Borel-Moore Functors and Algebraic Oriented Theories

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Abstract

Borel-Moore functors for singular schemes and algebraic oriented theories for smooth pairs of schemes are classical, but different tools. The aim of this paper is to give a canonical construction which explains when these two structures are equivalent, namely when a Borel-Moore functor arises from an algebraic oriented theory and what type of algebraic oriented theories can be constructed from a Borel-Moore functor.

Contents

1 Introduction

Let $(\mathbf{Sch_k})'$ be the category of quasi-projective schemes with morphisms just the projective morphisms. A Borel-Moore functor is an additive covariant functor $H : (\mathbf{Sch_k})' \longrightarrow \mathbf{Ab}$ endowed with pull-backs for some special type of morphisms, called embeddable maps, and which verifies a given set of axioms. Algebraic oriented theories are define on the category of smooth pairs, **SP**. Objects in this category are pairs (M, X) with M a smooth scheme and X a closed subscheme of M; a morphism $f : (M, X) \longrightarrow (N, Y)$ is given by a scheme morphism $f : M \longrightarrow N$ such that $f^{-1}(Y) \subset X$. Clearly the category of smooth pairs can be embedded in the above category, if for any smooth scheme M we consider the smooth pair (M, M). An algebraic oriented cohomology theory is an additive contravariant functor $A : \mathbf{SP} \longrightarrow \mathbf{Ab}$ endowed with push-forwards for some special type of diagrams, and that verifies a given set of axioms. The Chow groups, the Grothendieck groups and the algebraic cobordism theory the can be seen as Borel-Moore functors, while operational Chow groups form an algebraic oriented theory.

It looks reasonable to ask how much these two concepts are alike in general, since a Poincare-type duality holds for the standard theories. Moreover, such a duality would allow us to translate problems for singular schemes into a smooth set-up, where a solution may be easier to find. The goal of this paper is to construct a canonical connection between Borel-Moore functors with a Gysin structure and algebraic oriented theories.

While Chow groups have pull-backs for flat maps, this is not the set-up for a general Borel-Moore functor. For example, in the algebraic cobordism theory ([ML01]) we have pull-backs defined for smooth equidimensional maps while the Grothendieck groups have pull-backs defined for maps of finite Tor-dimension. The class of morphisms that need to have pull-backs under a general Borel-Moore functor is the class of *embeddable* morphisms, introduced in the first section.

The construction of a Borel-Moore functor induced by an algebraic oriented theory is given in the second section. We need to work in the category of quasi-projective schemes because any such scheme can be seen as a closed subscheme of a smooth one, which is not true in general for schemes of finite-type. The closed embedding $X \hookrightarrow M$, with M a smooth scheme, allows us to define the associated Borel-Moore functor by $H(X) = A_X(M)$. The push-forwards for projective morphisms are defined in the standard way, first for closed embeddings and projections and second for any $f = p \circ i$. The main difficulty in this construction is proving that all the definitions are independent of the choices made. This is the reason why axiom $(A5)$ (which is not a canonical axiom for a cohomology theory) appears in the list of axioms for an algebraic oriented theory. The remaining part of this section checks that H is indeed a Borel-Moore functor with an associated Gysin structure.

The next section gives the reverse process. Given a Borel-Moore functor H we can simply define the associated algebraic oriented theory by $A_X(M) = H(X)$. However, the pull-backs in this theory can be constructed only if the Borel-Moore functor has a Gysin structure associated. An algebraic oriented theory has push-forwards associated to any commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ M & \xrightarrow{F} & N \end{array}$$

with f a projective morphism, $X \hookrightarrow M, Y \hookrightarrow N$ closed embeddings; we denote this push-forward by f_A^F. In the above diagram the lower map F plays a minimal role, fact underlined by axiom $(A2b)$, which states that for any two morphisms F, G that make the given diagram commutative the corresponding push-forwards are equal. Nevertheless in the smooth pairs context this map F needs to be mentioned; however, if A is a theory induced by a Borel-Moore functor H, then we simply define $f_A^F = f_*$.

The main challenge for the second part lies not in the basic construction of A, which is canonical, but in proving that this theory has all the needed properties. For example, since the pull-backs of H are not functorial (it is not generally true that $(f \circ g)^* = g^* \circ f^*$, moreover, the composition of two embeddable maps needs not be embeddable), the functoriality of A is tedious to verify.

The last section describes the way in which these two constructions are reciprocal. More precisely, if one starts with a Borel-Moore functor with an associated Gysin structure, builds an algebraic oriented theory from which a second Borel-Moore functor is constructed, the these two functors are equal. This proves that any Borel-Moore functor with a Gysin structure can be obtained from an algebraic oriented theory. The reverse is not true in general; the description of those theories that can be constructed this way is given by:

Proposition 1.1. *Let M be a smooth scheme, X, Y two closed subschemes of M, such that $j : X \hookrightarrow Y$ is a closed embedding. In this set-up we have a well-defined map of pairs $F : (M, Y) \longrightarrow (M, X)$. If for any such map we have $F^A = j_A^{Id_M}$ then the theory $\tilde{A}$, constructed by applying the algorithm twice, $A \longrightarrow \tilde{H} \longrightarrow \tilde{A}$, is equivalent with the original theory A.*

Both Borel-Moore functors and algebraic oriented theories are not defined as graded theories in this paper; this choice was made to simplify the exposition. However, the construction and most of the proofs were done keeping track of a possible grading. Consequently one may add the standard grading conventions ($H_i(X) = A_X^{dim(M)-i}(M)$, pull-backs and Gysin morphisms change grading for H by codimension, push-forwards change grading for A) and the results would still hold.

I am especially grateful to Alexander Merkurjev, who suggested the problem and without whose involvement this paper would not be finished.

1.1 Notations and Conventions

1. $\mathbf{Sm_k}$ is the category of smooth schemes over a field k.
2. $\mathbf{Sch_k}$ is the category of quasi-projective schemes over a field k.
3. $(\mathbf{Sch_k})'$ is the subcategory with objects quasi-projective schemes and morphisms just the projective morphisms.
4. **SP** is the category of smooth pairs.
5. **Ab** is the category of abelian groups.
6. A closed embedding between two schemes is denoted by "$\hookrightarrow$".

1.2 Embeddable Morphisms

Let k be a field, and $\mathbf{Sch_k}$ the category of quasi-projective schemes over k. Consider further $(\mathbf{Sch_k})'$ the sub-category with objects quasi-projective schemes over k and morphisms only the projective morphisms.

Definition 1.2. Let X, X' be two quasi-projective schemes. A morphism $f : X \longrightarrow X'$ is called *embeddable* if f is smooth equidimensional and if there exist smooth schemes M, M', closed embeddings $i : X \hookrightarrow M$ and $i' : X' \hookrightarrow M'$ and a smooth morphism $g : M \longrightarrow M'$ such that the following square is a fibre square:

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \underset{i'}{\hookrightarrow} & M' \end{array}$$

Since we are mostly interested in a few special types of embeddable morphisms, it is worth listing them from the beginning.

Example 1.3. Any open embedding $j : X \longrightarrow X'$ is an embeddable morphism. To prove this we need to construct is fibre square associated to this map, since any open embedding is always smooth equidimensonal. For this consider $i' : X' \longrightarrow M'$ a closed embedding into some smooth scheme M'; then there exists an open subscheme M of M' such that $X \hookrightarrow M$ is a closed embedding, and the following is a fibre square:

$$\begin{array}{ccc} X & \hookrightarrow & M \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle f} \\ X' & \underset{i'}{\hookrightarrow} & M' \end{array}$$

therefore j is embeddable.

Example 1.4. Any vector bundle map $p : E \longrightarrow X$ is an embeddable morphism. First, it is smooth and equidimensional, and from [Ful84, Lemma 18.2] we can construct the associated fibre square. We are going to give an explicit construction of this fibre square in Theorem 2.13.

Example 1.5. If X is smooth, irreducible and $p : X \times Y \longrightarrow Y$ is the standard projection, then p is embeddable for any quasi-projective scheme Y. To prove this first note that since X is smooth p is smooth and since X is irreducible p is equidimensional. Now consider M a smooth scheme such that $i : Y \longrightarrow M$ is a closed embedding (since Y is quasi-projective such a smooth scheme exists always); then the following is a fibre square:

$$\begin{array}{ccc} X \times Y & \xhookrightarrow{(Id,i)} & X \times M \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle f} \\ Y & \xhookrightarrow{\quad i \quad} & M \end{array}$$

therefore the map p is embeddable.

Theoretically embeddable morphisms should play the role of flat maps for Chow groups, namely they should be well behaved under composition and under base change. This is not true in general, since the embeddable property depends on the existence of a special type of fibre square associated, but it works with special assumptions, as it is proven in the next proposition.

Proposition 1.6. *The pull-back of an embeddable map with respect to another embeddable map is embeddable; namely, for any fibre diagram*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\Big\downarrow & & \Big\downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

if both f and g are embeddable, then so is f' (and by symmetry g'). Moreover, the compositions $g \circ f'$, $f \circ g'$ are embeddable maps.

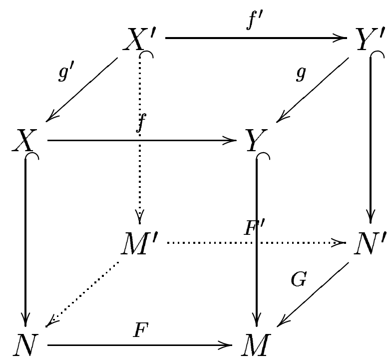
Proof. Consider two fibre squares corresponding to the maps f, g:

$$\begin{array}{ccc} X & \xhookrightarrow{j} & N \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle F} \\ Y & \xhookrightarrow{i} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} Y' & \xhookrightarrow{j'} & N' \\ {\scriptstyle g}\Big\downarrow & & \Big\downarrow{\scriptstyle G} \\ Y & \xhookrightarrow{i'} & M' \end{array}$$

with M, N, M', N' smooth schemes and F, G smooth equidimensional morphisms. Then we can build another pair of fibre squares associated to f and g, given by:

$$\begin{array}{ccc} X & \xhookrightarrow{(j,i'f)} & N \times M' \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle (F,Id)} \\ Y & \xhookrightarrow{(i,i')} & M \times M' \end{array} \quad \text{and} \quad \begin{array}{ccc} Y' & \xhookrightarrow{(ig,j)} & M \times N' \\ {\scriptstyle g}\Big\downarrow & & \Big\downarrow{\scriptstyle (Id,G)} \\ Y & \xhookrightarrow{(i,i')} & M \times M' \end{array}$$

which proves that we can consider $M = M'$ in the two fibre diagrams; under the assumption $M = M'$ from the initial data we can build the following fibre diagram



All faces of the cube are fibre squares. Since F' is the pull-back of F it is a smooth equidimensional morphism, and since N' is a smooth scheme we get that M' is smooth, and therefore the back face is precisely a 'good' fibre square associated to the morphism f', which ends the proof that f' is embeddable. The fact that $g \circ f'$ and $f \circ g'$ are embeddable follows from the associated fibre squares built. $\square$

1.3 Borel-Moore Functors

Recall that $(\mathbf{Sch_k})'$ is the subcategory of $\mathbf{Sch_k}$ where morphism are just projective morphisms.

Definition 1.7. We say that a covariant functor $H : (\mathbf{Sch_k})' \longrightarrow \mathbf{Ab}$, is a **Borel-Moore functor** if it has the following properties:

(H1) H is an additive functor, namely for any pair of schemes (X, Y) the canonical morphism $H(X) \times H(Y) \longrightarrow H(X \coprod Y)$ is an isomorphism.

(H2) Any embeddable morphism $f : X \longrightarrow X'$ has a pull-back $f^* : H(X') \longrightarrow H(X)$.

(H3) For any fibre diagram:

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & M \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle G} \\ X' & \overset{j}{\hookrightarrow} & M' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ X'' & \overset{k}{\hookrightarrow} & M'' \end{array}$$

with M, M', M'' smooth schemes, G, F smooth equidimensional morphisms and i, j, k closed embeddings we have f, g and $f \circ g$ embeddable morphisms (since they are pull-backs of smooth equidimensional morphisms) and

$$(f \circ g)^* = g^* \circ f^*$$

(H4) For any fibre square:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

if f, f' are projective maps and g, g' are embeddable then $f'_* \circ g'^* = g^* \circ f_*$.

(H5) H **is homotopic invariant**. If $p : E \longrightarrow X$ is a vector bundle then $p^* : H(X) \longrightarrow H(E)$ is an isomorphism. (Note: from example 1.4 we know that p is embeddable and therefore that it has a pull-back.)

(H6) **Normalization**: For any quasi-projective scheme X the identity map $Id : X \longrightarrow X$ is embeddable and $Id^* = Id_{H(X)}$.

Remark 1.8. Note that the pull-back f^* of an embeddable morphism f is defined independent of the fibre square associated with that particular morphism.

Remark 1.9. In general the composition of embeddable maps may be not embeddable, therefore the special set up of the functoriality axiom $(H3)$. There is though one special case, namely given embeddable maps f, g in a fibre square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

we know from Proposition 1.6 that we can build an associated cube diagram showing that $g \circ f', f \circ g'$ are embeddable morphisms, and from $g \circ f' = f \circ g'$ we get that $f'^* \circ g^* = g'^* \circ f^*$.

Definition 1.10. *(Refined Gysin Homomorphisms)* We say that a Borel-Moore functor is endowed with refined Gysin morphisms if for any fibre square

$$\begin{array}{ccc} X & \hookrightarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xhookrightarrow{i} & M' \end{array}$$

with M, M' smooth, f, g closed embeddings and i a regular embedding there exists a morphisms $i^! : H(X') \longrightarrow H(X)$, called the *refined Gysin homomorphism* associated to the diagram. Moreover, these morphisms verify the following conditions:

(G1) If the following is a fibre diagram with M, M', M'' smooth schemes and i, j regular embeddings

$$\begin{array}{ccccc} X & \longrightarrow & X' & \longrightarrow & X'' \\ \downarrow & & \downarrow & & \downarrow \\ M & \xhookrightarrow{i} & M' & \xhookrightarrow{j} & M'' \end{array}$$

then $(j \circ i)^! = i^! \circ j^!$.

(G2) For any fibre square

$$\begin{array}{ccc} X & \hookrightarrow & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \hookrightarrow & M' \end{array}$$

with g a smooth equidimensional morphism, one has f embeddable and

$$f^* = i^! \circ q^*$$

where $g = p \circ i$ and the maps q, p, i are from the fibre diagram:

$$\begin{array}{ccccc}
X & \hookrightarrow & M \times X' & \overset{q}{\hookrightarrow} & X' \\
\downarrow & & \downarrow & & \downarrow \\
M & \overset{i}{\hookrightarrow} & M \times M' & \overset{p}{\longrightarrow} & M'
\end{array}$$

(G3) For any fibre square

$$\begin{array}{ccc}
X & \hookrightarrow & M \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
X' & \hookrightarrow & M'
\end{array}$$

with g a regular embedding, one has

$$g^! = i^! \circ q^*$$

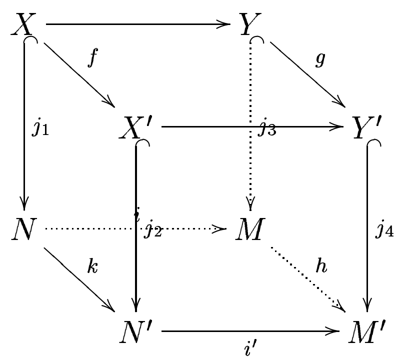
where $g = p \circ i$ and q, p, i are as in the previous axiom.

(G4) Consider a fibre diagram:

$$\begin{array}{ccc}
X' & \hookrightarrow & Y' \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\
X & \hookrightarrow & Y \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
M & \overset{i}{\hookrightarrow} & N
\end{array}$$

where M, N are smooth schemes, i is a regular embedding and p, q, f, g are closed embeddings. Then $p_* \circ i^! = i^! \circ q_*$.

(G5) Consider the commutative diagram



where all the squares are fibre squares, M, N, M', N' are smooth schemes, h, k are smooth equidimensional maps, i, i' are regular embeddings and all the maps j_i are closed embeddings. Then the resulting square for the H-groups is commutative, namely $i^! \circ g^* = f^* \circ i'^!$ from:

$$\begin{array}{ccc} H(X) & \xleftarrow{i^!} & H(Y) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle g^*} \\ H(X') & \xleftarrow{i'^!} & H(Y') \end{array}$$

(G6) Consider the fibre diagram

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'} \\ M & \xhookrightarrow{i} & N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M' & \xhookrightarrow{i'} & N' \end{array}$$

with M, N, M', N' smooth schemes, i, i' regular embeddings and j, j' closed embeddings such that $X \hookrightarrow M'$, $Y \hookrightarrow N'$ are still closed embeddings. If g is smooth equidimensional then $i^! = (i')^!$.

Example 1.11. For any quasi-projective scheme X, the Chow group $CH(X)$, together with the standard push-forwards and pull-backs for cycles, is a Borel-Moore functor, endowed with a refined Gysin structure.

(Push-forwards) Any projective map $f: X \longrightarrow Y$ is proper, so it has the standard push-forward $f_*: CH(X) \longrightarrow CH(Y)$, which is functorial, namely for composable maps one has $(f \circ g)_* = f_* \circ g_*$.

(H2) Any embeddable map $f: X \longrightarrow Y$ is smooth equidimensional, so it is flat of a given relative dimension, which means that we can build the pull-back $f^*: CH(Y) \longrightarrow CH(X)$.

(H3) $(f \circ g)^* = g^* \circ f^*$ is the known functoriality property of pull-backs for Chow groups.

(H4) $f'_* \circ g'^* = g^* \circ f_*$ is proven in [Ful84, Proposition 1.7].

(H5) For any vector bundle $p: E \longrightarrow X$ the pull-back p^* is an isomorphism. ([Ful84, Theorem 3.3]

(H6) From the definition of pull-backs for cycles, we have that $Id^*[V] = [Id^{-1}(V)] = [V]$, therefore $Id^* = Id_{CH(X)}$.

(Refined Gysin) For any fibre square

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \big\downarrow & & \big\downarrow \\ M & \xrightarrow{\ i\ } & M' \end{array}$$

if i is a regular embedding then we have the refined Gysin morphism $i^!: CH(X') \longrightarrow CH(X)$.

(G1) $(j \circ i)^! = i^! \circ j^!$ is proven in [Ful84, Theorem 6.5].

(G2), (G3) These properties are proven in [Ful84, Proposition 6.5]

(G5) We want to show that, in the given context, $i^! \circ g^* = f^* \circ i'^!$. But from the fibre diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ N & \xhookrightarrow{\ i\ } & M \\ {\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ N' & \xhookrightarrow{\ i'\ } & M' \end{array}$$

since k, h are smooth equidimensional we know that $i^! = i'^!$, and the equality $i^! \circ g^* = f^* \circ i^!$ is proven in [Ful84, Theorem 6.2].

(G4),(G6) These are proven in [Ful84, Theorem 6.2]

Example 1.12. Let X be a noetherian quasi-projective scheme. Then the Grothendieck group of X, $G_o(X)$ is a Borel-Moore functor.

(Push-forwards) For any projective morphism $f : X \longrightarrow Y$ define

$$\begin{aligned} f_* : \mathbf{Sheaf/X} \longrightarrow \mathbf{Sheaf/Y} \\ f_*(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)) \end{aligned}$$

Since f is projective, f takes coherent sheaves into coherent sheaves and f_* is exact, therefore we get a morphism $f_* : G_o(X) \longrightarrow G_o(Y)$.

(H2) For any $f : X \longrightarrow Y$ define

$$\begin{aligned} f^* : \mathbf{Sheaf/Y} \longrightarrow \mathbf{Sheaf/X} \\ f^*(\mathcal{E}) = \mathcal{O}_{\mathcal{X}} \otimes_{f^{-1}(\mathcal{O}_{\mathcal{Y}})} f^{-1}(\mathcal{E}) \end{aligned}$$

where $f^{-1} : \mathbf{Sheaf/Y} \longrightarrow \mathbf{Sheaf/X}$ is the left adjoint of f_*. Then, if f is embeddable then f is flat, therefore f^* is exact, and it takes coherent sheaves into coherent sheaves, and we can construct the morphism $f^* : G_o(Y) \longrightarrow G_o(X)$.

(H3) The pull-back for G_o-theory is functorial ([Sri93, property (5.8)]).

(H4) $f'_* \circ g'^* = g^* \circ f_*$ [Sri93, Proposition (5.13)].

(H5) G_o is homotopic invariant, ([Sri93, Proposition 5.17]).

(H6) For the normalization property, look at the definition:

$$\begin{aligned} Id^*(\mathcal{E}) &= \mathcal{O}_{\mathcal{X}} \otimes_{Id^{-1}(\mathcal{O}_{\mathcal{X}})} Id^{-1}(\mathcal{E}) \\ &= \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{E} \\ &= \mathcal{E} \end{aligned}$$

(Gysin Structure) For G_o we do not have a refined Gysin structure, but we still can build Gysin morphisms for regular embeddings $i : X \hookrightarrow Y$. All such maps have finite Tor-dimension, so we can define $i^! = f^*$ [Sri93, property (5.10)]. For these morphisms the properties $(G1) - (G6)$ follow:

(G1),(G2),(G3),(G5) All these properties follow from the functoriality of the pull-backs of G_o.

(G4) Consider a fibre diagram:

$$\begin{array}{ccc} X' & \stackrel{f}{\hookrightarrow} & Y' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \stackrel{g}{\hookrightarrow} & Y \\ \downarrow & & \downarrow \\ M & \stackrel{i}{\hookrightarrow} & N \end{array}$$

where M, N are smooth schemes, i is a regular embedding and all vertical maps are closed embeddings. If $\mathcal{O}_{\mathcal{Y}'}, \mathcal{O}_{\mathcal{X}}$ are Tor-independent over $\mathcal{O}_{\mathcal{Y}}$ then from ([Sri93, property (5.13)]) we get $p_* \circ f^* = g^* \circ q_*$.

(G6) This property follows from the definition of Gysin maps for G_o groups.

Example 1.13. Algebraic cobordism, Ω, the way it is defined in [ML01], is a Borel-Moore functor. Pull-backs in [ML01] are defined for smooth equidimensional morphisms, but any embeddable map is smooth equidimensional. The list of axioms in [ML01] includes axioms $(H1)-(H6)$, so there is nothing to prove.

1.4 Algebraic Oriented Theories

Along with the Borel-Moore covariant theories we are interested in a special type of contravariant theories, known as algebraic oriented theories. For this, consider **SP** the category of smooth pairs, where objects are (X, Z) with X a smooth scheme over k and Z closed subscheme in X. Morphisms are maps of pairs $f : (X, Z) \longrightarrow (X', Z')$ such that $f : X \longrightarrow X'$ is a morphism of schemes and $f^{-1}(Z') \subset Z$.

Remark 1.14. Another way to describe the smooth pairs category is to define objects as pairs (X, U) with X smooth scheme, U open subscheme of X and morphisms as $f : (X, U) \longrightarrow (Y, V)$ given by morphisms of schemes $f : X \longrightarrow Y$ such that $f(U) \subset V$; this is the set-up of [IP00] and [Pan02]. The two descriptions are equivalent; we choose to work with the first one because algebraic oriented theories correspond to reduced cohomology theories, hence we prefer to underline the closed support scheme rather than its open complement.

Definition 1.15. A contravariant functor $A : \mathbf{SP} \longrightarrow \mathbf{Ab}$ is an **algebraic oriented theory** if it has the following properties:

(A1) A is additive, namely for any pair of smooth schemes (X, Y) the canonical map $A(X \coprod Y) \longrightarrow A(X) \times A(Y)$ is an isomorphism.

(A2) a. For any commutative diagram of the form:

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \big\downarrow & & \big\downarrow \\ M & \xrightarrow{F} & M' \end{array}$$

with $X \hookrightarrow M$, $X' \hookrightarrow M'$ closed embeddings, M, M' smooth schemes, if $f : X \longrightarrow X'$ is projective then it has a push-forward $f_A^F : A_X(M) \longrightarrow A_{X'}(M')$.

b. For any two morphisms $F, G : M \longrightarrow M'$ such that the diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \big\downarrow & & \big\downarrow \\ M & \overset{G}{\underset{F}{\rightrightarrows}} & M' \end{array}$$

one has $f_A^F = f_A^G$.

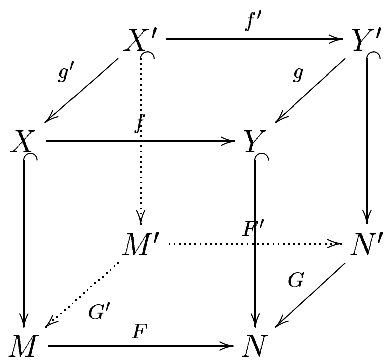
c. Id_A^F is an isomorphism for any morphism F.

(A3) For any commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{f} & X' & \xrightarrow{g} & X'' \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ M & \xrightarrow{F} & M' & \xrightarrow{G} & M'' \end{array}$$

if f and g are projective then so is $g \circ f$ and $(g \circ f)_A^{G \circ F} = g_A^G \circ f_A^F$.

(A4) Consider a commutative cube diagram



with X, X', Y, Y' closed subschemes of the smooth schemes M, M', N, N'. If f, f' are projective maps, G, G' are either smooth equidimensional morphisms or regular embeddings, and if the top, left and right faces are fibre squares, then

$$G^A \circ f_A^F = {f'}_A^{F'} \circ {G'}^A$$

Note: In order to have G, G' *well-defined maps of pairs we need* $G^{-1}(X) \subset X'$ *and* ${G'}^{-1}(Y) \subset Y'$*; this follows if the left and right faces are fibre squares.*

(A5) For any commutative diagram of the form

$$\begin{array}{ccc} (M, X) & \xrightarrow{g'} & (M', X) \\ \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\ (N, X) & \xrightarrow{g} & (N', X') \end{array}$$

with f, g smooth equidimensional maps, such that $f^{-1}(X') = X$ and $g^{-1}(X') = X$ one can build the following diagram at the level of A-groups

$$\begin{array}{ccc} A_X(M) & \xrightarrow{(Id_X)_A^{g'}} & A_X(M') \\ \Big\downarrow{\scriptstyle (Id_X)_A^{f'}} & & \Big\uparrow{\scriptstyle f^A} \\ A_X(N) & \xleftarrow{g^A} & A_{X'}(N') \end{array}$$

From axiom $(A2c)$ we have that both $(Id_X)_A^{g'}$ and $(Id_X)_A^{f'}$ are isomorphisms. This new axiom says that

$$(Id_A^{g'})^{-1} \circ f^A = (Id_A^{f'})^{-1} \circ g^A$$

(A6) **A is homotopic invariant**, namely for any fibre diagram

$$\begin{array}{ccc} Z & \hookrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ Z' & \hookrightarrow & X \end{array}$$

with E, X smooth schemes, Z, Z' closed subschemes, such that both $p : E \longrightarrow X$ and $p|_Z : Z \longrightarrow Z'$ are vector bundles, the pull-back $p^A : A_{Z'}(X) \longrightarrow A_Z(E)$ is an isomorphism.

(A7) **Normalization**. For any smooth pair (X, Z) we have $(Id_{(X,Z)})_A^{Id} = Id_{A_Z(X)}$.

Remark 1.16. For any pair of closed embeddings $i : X \hookrightarrow Y$, $f : Y \hookrightarrow M$, with M a smooth scheme one can build the following commutative diagram

$$\begin{array}{ccc} X & \xhookrightarrow{i} & Y \\ {\scriptstyle f\circ i}\downarrow & & \downarrow{\scriptstyle f} \\ M & = & M \end{array}$$

which gives a push-forward i_A^{Id}. We will consider this the natural push-forward associated to a closed embedding. Note that it still depends on the chosen smooth scheme M, so we will use such a notation only when there is a given closed embedding $Y \hookrightarrow M$.

Remark 1.17. As it is given, the axiom $(A4)$ connecting pull-backs and push-forwards may be difficult to check in real cases. In most computations the set up is a lot simpler; here are a couple of special forms in which axiom $(A4)$ may appear.

(A4a) If f and f' are closed embeddings, then from any fibre diagram

$$\begin{array}{ccccc} X' & \xhookrightarrow{f'} & Y' & \hookrightarrow & N' \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle G} \\ X & \xhookrightarrow{f} & Y & \hookrightarrow & N \end{array}$$

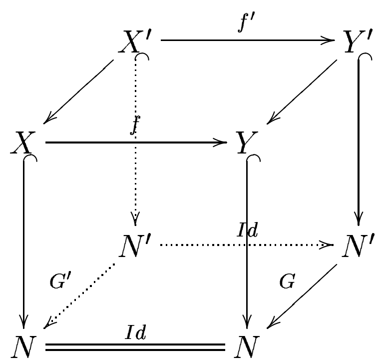
with Y, Y' closed subschemes of the smooth schemes M, M', and G a smooth equidimensional morphism or a regular embedding we have

$$f'^{Id}_A \circ G^A = G'^A \circ f^{Id}_A$$

where $G : (N', X') \longrightarrow (N, X)$ and $G' : (N', Y') \longrightarrow (N, Y)$ are well defined maps of pairs (since the squares are fibre), and f'^{Id}_A, f^{Id}_A are the push-forward maps corresponding to the commutative diagrams:

$$\begin{CD} X @>f>> Y \\ @VVV @VVV \\ N @= N \end{CD} \quad \text{and} \quad \begin{CD} X' @>f'>> Y' \\ @VVV @VVV \\ N' @= N' \end{CD}$$

Proof. From the above fibre diagram we can build the following cube diagram



which verifies all the conditions of $(A4)$, and therefore the equality holds. □

(A4b) Since any map between two smooth schemes is an l.c.i., we can generalize the above situation to the following: given f and f' closed embeddings inside a fibre diagram:

$$\begin{CD} X' @>f'>> Y' @>>> N' \\ @VVV @VVV @VV{G}V \\ X @>f>> Y @>>> N \end{CD}$$

with Y, Y' closed subschemes of the smooth schemes M, M', for any morphism G we have

$$f'^{Id}_A \circ G^A = G'^A \circ f^{Id}_A$$

Proof. Decompose G into a smooth map and a regular embedding:

$$G : N' \stackrel{I}{\hookrightarrow} N \times N' \stackrel{P}{\longrightarrow} N$$

Then we can apply the case $(A4a)$ twice in the following fibre diagram

$$\begin{array}{ccccc}
X' & \stackrel{f'}{\hookrightarrow} & Y' & \hookrightarrow & N' \\
\downarrow & & \downarrow & & \downarrow I \\
X \times N' & \stackrel{f''}{\hookrightarrow} & Y \times N' & \hookrightarrow & N \times N' \\
\downarrow & & \downarrow & & \downarrow P \\
X & \stackrel{f}{\hookrightarrow} & Y & \hookrightarrow & N
\end{array}$$

As before, we denote by I, P the maps of smooth pairs corresponding to closed subscheme $X', X \times N'$ and by I', P' the maps corresponding to $Y', Y \times N'$. From $(A4a)$ we get that:

$$\begin{aligned}
{f'}_A^{Id} \circ I^A &= {I'}^A \circ {f''}_A^{Id} \\
{f''}_A^{Id} \circ P^A &= {P'}^A \circ f_A^{Id}
\end{aligned}$$

and therefore

$$\begin{aligned}
{f'}_A^{Id} \circ G^A &= {f'}_A^{Id} \circ (P \circ I)^A = ({f'}_A^{Id} \circ I^A) \circ P^A \\
&= {I'}^A \circ {f''}_A^{Id} \circ P^A = {I'}^A \circ {P'}^A \circ f_A^{Id} \\
&= {G'}^A \circ f_A^{Id}
\end{aligned}$$

□

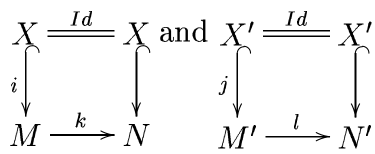
(A4c) Consider a fibre diagram:

$$\begin{array}{ccccc}
X & \stackrel{i}{\hookrightarrow} & M & \stackrel{k}{\longrightarrow} & N \\
g \downarrow & & \downarrow f & & \downarrow h \\
X' & \stackrel{j}{\hookrightarrow} & M' & \stackrel{l}{\longrightarrow} & N'
\end{array}$$

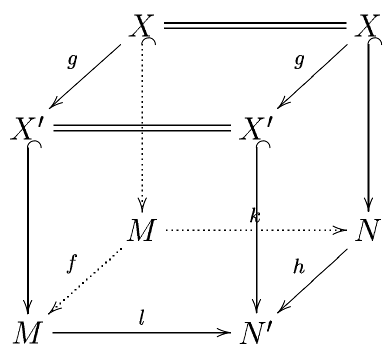
with M, N, M', N' smooth schemes, such that $k \circ i : X \hookrightarrow N$ and $l \circ j : X' \hookrightarrow N'$ are still closed embeddings. If h and f are both either smooth equidimensional or regular embeddings we have that:

$$Id_A^k \circ f^A = h^A \circ Id_A^l$$

Here Id_A^k, Id_A^l are the pushforwards associated to the diagrams



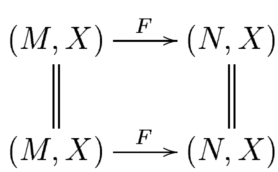
Proof. Again, from the above diagram we can build a cube diagram as in $(A4)$:



which proves that $h^A \circ Id_A^l = Id_A^k \circ f^A$. □

Remark 1.18. Suppose we have $F : (M, X) \longrightarrow (N, X)$ a well-defined map of pairs, such that $F^{-1}(X) = X$. If $F : M \longrightarrow N$ is smooth equidimensional then $F^A : A_X(N) \longrightarrow A_X(M)$ is an isomorphism.

Proof. Apply axiom $(A5)$ to the commutative diagram :



to get that:

$$(Id_A^F)^{-1} \circ Id^A = (Id_A^{Id})^{-1} \circ F^A$$

□

Remark 1.19. *Stronger normalization axiom* From the axioms listed above one gets a stronger normalization property for any algebraic oriented structure. For this, consider the fibre diagram

$$\begin{array}{ccccc} X & \xhookrightarrow{i} & U & = & U \\ \| & & \| & & \downarrow{\scriptstyle f} \\ X & \xhookrightarrow{i} & U & \xrightarrow{f} & M \end{array}$$

with i a closed embedding, f a smooth equidimensional morphism, of relative dimension zero, such that $(f \circ i) : X \longrightarrow M$ is still a closed embedding (for example, f may be an open embedding). Then from $(A4b)$ we get

$$\begin{aligned} Id_A^{Id} \circ (Id)^A &= f^A \circ Id_A^f \\ Id_{A_X(U)} &= f^A \circ Id_A^f \end{aligned}$$

Example 1.20. *The operational Chow group.* For any closed imbedding $i : X \hookrightarrow M$, with M smooth, X quasi-projective, let $A_X(M) = A(X \hookrightarrow M)$ be the operational Chow group as it is defined in [Ful84, Chapter 17]. Then $A_X(M)$ is an oriented algebraic cohomology theory. Since this is our main example of an algebraic oriented theory it is worth giving a detailed proof.

Proof. First, we need to construct pull-backs and push-forwards. These will not be the standard push-forward and pull-back for the operational Chow groups, the way they are defined in [Ful84], since those maps do not satisfy the axioms we require, but they will be closely related.

For the pull-backs, consider any map of pairs $f : (M, X) \longrightarrow (M', X')$ and let $\widetilde{X} = f^{-1}(X')$. Then define $f^A = i_* \circ f^*$ from

$$A(X' \hookrightarrow M') \xrightarrow{f^*} A(\widetilde{X} \hookrightarrow M) \xrightarrow{i_*} A(X \hookrightarrow M)$$

where f^*, i_* are the pull-back and push-forward defined in [Ful84].

First we need to show that this pull-back is functorial; consider two composable maps: $(M, X) \xrightarrow{g} (M', X') \xrightarrow{f} (M'', X'')$. Let

$$\begin{aligned} \widetilde{X} &= g^{-1}(X') \quad , \quad i : \widetilde{X} \hookrightarrow X \\ \widetilde{X'} &= f^{-1}(X'') \quad , \quad j : \widetilde{X'} \hookrightarrow X' \\ \bar{X} &= g^{-1}(\widetilde{X'}) \quad , \quad k : \bar{X} \hookrightarrow \widetilde{X} \end{aligned}$$

From our definitions we have:

$$\begin{aligned} g^A &= i_* \circ g^* \\ f^A &= j_* \circ f^* \\ (f \circ g)^A &= (i \circ k)_* \circ (f \circ g)^* \\ &= i_* \circ k_* \circ g^* \circ f^* \\ &= i_* \circ g^* \circ j_* \circ f^* \\ &= g^A \circ f^A \end{aligned}$$

We used the functoriality of standard pull-backs and push-forwards, and the fact that $k_* \circ g^* = g^* \circ j_*$, which follows from [Ful84, Property A23] for the fibre diagram:

$$\begin{array}{ccccc} \bar{X} & \xrightarrow{k} & \widetilde{X} & \hookrightarrow & M \\ \downarrow & & \downarrow & & \downarrow g \\ \widetilde{X'} & \xrightarrow{j} & X' & \hookrightarrow & M' \end{array}$$

Now, for the push-forwards, consider a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ M & \xrightarrow{F} & M' \end{array}$$

and define

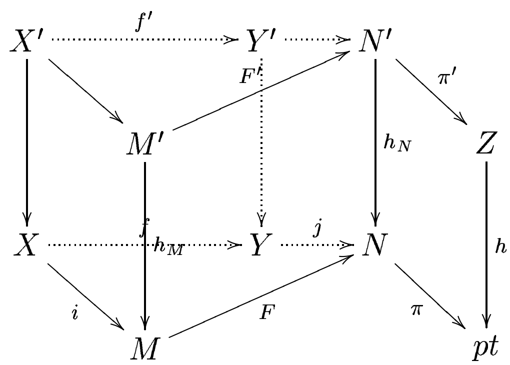
$$\begin{aligned} f_A^F &: A_X(M) \longrightarrow A_{X'}(M') \\ f_A^F(c) &= f_*(c \cdot [F]) \end{aligned}$$

Recall that $c \in A(X \hookrightarrow M)$, $f_* : A(X \longrightarrow M') \longrightarrow A(X' \hookrightarrow M')$ and $[F] \in A(M \longrightarrow M')$ is the orientation associated to the l.c.i. morphism F (since M, N are smooth any morphism $F : M \longrightarrow N$ is an l.c.i. morphism). Now that we have a contravariant functor A we need to check that all axioms are verified.

(A2 a.) The construction of push-forwards is given above

(A2 b.) If F, G are morphism such that the diagram commutes, then $f_A^F = f_A^G$.

Proof. Let $c \in A_X(M)$ and $d = f_A^F(c) \in A_Y(N)$. To know d means to know the morphism $d_{h_N} : CH(N') \longrightarrow CH(Y')$ induced by any map $h_N : N' \longrightarrow N$. All this data is contained in the following commutative diagram:



$$\begin{aligned} d_{h_N}(\alpha) &= f_*(c \cdot [F])(\alpha) \\ &= f_*(c([F](\alpha))) \\ &= f_*(c(F'(\alpha))) \\ &= f'_*(c_{h_M}(F'^*(\alpha))) \end{aligned}$$

Since $\cdot[\pi] : A(Y \hookrightarrow N) \longrightarrow A(Y \longrightarrow pt)$ is an isomorphism, it is enough to work with the image of d under this map, namely consider $\bar{d} = d \cdot [\pi] \in A(Y \longrightarrow pt)$. Then, for any morphism $h : Z \longrightarrow pt$, consider h_N its pull-back.

$$\begin{aligned} \bar{d}_h(\gamma) &= (d \cdot [\pi])(\gamma) \\ &= d_{h_N}(\pi'^*(\gamma)) \\ &= f'_*(c_{h_M}(F'^*(\pi'^*(\gamma)) \\ &= f'_*(c_{h_M}([\pi F](\gamma))) \\ &= f'_*(\bar{c}(\gamma)) \end{aligned}$$

where $\bar{c}$ is the image of c under the isomorphism

$$[\pi F] : A(X \hookrightarrow M) \longrightarrow A(X \longrightarrow pt)$$

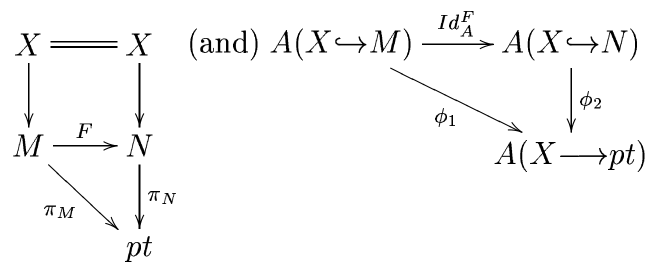
But this image does not depend on F, since $\pi \circ F = \pi \circ G$, and therefore $\bar{d}$, the image of $d = f_A^F(c)$ is the same for both maps f_A^F and f_A^G, which ends the proof. □

(A2 c.) $Id_A^F(c)$ is an isomorphism.

Proof. By definition we have $Id_A^F(c) = Id_*(c \cdot [F])$. Consider the following two morphisms: $\phi_1 : A(X \hookrightarrow M) \longrightarrow A(X \longrightarrow pt)$ and $\phi_2 : A(X \hookrightarrow N) \longrightarrow A(X \longrightarrow pt)$ given by

$$\phi_1(c) = c \cdot [\pi_M]$$
$$\phi_2(c) = c \cdot [\pi_N]$$

Then we have the following commutative diagrams:



The second diagram commutes because:

$$\begin{aligned}
(\phi_2 \circ Id_A^F)(c) &= \phi_2(Id_*(c \cdot [F])) = \phi_2(c \cdot [F]) \\
&= c \cdot [F] \cdot [\pi_N] \\
&= c \cdot [\pi_N \circ F] \\
&= c \cdot [\pi_M] = \phi_1(c)
\end{aligned}$$

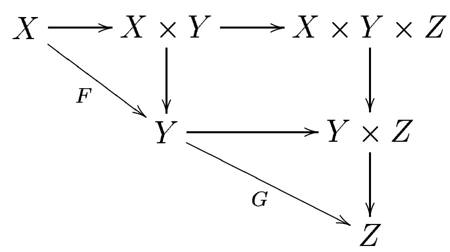
But both maps ϕ_1, ϕ_2 are isomorphisms, since M, N are smooth schemes ([Ful84, Proposition 17.4.2]), therefore Id_A^F is an isomorphism. ☐

(A3) The push-forwards are functorial.

Proof.

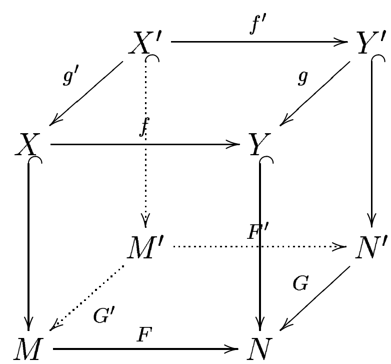
$$\begin{aligned}
(g_A^G \circ f_A^F)(c) &= g_A^G(f_*(c \cdot [F]) \\
&= g_*(f_*(c \cdot [F]) \cdot [G]) \\
&= g_*(f_*(c \cdot [F] \cdot [G]))\text{[Ful84, Property A12]} \\
&= g_*(f_*(c \cdot [G \circ F])) \\
&= (g \circ f)_*(c \cdot [G \circ F]) \\
&= (g \circ f)_A^{G \circ F}(c)
\end{aligned}$$

What is needed here is the equality $[F] \cdot [G] = [G \circ F]$ for our l.c.i. morphisms; this equality holds ([Ful84, Section 17.4]) as long as F and G have compatible factorizations; in our case this is true, since we have:



□

(A4) Consider a commutative cube diagram



with X, X', Y, Y' closed subschemes of the smooth schemes M, M', N, N'. If f, f' are projective maps, G, G' are either smooth equidimensional morphisms or regular embeddings, and if the top, left and right faces are fibre squares, then

$$G^A \circ f_A^F = {f'}_A^{F'} \circ {G'}^A$$

Proof. First of all, since the left and right squares are fibre we have $G^{-1}(Y) = Y', {G'}^{-1}(X) = X'$, therefore by definition $G^A = G^*, {G'}^A =$

G'^*.

$$\begin{aligned}(G^A \circ f_A^F)(c) &= G^*(f_*(c \cdot [F]) \\ &= f'_*(G^*(c \cdot [F]))\text{[Ful84, Property A23]} \\ &= f'_*(G'^*(c) \cdot G^*[F])\text{[Ful84, Property A13]} \\ &= f'_*(G'^*(c) \cdot [F'])\text{[Ful84, Proposition 17.4.1]} \\ &= (f'^{F'}_A \circ G'^A)(c)\end{aligned}$$

We use the equality $G^*[F] = [F']$; since the excess intersection formula is $G^*[F] = c_e(E)[F']$, we need to show that $e = 0$ (e is the difference between the codimensions of F and F'). First, note that:

$$\begin{aligned}e &= codim_F - codim_{F'} \\ &= dim(N) - dim(M) - dim(N') + dim(M') \\ &= codim_G - codim_{G'}\end{aligned}$$

and since the left, right and top squares are fibre we get that $codim_G = codim_{G'}$, therefore $e = 0$. □

(A5) For any commutative diagram of the form

$$\begin{array}{ccc}(M, X) & \xrightarrow{g'} & (M', X) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ (N, X) & \xrightarrow{g} & (N', X')\end{array}$$

with f, g smooth equidimensional maps, such that $f^{-1}(X') = X$ and $g^{-1}(X') = X$, one has

$$(Id_A^{g'})^{-1} \circ f^A = (Id_A^{f'})^{-1} \circ g^A$$

Proof.

$$Id_A^{g'}(c) = Id_*(c \cdot [g']) = c \cdot [g']$$

From [Ful84, Chapter 17.4] we know that $(Id_A^{g'})^{-1}(c) = [\gamma] \cdot g'^*(c)$; since in our case $X = X \times_N M$, the map γ comes from the following diagram:

$$
\begin{array}{ccccc}
X & \hookrightarrow & M & & \\
\gamma \Vert & & \downarrow \delta & & \\
X & \longrightarrow & M \times_N M & \longrightarrow & M \\
\Vert & & \downarrow & & \downarrow g' \\
X & \hookrightarrow & M & \xrightarrow{g'} & N
\end{array}
$$

Therefore $\gamma = Id$ so $[\gamma] = 1$, and since $f^A = f^*, g^A = g^*$ the equality boils down to

$$
\begin{array}{rcl}
(Id_A^{g'})^{-1} \circ f^A(c) & = & (Id_A^{f'})^{-1} \circ g^A(c) \\
[\gamma] \cdot g'^*(f^*(c)) & = & [\gamma] \cdot f'^*(g^*(c)) \\
g'^* \circ f^* & = & f'^* \circ g^*
\end{array}
$$

which follows from the functoriality of the standard pull-backs for operational groups, since $f \circ g' = g \circ f'$. □

(A6) For any fibre square

$$
\begin{array}{ccc}
Z & \hookrightarrow & E \\
\downarrow & & \downarrow p \\
Z' & \hookrightarrow & X
\end{array}
$$

we have $p^A = p^*$, and $p^* : A(Z' \longrightarrow X) \longrightarrow A(Z \longrightarrow E)$ is an isomorphism.

□

2 The Borel-Moore Functor Induced by an Algebraic Oriented Theory

Our goal is to construct a natural connection between these two structures.

2.1 The Basic Construction for $H(X), f_*, f^*, i^!$

Let $A : \mathbf{SP} \longrightarrow \mathbf{Ab}$ be an algebraic oriented theory. Our claim is that A induces an associated Borel-Moore functor H endowed with refined Gysin morphisms for regular embeddings.

First, for any quasi-projective scheme of finite type over k, X, there exists a smooth scheme M such that $X \hookrightarrow M$ is a closed embedding. Define:

$$H(X) = A_X(M)$$

Claim 2.1. *For two different closed embeddings $X \hookrightarrow M$ and $X \hookrightarrow M'$ the groups $A_X(M)$ and $A_X(M')$ are equal up to a canonical isomorphism.*

Proof. Suppose $i : X \hookrightarrow M$ and $j : X \hookrightarrow M'$ are two such closed embeddings and consider the diagram:

$$\begin{array}{ccc} X & \overset{Id}{=\!=\!=} & X \\ {\scriptstyle (i,j)}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ M \times M' & \xrightarrow{pr_1} & M \end{array}$$

From property $(A2c)$ we have an isomorphism $Id_A^{pr_1} : A_X(M \times M') \longrightarrow A_X(M)$ and similarly $Id_A^{pr_2} : A_X(M \times M') \longrightarrow A_X(M')$. Therefore we have a canonical isomorphism:

$$Id_A^{pr_2} \circ (Id_A^{pr_1})^{-1} : A_X(M) \simeq A_X(M')$$

which proves that the definition of $H(X)$ does not depend on the choice of the smooth scheme M. □

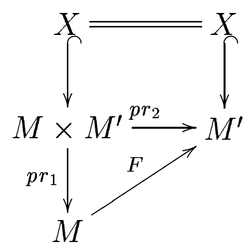
Remark 2.2. For any commutative diagram of the form:

$$\begin{array}{ccc} X & =\!=\!= & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ M & \xrightarrow{F} & M' \end{array}$$

such that i, j are closed embeddings, M, M' smooth schemes and F any morphism, the map induced by Id_A^F at the level of H-groups is not only an isomorphism, but the identity map, namely

$$\begin{array}{ccc} A_X(M) & \xrightarrow{Id_A^F} & A_X)(M') \\ \Vert & & \Vert \\ H(X) & \xrightarrow{Id_{H(X)}} & H(X) \end{array}$$

Indeed, if we consider:



then from properties $(A2b)$ and $(A2c)$ we have

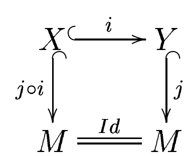
$$\begin{aligned} Id_A^{pr_2} &= Id_A^{F\circ pr_1} = Id_A^F \circ Id_A^{pr_1} \\ Id_A^{pr_2} \circ (Id_A^{pr_1})^{-1} &= Id_A^F \\ Id_{H(X)} &= Id_A^F \end{aligned}$$

This property of the push-forward for identity maps is very useful when trying to identify maps in the H-theory, and it will show up many times in the following computations.

To define push-forwards for projective maps, consider $f : X \longrightarrow Y$ a projective map, and take $f = p \circ i$ a standard decomposition, with $i : X \longrightarrow \mathbb{P}^n_Y$ the closed embedding and $p : \mathbb{P}^n_Y \longrightarrow Y$ the projection. We define the push-forward in steps, first for closed embeddings and projections, then for the general case.

Step 1

Let $i : X \longrightarrow Y$ be a closed embedding. Choose M a smooth scheme such that $j : Y \hookrightarrow M$ is a closed embedding. Then one has:



From $(A2a)$ we have the push-forward $i_A^{Id} : A_X(M) \longrightarrow A_Y(M)$ and we define $i_* = i_A^{Id} : H(X) \longrightarrow H(Y)$.

Claim 2.3. *The definition does not depend on the closed embedding $j : Y \hookrightarrow M$.*

Proof. Consider another closed embedding $j' : Y \hookrightarrow M'$ and the diagrams:

$$\begin{array}{ccccc} X & \stackrel{i}{\hookrightarrow} & Y & \stackrel{Id_Y}{=\!=} & Y \\ \downarrow & & \downarrow (j,j') & & \downarrow j \\ M \times M' & \stackrel{Id_{M\times M'}}{=\!=} & M \times M' & \stackrel{pr_1}{\longrightarrow} & M \end{array} \quad \text{and} \quad \begin{array}{ccccc} X & =\!= & X & \stackrel{i}{\hookrightarrow} & Y \\ \downarrow & & \downarrow & & \downarrow j \\ M \times M' & \stackrel{pr_1}{\longrightarrow} & M & \stackrel{Id_M}{=\!=} & M \end{array}$$

Since A is functorial for push-forwards $(A3)$, from the above diagrams we get

$$\begin{aligned} i_A^{pr_1} &= (Id_Y \circ i)_A^{pr_1 \circ Id_{M\times M'}} = (Id_Y)_A^{pr_1} \circ i_A^{Id_{M\times M'}} \\ i_A^{pr_1} &= (i \circ Id_X)_A^{Id_M \circ pr_1} = i_A^{Id_M} \circ (Id_X)_A^{pr_1} \end{aligned}$$

But from the previous remark we know that $Id_A^{pr_1}$ is the identity for the H-groups, so at the H-theory level we have that $i_A^{Id_M} = i_A^{Id_{M\times M'}}$, and similarly $i_A^{Id_{M'}} = i_A^{Id_{M\times M'}}$, which ends the proof. □

Step 2

Let X be a smooth projective scheme, Y any quasi-projective scheme, and $p : X \times Y \longrightarrow Y$ the projection map. Let M be a smooth scheme such that $Y \hookrightarrow M$. Consider the diagram:

$$\begin{array}{ccc} X \times Y & \stackrel{p}{\longrightarrow} & Y \\ \downarrow & & \downarrow \\ X \times M & \stackrel{q}{\longrightarrow} & M \end{array}$$

since X is a projective scheme p is a projective map and from $(A2a)$ one has $p_A^q : A_{X\times Y}(X \times M) \longrightarrow A_Y(M)$. Now we can define $p_* = p_A^q$.

Claim 2.4. *The definition does not depend on the closed embedding* $Y \hookrightarrow M$.

Proof. Consider $Y \hookrightarrow M$ and $Y \hookrightarrow M'$ two closed embeddings, and let $N = M \times M'$. Apply property $(A3)$ to the commutative diagrams

$$\begin{array}{ccccc} X \times Y & \stackrel{p}{\longrightarrow} & Y & =\!= & Y \\ \downarrow & & \downarrow & & \downarrow \\ X \times N & \stackrel{Q}{\longrightarrow} & N & \stackrel{f}{\longrightarrow} & M \end{array} \quad \text{and} \quad \begin{array}{ccccc} X \times Y & =\!= & X \times Y & \stackrel{p}{\longrightarrow} & Y \\ \downarrow & & \downarrow & & \downarrow \\ X \times N & \stackrel{g}{\longrightarrow} & X \times M & \stackrel{q}{\longrightarrow} & M \end{array}$$

where f, g and Q are just standard projections.

$$\begin{aligned} (Id \circ p)_A^{f\circ Q} &= (p \circ Id)_A^{q\circ g} \\ Id_A^f \circ p_A^Q &= p_A^q \circ Id_A^g \end{aligned}$$

Which gives at the level of A and H groups the diagrams:

$$\begin{array}{ccc} A_{X\times Y}(X\times N) & \xrightarrow{p_A^Q} & A_Y(N) \\ {\scriptstyle Id_A^g}\downarrow & & \downarrow{\scriptstyle Id_A^f} \\ A_{X\times Y}(X\times M) & \xrightarrow{p_A^q} & A_Y(M) \end{array} \quad \text{respectively} \quad \begin{array}{ccc} H(X\times Y) & \xrightarrow{p_A^Q} & H(Y) \\ \| & & \| \\ H(X\times Y) & \xrightarrow{p_A^q} & H(Y) \end{array}$$

therefore $p_A^Q = p_A^q$ (we have $Id_A^g = Id_{H(X\times Y)}$, $Id_A^f = Id_{H(Y)}$ from remark 2.2). Similarly one can show that $p_A^Q = p_A^{q'}$ which shows that the definition $p_* = p_A^q$ does not depend on the chosen embedding. ☐

Step 3

For the initial projective map $f = p \circ i$ let $f_* = p_* \circ i_*$

Claim 2.5. *The definition of f_* does not depend on the decomposition.*

Proof. Consider $f = p \circ i = p' \circ i'$ two different decompositions; let $Y \hookrightarrow M$ be a closed embedding of Y into a smooth scheme. Then we have

$$\begin{array}{ccccc} X & \xhookrightarrow{i} & \mathbb{P}_Y^n & \xrightarrow{p} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}_M^n & = & \mathbb{P}_M^n & \xrightarrow{q} & M \end{array} \quad \text{and} \quad \begin{array}{ccccc} X & \xhookrightarrow{i'} & \mathbb{P}_Y^m & \xrightarrow{p'} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}_M^m & = & \mathbb{P}_M^m & \xrightarrow{q'} & M \end{array}$$

$$\begin{aligned} p_* \circ i_* &= p_A^q \circ i_A^{Id} \\ p'_* \circ i'_* &= {p'}_A^{q'} \circ {i'}_A^{Id} \end{aligned}$$

Now look at the commutative diagrams

$$\begin{array}{ccccc} X & \xhookrightarrow{I=(i,i')} & \mathbb{P}_Y^n \times \mathbb{P}_Y^m & \xrightarrow{P_1} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}_M^n \times \mathbb{P}_M^m & = & \mathbb{P}_M^n \times \mathbb{P}_M^m & \xrightarrow{Q_1} & M \end{array} \quad \text{and} \quad \begin{array}{ccccc} X & \xhookrightarrow{i} & \mathbb{P}_Y^n & \xrightarrow{p} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}_M^n \times \mathbb{P}_M^m & \xrightarrow{h} & \mathbb{P}_M^n & \xrightarrow{q} & M \end{array}$$

where $P_1 : \mathbb{P}^n \times Y \times \mathbb{P}^m \times Y \longrightarrow Y$ is defined by $P_1(p, y_1, q, y_2) = y_1$.

$$\begin{aligned} {P_1}_A^{Q_1} \circ I_A^{Id} &= (P_1 \circ I)_A^{Q_1} \\ &= (p \circ i)_A^{q \circ h} \\ &= p_A^q \circ i_A^h \end{aligned}$$

But since i is a closed embedding one has a commutative diagram

$$\begin{array}{ccccc} X & = & X & \overset{i}{\hookrightarrow} & \mathbb{P}^n_Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^n_M \times \mathbb{P}^m_M & \xrightarrow{h} & \mathbb{P}^n_M & = & \mathbb{P}^n_M \end{array}$$

therefore

$$i_A^h = (i \circ Id)_A^{Id \circ h} = i_A^{id} \circ Id_A^h$$

By a similar argument for the second decomposition one gets

$$\begin{aligned} {P_1}_A^{Q_1} \circ I_A^{Id} &= p_A^q \circ i_A^{Id} \circ Id_A^h \\ {P_2}_A^{Q_2} \circ I_A^{Id} &= {p'}_A^{q'} \circ {i'}_A^{Id} \circ Id_A^{h'} \end{aligned}$$

where P_2, Q_2 are the maps corresponding to the second decomposition.

On the other hand

$$\begin{aligned} {P_1}_A^{Q_1} \circ I_A^{Id} &= (P_1 \circ I)_A^{Q_1} = f_A^{Q_1} \\ {P_2}_A^{Q_2} \circ I_A^{Id} &= (P_2 \circ I)_A^{Q_2} = f_A^{Q_2} \end{aligned}$$

And since the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \mathbb{P}^n_M \times \mathbb{P}^m_M & \overset{Q_2}{\underset{Q_1}{\rightrightarrows}} & M \end{array}$$

from axiom $(A2b)$ we have that $f_A^{Q_1} = f_A^{Q_2}$, therefore

$$\begin{aligned} p_A^q \circ i_A^{Id} \circ Id_A^h &= {p'}_A^{q'} \circ {i'}_A^{Id} \circ Id_A^{h'} \\ p_* \circ i_* \circ Id_* &= p'_* \circ q'_* \circ Id_* \\ p_* \circ i_* &= p'_* \circ i'_* \end{aligned}$$

□

To define the pull-back for an embeddable map $f : X \longrightarrow X'$ consider a fibre square associated to this map

$$\begin{array}{ccc} X & \hookrightarrow & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \hookrightarrow & M' \end{array}$$

Since $g^{-1}(X') = X$ we have a well-defined map of smooth pairs $g : (M, X) \longrightarrow (M', X')$, and we can define $f^* = g^A$ from the diagram

$$\begin{array}{ccc} A_{X'}(M') & \xrightarrow{g^A} & A_X(M) \\ \| & & \| \\ H(X') & \overset{f^*}{\dashrightarrow} & H(X) \end{array}$$

Claim 2.6. *The definition of f^* does not depend on the choice of the associated fibre square.*

Proof. Suppose f belongs to two different fibre squares

$$\begin{array}{ccc} X & \hookrightarrow & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \hookrightarrow & M' \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \hookrightarrow & N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \hookrightarrow & N' \end{array}$$

Then we can build the following fibre diagrams

$$\begin{array}{ccccc} X & \hookrightarrow & M \times N' & \xrightarrow{p_1} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle g} \\ X' & \hookrightarrow & M' \times N' & \xrightarrow{q_1} & M' \end{array} \quad \text{and} \quad \begin{array}{ccccc} X & \hookrightarrow & M' \times N & \xrightarrow{p_2} & N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle h} \\ X' & \hookrightarrow & M' \times N' & \xrightarrow{q_2} & N' \end{array}$$

axiom $(A4c)$ applied to these diagrams gives that

$$g^A \circ Id_A^{q_1} = Id_A^{p_1} \circ G^A \qquad (1)$$
$$h^A \circ Id_A^{q_2} = Id_A^{p_2} \circ H^A \qquad (2)$$

Let P_1, P_2 be the standard projections of $M \times N$ on the first, respectively on the second term. Then from the commutative diagrams

$$\begin{array}{ccccc} X & = & X & = & X \\ \downarrow & & \downarrow & & \downarrow \\ M \times N & \xrightarrow{\bar{h}} & M \times N' & \xrightarrow{p_1} & M \end{array} \quad (P_1 : M \times N \to M) \quad \text{and} \quad \begin{array}{ccccc} X & = & X & = & X \\ \downarrow & & \downarrow & & \downarrow \\ M \times N & \xrightarrow{\bar{g}} & M' \times N & \xrightarrow{p_2} & M \end{array} \quad (P_2 : M \times N \to M)$$

we get

$$\begin{aligned} Id_A^{P_1} &= Id_A^{p_1 \circ \bar{h}} = Id_A^{p_1} \circ Id_A^{\bar{h}} && (3) \\ Id_A^{P_2} &= Id_A^{p_2 \circ \bar{g}} = Id_A^{p_2} \circ Id_A^{\bar{g}} && (4) \end{aligned}$$

Finally, from axiom $(A5)$ applied to the commutative square

$$\begin{array}{ccc} M \times N & \xrightarrow{\bar{h}} & M \times N' \\ \downarrow \bar{g} & & \downarrow G \\ M' \times N & \xrightarrow{H} & M' \times N' \end{array}$$

we know that

$$(Id_A^{\bar{h}})^{-1} \circ G^A = (Id_A^{\bar{g}})^{-1} \circ H^A \qquad (5)$$

Now we can put all these together to get the desired result, namely:

$$\begin{aligned} g^A \circ Id_A^{q_1} &= Id_A^{p_1} \circ G^A \text{ — equation (1)} \\ &= Id_A^{P_1} \circ (Id_A^{\bar{h}})^{-1} \circ G^A \text{ — equation (3)} \\ &= Id_A^{P_1} \circ (Id_A^{\bar{g}})^{-1} \circ H^A \text{ — equation (5)} \\ &= Id_A^{P_1} \circ (Id_A^{P_2})^{-1} \circ Id_A^{p_2} \circ H^A \text{ — equation(4)} \\ &= Id_A^{P_1} \circ (Id_A^{P_2})^{-1} \circ h^A \circ Id_A^{q_2} \text{ — equation(2)} \\ g^A \circ Id_A^{q_1} \circ (Id_A^{q_2})^{-1} &= Id_A^{P_1} \circ (Id_A^{P_2})^{-1} \circ h^A \end{aligned}$$

And since $Id_A^{q_1} \circ (Id_A^{q_2})^{-1}$ and $Id_A^{P_1} \circ (Id_A^{P_2})^{-1}$ are exactly the standard isomorphisms for the H-groups we have proved that $g^A = h^A$ at the level of H-groups, hence f^* is well-defined. □

The last construction is that of refined Gysin morphisms. Again, consider a fibre square

$$\begin{array}{ccc} X & \hookrightarrow & X' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ M & \overset{i}{\hookrightarrow} & M' \end{array}$$

with M, M' smooth, f, g closed embeddings and i a regular embedding. Then the map $i : (M, X) \longrightarrow (M', X')$ is a well-defined map of pairs, so one has $i^A : A_{X'}(M') \longrightarrow A_X(M)$. Now we can simply define $i^! = i^A$ from the diagram

$$\begin{array}{ccc} A_{X'}(M') & \xrightarrow{i^A} & A_X(M) \\ \| & & \| \\ H(X') & \xrightarrow{i^!} & H(X) \end{array}$$

2.2 Properties for the Functor H

Proposition 2.7. *(Functoriality) For any pair of composable projective maps* $X \xrightarrow{g} X' \xrightarrow{f} X''$ *we have* $(f \circ g)_* = f_* \circ g_*$.

Proof. Consider standard decompositions for the given projective maps $g = p \circ i$ and $f = p' \circ i'$ and build the commutative diagram

$$\begin{array}{ccccccccc} & & & & M \times M' \times X'' & & & & \\ & & & {\scriptstyle I}\nearrow & & \searrow{\scriptstyle \widetilde{P}} & & & \\ & & M \times X' & & & & M' \times X'' & & \\ & {\scriptstyle i}\nearrow & & \searrow{\scriptstyle p} & & {\scriptstyle i'}\nearrow & & \searrow{\scriptstyle p'} & \\ X & & \xrightarrow{\quad g \quad} & & X' & & \xrightarrow{\quad f \quad} & & X'' \end{array}$$

where $I = (Id, i')$ and $\tilde{P}(m, m', x'') = (m', x'')$. Then for $f \circ g$ we have the decomposition $f \circ g = (p' \circ \tilde{P}) \circ (I \circ i)$.

To define the push-forwards for the projection maps $p, p', (p' \circ \tilde{P})$ consider N', N'' be smooth schemes such that $X' \hookrightarrow N'$, $X'' \hookrightarrow N''$ are closed embeddings. From the definition of push-forwards for H and the commutative diagrams:

$$
\begin{array}{ccc}
M \times X' & \xrightarrow{p} & X' \\
\big\downarrow & & \big\downarrow \\
M \times N' & \xrightarrow{Q} & N'
\end{array}
\text{ and }
\begin{array}{ccc}
M' \times X'' & \xrightarrow{p'} & X'' \\
\big\downarrow & & \big\downarrow \\
M' \times N'' & \xrightarrow{Q'} & N''
\end{array}
\text{ and }
\begin{array}{ccccc}
M \times M' \times X'' & \xrightarrow{\tilde{P}} & M'\tilde{X}'' & \xrightarrow{p'} & X'' \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
M \times M' \times N'' & \xrightarrow{\tilde{Q}} & M' \times N'' & \xrightarrow{Q'} & N''
\end{array}
$$

we have that

$$
\begin{aligned}
g_* &= p_* \circ i_* = p_A^Q \circ i_A^{id} \\
f_* &= p'_* \circ i'_* = p'^{Q'}_A \circ i'^{Id}_A \\
(f \circ g)_* &= (p' \circ \tilde{P})_* \circ (I \circ i)_* = (p' \circ \tilde{P})_A^{Q' \circ \tilde{Q}} \circ (I \circ i)_A^{Id} \\
&= p'^{Q'}_A \circ \tilde{P}_A^{\tilde{Q}} \circ I_A^d \circ i_A^{Id}
\end{aligned}
$$

on the other hand

$$
\begin{aligned}
\tilde{P}_A^{\tilde{Q}} \circ I_A^I d &= (\tilde{P} \circ I)_A^{\tilde{Q}} = (i' \circ p)_A^{\tilde{Q}} \\
&= i'^{Id}_A \circ p_A^{\tilde{Q}}
\end{aligned}
$$

And $p_A^{\tilde{Q}} = p_A^P$ since the definition of the push-forward for projections does not depend on the closed embedding, therefore

$$
\begin{aligned}
(f \circ g)_* &= p'^{Q'}_A \circ i'^{id}_A \circ p_A^Q \circ i_A^{Id} \\
&= f_* \circ g_*
\end{aligned}
$$

□

Now that H is indeed a functor we can proceed to check that H verifies the set of axioms $(H1) - (H6)$.

Proposition 2.8. *(H1) H is an additive functor, namely for any pair of schemes (X, Y) the canonical morphism $H(X) \times H(Y) \longrightarrow H(X \coprod Y)$ is an isomorphism.*

Proof. Let $i : X \hookrightarrow X \coprod Y, j : Y \longrightarrow X \coprod Y$ be the inclusions and consider $I : (M, X) \longrightarrow (M \coprod N, X \coprod Y)$, $J : (N, Y) \longrightarrow (M \coprod N, X \coprod Y)$ be the induced well-defined maps of pairs for the A theory.

We want to show that $(i_* + j_*) : H(X) \times H(Y) \longrightarrow H(X \coprod Y)$ is an isomorphism. From the definition of push-forwards, we have that $i_* = i_A^{Id}$, $j_* = j_A^{Id}$.

Consider the commutative diagram:

$$\begin{array}{ccc} A_{X\coprod Y}(M\coprod N) & \underset{i_A^{Id}\circ Id_A^I + j_A^{Id}\circ Id_A^J}{\overset{(I^A,J^A)}{\rightleftarrows}} & A_X(M)\times A_Y(N) \\ \downarrow \simeq & & \downarrow \simeq \\ H(X\coprod Y) & \underset{i_*+j_*}{\longleftarrow} & H(X)\times H(Y) \end{array}$$

Note that at the level of H-theory both maps Id_A^I, Id_A^J are indentity maps so for the H-groups we have $i_A^{Id}\circ Id_A^I + j_A^{Id}\circ Id_A^J = i_* + j_*$. From axiom $(A4b)$ applied to the fibre squares:

$$\begin{array}{ccccc} \emptyset & \hookrightarrow & Y & \hookrightarrow & N \\ \downarrow & & j\downarrow & & \downarrow J \\ X & \overset{i}{\hookrightarrow} & X\coprod Y & \hookrightarrow & M\coprod N \end{array} \quad \text{and} \quad \begin{array}{ccccc} X & = & X & \hookrightarrow & M \\ \| & & i\downarrow & & \downarrow I \\ X & \overset{i}{\hookrightarrow} & X\coprod Y & \hookrightarrow & M\coprod N \end{array}$$

we know that

$$\begin{aligned} J^A\circ i_A^{Id} &= J^A\circ i_A^{Id} = 0 \\ I^A\circ i_A^{Id} &= I'^A\circ Id_A^{Id} \end{aligned}$$

where $I' : (M,X)\longrightarrow(M\coprod N, X)$. We can now compute the composition:

$$\begin{aligned} ((I^A,J^A)\circ(i_A^{Id}\circ Id_A^I + j_A^{Id}\circ Id_A^J)(x,y) &= (I^A(i_A^{Id}(Id_A^I(x))), J^A(j_A^{Id}(Id_A^J(x)))) \\ &= (I'^A(Id_A^I(x)), J'^A(Id_A^J(y))) \end{aligned}$$

From axiom $(A2c)$ and remark 1.18 the maps $(I'^A\circ Id_A^I, J'^A\circ Id_A^J)$ are isomorphisms. If we start with an additive theory A we have (I^A, J^A) an isomorphism, and therefore $(i_A^{Id}\circ Id_A^I + j_A^{Id}\circ Id_A^J) = i_*\circ j^*$ is an isomorphism. □

Axiom $(H2)$ states that pull-backs exist for all embeddable maps, but we have already constructed these in the previous section.

Proposition 2.9. *(H3) For any fibre diagram:*

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & M \\ g\downarrow & & \downarrow G \\ X' & \overset{j}{\hookrightarrow} & M' \\ f\downarrow & & \downarrow F \\ X'' & \overset{k}{\hookrightarrow} & M'' \end{array}$$

with M, M', M'' smooth schemes, F, G smooth equidimensional morphisms and i, j, k closed embeddings we have

$$(f \circ g)^* = g^* \circ f^*$$

Proof.

$$\begin{aligned} g^* &= G^A : A_{X'}(M') \longrightarrow A_X(M) \\ f^* &= F^A : A_{X''}(M'') \longrightarrow A_{X'}(M') \\ (f \circ g)^* &= (F \circ G)^A = G^A \circ F^A \\ &= g^* \circ f^* \end{aligned}$$

just from the functoriality of A. □

Proposition 2.10. *(H4) For any fibre diagram:*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

if f, f' are projective maps and g, g' are embeddable then $f'_* \circ g'^* = g^* \circ f_*$.

Proof. Let $f = p \circ i$ be a standard decomposition of f into a closed embedding followed by a projection. Then consider the fibre diagram:

$$\begin{array}{ccccc} X' & \xhookrightarrow{i'} & \mathbb{P}^n_{Y'} & \xrightarrow{p'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle \bar{g}} & & \downarrow{\scriptstyle g} \\ X & \xhookrightarrow{i} & \mathbb{P}^n_Y & \xrightarrow{p} & Y \end{array}$$

which gives a decomposition $f' = p' \circ i'$ for the second projective morphism. Since $\bar{g}$ is the pull-back of an embeddable map g under another embeddable map p we have $\bar{g}$ embeddable (Proposition 1.6), therefore it is enough to prove the property for closed embeddings and projective morphisms.

Since g is embeddable, there exists a fibre square

$$\begin{array}{ccc} Y' & \hookrightarrow & M' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle G} \\ Y & \hookrightarrow & M \end{array}$$

First, from the fibre diagram

$$\begin{array}{ccccc} X' & \xhookrightarrow{i'} & \mathbb{P}^n_{Y'} & \hookrightarrow & \mathbb{P}^n_{M'} \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle \bar{g}=(Id,g)} & & \downarrow{\scriptstyle G'=(Id,G)} \\ X & \xhookrightarrow{i} & \mathbb{P}^n_{Y} & \hookrightarrow & \mathbb{P}^n_{M} \end{array}$$

since G is smooth so is G' and from $(A4b)$ we get

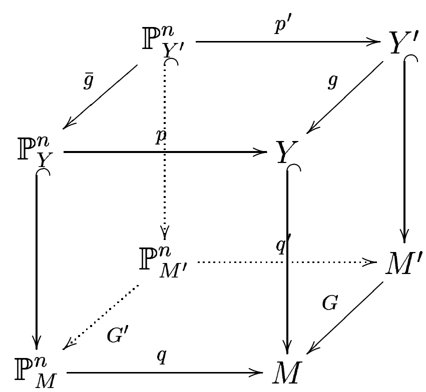
$$i'^{Id}_A \circ G'^A = G''^A \circ i^{Id}_A$$

where $G'' : (\mathbb{P}^n_{M'}, X') \longrightarrow (\mathbb{P}^n_M, X)$. Now we can conclude the proof for the closed embeddings case, from:

$$\begin{array}{rcl} i'^{Id}_A & = & i'_* \\ i^{Id}_A & = & i_* \\ G'^A & = & g'^* \\ G''^A & = & \bar{g}^* \end{array}$$

we have that $i'_* \circ g'^* = \bar{g}^* \circ i_*$.

Second, for the projective maps we can build the commutative cube diagram:



So from axiom $(A4)$ we get that

$$\begin{array}{rcl} G^A \circ p^q_A & = & p'^{q'}_A \circ G'^A \\ g^* \circ p_* & = & p'_* \circ \bar{g}^* \end{array}$$

Now we are almost done, since by functoriality $f_* = p_* \circ i_*$:

$$\begin{aligned} g^* \circ f_* &= g^* \circ p_* \circ i_* \\ &= {p'}_* \circ \bar{g}^* \circ i_* \\ &= {p'}_* \circ i'_* \circ {g'}^* \\ &= f'_* \circ {g'}^* \end{aligned}$$

□

Proposition 2.11. *(H5) If $p : E \longrightarrow X$ is a vector bundle, then there exists a smooth scheme M, a closed embedding $i : X \hookrightarrow M$ and a vector bundle $q : F \longrightarrow M$ such that the following is a fibre square:*

$$\begin{array}{ccc} E & \hookrightarrow & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \hookrightarrow & M \end{array}$$

and the pullback associated to this fibre square $p^* : H(X) \longrightarrow H(E)$ is an isomorphism.

The existence of such a fibre square is given in [Ful84, Lemma 18.2], but since we need to explain why the map is an isomorphism here is an explicit construction. First we consider the special case of vector bundles generated by global sections.

Lemma 2.12. *If $p : K \longrightarrow X$ is a vector bundle of rank k, such that K^* is generated by global sections then there exists a smooth scheme M, a closed embedding $i : X \hookrightarrow M$ and a vector bundle $q : F \longrightarrow M$ such that the following square is fibre*

$$\begin{array}{ccc} E & \hookrightarrow & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \overset{i}{\hookrightarrow} & M \end{array}$$

moreover, the associated pullback p^* is an isomorphism.

Proof. If K^* is generated by global sections then there exists an n-dimensional vector space V and a surjective map $V^* \times X \longrightarrow K^*$; this map induces an injection

$$F : K \hookrightarrow V \times X$$
$$F(y) = (f(y), p(y))$$

Now let $M = Gr(k, V)$, where k is the rank of K. Then X is a closed subscheme of M, with the closed embedding given by

$$\begin{array}{rcl} i : X & \longrightarrow & Gr(k, V) \\ i(x) & = & p^{-1}(x) \end{array}$$

Let $F = \{(v, L) | v \in V, L \in Gr(k, V) \text{ such that } v \in L\ \}$, then F is a vector bundle over $Gr(k, V)$ of rank k; moreover, the map

$$\begin{array}{rcl} j : K & \longrightarrow & F \\ j(y) & = & (f(y), p^{-1}(p(y))) \end{array}$$

is a well defined closed embedding, since the line $p^{-1}(p(k))$ goes through the image of k in V, which is $f(k)$, and we have the fibre square:

$$\begin{array}{ccc} K & \overset{j}{\hookrightarrow} & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \overset{i}{\hookrightarrow} & Gr(k, V) \end{array}$$

Now consider the map of smooth pairs $q : (F, K) \longrightarrow (Gr(k, V), X)$; we have $q|_K = p$ from the construction, and $F/Gr(k, V)$, K/X are vector bundles; from the strong homotopy axiom for A we get a canonical isomorphism $q^A : A_X(Gr(k, V)) \longrightarrow A_K(F)$.

But $p^* = q^A$ so the associated pullback $p^* : H(X) \longrightarrow H(K)$ is an isomorphism. □

Now we are ready to prove the property for the general case:

Theorem 2.13. *If $p : E \longrightarrow X$ is a vector bundle of rank k then there exists a smooth scheme M, a closed embedding $i : X \hookrightarrow M$ and a vector bundle $q : \widetilde{F} \longrightarrow M$ such that the following square is fibre*

$$\begin{array}{ccc} E & \hookrightarrow & \widetilde{F} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \tilde{q}} \\ X & \overset{i}{\hookrightarrow} & M \end{array}$$

moreover, the associated pullback p^* is an isomorphism.

Proof. Consider $\mathcal{E}^*$ be the dual of the sheaf associated to E. This is a coherent sheaf, so from Serre's theorem we know that there is an integer n such that $\mathcal{E}^* \otimes \mathcal{O}(n)$ is a sheaf generated by global sections.

Then if K is the vector bundle associated to $\mathcal{E} \otimes \mathcal{O}(-n)$ we have K^* associated to $(\mathcal{E} \otimes \mathcal{O}(-n))^* = \mathcal{E}^* \otimes \mathcal{O}(n)$, so K^* is generated by global sections, and from the previous lemma we have the following commutative diagram:

$$\begin{array}{ccc} K & \hookrightarrow & F \\ \downarrow & & \downarrow \\ X & \overset{i}{\hookrightarrow} & Gr(k,V) \end{array}$$

But $\mathcal{K} = \mathcal{E} \otimes \mathcal{O}(-n)$ hence $E \simeq K \times L$ where $L = j^*(\mathcal{O}(n)$, $j : X \longrightarrow \mathbb{P}^n$ so we have the fibre square:

$$\begin{array}{ccc} E \simeq K \otimes L & \hookrightarrow & F \otimes \mathcal{O}(n) \\ \downarrow & & \downarrow \tilde{q} \\ X & \xrightarrow{(i,j)} & Gr(k,V) \times \mathbb{P}^n \end{array}$$

therefore the smooth scheme we need is $M = Gr(k,V) \times \mathbb{P}^n$ and the vector bundle is $\widetilde{F} = F \otimes \mathcal{O}(n)$.

As before, from the strong homotopy axiom for A we know that $\tilde{q}^A$ is an isomorphism, so $p^* = \tilde{q}^A$ is an isomorphism. □

Proposition 2.14. *(H6) If the following is a fibre square*

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & U \\ {\scriptstyle Id}\parallel & & \downarrow {\scriptstyle f} \\ X & \overset{j}{\hookrightarrow} & M \end{array}$$

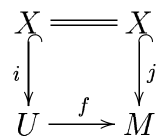
with i, j closed embeddings and f smooth of relative dimension zero then

$$(Id_X)^* = Id_{H(X)}$$

Proof. Use the remark about the stronger normalization property for A.

$$\begin{aligned} Id^* &= f^A (\text{by definition}) \\ f^A \circ Id_A^f &= Id_{A_X(U)} (\text{from remark 1.19}) \\ f^A &= (Id_A^f)^{-1} \end{aligned}$$

But we already proved that for any diagram

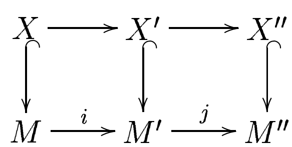


the morphism Id_A^f is the canonical isomorphism of $A_X(M) \simeq A_X(U)$; therefore, at the H level, $Id_A^f = Id_H(X)$. ☐

2.3 Properties for the Refined Gysin Structure

In the last part of this section we check that the constructed Gysin structure verifies properties $(G1) - (G6)$.

Proposition 2.15. *(G1) If the following is a fibre diagram with M, M', M'' smooth schemes and i, j regular embeddings:*



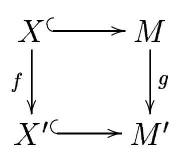
then $(j \circ i)^! = i^! \circ j^!$.

Proof. This property follows directly from the functoriality property of A:

$$\begin{aligned} (j \circ i)^! &= (j \circ i)^A = i^A \circ j^A \\ &= i^! \circ j^! \end{aligned}$$

☐

Proposition 2.16. *(G2) For any fibre square:*



with g a smooth equidimensional morphism, one has

$$f^* = i^! \circ q^*$$

where $g = p \circ i$ and the maps p, i are from the commutative diagram:

$$\begin{array}{ccccc} X & \longrightarrow & M \times X' & \xrightarrow{q} & X' \\ \downarrow & & \downarrow & & \downarrow \\ M & \xhookrightarrow{i} & M \times M' & \xrightarrow{p} & M' \end{array}$$

Proof. Again, the functoriality of A:

$$\begin{aligned} i^! \circ q^* &= i^A \circ p^A = (p \circ i)^A \\ &= g^A = f^* \end{aligned}$$

□

Proposition 2.17. *(G3) For any fibre square:*

$$\begin{array}{ccc} X & \hookrightarrow & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \hookrightarrow & M' \end{array}$$

with g a regular embedding, one has

$$g^! = i^! \circ q^*$$

where $g = p \circ i$ and the maps p, i are the standard projection and closed embedding, as in the previous proposition.

Proof. As before:

$$\begin{aligned} i^! \circ q^* &= i^A \circ p^A = (p \circ i)^A \\ &= g^A = g^! \end{aligned}$$

□

Proposition 2.18. *(G4) Consider a fibre diagram with M, N, M', N' smooth, i a regular embedding, p, q, f, g closed embeddings:*

$$\begin{array}{ccc} X' & \hookrightarrow & Y' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xhookrightarrow{i'} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{i} & N \end{array}$$

Then $p_* \circ i^! = i^! \circ q_*$.

Proof. First, since both p and q are closed embeddings, from the definition of push-forwards one has

$$\begin{aligned} p_* &= p_A^{Id_M} \\ q_* &= q_A^{Id_N} \end{aligned}$$

And from the above diagram, using property $(A4b)$ we get the following commutative diagram for the A groups:

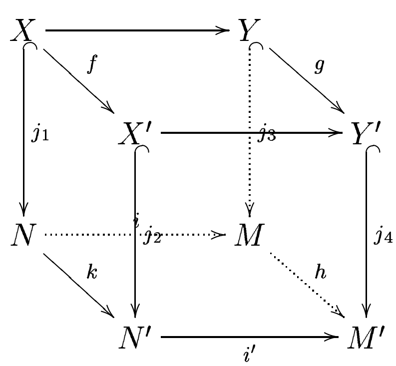
$$\begin{array}{ccc} A_{X'}(M) & \xleftarrow{i^A} & A_{Y'}(N) \\ {\scriptstyle p_A^{Id}}\downarrow & & \downarrow{\scriptstyle q_A^{Id}} \\ A_X(M) & \xleftarrow{i'^A} & A_Y(N) \end{array}$$

therefore

$$\begin{aligned} p_A^{Id} \circ i^A &= i'^A \circ q_A^{id} \\ p_* \circ i^! &= i'^! \circ q_* \end{aligned}$$

□

Proposition 2.19. *(G5) Consider the fibre diagram:*



where all the squares are fibre squares, M, N, M', N' are smooth schemes, h, k are smooth equidimensional maps, i, i' are regular embeddings and all the maps j_i are closed embeddings. Then the resulting square for the H-groups is commutative, namely $i^! \circ g^* = f^* \circ i'^!$.

Proof. The diagram:

$$\begin{array}{ccc} H(X) & \xleftarrow{i^!} & H(Y) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle g^*} \\ H(X') & \xleftarrow{i'^!} & H*(Y') \end{array}$$

comes from the following diagram for A groups:

$$\begin{array}{ccc} A_X(N) & \xleftarrow{i^A} & A_XY(NM) \\ {\scriptstyle k^A}\uparrow & & \uparrow{\scriptstyle h^A} \\ A_{X'}(N') & \xleftarrow{i'^A} & A_{Y'}(M') \end{array}$$

But this second diagram is commutative, since $(h \circ i) = (i' \circ k)$, so the first diagram is commutative. ☐

Proposition 2.20. *(G6) Consider the fibre diagram:*

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ M & \xhookrightarrow{i} & N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M' & \xhookrightarrow{i'} & N' \end{array}$$

If M, N, M', N' are smooth and i, i' are regular embeddings then $i^! = (i')^!$.

Proof. From the above diagram, using property $(A4c)$ we get at the level of A groups the following commutative diagram

$$\begin{array}{ccc} A_Y(N) & \xrightarrow{i^A} & A_X(M) \\ {\scriptstyle Id_A^g}\downarrow & & \downarrow{\scriptstyle Id_A^f} \\ A_Y(N') & \xrightarrow{i'^A} & A_X(M') \end{array}$$

therefore, since both f and g are smooth equidimensional

$$\begin{aligned} Id_A^f \circ i^A &= i'^A \circ Id_A^g \\ Id_* \circ i^! &= i'^! \circ Id_* \text{(from remark 2.2)} \\ i^! &= i'^! \end{aligned}$$

☐

3 The Algebraic Oriented Theory Induced by a Borel-Moore Functor

Our claim is that, given a Borel-Moore functor with refined Gysin homomorphism H, one can construct an associated algebraic oriented theory A.

3.1 The Basic Construction for $A_X(M), f^A, f_A^F$

For any smooth pair (M, X) define

$$A_X(M) = H(X)$$

To define the pullback f^A for any morphism of smooth pairs $f : (M, X) \longrightarrow (M', X')$ consider $\widetilde{X} = f^{-1}(X')$. Then $j : \widetilde{X} \hookrightarrow X$ is a closed embedding and we have a commutative diagram:

$$\begin{array}{ccccccc}
X & \overset{j}{\hookleftarrow} & \widetilde{X} & \longrightarrow & M \times X' & \overset{q}{\longrightarrow} & X' \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & M & \overset{i}{\hookrightarrow} & M \times M' & \overset{p}{\longrightarrow} & M'
\end{array}$$

Both squares are fibre squares, the map j is a closed embedding, i is a regular embedding and p is a smooth map; define $f^A : A_{X'}(M') \longrightarrow A_X(M)$ by:

$$\begin{array}{ccccccc}
H(X') & \overset{q^*}{\longrightarrow} & H(M \times X') & \overset{i^!}{\longrightarrow} & H(\widetilde{X}) & \overset{j_*}{\longrightarrow} & H(X) \\
\| & & & & & & \| \\
A_{X'}(M') & & & \overset{f^A}{\longrightarrow} & & & A_X(M)
\end{array}$$

The push-forward associated to a diagram:

$$\begin{array}{ccc}
X & \overset{f}{\longrightarrow} & X' \\
\downarrow & & \downarrow \\
M & \overset{F}{\longrightarrow} & M'
\end{array}$$

where f is projective, $X \hookrightarrow M, X' \hookrightarrow M'$ are closed embeddings and M, M' are smooth schemes is defined by $f_A^F = f_*$, since:

$$\begin{array}{ccc} A_X(M) & \xrightarrow{f_A^F} & A_{X'}(M') \\ \| & & \| \\ H(X) & \xrightarrow{f_*} & H(X') \end{array}$$

3.2 Properties for the Functor A

The first remarks describe the pull-backs of some special type of maps.

Remark 3.1. Let M be a smooth scheme and X, X' two closed subschemes of M such that we have $j : X' \hookrightarrow X$ a closed embedding. Then j induces a well-defined map of pairs $J : (M, X) \longrightarrow (M, X')$ and $J^A = j_*$.

Proof. From the definition we have

$$\begin{array}{ccccccc} X & \xleftarrow{j} & X' & \longrightarrow & M \times X' & \xrightarrow{q} & X' \\ & & \downarrow & & \downarrow & & \downarrow \\ & & M & \xrightarrow{i} & M \times M & \xrightarrow{p} & M \end{array}$$

$$\begin{aligned} J^A &= j_* \circ i^! \circ q_p^* \\ J^A &= j_* \circ Id_{X'}^* (\text{from axiom (G2)}) \\ J^A &= j_* \end{aligned}$$

□

Remark 3.2. Let $f : (M, X) \longrightarrow (M', X')$ be a morphism of pairs with $f^{-1}(X') = X$. If $f : M \longrightarrow M'$ is a smooth morphism of schemes then $f^A = (f|_X)^*$.

Proof. By definition

$$f^A = i^! \circ q^* = (f|_X)^*$$

from axiom $(G2)$. □

Remark 3.3. Let $f : (M, X) \longrightarrow (M', X')$ be a morphism of pairs with $f^{-1}(X') = X$. If $f : M \longrightarrow M'$ is a regular embedding of schemes then $f^A = f^!$.

Proof. By definition

$$f^A \;=\; i^! \circ q^* = f^!$$

from axiom $(G3)$. □

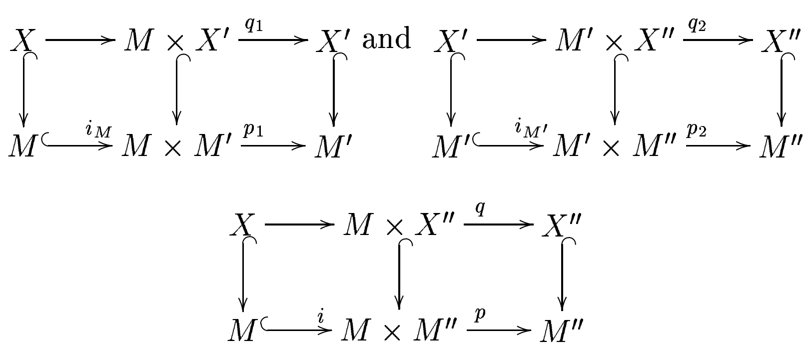
Proposition 3.4. *(Functoriality of A) If* $(M, X) \xrightarrow{g} (M', X') \xrightarrow{f} (M'', X'')$ *are composable maps of smooth pairs then* $(f \circ g)^A = g^A \circ f^A$.

Proof. Step 1

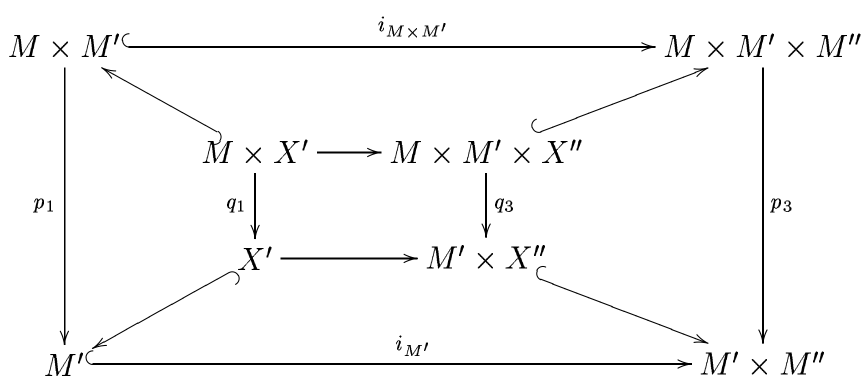
Suppose $f^{-1}(X'') = X'$ and $g^{-1}(X') = X$. Then

$$\begin{aligned} g^A &= i_M^! \circ {q_1}^* \\ f^A &= i_{M'}^! \circ {q_2}^* \\ (f \circ g)^A &= i^! \circ q^* \\ g^A \circ f^A &= i_M^! \circ {q_1}^* \circ i_{M'}^! \circ {q_2}^* \end{aligned}$$

where all the above maps are described in the fibre diagrams:



First, from the diagram



using property $(G5)$ we have:

$$q_1{}^* \circ i'^!_M = i^!_{M\times M'} \circ q_3{}^*$$

therefore we get that

$$g^A \circ f^A = i^!_M \circ i^!_{M\times M'} \circ q_3{}^* \circ q_2{}^*$$

Now, for $i^!_M \circ i^!_{M\times M'}$ consider the fibre diagram

$$\begin{array}{ccc} M & \xhookrightarrow{i_M} & M\times M' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_{M\times M'}} \\ M\times M'' & \xhookrightarrow{j} & M\times M'\times M'' \end{array}$$

$$\begin{aligned} i_{M\times M'} \circ i_M &= j \circ i \\ i^!_M \circ i^!_{M\times M'} &= i^! \circ j^! \text{(from property } (G1) \end{aligned}$$

Since pull-backs for H are not functorial in full generality, in order to prove that $q_3{}^* \circ q_2{}^* = q_5{}^* \circ q^*$ we need to consider fibre diagrams as in axiom $(H3)$:

$$\begin{array}{ccccc} M\times M'\times X'' & \xrightarrow{q_3} & M'\times X'' & \xrightarrow{q_2} & X'' \\ \downarrow & & \downarrow & & \downarrow \\ M\times M'\times M'' & \xrightarrow{p_3} & M'\times M'' & \xrightarrow{p_2} & M'' \end{array} \text{ and } \begin{array}{ccccc} M\times M'\times X'' & \xrightarrow{q_5} & M\times X'' & \xrightarrow{q} & X'' \\ \downarrow & & \downarrow & & \downarrow \\ M\times M'\times M'' & \xrightarrow{p_5} & M\times M'' & \xrightarrow{p} & M'' \end{array}$$

$$\begin{aligned} q_3{}^* \circ q_2{}^* &= (q_2 \circ q_3)^* \\ &= (q \circ q_5)^* \\ &= q_5{}^* \circ q^* \end{aligned}$$

So far, we have that:

$$\begin{aligned} g^A \circ f^A &= i^!_M \circ i^!_{M\times M'} \circ q_3{}^* \circ q_2{}^* \\ &= i^! \circ j^! \circ q_5{}^* \circ q^* \end{aligned}$$

Now we are almost done; to compute $j^! \circ q_5{}^*$ look at the diagram:

$$\begin{array}{ccccc} M\times X'' & \longrightarrow & M\times M'\times X'' & \xrightarrow{q_5} & M\times X'' \\ \downarrow & & \downarrow & & \downarrow \\ M\times M'' & \xhookrightarrow{j} & M\times M'\times M'' & \xrightarrow{p_5} & M\times M'' \end{array}$$

From axiom $(G2)$ we know that $j^! \circ {q_5}^* = (Id_{M\times X''})^* = Id_{H(M\times X'')}$ by the normalization property, which plugged in the previous equation gives the needed equality

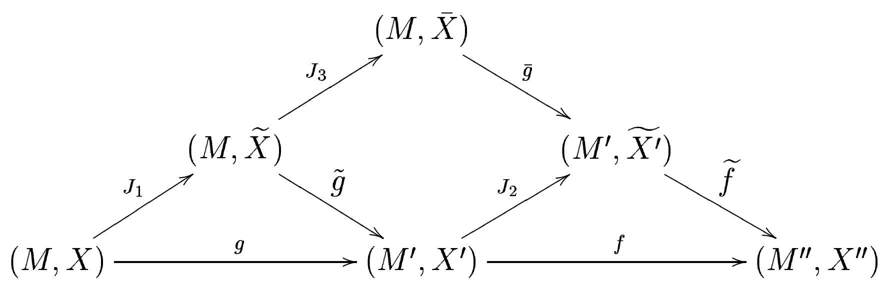
$$g^A \circ f^A = i^! \circ Id \circ q^* = (f \circ g)^A$$

Step 2 - The General Case

Let $\widetilde{X} = g^{-1}(X')$, $\widetilde{X'} = f^{-1}(X'')$ and $\bar{X} = g^{-1}(\widetilde{X'})$; if $j_1 : \widetilde{X} \hookrightarrow X, j_2 : \widetilde{X'} \hookrightarrow X', j_3 : \bar{X} \hookrightarrow \widetilde{X}, j = j_1 \circ j_3 : \bar{X} \hookrightarrow X$ are the corresponding closed embeddings, by the definition of pull-backs for A we have that:

$$\begin{aligned} g^A &= j_{1*} \circ i^!_M \circ {q_1}^* \\ f^A &= j_{2*} \circ i^!_{M'} \circ {q_2}^* \\ (f \circ g)^A &= j_* \circ i^! \circ q^* \end{aligned}$$

Consider the following commutative diagram



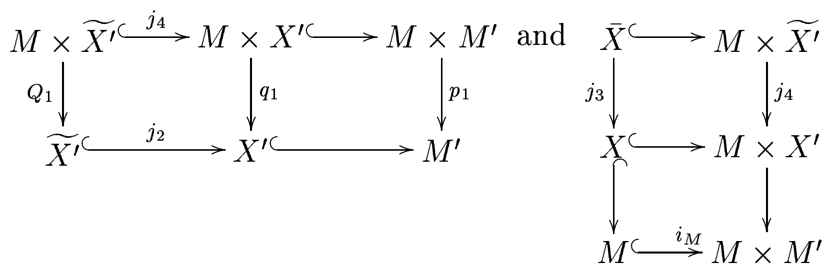
Where the maps J_1, J_2, J_3 are induced by the closed embeddings $j_1 : \widetilde{X} \hookrightarrow X, j_2 : \widetilde{X'} \hookrightarrow X', j_3 : \bar{X} \hookrightarrow \widetilde{X}$, and the maps $\tilde{g}, \widetilde{f}, \bar{g}$ are the restrictions of the maps f, g at the respective smooth pairs. First, from the previous remark we have $(j_i)_* = J_i^A$, therefore:

$$\begin{aligned} g^A &= {J_1}^A \circ \tilde{g}^A \\ f^A &= {J_2}^A \circ \widetilde{f}^A \\ g^A \circ f^A &= {J_1}^A \circ \tilde{g}^A \circ {J_2}^A \circ \widetilde{f}^A \end{aligned}$$

Second, since $\tilde{f} \circ \bar{g} : (M, \bar{X}) \longrightarrow (M'', X'')$ and $\tilde{f} \circ \bar{g}^{-1}(X'') = \bar{X}$ we have that $\widetilde{f} \circ \bar{g}^A = i^! \circ q^*$ (from the definition of pull-backs), which translates into:

$$(f \circ g)^A = J^A \circ (\widetilde{f} \circ \bar{g})^A$$

We claim that $\tilde{g}^A \circ {J_2}^A = J_3^A \circ \bar{g}^A$. To prove this look at the diagrams:



$$\begin{aligned}
\tilde{g}^A \circ J_2^A &= i_M^! \circ ({q_1}^* \circ (j_2)_*) \\
&= i_M^! \circ (j_4)_* \circ {Q_1}^* \text{(from axiom (H4) in 1st diagram)} \\
&= (j_3)_* \circ i_M^! \circ {Q_1}^* \text{(from (G4), 2nd diagram)} \\
&= J_3^A \circ \bar{g}^A
\end{aligned}$$

So we can replace this in our equation to get

$$\begin{aligned}
g^A \circ f^A &= {J_1}^A \circ \tilde{g}^A \circ {J_2}^A \circ \widetilde{f}^A \\
&= J_1^A \circ J_3^A \circ \bar{g}^A \circ \widetilde{f}^A \\
&= (j_1)_* \circ (j_3)_* \circ (\widetilde{f} \circ \bar{g})^A \text{(from } \textit{Step 1}\text{)} \\
&= j_* \circ (\widetilde{f} \circ \bar{g})^A \\
&= (f \circ g)^A
\end{aligned}$$

□

Once we know that A is a well-defined contravariant functor we need to check that A has the required properties $((A1) - (A7)$.

Proposition 3.5. *(A1) A is additive, namely for any two smooth pairs $(M, X), (N, Y)$ the canonical map $A_{X \coprod Y}(M \coprod N) \longrightarrow A_X(M) \times A_Y(N)$ is an isomorphism.*

Proof. Let $i : X \hookrightarrow X \coprod Y, j : Y \longrightarrow X \coprod Y$ be the inclusions and consider $I : (M, X) \longrightarrow (M \coprod N, X \coprod Y)$, $J : (N, Y) \longrightarrow (M \coprod N, X \coprod Y)$ be the induced well-defined maps of pairs.

We want to show that

$$(I^A, J^A) : A_{X \coprod Y}(M \coprod N) \longrightarrow A_X(M) \times A_Y(N)$$

is an isomorphism. Form Remark 3.2 we know that $I^A = i^*$, $J^A = j^*$. As in the previous section, look at the diagram:

$$\begin{array}{ccc} A_{X\coprod Y}(M\coprod N) & \xrightarrow{(I^A,J^A)} & A_X(M)\times A_Y(N) \\ \downarrow\simeq & & \downarrow\simeq \\ H(X\coprod Y) & \underset{i_*+j_*}{\overset{(i^*,j^*)}{\rightleftarrows}} & H(X)\times H(Y)\end{array}$$

From axiom $(H4)$ applied to the fibre squares:

$$\begin{array}{ccc} \emptyset & \hookrightarrow & Y \\ \downarrow & & \downarrow j \\ X & \overset{i}{\hookrightarrow} & X\coprod Y \end{array} \quad \text{and} \quad \begin{array}{ccc} X & = & X \\ \| & & \downarrow i \\ X & \overset{i}{\hookrightarrow} & X\coprod Y \end{array}$$

we know that

$$\begin{aligned} i^*\circ j_* &= j^*\circ i_* = 0 \\ i^*\circ i_* &= Id_{H(X)} \end{aligned}$$

So we can compute the composition:

$$\begin{aligned} ((i^*,j^*)\circ(i_*+j_*))(x,y) &= (i^*,j^*)(i_*(x)+j_*(y)) \\ &= (i^*(i_*(x))+i^*(j_*(y)), j^*(i_*(x)+j^*(j_*(y))) \\ &= (x,y) \end{aligned}$$

If we start with an additive H functor then from axiom $(H1)$ the morphism i_*+j_* is an isomorphism, therefore $(i^*,j^*) = (I^A,J^A)$ is an isomorphism. $\square$

Proposition 3.6. *(A2) Consider a commutative diagram:*

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ M & \xrightarrow{F} & M' \end{array}$$

with $X \hookrightarrow M$, $X' \hookrightarrow M'$ closed embeddings, M, M' smooth schemes. If $f : X\longrightarrow X'$ is projective then it has a push-forward $f_A^F : A_X(M)\longrightarrow A_{X'}(M')$.

b. For any two morphisms $F, G : M \longrightarrow M'$ such that the diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \big\downarrow & & \big\downarrow \\ M & \overset{G}{\underset{F}{\rightrightarrows}} & M' \end{array}$$

one has $f_A^F = f_A^G$.

c. Id_A^F is an isomorphism for any morphism F.

Proof. Part a) is solved in the construction of the push-forward. From this construction parts b) and c) follow, since

$$\begin{aligned} f_A^F &= f_* = f_A^G \\ Id_A^F &= Id_* = Id_{H(X)} \end{aligned}$$

□

Proposition 3.7. *(A3) For any commutative diagram:*

$$\begin{array}{ccccc} X & \xrightarrow{f} & X' & \xrightarrow{g} & X'' \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ M & \xrightarrow{F} & M' & \xrightarrow{G} & M'' \end{array}$$

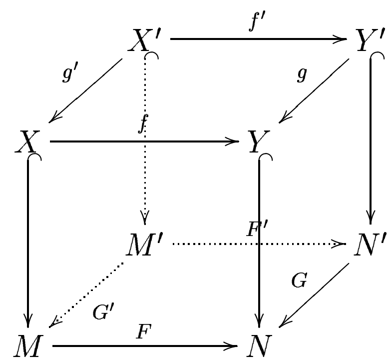
if f and g are projective then so is $g \circ f$ and $(g \circ f)_A^{G \circ F} = g_A^G \circ f_A^F$.

Proof. This follows directly from the definitions:

$$\begin{aligned} (g \circ f)_A^{G \circ F} &= (g \circ f)_* \\ &= g_* \circ f_* \\ &= g_A^G \circ f_A^F \end{aligned}$$

□

Proposition 3.8. *(A4) Consider a commutative cube diagram:*



with X, X', Y, Y' closed subschemes of the smooth schemes M, M', N, N'. If f, f' are projective maps, G, G' are either smooth equidimensional morphisms or regular embeddings, and if the top, left and right faces are fibre squares, then

$$G^A \circ f_A^F = {f'}_A^{F'} \circ {G'}^A$$

Proof. First, note that $f_A^F = f_*$ and ${f'}_A^{F'} = f'_*$. First, in the case of G, G' smooth morphisms, since the left and right squares are fibre squares from 3.2 we have that

$$G^A = g^*, {G'}^A = {g'}^*$$

therefore the equation we want is $g^* \circ f_* = {f'}_* \circ {g'}^*$, which follows from axiom $(H4)$ applied to the top fibre square.

In the second case, if G, G' are regular embeddings we have

$$G^A = G^!, {G'}^A = {G'}^!$$

and then the equation becomes $G^! \circ f_* = {f'}_* \circ {G'}^!$.

But from axiom $(G6)$ we get that ${G'}^! = G^!$, and then the equality follows from axiom $(G4)$. □

Proposition 3.9. *(A5) For any commutative diagram of the form:*

$$\begin{array}{ccc} (M, X) & \xrightarrow{g'} & (M', X) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ (N, X) & \xrightarrow{g} & (N', X') \end{array}$$

with f, g smooth equidimensional maps, such that $f^{-1}(X') = X$ and $g^{-1}(X') = X$

$$(Id_A^{g'})^{-1} \circ f^A = (Id_A^{f'})^{-1} \circ g^A$$

Proof. Let us first note that, since $Id_A^{g'} = Id_*$ and $Id_A^{f'} = Id_*$, all we need to show is that $f^A = g^A$. Now, from the definition on pull-backs we know that

$$\begin{aligned} f^A &= i^! \circ q^* \\ g^A &= j^! \circ p^* \end{aligned}$$

where i, q, j, p are the maps from the standard decompositions:

$$\begin{array}{ccccc} X & \longrightarrow & M \times X' & \xrightarrow{q} & X' \\ \downarrow & & \downarrow & & \downarrow \\ M' & \xhookrightarrow{i} & M' \times N' & \longrightarrow & N' \end{array} \quad \text{and} \quad \begin{array}{ccccc} X & \longrightarrow & N \times X' & \xrightarrow{p} & X' \\ \downarrow & & \downarrow & & \downarrow \\ N & \xhookrightarrow{j} & N \times N' & \longrightarrow & N' \end{array}$$

On the other hand, consider the map $h : X \longrightarrow X'$ between the quasi-projective schemes (h corresponds to $f \circ g' = g \circ f'$). It is an embeddable map, and it can be placed into two different fibre squares

$$\begin{array}{ccc} X & \hookrightarrow & M' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \hookrightarrow & N' \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \hookrightarrow & N \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \hookrightarrow & N' \end{array}$$

From axiom $(G2)$ applied to both diagrams we have that

$$\begin{aligned} h^* = i^! \circ q^* \\ h^* = j^! \circ p^* \end{aligned}$$

hence the two compositions are equal, therefore $f^A = g^A$ and we are done. □

Proposition 3.10. *(A6) A is homotopic invariant.*

Proof. We want to show that p^A is an isomorphism, where p is the vector bundle map from the fibre square

$$\begin{array}{ccc} Z & \hookrightarrow & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Z' & \hookrightarrow & X \end{array}$$

But from the definition $p^A = i^! \circ r^*$, where

$$\begin{array}{ccccc} Z & \longrightarrow & E \times Z' & \xrightarrow{r} & Z' \\ \downarrow & & \downarrow & & \downarrow \\ E & \xhookrightarrow{i} & E \times X & \longrightarrow & X \end{array}$$

And from property $(H5)$ we know that $i^! \circ r^* = q^*$, hence $p^A = q^*$, and since H is homotopic invariant q^* is an isomorphism, therefore p^A is an isomorphism. ☐

Proposition 3.11. *(A7)* ***Normalization:*** $(Id_{(X,M)})_A^{Id} = Id_{A_X(M)}$.

Proof.

$$\begin{aligned} Id_A^{Id} &= Id_* = Id_{H(Z)} \text{(functoriality of H)} \\ &= Id_{A_X(M)} \end{aligned}$$

☐

4 The Two Constructions Are Reciprocal

One may ask what happens if we apply the above algorithms twice. One way, if we start with a Borel-Moore functor H, build the associated algebraic oriented cohomology $\tilde{A}$, and from this theory build a second Borel-Moore functor $\widetilde{H}$, we get the same functor we started with.

However, if we start with and algebraic oriented theory A and apply the construction twice we do not return to the exact same theory. The next subsections describe what works, what does not work, and why this happens.

4.1 $H \longrightarrow \tilde{A} \longrightarrow \widetilde{H}$

Let us denote by $\tilde{A}$ the cohomology theory built and by $\widetilde{H}$ the second Borel-Moore functor. We will prove that $H = \widetilde{H}$, namely that for any quasi-projective scheme X we have $H(X) = \widetilde{H}(X)$, and that the push-forwards and pull-backs of the theory H coincide with the ones for the theory $\widetilde{H}$, denoted by $\widetilde{f}_*$, $\widetilde{f}^*$.

1. *Objects*: let X be a quasi-projective scheme and consider M a smooth scheme such that $X \hookrightarrow M$ is a closed embedding. Then by definition we have:

$$\widetilde{H}(X) = \tilde{A}_X(M) = H(X)$$

2. *Push-forwards*: Let $f : X \longrightarrow Y$ be a projective morphism with $f = p \circ i$ a standard decomposition with p a projection, i a closed embedding. Then in the theory $\tilde{H}$ we have $\tilde{f}_* = \tilde{p}_* \circ \tilde{i}_*$. If M is a smooth scheme such that $Y \hookrightarrow M$ is a closed embedding then we can define $\tilde{p}_*, \tilde{i}_*$ using the following commutative diagram:

$$\begin{array}{ccccc} X & \stackrel{i}{\hookrightarrow} & \mathbb{P}^n_Y & \stackrel{p}{\longrightarrow} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^n_M & = & \mathbb{P}^n_M & \stackrel{P}{\longrightarrow} & M \end{array}$$

$$\begin{aligned} \tilde{p}_* &= p^P_{\tilde{A}} \\ \tilde{i}_* &= i^{Id}_{\tilde{A}} \end{aligned}$$

Therefore $\tilde{f}_* = p^P_{\tilde{A}} \circ p^P_{\tilde{A}} = p_* \circ i_* = f_*$.

3. *Pull-Backs*: For any embeddable map $f : X \longrightarrow X'$ consider an associated fibre diagram:

$$\begin{array}{ccc} X & \hookrightarrow & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ X' & \hookrightarrow & M' \end{array}$$

Then from the definitions and remark 3.2 we get that:

$$\widetilde{f}^* = F^{\tilde{A}} = f^*$$

4. *Gysin Structure*: For any fibre square:

$$\begin{array}{ccc} X & \hookrightarrow & X' \\ \downarrow & & \downarrow \\ M & \stackrel{i}{\hookrightarrow} & M' \end{array}$$

from remark 3.3 we get that:

$$\tilde{i}^! = i^{\tilde{A}} = i^!$$

Therefore the constructed Borel-Moore functor $\widetilde{H}$ is equal to the Borel-Moore functor we started with.

Example 4.1. If we start with $H(X) = CH(X)$, the Chow group, then the constructed $\tilde{A}$ is isomorphic to the operational Chow group, with the structure given in example 1.20.

Proof. For any closed embedding $X \hookrightarrow M$ we know [Ful84, Chapter17] that $A(X \hookrightarrow M) \simeq CH(X)$, the isomorphism being given by

$$A(X \hookrightarrow M) \xrightarrow{\cdot[\pi]} A(X \longrightarrow pt) \xrightarrow{\phi} CH(X)$$

Therefore the two theories are isomorphic on objects:

$$\tilde{A}_X(M) = CH(X) \simeq A_X(M)$$

For any map of smooth pairs $f : (M, X) \longrightarrow (M', X')$ we can build the commutative diagram

$$\begin{array}{ccccccc} X & \xleftarrow{j} & \widetilde{X} & \longrightarrow & M \times X' & \xrightarrow{q} & X' \\ & & \downarrow & & \downarrow & & \downarrow \\ & & M & \xrightarrow{i} & M \times M' & \xrightarrow{p} & M' \end{array}$$

Then

$$\begin{aligned} f^{\tilde{A}} &= j_* \circ i^! \circ q^* \\ &= j_* \circ (p \circ i)^A \\ &= f^A \end{aligned}$$

For push-backs consider a projective morphism $f : X \longrightarrow Y$; then $f_A^F = f_*$ from the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ M & \xrightarrow{F} & N \end{array}$$

We need to show that $f_A^F(c) = f_*(c \cdot [F])$. For this, we need to take into consideration the canonical isomorphism between $\tilde{A}_X(M)$ and $A(X \hookrightarrow M)$. If

$\pi : N \longrightarrow pt$ is the structure morphism for N then $\pi \circ F : M \longrightarrow pt$ is the structure morphism for M and we have:

$$A(X \hookrightarrow M) \xrightarrow{\phi\circ(\cdot[\pi F])} \tilde{A}_X(M) \xrightarrow{f_*} \tilde{A}_Y(N) \xrightarrow{(\phi\circ\cdot[\pi])^{-1}} A(Y \hookrightarrow N)$$

$$c \longrightarrow \phi(c \cdot [\pi f]) \longrightarrow f_*(\phi(c \cdot [\pi f])) \longrightarrow (\phi \circ \cdot[\pi])^{-1}(f_*(\phi(c \cdot [\pi f])))$$

But

$$\begin{aligned}(\phi \circ \cdot[\pi])^{-1}(f_*(\phi(c \cdot [\pi f]))) &= f_*((\phi \circ \cdot[\pi])^{-1}((\phi(c \cdot [\pi f]))) \\ &= f_*(\cdot[\pi]^{-1}(\phi^{-1}((\phi(c \cdot [\pi f]))) \\ &= f_*(c \cdot [F])\end{aligned}$$

□

4.2 $A \longrightarrow \widetilde{H} \longrightarrow \tilde{A}$

As before, let us denote by $\tilde{H}$ the constructed Borel-Moore functor, and by $\tilde{A}$ the second oriented theory. If the theory A has an extra property, described in Proposition 4.2, we will prove that $A = \tilde{A}$. Therefore any algebraic oriented theory A with this extra property can be constructed out of a Borel-Moore functor H.

First we want to show that $A_X(M) = \tilde{A}_X(M)$

Let (M, X) be a smooth pair, with X a closed subscheme of M. Then by construction

$$\tilde{A}_X(M) = \widetilde{H}(X) \simeq A_X(M)$$

Proposition 4.2. *(Conditions for* $f^A = f^{\tilde{A}}$*)*

Any closed embedding $j : X \hookrightarrow Y$ *of two subschemes of a smooth scheme* M *induces a well-defined map of pairs* $F : (M, Y) \longrightarrow (M, X)$*. If for any such map we have* $F^A = j_A^{Id_M}$ *then the pull-backs of the theory* $\tilde{A}$ *coincide with the pull-backs of the theory* A*.*

Proof. Consider $f : (M, X) \longrightarrow (N, Y)$ a map of smooth pairs, and let $\widetilde{X} = f^{-1}(Y)$. Then $f^{\tilde{A}} = j_* \circ i^! \circ q^*$ from the commutative diagram:

$$\begin{array}{ccccccc} X & \xleftarrow{j} & \widetilde{X} & \longrightarrow & M \times Y & \xrightarrow{q} & Y \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M & \overset{Id}{=\!=} & M & \xrightarrow{i} & M \times N & \xrightarrow{P} & N \end{array}$$

On the other hand, we have

$$\begin{aligned} j_* &= j_A^{Id} \\ i^! &= i^A \\ q^* &= p^A \end{aligned}$$

Therefore, if $\bar{f} : (M, \widetilde{X}) \longrightarrow (N, Y)$ then

$$\begin{aligned} f^{\tilde{A}} &= j_A^{id} \circ i^A \circ p^A \\ &= j_A^{Id} \circ \bar{f}^A \end{aligned}$$

Let $F : (M, X) \longrightarrow (M, \widetilde{X})$ the map induced by $j : \widetilde{X} \hookrightarrow X$. Then $f = \bar{f} \circ F$, therefore

$$\begin{aligned} f^A &= (\bar{f} \circ F)^A \\ &= F^A \circ \bar{f}^A \\ &= j_A^{Id} \circ \bar{f}^A \\ &= f^{\tilde{A}} \end{aligned}$$

□

Proposition 4.3. *The push-forwards of the theory $\tilde{A}$ coincide with the ones of the theory A.*

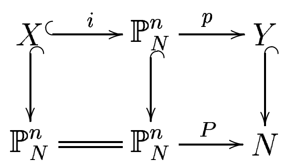
Proof. Consider a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ M & \xrightarrow{F} & N \end{array}$$

and a standard decomposition $f = p \circ i$ for the projective map f. Then according to the two constructions we have that

$$f^F_{\tilde{A}} = f_* = (p \circ i)_* = p^P_A \circ i^{Id}_A = (p \circ i)^{P \circ Id}_A = f^P_A$$

where the map $P : \mathbb{P}^n_N \longrightarrow N$ is the standard projection map, and the last equality comes from applying axiom $(A3)$ (functoriality for push-forwards) to the diagram:



Since $f^F_{\tilde{A}} : A_X(M) \longrightarrow A_Y(N)$ and $f^P_A : A_X(\mathbb{P}^n_N) \longrightarrow A_Y(N)$ we need to consider also the canonical isomorphism $Id^{pr_1}_A \circ (Id^{pr_2}_A)^{-1} : A_X(M) \longrightarrow A_X(\mathbb{P}^n_N)$ (where $pr_1 : \mathbb{P}^n_N \times M \longrightarrow \mathbb{P}^n_N, pr_2 : \mathbb{P}^n_N \times M \longrightarrow M$), therefore we have that:

$$f^F_{\tilde{A}} = f^P_A \circ Id^{pr_1}_A \circ (Id^{pr_2}_A)^{-1}$$

and our goal is to prove that

$$\begin{aligned} f^F_A &= f^P_A \circ Id^{pr_1}_A \circ (Id^{pr_2}_A)^{-1} \\ f^F_A \circ Id^{pr_2}_A &= f^P_A \circ Id^{pr_1}_A \end{aligned}$$

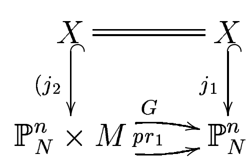
Since $F \circ pr_2$ may differ from $P \circ pr_1$ we need an extra step. For this, define $G : \mathbb{P}^n_N \times M \longrightarrow \mathbb{P}^n_N$ by the formula:

$$G(t, n, m) = (t, F(m))$$

Then $F \circ pr_2 = P \circ G$, therefore

$$\begin{aligned} f^F_A \circ Id^{pr_2}_A &= f^{F \circ pr_2}_A \\ &= f^{P \circ G}_A \\ &= f^P_A \circ Id^G_A \end{aligned}$$

and all left to prove is that $Id^{pr_1}_A = Id^G_A$, which follows from axiom $(A2b)$ applied to the commutative diagram:



One last computation is checking the commutativity of this diagram. For this let $x \in M$ and suppose $j_1(x) = (t, f(x))$ (since $p \circ i = f$ the second term has to be $f(x)$). Then:

$$\begin{aligned} (pr_1 \circ j_2)(x) &= pr_1(t, f(x), x) = (t, f(x)) \\ (G \circ j_2)(x) &= G(t, f(x), x) = (t, F(x)) \end{aligned}$$

and since $f(x) = F(x)$ in N the proof is complete. □

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