

Lettre de Beilinson à Soulé, 1/11/82

.. I wrote a paper on higher regulators, but unfortunately it will be typed only to the end of this week, so I'll send it to you with the first next occasion. There are small number of facts you may not know : first, I wrote down the complete constructions for cyclotomic fields. Secondly, the hypothesis for the value of pr. part of L-functions now are formulated for any integral points. My objections against them near the middle of the critical strip were false. Bloch in his letter to me (March 82) wrote some supporting example for $L(2, M)$, where $M = J(X_0(37))$; in the paper I extended his calculations to cover the case of products of any two modular curves. The middle of the critical strip needs somewhat another constructions : the key point is that \mathcal{D} -cohomology permits to define the global intersection pairing for cycles of complementary dimensions on arithmetical schemes. This constructions generalises Arakelov's approach to heights on curves (cf. his Vancouver ICM-report). But I have no calculations off Birch-Sw-D case. At last, look at the following travestie of Hodge conj. (for varieties defined over number fields this is an implication of regulator conj. for K_1) Let X be a smooth projective variety over \mathbb{C} . Let $\{y_i\}$ be a set of 1-dimensional irreducible subschemes of X , $\{\varphi_i\}$ be a set of non-zero rational functions at generic points of y_i s.t. the $(1,1)$ -cycle on X $\sum \text{div}(\varphi_i)$ is zero (i.e. $C = \{y_i, \varphi_i\}$ is an element of E_1 for niveau filtration of K_1). Consider the distribution $[C]$ defined by the formula

$$\langle [C], \omega \rangle = \sum_i \int_{C_i} \ln |\varphi_i| \omega$$

for $\omega \in \Omega_{X^{\text{an}}}^{2,1}$ - note that integral converges. Obviously this distribution has type $(1,1)$ and is not closed; but you may add to $[C]$ a certain sum of distr. $\varphi_{p,q}$ of type (p,q) , $p \neq q$, $p+q=2,1$, s.t. $[C] + \sum \varphi_{p,q}$ becomes closed. It is easy to see, that the $(1,1)$ -component of cohom. class $([C] + \sum \varphi_{p,q})$ depends only on C ; denote it say (C) . Now I conjecture that any $(1,1)$ -cohomology class is a \mathbb{C} -linear combination of cycles of type (C) . I could not prove it for surfaces (off the case of the product of modular curves). This conj. implies the following fact, seeming excitingly simple but I do not know the proof : let X be a surface. Then one may find Zariski open $U \subset X$ s.t. the image of $H^2(X, \mathbb{C})$ in $H^2(U, \mathbb{C})$ is contained in $F^2 + F^2$ (F^2 is Deligne-Hodge filtration).

periodical K-th. tensored by \mathbb{Z}/n - so $\pi_i | K = \mathbb{Z}/n (i/2)$
 if i is even and 0 if i is odd. Consider the Postnikov filtration
 $\tau_{\leq i}$ on K_j . Then $K_i(X, \mathbb{Z}/n)$ should coincide with the
 cohomology of X_{Zar} with values in (sheaf of spectra of maps from
 X_{et} to $\tau_{\leq i} K$).

Because of Gersten's conj. it suffices to look at the case
 of fields. The conj. stands, that up to factorials

$$K_j(F, \mathbb{Z}/n\mathbb{Z}) = \bigoplus_{i \leq j} H_{\text{gal}}^{2i-j}(F, \mathbb{Z}/n\mathbb{Z}(i))$$

By Thomason's theorem, this is equivalent to the following :

- a) Periodicity map $\nu_{\beta} : K(F, \mathbb{Z}/n\mathbb{Z}) \rightarrow K(F, \mathbb{Z}/n\mathbb{Z})$ is injective.
- b) $K/\beta \cup K = \bigoplus_{i \geq 0} H_{\text{gal}}^i(F, \mathbb{Z}/n\mathbb{Z}(i))$.

This is quite analogous to Bott's periodicity in topology !

Namely, it says, that if K^{top} is non-periodised top. K-th
 (i.e. $K_i^{\text{top}} = 0$ for $i < 0$) then $\nu_{\beta} : K^{\text{top}} \rightarrow K^{\text{top}}$ is injection,
 and $K^{\text{top}}/\beta \cup K^{\text{top}} = K_0^{\text{top}} \oplus K_1^{\text{top}} = \bigoplus H^i(\mathbb{Z}[\frac{1}{n}]))$ (up to
 factorials). So the situation should be the following. By Thomason,
 alg. and top, K-th has the same periodical K-th. And to obtain
 either of ones you have to cut by halves in two different ways.

Another (global) conj. Suppose that X is smooth projective over
 finite field. Then $K_j(X, \mathbb{Q}_\ell) (= K_j(X, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell)$ equals
 to the subspace of $\bigoplus H^{2i-j}(X, \mathbb{Q}_\ell(i))$ on which
 arithm. Frobenius acts with integral ~~xxxx~~ eigenvalues.