf{P}(E) - \mathbf{P}(M))$. As $\mathbf{P}(E) - \mathbf{P}(M)$ can be regarded as vector bundle over $\mathbf{P}(N)$ we may replace $A_*(\mathbf{P}(E) - \mathbf{P}(M))$ by $A_*(\mathbf{P}(N))$:

$$\cdots \to A_*(\mathbf{P}(N)) \xrightarrow{i_*} A_*(\mathbf{P}(E)) \xrightarrow{\beta} A_*^{\mathbf{P}(M)}(\mathbf{P}(E)) \to \ldots$$

By projective bundle theorem $i_* : A_*(\mathbf{P}(N)) \to A_*(\mathbf{P}(E))$ has an obvious splitting. Indeed, let $\mathrm{rk} N = n$, $\xi_E = e(\mathcal{O}_E(-1))$ and $\xi_N = e(\mathcal{O}_N(-1))$. Since $i_*\xi_N = \xi_E i_*$ then the homomorphism

$$(p_*^E, p_*^E \circ \xi_E, \ldots, p_*^E \circ \xi_E^{n-1}) : A_*(\mathbf{P}(E)) \longrightarrow A_*(pt) \oplus \cdots \oplus A_*(pt) \cong A_*(\mathbf{P}(N))$$

is a desired splitting. □

2.3.9 Lemma. *For each line bundle L over a smooth X and closed subset $Z \subset X$ the endomorphism $e^Z(L)$ of $A_*^Z(X)$ is nilpotent.*

Proof. Recall that the assertion holds in the case $L = \mathcal{O}_{\mathbf{P}^n}(1)$ by the lemma 2.3.5 applied to the dual Chern structure.

To prove the general case recall that by Claim **3.4.4** from [P2] one can find for any smooth variety X a diagram of the form

$$X \xleftarrow{p} X' \xrightarrow{f} \mathbf{P}(V) \tag{14}$$

with a torsor under a vector bundle $p : X' \to X$ and a morphism $f : X' \to \mathbf{P}(V)$ such that the line bundles $L' = p^*(L)$ and $f^*(\mathcal{O}_V(1))$ over X' are isomorphic. By the strong homotopy invariance we can replace X by X' and regard that $L = f^*\mathcal{O}_V(1)$.

The homomorphism $e(\mathcal{O}_V(1)) \in A(\mathbf{P}(V))$ is nilpotent as just mentioned above. Thus the homomorphism $e_X = e(\mathcal{O}_{X\times\mathbf{P}^n}(1))$ is nilpotent as well.

Let $g : X \xrightarrow{(id, \quad f)} X \times \mathbf{P}^n$. Then $L = g^*\mathcal{O}_{X\times\mathbf{P}^n}(1)$ and the following diagram commutes:

$$\begin{array}{ccc} A_*(X) & \xrightarrow{g_*} & A_*(X \times \mathbf{P}^n) \\ \downarrow{\scriptstyle e(L)} & & \downarrow{\scriptstyle e_X} \\ A_*(X) & \xrightarrow{g_*} & A_*(X \times \mathbf{P}^n). \end{array}$$

Since $pg = id_X$ the natural homomorphism g_* is injective. Therefore it is sufficient to check that $g_*e(L)^{n+1} = 0$.

As $g_*e(L)^{n+1} = e_X^{n+1}g_* = 0$ the Lemma is proved. □

2.3.10 Remark. *Define a group $A_*^{Z\times\mathbf{P}^\infty}(X \times \mathbf{P}^\infty)$ as an injective limit of the groups $A_*^{Z\times\mathbf{P}^n}(X \times \mathbf{P}^n)$. Then by Remark to the lemma 2.3.5 and the lemma 2.3.7 one has that $e_n^{Z\times\mathbf{P}^n}(A_*^{Z\times\mathbf{P}^n}(X \times \mathbf{P}^n)) = i_*^{n-1,n}(A_*^{Z\times\mathbf{P}^{n-1}}(X \times \mathbf{P}^{n-1}))$ where $e_n = e(\mathcal{O}_{X\times\mathbf{P}^n}(-1))$ and $i_{n-1,n} : X \times \mathbf{P}^{n-1} \hookrightarrow X \times \mathbf{P}^n$ is a linear imbedding. Particularly $e^{Z\times\mathbf{P}^\infty}(q^*L)$ is a surjection where L is a tautological line bundle over $\mathbf{P}^\infty$ and $q : X \times \mathbf{P}^\infty \to \mathbf{P}^\infty$ is a projection.*

2.3.11 Lemma. *Let A_* be a homology theory equipped with the Thom structure $\{th^Z(L)\}$ and let $\{c^Z(L)\}$ be the Chern structure corresponding to this Thom structure as define in the lemma 2.2.11. Let us take the Thom structure $\{\alpha^Z(L)\}$ corresponding to the Chern structure $\{c^Z(L)\}$ by the construction from lemma 2.2.9. Then for any (L, X, Z) one has $\alpha^Z(L) = th^Z(L)$.*

Proof. In the proof of the lemma we will use the notation from lemma 2.2.7. Since $\overline{\alpha^Z(L)}$ is defined by formula 10 one has to check a relation

$$p_* \circ \widehat{c^{p^{-1}(Z)}}(M) = \overline{th^Z(L)} \tag{15}$$

First of all to prove a relation $\alpha^Z(L) = th^Z(L)$ we assume the homomorphism $c^Z(L)$ to be a surjection.

Consider a diagram:

$$\begin{array}{ccc} A^Z_*(X) & \xrightarrow{c^Z(L)} & A^Z_*(X) \ . \\ \Big\downarrow{\scriptstyle \beta\circ s_*} & & \Big\downarrow{\scriptstyle id} \\ A^{s_*(Z)}_*(\mathbf{P}(1\oplus L)) & \xrightarrow{\overline{th^Z(L)}} & A^Z_*(X) \end{array}$$

where $\beta : A^{p^{-1}(Z)}_*(\mathbf{P}(1\oplus L)) \to A^{s_*(Z)}_*(\mathbf{P}(1\oplus L))$ is a support restriction. The diagram commutes by the very construction of the homomorphism $c^Z(L)$ from lemma 2.2.11. Since $c^Z(L)$ is a surjection by assumption and $\overline{th^Z(L)}$ is an isomorphism then $\beta\circ s_*$ is a surjection.

Therefore an equality 15 follows from $p_* \circ \widehat{c^{p^{-1}(Z)}}(M) \circ \beta \circ s_* = th^Z(L) \circ \beta \circ s_*$. Since $\widehat{c^{p^{-1}(Z)}}(M)\beta = c^{p^{-1}(Z)}(M)$ and $s^*M = L$ by lemma 2.2.7 then the left hand is equal to $p_*c^{p^{-1}(Z)}(M)s_* = p_*s_*c^Z(s^*M) = c^Z(L)$. Right hand is equal to $c^Z(L)$ from the diagram above.

Lemma 2.3.10 and following remark imply that the lemma holds in the (universal) case: for the line bundle $L = \mathcal{O}(-1)$ over $X \times \mathbf{P}^\infty$.

The general case can be be reduced to the universal one as follows.

By Jouanolou trick we can find a diagram of the form 14. Therefore one can regard X to be an affine variety.

Let $f : X \to \mathbf{P}^\infty$ be a morphism such that $L = f^*\mathcal{O}_{\mathbf{P}^\infty}(-1)$. Set $g : X \xrightarrow{(id,f)} X \times \mathbf{P}^\infty$. Denoting $\mathcal{O}_{X\times\mathbf{P}^\infty}(-1)$ by L' one has $L = g^*L'$. Let $G_* : \mathbf{P}(1\oplus L) \to \mathbf{P}(1\oplus L')$ be an induced morphism of projective bundles. For a closed subset $Z \subset X$ denote $Z \times \mathbf{P}^\infty \subset X \times \mathbf{P}^\infty$ by Z'. Recall that $M = \mathcal{O}_{1\oplus L}(-1) \otimes p^*L$ is a line bundle over $\mathbf{P}(1\oplus L)$ and $M' = \mathcal{O}_{1\oplus L'}(-1) \otimes P^*L'$ is a line bundle over $\mathbf{P}(1\oplus L')$. Then one has a commutative diagram:

$$\begin{array}{ccccccc} A^{p^{-1}(Z)}_*(\mathbf{P}(1\oplus L)) & \xrightarrow{\overline{th^Z(L)}\circ\beta} & A^Z_*(X) & \xrightarrow{s_*} & A^{p^{-1}(Z)}_*(\mathbf{P}(1\oplus L)) & \xrightarrow{p_*} & A^Z_*(X) \quad . \\ \Big\downarrow{\scriptstyle G_*} & & \Big\downarrow{\scriptstyle g_*} & & \Big\downarrow{\scriptstyle G_*} & & \Big\downarrow{\scriptstyle g_*} \\ A^{P^{-1}(Z')}_*(\mathbf{P}(1\oplus L')) & \xrightarrow{\overline{th^{Z'}(L')}\circ\beta'} & A^{Z'}_*(X\times\mathbf{P}^\infty) & \xrightarrow{S_*} & A^{P^{-1}(Z')}_*(\mathbf{P}(1\oplus L')) & \xrightarrow{P_*} & A^{Z'}_*(X\times\mathbf{P}^\infty) \end{array}$$

One has to check the relation $\overline{th^Z(L)} \circ \beta = p_* \circ c^{p^{-1}(Z)}(M)$.

Since g_* is mono then it is sufficient to check a relation $g_*\overline{th^Z(L)} \circ \beta = g_* p_* \circ c^{p^{-1}(Z)}(M)$. The right hand is equal to $P_* G_* \circ c^{p^{-1}(Z)}(M) = P_* \circ c^{P^{-1}(Z')}(M')G_*$ because $G^* M' = M$. Since for the universal case the lemma holds then $P_* \circ c^{P^{-1}(Z')}(M') = \overline{th^{Z'}(L')} \circ \beta'$. Therefore the right hand is equal to $\overline{th^{Z'}(L')} \circ \beta' G_*$. Since the diagram commutes the last is equal to the left hand. The lemma follows.

□

2.3.12 Definition. *The Chern structure on a homology theory A_* is said to be COMMUTATIVE if for any smooth variety X, closed subset $Z \subset X$ and line bundles L_1/X and L_2/X the Chern homomorphisms $e^Z(L_1)$ and $e^Z(L_2)$ commute with each other.*

2.4 Chern classes

2.4.1 Definition. *Let A_* be a homology theory. We say that A_* is endowed with a Chern class theory if for any vector bundle E/X and any closed subset $Z \subset X$ there are given homomorphisms $c_i^Z(E) : A_*^Z(X) \to A_*^Z(X)$ such that*

1. *c_i^Z depends only on an isomorphism class of E;*

   For any morphism $f : X' \to X$ and vector bundle E/X the following diagram is commutative:

$$\begin{array}{ccc} A_*(X') & \xrightarrow{c_i^{Z'}(E')} & A_*(X') \\ \downarrow f_* & & \downarrow f_* \\ A_*(X) & \xrightarrow{c_i^Z(E)} & A_*(X) \end{array}$$

   where $E' = E \times_X X'$ is a pullback vector bundle over X'; $Z \subset X$ is a closed subset and $Z' = f^{-1}(Z)$;

2. *$c_0^Z(E) = id_{A_*^Z(X)}$;*

   the restriction of the assignment $(L, X, Z) \mapsto c_1^Z(L)$ to line bundles is a Chern structure on A_*;

3. *Any two homomorphisms $c_i^Z(E)$ and $c_j^Z(F)$ commute with each other;*

4. *Cartan formula: For each short exact sequence of vector bundles $0 \to E_1 \to E \to E_2 \to 0$ we have $c_r^Z(E) = c_r^Z(E_1)c_0^Z(E_2) + c_{r-1}^Z(E_1)c_1^Z(E_2) + ... + c_0^Z(E_1)c_r^Z(E_2)$;*

5. *Vanishing property: $c_i^Z(E) = 0$ for $i > rkE$.*

2.4.2 Theorem. *Let A_* be endowed with a commutative Chern structure $(L, X, Z) \mapsto e^Z(L)$. Then there exists a unique Chern class theory on A such that for each line bundle L one has $c_1^Z(L) = e^Z(L)$. Moreover the Chern class homomorphisms $c_i^Z(E)$ are nilpotent for $i > 0$.*

Proof. For simplicity of notation we proceed only the absolute case.

First of all we prove the uniqueness assertion. If there are two assignements $E/X \mapsto c'_i(E)$ and $E/X \mapsto c''_i(E)$ satisfying the required properties. Then they coincide on line bundles by the properties 2 and 5. Therefore they coincide on direct sums of line bundles by the Cartan formula 4. Thus they coincide on all vector bundles by the splitting principle from Lemma **3.5.1** from [P2].

It remains to construct a Chern classes theory. Let X be a smooth variety and E/X be a vector bundle with $\mathrm{rk}E = n$. Set $\xi = e(\mathcal{O}_E(-1)) \in Hom(A_*(\mathbf{P}(E), A_*(\mathbf{P}(E)))$. By theorem (2.3.1) the homomorphism ξ can be uniquely represented by $n \times n$ matrix $(\xi_{ij})_{i,j=0}^{n-1}$ with entries in $Hom(A_*(X), A_*(X))$. More detally, any element $a \in A_*(\mathbf{P}(E))$ can be written by theorem 2.3.1 as a column $(a_0, \dots, a_{n-1})^T$ where $a_k = p_*(\xi^k(a)) \in A_*(X)$. Then by definition of ξ_{ij} one has formula (the usually matrix multiplication) $(\xi(a))_k = \sum_{j=0}^{n-1} \xi_{kj}(a_j)$.

Taking $k = n-1$ one can write:.

$$p_*(\xi^n(a)) = \xi_{n,n-1}(p_*(a)) + \xi_{n,n-2}(p_*(\xi(a))) + \cdots + \xi_{n,0}(p_*(\xi(a))). \tag{16}$$

Set $c_0(E) = id$, $c_k(E) = (-1)^{k+1}\xi_{n,k-1}$ and $c_m(E) = 0$ if $m > n$. In order to define Chern class homomorphism with supports one has to apply Projective Bundle theorem with supports 2.3.7.

2.4.3 Claim. *Homomorphisms $c_i(E)$ satisfy the theorem.*

The rest of the proof is devoted to the proof of this Claim. The property $c_0(E) = 1$ holds by the very definition. To prove the property $c_1(L) = e(L)$ for a line bundle L observe that $\mathbf{P}(L) = X$ and $\mathcal{O}_L(-1) = L$ over X. Thus $\xi = e(L) \in Hom(A_*(X), A_*(X))$ and the relation (16) shows that $c_1(L) = e(L)$.

Now prove the property 1 of 2.4.1. A vector bundle isomorphism $\varphi : E \to E'$ induces an isomorphism $\Phi : \mathbf{P}(E) \to \mathbf{P}(E')$ of the projective bundles and a line bundle isomorphism $\Phi^*(\mathcal{O}_{E'}(-1)) \to \mathcal{O}_E(-1)$ over $\mathbf{P}(E)$. Therefore $\Phi_* \circ \xi = \xi' \circ \Phi_*$ and for all k one has

$$p'_*((\xi')^k \Phi_*(a)) = p_*(\xi^k(a)). \tag{17}$$

By the formula 16 one has equations

$$p_*(\xi^n(a)) = (-1)^{n+1} c_n(E)(p_*(a)) + \cdots + c_1(E)(p_*(\xi^{n-1}(a)))$$

and

$$p'_*((\xi')^n(\Phi_*(a))) = (-1)^{n+1} c_n(E')(p'_*\Phi_*(a)) + \cdots + c_1(E')(p'_*(\xi')^{n-1}(\Phi_*(a))).$$

But the formula 17 allows us to rewrite the last equation as $p_*(\xi^n(a)) = (-1)^{n+1}c_n(E')(p_*(a)) + \cdots + c_1(E')(p_*(\xi^{n-1}(a)))$. Comparing with the previous equation one gets $c_k(E') = c_k(E)$.

The property $f_* \circ c_i(E) = c_i(f^*(E)) \circ f_*$ is proved similarly.

For the rest of the proof we need in the following

2.4.4 Claim. *For a rank r vector bundle E set $c_t(E) = id + c_1(E)t + \cdots + c_r(E)t^r \in End(A_*(X))[t]$. Let T be a smooth variety and let $E = \oplus_{i=1}^r L_i$ for certain line bundles L_i over T. Then one has*

$$c_t(E) = \prod_{i=1}^{r} c_t(L_i).$$

Proof. Let $\xi_E = e(\mathcal{O}_E(-1))$ where $\mathcal{O}_E(-1)$ is the tautological line bundle on $\mathbf{P}(E)$.

First of all we shall prove the relation

$$\prod_{i=1}^{r} (\xi - e(p_E^* L_i)) = 0 \tag{18}$$

where $p_E : \mathbf{P}(E) \to T$ is a projection. Since all first Chern class homomorphisms commute the formula above has sense.

To prove the very last relation set $F = L_1 \oplus \cdots \oplus L_{r-1}$. The canonical projection $\mathbf{P}(E) - \mathbf{P}(F) \to \mathbf{P}(L_r)$ makes $\mathbf{P}(E) - \mathbf{P}(F)$ into a vector bundle over $X = \mathbf{P}(L_r)$ with the zero section $\mathbf{P}(L_r)$. Therefore the natural mapping $A_*(\mathbf{P}(L_r)) \to A_*(\mathbf{P}(E) - \mathbf{P}(F))$ is an isomorphism. Let $j : \mathbf{P}(E) - \mathbf{P}(F) \to \mathbf{P}(E)$ be an open inclusion. Since $\mathcal{O}_E(-1)|_{\mathbf{P}(L_r)} = L_r$ then the bundles $j^*\mathcal{O}_E(-1)$ and $j^* p_E^* L_r$ coincide.

One has a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_*(\mathbf{P}(F)) & \xrightarrow{i_*} & A_*(\mathbf{P}(E)) & \xrightarrow{\beta} & A_*^{\mathbf{P}(L_r)}(\mathbf{P}(E)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \xi_F - e(p_F^* L_r)} & & \downarrow{\scriptstyle \xi_E - e(p_E^* L_r)} & & \downarrow{\scriptstyle \xi_E^{\mathbf{P}(L_r)} - e^{\mathbf{P}(L_r)}(p_E^* L_r)} & & \\
0 & \longrightarrow & A_*(\mathbf{P}(F)) & \xrightarrow{i_*} & A_*(\mathbf{P}(E)) & \xrightarrow{\beta} & A_*^{\mathbf{P}(L_r)}(\mathbf{P}(E)) & \longrightarrow & 0
\end{array}$$

where strings are the short exact sequences of type 2.3.8. The right vertical arrow is null by following reasons. $A_*^{\mathbf{P}(L_r)}(\mathbf{P}(E)) = A_*^{\mathbf{P}(L_r)}(\mathbf{P}(E) - \mathbf{P}(F))$ by exscision property; but the bundles $\mathcal{O}(-1)$ and $p^* L_r$ coincide being restricted to $\mathbf{P}(E) - \mathbf{P}(F)$.

So we get that the image of $\xi_E - e(p_E^* L_r)$ lies in the image of i_*.

Since $(\prod_{i=1}^{r-1} \xi_F - e(p_F^* L_i)) = 0$ by induction hypothesis then

$$(\prod_{i=1}^{r-1} (\xi_E - e(p_E^* L_i))) \circ i_* = i_* \circ \prod_{i=1}^{r-1} (\xi_F - e(p_F^* L_i)) = 0$$

and the relation 18 is proved.

We need to check that $c_k(E)$ is just a symmetric polynomial $\sigma_k(e(L_1), \ldots, e(L_r))$. Expanding brackets in the equation 18 one has $\xi^n - \sigma_1' \xi^{n-1} + \cdots + (-1)^n \sigma_n' = 0$ where $\sigma_k' = \sigma_k(e(p^* L_1), \ldots, e(p^* L_r))$. Since $p_* \sigma_k' = \sigma_k p_*$ then $p_* \xi^n = \sigma_1 p_* \xi^{n-1} - \sigma_2 p_* \xi^{n-2} + \cdots + (-1)^{n+1} p_*$. Comparing last formula with the formula 16 one completes the proof.

□

Claim 2.4.4 and splitting principle allow us to complete the proof of Claim 2.4.3.

Let E and F be a vector bundles over X. Prove that the homomorfhisms $c_k(E)$ and $c_l(F)$ commute. Indeed, by the splitting principle there exists a smooth variety T and a morphism $r : T \to X$ such that each the vector bundle r^*E and r^*F is a sum of line bundles and $r_* : A_*(T) \to A_*(X)$ is a split surjection.

The homomorphisms $c_k(r^*E)$ and $c_l(r^*F)$ commute as symmetric polynomes of line bundles chern classes , which are commute because Chern structure is commutative.

Then $c_k(E)c_l(F)r_* = r_*c_k(r^*E)c_l(r^*F) = r_*c_l(r^*F)c_k(r^*E) = c_l(F)c_k(E)r_*$. Last implies commutativity of higher chern classes because r_* is onto.

Cartan formula and nilpotence are easily can be proved essentially in the same way. $\square$

Below we will need Proposition

2.4.5 Proposition. . *Namely, let X be a smooth variety and let E be a vector bundles over X of constant rank n. Let $p : \mathbf{P}(E) \to X$ be the projection. Then the homomorphism $c_n(\mathcal{O}_E(1) \otimes p^*E)$ is null.*

Proof. Define a rank $n-1$ vector bundle Q via the short exact sequence $0 \to \mathcal{O}_E(-1) \to p^*E \to Q \to 0$.

Tensoring with line bundle $\mathcal{O}_E(1)$ and applying Cartan formula we get $c_n(\mathcal{O}_E(1) \otimes p^*E) = c_1(\mathcal{O})c_{n-1}(Q) = 0$. $\square$

2.5 Orienting a theory

In this subsection A_* is a homology theory. Two theorems in this subsection shows how one can construct an orientation using a commutative Chern structure (or a Thom structure) on A and how one can construct a commutative Chern structure (or a Thom structure) using an orientation.

Let us recall that for a vector bundle E over a variety X we identify X with $z(X)$ where $z : X \to E$ is the zero section.

2.5.1 Definition. *An orientation on the theory A is a rule assigning to each smooth variety X, to each its closed subset Z and to each vector bundle E/X an isomorphism*

$$th^Z(E) : A_*^Z(E) \to A_*^Z(X)$$

which satisfies the following properties

1. *invariance: for each vector bundle isomorphism $\varphi : E \to F$ the diagram commutes*

$$\begin{array}{ccc} A_*^Z(E) & \xrightarrow{th^Z(E)} & A_*^Z(X) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle id} \\ A_*^Z(F) & \xrightarrow{th^Z(F)} & A_*^Z(X) \end{array}$$

2. *base change: for each morphism $f : (X', X' - Z') \to (X, X - Z)$ with closed subsets $Z \hookrightarrow X$ and $Z' \hookrightarrow X'$ and for each vector bundle E/X and for its pull-back E' over X' and for the projection $g : E' = E \times_X X' \to E$ the diagram commutes*

$$\begin{array}{ccc} A_*^{Z'}(E') & \xrightarrow{th^{Z'}(E')} & A_*^{Z'}(X') \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ A_*^{Z}(E) & \xrightarrow{th^{Z}(E)} & A_*^{Z}(X) \end{array}$$

3. for each vector bundles $p : E \to X$ and $q : F \to X$ the following diagram commutes

$$\begin{array}{ccc} A_*^{Z}(E \oplus F) & \xrightarrow{th^{Z}(p^*F)} & A_*^{Z}(E) \\ {\scriptstyle th^Z(q^*E)}\downarrow & & \downarrow{\scriptstyle th^Z(E)} \\ A_*^{Z}(F) & \xrightarrow{th^{Z}(F)} & A_*^{Z}(X) \end{array}$$

and both compositions coincide with the operator $th^Z(E \oplus F)$.

The operators $th^Z(E)$ are called Thom isomorphisms. The theory A_* is called orientable if there exists an orientation of A_*. The theory A_* is called oriented if an orientation is chosen and fixed.

2.5.2 Lemma. *If an assignment $(E, X, Z) \mapsto th^Z(E)$ is an orientation on A_*, then its restriction to line bundles is a Thom structure on A_*.*

If two orientations coinside on each line bundle then they coincide.

Proof. The first assertion is obvious. To prove the second assertion consider two Thom orientations $th(-)$ and $th'(-)$ which coincide on line bundles. To prove that for a vector bundle E one has the relation $th^Z(E) = (th')^Z(E)$ one may assume by the splitting principle (c.f. Lemma **3.3.5** from [P2]) that $E = \oplus L_i$ is a direct sum of line bundles. As each of isomorphisms $th^Z(E)$ and $(th')^Z(E)$ can be described in terms of coinciding line bundles Thom isomorphisms then $th^Z(E) = (th')^Z(E)$. □

2.5.3 Theorem. *Given a commutative Chern structure $(L, X, Z) \mapsto e^Z(L)$ on A_* and corresponding Thom structure $(L, X, Z) \mapsto \alpha^Z(L)$ there exists an orientation $(X, Z, E) \mapsto th^Z(E)$ on A_* such that the following properties hold*

1. *for each smooth variety X, each line bundle L/X and each closed subset $Z \subset X$ one has $\alpha^Z(L) = th^Z(L)$;*
2. *for each smooth X, closed subset $Z \subset X$ and each line bundle L/X one has $e^Z(L) = th^Z(L) \circ z_*$ where $z : (X, X - Z) \to (L, L - Z)$ is the zero section.*

Moreover the required orientation is uniquely determined both by the property (1) and by the property (2).

2.5.4 Theorem. *If $(X, Z, E) \mapsto th^Z(E)$ is an orientation on A then the assignment $(L, X, Z) \mapsto th^Z(L) \circ z_*$ is a COMMUTATIVE Chern structure on A_*, the assignment $(L, X, Z) \mapsto th^Z(L)$ is a Thom structure on A_* and so constructed Chern and Thom structures correspond to each other.*

Moreover the construction of an orientation by means of a Chern (or a Thom) structure given by Theorem 2.5.3 and the construction of a Chern and a Thom structure by means of an orientation are inverse of each other.

Proof of Theorem 2.5.3. Let E/X be a rank n vector bundle, $Z \subset X$ be a closed subset and let $F = E \oplus 1$. Recall that the sequence 2.3.8 is exact:

$$0 \to A_*^{p_E^{-1}(Z)}(\mathbf{P}(E)) \xrightarrow{i_*} A_*^{p_F^{-1}(Z)}(\mathbf{P}(F)) \xrightarrow{\beta} A_*^{s_*(Z)}(\mathbf{P}(F)) \to 0.$$

Since

$$c_n^{p_F^{-1}(Z)}(\mathcal{O}_F(1) \otimes p_F^* E) i_* = i_* c_n^{p_E^{-1}(Z)}(\mathcal{O}_E(1) \otimes p_E^* E) = 0$$

by Proposition 2.4.5 then $c_n^{p_F^{-1}(Z)}(\mathcal{O}_F(1) \otimes p^* E)$ factors through support restriction β:

$$c_n^{p_F^{-1}(Z)}(\mathcal{O}_F(1) \otimes p_F^* E) = \widehat{c}_n^{\,p_F^{-1}(Z)}(\mathcal{O}_F(1) \otimes p_F^* E) \circ \beta.$$

Set

$$\overline{th^Z(E)} = p_*^F \circ \widehat{c}_n^{\,p_F^{-1}(Z)}(\mathcal{O}_F(1) \otimes p^* E) : A_*^{s_*(Z)}(\mathbf{P}(F)) \to A_*^Z(X). \tag{19}$$

and define the homomorphism $th^Z(E) : A_*^Z(E) \to A_*^Z(X)$ as follows

$$th^Z(E) = \overline{th^Z(E)} \circ j_*. \tag{20}$$

(here $p : \mathbf{P}(F) \to X$ is the projection and $j : E = \mathbf{P}(F) - \mathbf{P}(1) \hookrightarrow \mathbf{P}(F)$ is the open inclusion). To show that the assignment $(X, Z, E) \mapsto th^Z(E)$ is an orientation it remains to check the properties from 2.5.1.

The second and the first property follows immidiately from the functoriality of the Chern classes 2.4.2.

To prove that the operator $th^Z(E)$ is an isomorphism it is enough to check that $\overline{th(E)}$ is an isomorphism.

For that consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_*(\mathbf{P}(E)) & \longrightarrow & A_*(\mathbf{P}(F)) & \longrightarrow & A_*^{\mathbf{P}(1)}(\mathbf{P}(F)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle (\overline{th(E)}\circ\beta,\ \alpha)} & & \downarrow{\scriptstyle \overline{th(E)}} & & \\
0 & \longrightarrow & A(X)^n & \longrightarrow & A(X) \oplus A(X)^n & \longrightarrow & A(X) & \longrightarrow & 0
\end{array}$$

where $\gamma = (p_*^E, p_*^E \zeta_E, \ldots, p_*^E \zeta_E^{n-1})$, $\zeta_E = e(\mathcal{O}_E(1))$ and $\alpha = (p_*^F, p_*^F \zeta_F, \ldots, p_*^F \zeta_F^{n-1})$, $\zeta_F = e(\mathcal{O}_F(1))$.

The map γ is an isomorphism by the projective bundle theorem. Thus to prove that the right vertical arrow is an isomorphism it suffices to check that the map $(\overline{th(E)} \circ \beta, \alpha)$ is an isomorphism. Using the Mayer-Vietoris property and Remark 2.2.5 one may assume that the bundle E is the trivial rank n bundle. In this case one has $c_n(\mathcal{O}_F(1)^n) = \zeta_F^n$. Thus the map $(\overline{th(E)} \circ \beta, \quad \alpha) = (p_*^F c_n(\mathcal{O}_F(1) \otimes p_*^F E), \quad \alpha)$ coincides in this case with the map $(p_*^F \zeta_F^n, \quad p_*^F, \ldots, p_*^F \zeta_F^{n-1})$ and it is an isomorphism by the projective bundle theorem.

Basically the property (3) follows from the Cartan formula for Chern classes. But to give a detailed prove one needs certain preliminaries. For simplicity we write the proof below without supports. It will be an easy exercise to give a complete proof in usual way.

Let $E = E_1 \oplus E_2$ be a vector bundle over a smooth variety X and let $F_r = E_r \oplus 1$ $(r = 1, 2)$ and let $F = E \oplus 1$ and let $p : \mathbf{P}(F) \to X$ be the projection.

We will identify E with the open subset $\mathbf{P}(F) - \mathbf{P}(E)$ of $\mathbf{P}(F)$ and identify E_i with the open subset $\mathbf{P}(F_r) - \mathbf{P}(E_i)$ of $\mathbf{P}(F_r)$. Let $\mathbf{P}(F_i)$ be the subvariety in $\mathbf{P}(F)$ defined by the direct summund F_r of F. Let $i_r : \mathbf{P}(F_i) \hookrightarrow \mathbf{P}(F)$ be the closed embedding.

Let $p_r : \mathbf{P}(F_r) \to \mathbf{P}(1)$ be the projection and let $q_r : \mathbf{P}(F) \to \mathbf{P}(F_r)$ be the projection. Then we have a diagram

$$\begin{array}{ccc} \mathbf{P}(F) & \xrightarrow{q_2} & \mathbf{P}(F_1) \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle p_1} \\ \mathbf{P}(F_2) & \xrightarrow{p_2} & \mathbf{P}(1). \end{array}$$

In the diagram we can regard $\mathbf{P}(F) \xrightarrow{q_2} \mathbf{P}(F_1)$ as a projectivization of a vector bundle $p_1^* F_2 = 1 \oplus p_1^* E_2$ over $\mathbf{P}(F_1)$ and similarly can treate $\mathbf{P}(F) \xrightarrow{q_1} \mathbf{P}(F_2)$.

The support restriction operators

$$\beta_r : A_*(\mathbf{P}(F)) \to A_*^{\mathbf{P}(F_r)}(\mathbf{P}(F))$$

are surjective for $(r = 1, 2)$ because they are the operators from the short exact sequences of the form (2.3.8).

Denote the homomorphisms

$$c^Z_{n_1}(\mathcal{O}_F(1) \otimes p^*(E_1)), c^Z_{n_2}(\mathcal{O}_F(1) \otimes p^*(E_2)) \in Hom(A_*(\mathbf{P}(F)), A_*(\mathbf{P}(F)))$$

by α_1^Z and α_2^Z respectively.

Consider a following diagram:

$$\begin{array}{ccccc} A_*(\mathbf{P}(F)) & \xrightarrow{\alpha_1} & A_*(\mathbf{P}(F)) & \xrightarrow{\alpha_2} & A_*(\mathbf{P}(F)) \\ {\scriptstyle \beta_1}\downarrow & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle q^2_*} \\ A_*^{\mathbf{P}(F_1)}(\mathbf{P}(F)) & \xrightarrow{\alpha_1^{\mathbf{P}(F_1)}} & A_*^{\mathbf{P}(F_1)}(\mathbf{P}(F)) & \xrightarrow{\overline{th(p_1^* E_2)}} & A_*(\mathbf{P}(F_1)) \\ {\scriptstyle \beta'}\downarrow & & \downarrow{\scriptstyle q^1_*} & & \downarrow{\scriptstyle p^1_*} \\ A_*^{\mathbf{P}(1)}(\mathbf{P}(F)) & \xrightarrow{\overline{th^{\mathbf{P}(1)}(p_2^* E_1)}} & A_*^{\mathbf{P}(1)}(\mathbf{P}(F_2)) & \xrightarrow{\overline{th(E_2)}} & A_*(\mathbf{P}(1)). \end{array}$$

Left bottom square and top right square are just from definition of respective Thom isomorphisms. Right bottom square is commutative from the functoriality of Thom isomorphisms (recall that q_1 is a projection in a projective bundle associated with $p_1^* E_2$).

As $\beta : A_*(\mathbf{P}(F)) \to A_*^{\mathbf{P}(1)}(\mathbf{P}(F))$ is onto then it is enough to prove a relation

$$\overline{th(E_2)} \circ \overline{th^{\mathbf{P}(1)}(p_2^* E_1)}\beta = \overline{th^{\mathbf{P}(1)}(E)}\beta \tag{21}$$

Diagram above shows that left hand is equal to $p_* \alpha_1 \alpha_2$. But the right hand is $p_* c_{n_1+n_2}(\mathcal{O}_F(1) \otimes p^* E)$. By cartan formula $\alpha_1 \alpha_2 = c_{n_1+n_2}((\mathcal{O}_F(1) \otimes p^* E_1) \oplus (\mathcal{O}_F(1) \otimes p^* E_2)) = c_{n_1+n_2}(\mathcal{O}_F(1) \otimes p^* E)$ as required to prove the property (3). Thus the assignment $(X, Z, E) \mapsto th^Z(E)$ is indeed an orientation on A_*.

We still have to check that the orientation 1 and 2 of Theorem 2.5.3.

The first is obvious by very definition of higher Thom isomorphisms coinciding with that of Thom isomorphism for line bundle.

The requirement 2 is satisfied from the first one and Lemma 2.2.11.

To complete the proof of Theorem it remains to prove the uniqueness of the orientation. To prove the uniqueness of the orientation satisfying the property 1 take two orientations ω and ω' on A_* satisfying the property 1. To check that they coincide it suffices by Lemma 2.5.2 to check that their restrictions to line bundles coincide. This is the case by the requirement 1. Thus $\omega = \omega'$.

Now prove the uniqueness of the orientation satisfying the requirement 2. Let $L \mapsto e(L)$ be a Chern structure and let ω and ω' be two orientations satisfying the requirement 2. We will show that $\omega = \omega'$.

The restriction of these orientations to line bundles are Thom structures on A. By the same Lemma the orientations coincide if the mentioned Thom structures on A coincide. To prove that the two Thom structures on A coincide it suffices by Lemma 2.2.9 to check that the two corresponding Chern structures $L \mapsto e_\omega(L)$ and $L \mapsto e_{\omega'}(L)$ coincide. But that is the case. The proof of the relation $\omega = \omega'$ is completed. □

Proof of Theorem 2.5.4. The restriction of orientation on line bundles is a Thom structure.

By lemma 2.2.11 assignment $L \mapsto e(L) = th(L) \circ z_*$ is a Chern structure. The only to prove, that this Chern structure is commutative.

Let L_1 and L_2 be line bundles over X, $Z \subset X$ is a closed subset. We need to check $e^Z(L_1)e^Z(L_2) = e^Z(L_2)e^Z(L_1)$.

Consider a rank 2 bundle $E = L_1 \oplus L_2$. Let $z_E : X \to E$, $z_1 : X \to L_1$ and $u_2 : L_1 \to E$ be a zero sections of respective bundles.

Let us calculate a composition $A_*^Z(X) \xrightarrow{s_*^E} A_*^Z(E) \xrightarrow{th^Z(E)} A_*^Z(X)$ in two ways.

For the first one, $th^Z(E)s_*^E = th^Z(L)th^Z(p_1^*L_2)s_*^E$ where $p_1 : L_1 \to X$ is a projection. Consider a diagram

$$\begin{array}{ccccc} A_*^Z(X) & \xrightarrow{z_*^1} & A_*^Z(L_1) & \xrightarrow{u_*^2} & A_*^Z(E) \\ \downarrow{\scriptstyle e^Z(L_2)} & & \downarrow{\scriptstyle e^Z(p_1^*L_2)} & & \downarrow{\scriptstyle th^Z(p_1^*L_2)} \\ A_*^Z(X) & \xrightarrow{s_*^1} & A_*^Z(L) & \xrightarrow{id} & A_*^Z(L). \end{array}$$

Right square is commutative by definition Chern by Thom (cf. Lemma 2.2.11). Left square is commutative by functoriality of Chern.

Recall that also by Lemma 2.2.9 $th^Z(L_1)z_*^1 = e^Z(L_1)$.

As $u_2 s_1 = s_E$ we get from diagram above $th^Z(L)th^Z(p_1^*L_2)s_*^E = th^Z(L_1)s_*^1 e^Z(L_2) = e^Z(L_1)e^Z(L_2)$.

Commutativity of Chern is proven because left hand is stable under exchanging L_1 and L_2.

Now verify that the correspondences between orientations and Chern structures in the two theorems are inverse to each other. We know that the correspondences between Thom structures and Chern structures are inverse to each other. By theorem 2.5.3 an orientation is uniquely defined by corresponding Thom structure. Moreover, the correspondences between orientation being restricted to line bundles and Chern structure defined by theorem 2.5.3 are the same as the correspondences between Thom structures and Chern structures described in Lemmas 2.2.9 and 2.2.11. Since the last correspondences are inverse to each other then the correspondences described by theorem 2.5.3 are inverse to each other.

The theorem is proved.

□

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