

The Postnikov tower  
in motivic stable homotopy theory

Great Lakes  $K$ -theory X  
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## Goals:

- Give a concrete description of Voevodsky' slice tower in terms of “generalized cycle complexes” .
- Compute the slices of algebraic  $K$ -theory: motivic cohomology.
- Describe the building blocks of  $\mathbb{A}^1$ -stable homotopy theory: birational motives.

# Outline

- Voevodsky's slice tower
- The homotopy coniveau tower
- The unstable theory
- The stable theory
- The stable homotopy motives

**Voevodsky's slice tower**

## Categories of spaces

Fix a field  $k$  (perfect?).

- $\mathbf{Spc} :=$  the category of simplicial sets: “spaces” .
- $\mathbf{Spc}(k) :=$  the category of presheaves of spaces on  $\mathbf{Sm}/k$ : “spaces over  $k$ ” .
- Pointed versions  $\mathbf{Spc}_*$ ,  $\mathbf{Spc}_*(k)$ .

$\mathbf{Spc} \rightarrow \mathbf{Spc}(k)$ :  $S \mapsto$  “constant presheaf”  $S$ .

$\mathbf{Sm}/k \rightarrow \mathbf{Spc}(k)$ :  $X \mapsto$  “representable presheaf”  $Y \mapsto \mathrm{Hom}(Y, X)$ .

## Categories of spectra.

- $\mathbf{Spt} :=$  the category of “standard” spectra of pointed spaces.
- $\mathbf{Spt}(k) :=$  the category of presheaves of spectra on  $\mathbf{Sm}/k$   
= the category of spectra of pointed spaces over  $k$ .
- $\mathbf{Spt}_{\mathbb{P}^1}(k) :=$  the category of  $\mathbb{P}^1$ -spectra over  $k$ :

$$\begin{aligned} \mathcal{E} &:= (E_0, E_1, \dots, E_n, \dots), \quad E_n \in \mathbf{Spt}(k) \\ &+ \\ \epsilon_n &: E_n \rightarrow \Omega_{\mathbb{P}^1} E_{n+1} \end{aligned}$$

$$\Omega_{\mathbb{P}^1} E(X) := E(X \times \mathbb{P}^1 / X \times \infty) := \text{fib}[E(X \times \mathbb{P}^1) \rightarrow E(X \times \infty)].$$

## Homotopy categories

- $\mathbf{Spt} \rightarrow \mathcal{SH}$  the stable homotopy category
- $\mathbf{Spt}(k) \rightarrow \mathcal{SH}_{S^1}^{\mathrm{Nis}}(k) =: \mathcal{SH}_s(k)$  the stable homotopy category of  $S^1$ -spectra on  $\mathbf{Sm}/k_{\mathrm{Nis}}$ .
- $\mathbf{Spt}(k) \rightarrow \mathcal{SH}_{S^1}^{\mathrm{Nis}}(k) \rightarrow \mathcal{SH}_{\mathbb{A}^1}^{\mathrm{Nis}}(k) =: \mathcal{SH}_{\mathbb{A}^1}(k)$  the  $\mathbb{A}^1$ -stable homotopy category of  $S^1$ -spectra on  $\mathbf{Sm}/k_{\mathrm{Nis}}$ .
- $\mathbf{Spt}_{\mathbb{P}^1}(k) \rightarrow \mathcal{SH}_{\mathbb{P}^1}^{\mathrm{Nis}}(k) =: \mathcal{SH}(k)$  the  $\mathbb{A}^1$ -stable homotopy category of  $\mathbb{P}^1$ -spectra on  $\mathbf{Sm}/k_{\mathrm{Nis}}$ .

## Suspension spectra and infinite-loop spectra

We have infinite suspension and 0th-space functors

$$\mathbf{Spc}_* \begin{array}{c} \xrightarrow{\Sigma^\infty} \\ \xleftarrow{\Omega^\infty} \end{array} \mathbf{Spt}$$

$$\mathbf{Spc}_*(k) \begin{array}{c} \xrightarrow{\Sigma^\infty} \\ \xleftarrow{\Omega^\infty} \end{array} \mathbf{Spt}(k)$$

$$\mathbf{Spt}(k) \begin{array}{c} \xrightarrow{\Sigma_{\mathbb{P}^1}^\infty} \\ \xleftarrow{\Omega_{\mathbb{P}^1}^\infty} \end{array} \mathbf{Spt}_{\mathbb{P}^1}(k)$$

which are (left/right) adjoints on the respective homotopy categories.

Write  $\Sigma_{\mathbb{P}^1}^\infty$  for  $\Sigma_{\mathbb{P}^1}^\infty \circ \Sigma^\infty : \mathbf{Spc}_*(k) \rightarrow \mathbf{Spt}_{\mathbb{P}^1}(k)$ , e.g.  $\Sigma_{\mathbb{P}^1}^\infty X_+$ , for  $X \in \mathbf{Sm}/k$ .

**Definition 1**  $\mathcal{SH}_{\mathbb{P}^1}^{\text{eff}}(k) \subset \mathcal{SH}_{\mathbb{P}^1}(k)$  is the image of

$$\Sigma_{\mathbb{P}^1}^{\infty} : \mathcal{SH}_{\mathbb{A}^1}(k) \rightarrow \mathcal{SH}_{\mathbb{P}^1}(k).$$

This gives the tower of full triangulated subcategories

$$\dots \subset \Sigma_{\mathbb{P}^1}^{n+1} \mathcal{SH}_{\mathbb{P}^1}^{\text{eff}}(k) \subset \Sigma_{\mathbb{P}^1}^n \mathcal{SH}_{\mathbb{P}^1}^{\text{eff}}(k) \subset \dots \subset \mathcal{SH}_{\mathbb{P}^1}^{\text{eff}}(k) \subset \dots \subset \mathcal{SH}_{\mathbb{P}^1}(k).$$

**Lemma 1** The inclusion  $i_n : \Sigma_{\mathbb{P}^1}^n \mathcal{SH}_{\mathbb{P}^1}^{\text{eff}}(k) \rightarrow \mathcal{SH}_{\mathbb{P}^1}(k)$  admits a right adjoint  $r_n : \mathcal{SH}_{\mathbb{P}^1}(k) \rightarrow \Sigma_{\mathbb{P}^1}^n \mathcal{SH}_{\mathbb{P}^1}^{\text{eff}}(k)$ .

This yields the exact functor  $f_n := i_n \circ r_n : \mathcal{SH}_{\mathbb{P}^1}^{\text{eff}}(k) \rightarrow \Sigma_{\mathbb{P}^1}^n \mathcal{SH}_{\mathbb{P}^1}^{\text{eff}}(k)$  and the natural *slice* tower (introduced by Voevodsky)

$$\dots \rightarrow f_{n+1} \mathcal{E} \rightarrow f_n \mathcal{E} \rightarrow \dots \rightarrow f_0 \mathcal{E} \rightarrow f_{-1} \mathcal{E} \rightarrow \dots \rightarrow \mathcal{E}.$$

## The slices

**Definition 2** For  $\mathcal{E} \in \mathcal{SH}(k)$ , set  $s_n \mathcal{E} := \text{cofib}(f_{n+1} \mathcal{E} \rightarrow f_n \mathcal{E})$ : the  $n$ th slice of  $\mathcal{E}$ .

**Conjecture 1 (Voevodsky)** The  $s_n \mathcal{E}$  are motives, i.e., have a natural  $\mathcal{H}\mathbb{Z}$ -module structure.

## Other problems

- What motives occur as slices?
- What are the slices of familiar objects in  $\mathcal{SH}(k)$ , e.g.,  $\mathcal{K}$ ,  $MGL$ ,  $\mathcal{K}O$ ,  $\mathcal{W}$ ,  $\mathcal{S}$ .

## Main results

**Theorem 1 (Voevodsky, L.)**  $s_0\mathcal{S} = \mathcal{HZ}$

This result implies Conjecture 1.

**Theorem 2 (L.)**  $s_n\mathcal{K} = \Sigma_{\mathbb{P}^1}^n \mathcal{HZ}$ .

**Theorem 3 (Hopkins-Morel)**  $s_n MGL = \Sigma_{\mathbb{P}^1}^n \mathcal{HL}^{-2n}$ ,  $\mathbb{L}^*$  is the Lazard ring:

$$\mathbb{L}^* = MU^*(pt.).$$

**Theorem 4 (L.)** *The slices  $s_n\mathcal{E}$ ,  $\mathcal{E} \in \mathcal{SH}(k)$  are exactly the motives of the form  $M(n)$ , where  $M \in DM(k)$  is a birational motive in the sense of Kahn-Sujatha.*

## The program:

- The construction of the spectral sequence from motivic cohomology to  $K$ -theory (Bloch-Lichtenbaum, Friedlander-Suslin) generalizes to the *homotopy coniveau tower* for an arbitrary  $E \in \mathbf{Spt}(k)$ ,  $X \in \mathbf{Sm}/k$ :

$$\dots \rightarrow E^{(n+1)}(X) \rightarrow E^{(n)}(X) \rightarrow \dots \rightarrow E^{(0)}(X) = E^{(-1)}(X) = \dots$$

- For “good”  $E$ ,  $X \mapsto E^{(n)}(X)$  is functorial in  $X$ , and one can describe the layers  $E^{(n/n+1)}(X)$  as a “generalized codimension  $n$  cycle complex”, with coefficients in  $(\Omega_{\mathbb{P}^1}^n E)^{(0/1)}$ .

- For  $\mathcal{E} = (E_0, E_1, \dots, E_m, \dots) \in \mathcal{SH}(k)$ , the sequence

$$(E_0^{(n)}, E_1^{(n+1)}, \dots, E_m^{(n+m)}, \dots)$$

gives a model for  $f_n \mathcal{E}$ .

**The homotopy coniveau tower**

*Notation:*

$$\Delta^n := \text{Spec } k[t_0, \dots, t_n] / \sum_i t_i - 1$$

A *face*  $F$  of  $\Delta^n$  is a closed subscheme defined by  $t_{i_1} = \dots = t_{i_r} = 0$ .

$n \mapsto \Delta^n$  extends to the cosimplicial scheme

$$\Delta^* : \mathbf{Ord} \rightarrow \mathbf{Sm}/k.$$

For  $E \in \mathbf{Spt}(k)$ ,  $X \in \mathbf{Sm}/k$ ,  $W \subset X$  closed, set

$$E^W(X) := \text{fib}(E(X) \rightarrow E(X \setminus W)).$$

- For  $X \in \mathbf{Sm}/k$ :

$$\mathcal{S}_X^{(p)}(n) := \{W \subset X \times \Delta^n, \text{ closed, } \text{codim}_{X \times F} W \cap (X \times F) \geq p\}.$$

- For  $E \in \mathbf{Sm}/k$ :

$$E^{(p)}(X, n) := \text{hocolim}_{W \in \mathcal{S}_X^{(p)}(n)} E^W(X \times \Delta^n).$$

- This gives the simplicial spectrum  $E^{(p)}(X)$ :  $n \mapsto E^{(p)}(X, n)$ , and the *homotopy coniveau tower*

$$\dots \rightarrow E^{(p+1)}(X) \rightarrow E^{(p)}(X) \rightarrow \dots \rightarrow E^{(0)}(X) = E^{(-1)}(X) = \dots$$

*Remark:*  $X \mapsto E^{(p)}(X)$  is functorial in  $X$  for *flat* maps.

# **The unstable theory**

**The axioms.** Fix an  $E \in \mathbf{Spt}(k)$ . We will impose 3 basic axioms:

1. *homotopy invariance:* For all  $X \in \mathbf{Sm}/k$ ,  $E(X) \rightarrow E(X \times \mathbb{A}^1)$  is a stable weak equivalence.
2. *Nisnevic excision:* Let  $f : Y \rightarrow X$  be an étale map in  $\mathbf{Sm}/k$ . Suppose  $W \subset X$  is a closed subset such that  $f : f^{-1}(W) \rightarrow W$  is an isomorphism. Then  $f^* : E^W(X) \rightarrow E^{f^{-1}(W)}(Y)$  is a stable weak equivalence.
3.  *$m$ -fold delooping:* There is an  $E_m \in \mathbf{Spt}(k)$  satisfying axioms 1 and 2 and a stable weak equivalence

$$E \sim \Omega_{\mathbb{P}^1}^m E_m.$$

We also assume that  $k$  is an infinite field.

**Theorem 5** *Let  $E$  be in  $\mathbf{Spt}(k)$  satisfying axioms 1, 2 and 3.m with  $m \geq 2$ . Then*

(1)  $X \mapsto E^{(p)}(X)$  extends (up to weak equivalence) to a functor  $E^{(p)} : \mathbf{Sm}/k^{\text{op}} \rightarrow \mathbf{Spt}$  satisfying axioms 1, 2 and 3.( $m - 2$ ).

(2) Localization. Let  $i : W \rightarrow X$  be a closed codimension  $d$  closed embedding in  $\mathbf{Sm}/k$ , with trivial normal bundle, and open complement  $j : U \rightarrow X$ . There is a natural homotopy fiber sequence in  $\mathcal{SH}$

$$(\Omega_{\mathbb{P}^1}^d E)^{(p-d)}(W) \rightarrow E^{(p)}(X) \xrightarrow{j^*} E^{(p)}(U)$$

(3) Delooping. Suppose  $E$  satisfies axioms 1,2 and 3.2. There is a natural weak equivalence

$$(\Omega_{\mathbb{P}^1}^m E)^{(n)} \xrightarrow{\sim} \Omega_{\mathbb{P}^1}^m (E^{(n+m)})$$

## Birationality:

**Corollary 1** Take  $E \in \mathbf{Spt}(k)$ ,  $X \in \mathbf{Sm}/k$ ,  $Z \subset X$  closed,  $\text{codim}_X Z \geq d$ . Then

$$(E^{(d/d+1)})^Z(X) \cong \bigoplus_{z \in Z \cap X^{(d)}} (\Omega_{\mathbb{P}^1}^d E)^{(0/1)}(k(z)).$$

In particular, for  $d = 0$ ,  $Z = X$  irreducible, this gives

$$E^{(0/1)}(X) \cong E^{(0/1)}(k(X)).$$

*Proof:* Localization +

For  $W \subset X$  smooth, codimension  $q > d$  with trivial normal bundle

$$(\Omega_{\mathbb{P}^1}^q E)^{(d-q/d-q+1)}(W) = (\Omega_{\mathbb{P}^1}^q E)^{(0/0)}(W) = 0.$$

## The idempotence theorem.

We iterate the operation  $E \mapsto E^{(p)}$ .

**Theorem 6** *Suppose  $E$  satisfies axioms 1, 2 and 3 (for  $m = 4$ ).  
Then*

$$(E^{(p)})^{(q)} \cong E^{(\max\{p,q\})}$$

*in  $\mathcal{SH}_{S^1}(k)$ .*

## Generalized cycle complexes

Recall Bloch's cycle complexes:

Let  $X^{(p)}(n) = \{z \in (X \times \Delta^n)^{(p)} \mid \bar{z} \in \mathcal{S}_X^{(p)}(n)\}$

Set

$$z^p(X, n) := \bigoplus_{z \in X^{(p)}(n)} \mathbb{Z}.$$

This gives the simplicial abelian group  $z^p(X, -)$ :

$$n \mapsto z^p(X, n).$$

Set:

$$\mathrm{CH}^p(X, n) := \pi_n(z^p(X, -)).$$

For  $X \in \mathbf{Sm}/k$ ,  $E^{(p/p+1)}(X)$  is a “generalized cycle complex” with coefficients  $(\Omega_{\mathbb{P}^1}^p E)^{(0/1)}$ :

**Corollary 2** *Suppose  $E$  satisfies axioms 1,2 and 3 (with  $m = 4$ ). There is a simplicial spectrum  $E_{s.l.}^{(p)}(X)$ , with*

$$E_{s.l.}^{(p)}(X)(n) \cong \bigoplus_{z \in X^{(p)}(n)} (\Omega_{\mathbb{P}^1}^p E)^{(0/1)}(z)$$

*and with  $E^{(p/p+1)}(X)$  is isomorphic in  $\mathcal{SH}$  to  $E_{s.l.}^{(p)}(X)$ .*

*Idea of proof:*

From the idempotence theorem:  $E^{(p/p+1)} = (E^{(p/p+1)})^{(p)}$ .

By definition of  $(E^{(p/p+1)})^{(p)}$ :

$(E^{(p/p+1)})^{(p)}(X)$  is the total spectrum of

$$n \mapsto \operatorname{hocolim}_{W \in \mathcal{S}_X^{(p)}(n)} (E^{(p/p+1)})^W(X \times \Delta^n).$$

From the birationality theorem:

$$(E^{(p/p+1)})^W(X \times \Delta^n) = \bigoplus_{w \in W \cap (X \times \Delta^n)^{(p)}} (\Omega_{\mathbb{P}^1}^p E)^{(0/1)}(w).$$

## ***K*-theory**

To compute  $K^{(p/p+1)}$ :  $\Omega_{\mathbb{P}^1} K = K$ , so just need to compute the “coefficients”  $K^{(0/1)}$ .

Note:  $E^{(0/1)}(X) = E(\Delta_{s.l.,k(X)}^*)$ , where

$$\Delta_{s.l.,F}^n = \text{Spec}(\mathcal{O}_{\Delta_F^n, \{v_0, \dots, v_n\}}).$$

$K_1(\mathcal{O}) = \mathcal{O}^\times$  for  $\mathcal{O}$  semi-local +  $K_1$ -regularity of NCD's  
 $\rightsquigarrow K(\Delta_{s.l.,F}^*) = EM(K_0(F)) = EM\mathbb{Z}$ .

By the generalized cycle complex theorem:

$$K^{(p/p+1)}(X) \cong K_{s.l.}^{(p)}(X) = |n \mapsto \bigoplus_{z \in X^{(p)}(n)} (\Omega_{\mathbb{P}^1} K)^{(0/1)}(z)|$$

Since  $(\Omega_{\mathbb{P}^1} K)^{(0/1)}(z) = EM\mathbb{Z}$ ,  $K^{(p/p+1)}(X) \cong z^p(X, -)$ .

# **The stable theory**

- The stable homotopy coniveau tower
- The comparison theorem
- The 0th slice of the sphere spectrum
- The layers as motives

## The stable homotopy coniveau tower

Let

$$\begin{aligned}\mathcal{E} &:= (E_0, E_1, \dots, E_n, \dots) \\ \epsilon_n &: E_n \rightarrow \Omega_{\mathbb{P}^1} E_{n+1}\end{aligned}$$

be a  $\mathbb{P}^1$ -spectrum over  $k$ . We assume that the  $\epsilon_n$  are weak equivalences.

For each  $n, m$  we have the weak equivalence  $\epsilon_n^{<m>}$ :

$$E_n^{(n+m)} \xrightarrow{(\epsilon_n)^{(n+m)}} (\Omega_{\mathbb{P}^1} E_{n+1})^{(n+m)} \xrightarrow{\text{deloop}} \Omega_{\mathbb{P}^1} (E_{n+1}^{(n+m+1)})$$

Set:

$$\mathcal{E}^{<m>} := (E_0^{(m)}, E_1^{(m+1)}, \dots, E_n^{(m+n)}, \dots)$$

with attaching maps  $\epsilon_n^{<m>}$ .

The homotopy coniveau towers

$$\dots \rightarrow E_n^{(m+n+1)} \rightarrow E_n^{(m+n)} \rightarrow \dots$$

fit together to form the  $\mathbb{P}^1$ -stable homotopy coniveau tower

$$\dots \rightarrow \mathcal{E}\langle m+1 \rangle \rightarrow \mathcal{E}\langle m \rangle \rightarrow \dots \rightarrow \mathcal{E}\langle 0 \rangle \rightarrow \mathcal{E}\langle -1 \rangle \rightarrow \dots \rightarrow \mathcal{E}.$$

in  $\mathcal{SH}(k)$ .

## The comparison theorem

**Proposition 1** *For  $\mathcal{E} \in \mathcal{SH}(k)$ ,  $\mathcal{E}\langle m \rangle$  is in  $\Sigma_{\mathbb{P}^1}^m \mathcal{SH}^{\text{eff}}(k)$*

The universal property of  $f_n \mathcal{E} \rightarrow \mathcal{E}$ , applied to  $\mathcal{E}\langle m \rangle \rightarrow \mathcal{E}$ , gives the map  $h : \mathcal{E}\langle m \rangle \rightarrow f_n \mathcal{E}$ .

**Theorem 7** *For each  $\mathcal{E} \in \mathcal{SH}(k)$ , the map  $h : \mathcal{E}\langle m \rangle \rightarrow f_n \mathcal{E}$  is an isomorphism.*

Main idea: First show  $(\Sigma_{\mathbb{P}^1}^\infty X_+)^{\langle 0 \rangle} = \Sigma_{\mathbb{P}^1}^\infty X_+$   
 $\rightsquigarrow \mathcal{E}^{\langle m \rangle} = \mathcal{E}$  for  $\mathcal{E} \in \Sigma_{\mathbb{P}^1}^m \mathcal{SH}^{\text{eff}}$ .

Using the universal property of  $f_m$ , and the definition of  $\mathcal{E}^{\langle m \rangle}$  in terms of spectra with support, get  $(f_m \mathcal{E})^{\langle m \rangle} = \mathcal{E}^{\langle m \rangle}$ .

Idempotence theorem  $\rightsquigarrow (\mathcal{E}^{\langle m \rangle})^{\langle m \rangle} = \mathcal{E}^{\langle m \rangle}$

Consider

$$\begin{array}{ccccc}
 (\mathcal{E}^{\langle n \rangle})^{\langle n \rangle} & \xrightarrow{p^{\langle n \rangle}} & \mathcal{E}^{\langle n \rangle} & \xrightarrow{p} & \mathcal{E} \\
 h^{\langle n \rangle} \downarrow & \nearrow q^{\langle n \rangle} & h \downarrow & & \parallel \\
 f_n(\mathcal{E})^{\langle n \rangle} & \xrightarrow{j} & f_n(\mathcal{E}) & \xrightarrow{q} & \mathcal{E}
 \end{array}$$

$p^{\langle n \rangle}$ ,  $j$  and  $q^{\langle n \rangle}$  are isomorphisms.  $\rightsquigarrow h^{\langle n \rangle}$  is an isomorphism  
 $\rightsquigarrow h$  is an isomorphism.

## The 0th slice of $\mathbb{S}$

**Theorem 8**  $s_0\mathbb{S} \cong \mathcal{HZ}$ .

*Idea:*

$$s_0\mathbb{S} \cong \mathbb{S}\langle 0/1 \rangle := ((\Sigma_{\mathbb{P}^1}^\infty \text{Spec } k_+)_0^{(0/1)}, (\Sigma_{\mathbb{P}^1}^\infty \text{Spec } k_+)_1^{(1/2)}, \dots)$$

$$\Sigma_{\mathbb{P}^1}^\infty (\text{Spec } k_+)_p^{(p/p+1)} = \text{codim } p \text{ cycle complex} \\ \text{with coefficients } (\Omega^\infty \Sigma_{\mathbb{P}^1}^\infty \text{Spec } k_+)^{(0/1)}.$$

This description allows one to construction a “reverse cycle map”

$$\text{rev} : \mathcal{HZ} \rightarrow \mathbb{S}\langle 0/1 \rangle$$

inverse to the canonical map  $\mathbb{S}\langle 0/1 \rangle \rightarrow \mathcal{HZ}\langle 0/1 \rangle = \mathcal{HZ}$ .

# **The stable homotopy motive**

**Motivic stable  $\pi_n$ :**

Theorem 8  $\rightsquigarrow$  the slices  $\mathcal{E}\langle n/n + 1 \rangle$  are *motives*.

**Definition 3** For  $\mathcal{E}$  in  $\mathcal{SH}(k)$ , let  $\pi_n^\mu(\mathcal{E}) \in DM(k)$  be the motive with

$$\mathcal{H}(\pi_n^\mu \mathcal{E}[n]) = \mathcal{E}\langle n/n + 1 \rangle.$$

$\pi_n^\mu(\mathcal{E})$  is the  $n$ th stable homotopy motive of  $\mathcal{E}$ .

Atiyah-Hirzebruch spectral sequence:

$$E_2^{p,q} = H^p(X, (\pi_{-q}^\mu \mathcal{E})(n)) \implies \hat{\mathcal{E}}^{p+q,n}(X)$$

**Theorem 9**  $\pi_n^\mu \mathcal{K} = \mathbb{Z}(n)[n]$  for all  $n$ .

## Birationality

**Definition 4** A motive  $M$  is birational if  $\text{Hom}_{DM(k)}(\mathbb{Z}_{tr}(X), M[n]) \rightarrow \text{Hom}_{DM(k)}(\mathbb{Z}_{tr}(U), M[n])$  is an isomorphism for all open immersions  $U \rightarrow X$  and all  $n$ .

**Proposition 2** (1)  $(\pi_n^\mu \mathcal{E})(-n)$  is birational for all  $\mathcal{E}, n$ .

(2) If  $M$  is birational, then  $M = \pi_0^\mu \mathcal{H}(M)$ .

In fact, the motive  $(\pi_n^\mu \mathcal{E})(-n)[-n]$  is the coefficient spectrum  $(\Omega_{\mathbb{P}^1}^n \mathcal{E})^{(0/1)}$  for  $\mathcal{E} \langle n/n+1 \rangle$ .  $(\Omega_{\mathbb{P}^1}^n \mathcal{E})^{(0/1)}$  is birational by the birationality theorem.

Conclusion: all objects in  $\mathcal{SH}(k)$  are built out of birational motives and Tate twists.

## New cycle theories

(1)  $K$ -theory:  $\pi_n^\mu(\mathcal{K})[-n] = \mathbb{Z}(n) \rightsquigarrow$  motivic cohomology  $H^p(-, \mathbb{Z}(n))$ .

(2)  $K$ -theory of a c.s. algebra  $A/k$  (w. Bruno Kahn):  $\pi_n^\mu(\mathcal{K}_A)[-n] = \mathbb{Z}_A(n) \rightsquigarrow$  twisted motivic cohomology  $H^p(-, \mathbb{Z}_A(n))$ .  $\mathbb{Z}_A$  is the birational motive  $\mathbb{Z}_A(X) = K_0(A \otimes_k k(X))$ .

(3)  $G$ -equivariant  $K$ -theory (w. Christian Serpé): Take a finite group  $G$  acting on a smooth  $k$ -scheme  $X$ .  $K^G(X) = K$ -theory of locally free sheaves on  $X$  with  $G$ -action.

Associated cycle-theory: For  $W \subset X \times_k Y$  irreducible,  $w \in W$  generic point,  $G_w =$  isotropy group.  $\mathbb{Z}_G(w) = K_0(k(w) * G_w)$ .

This leads to a  $G$ -Bredon motivic cohomology  $H^*(X, \mathbb{Z}_G(n))$ , and a spectral sequence converging to  $K^G(X)$ .

**Problem:**

Compute the birational motives  $\pi_n^\mu \mathcal{E}$  for known  $\mathcal{E}$ , e.g.  $\mathcal{K}\mathcal{O}$ ,  $\mathcal{W}$ .  
Understand these new cycle theories.

**Thank you**