Homotopy invariance of the sheaf W_{Nis} and of its cohomology

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Abstract

A conjecture of F.Morel states that the motivic group $\pi_{0,0}(k)$ of a perfect field k coincides with the Grothendieck-Witt group $GW(k)$ of quadratic forms over k provided that $\text{char}(k) \neq 2$. Morel's proof of the conjecture requires the following result: the Nisnevich sheaf W_{Nis} associated with the presheaf $X \mapsto W(X)$ is homotopy invariant and all its Nisnevich cohomology are homotopy invariant too. A rather short and self-contained proof of the result is given here. Another proof is due to J.Hornbostel [H].

Introduction

We consider a perfect field of characteristic different of 2. For an affine scheme S we write $W(S)$ for the Witt group of quadratic spaces over the ring $k[S]$ of regular functions on S . We consider the big Nisnevich site $\mathcal{S}m_{Nis}$ of k -smooth schemes and the Nisnevich sheaf W_{Nis} associated with the presheaf $X \mapsto W(S)$ on $\mathcal{S}m_{Nis}$. Furthermore consider the big Zarisky site $\mathcal{S}m_{Zar}$ of k -smooth schemes and the Zarisky sheaf W_{Zar} associated with the presheaf $X \mapsto W(S)$ on $\mathcal{S}m_{Zar}$. The main aim of the preprint is to prove the following result.

0.0.1 Theorem (Main). *The Nisnevich sheaf W_{Nis} and the Zarisky sheaves are strictly homotopy invariant that is for each k -smooth variety X one has*

- (1) $W_{Nis}(X) = W_{Nis}(X \times \mathbf{A}^1)$,
- (2) $H_{Nis}^p(X, W_{Nis}) = H_{Nis}^p(X \times \mathbf{A}^1, W_{Nis})$ ($p > 0$).

Moreover

- (3) $W_{Zar} = W_{Nis}$, that is for any k -smooth X one has $W_{Zar}(X) = W_{Nis}(X)$,

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(4) $H_{Zar}^i(X, W_{Zar}) = H_{Nis}^i(X, W_{Nis})$ for any $i > 0$.

In particular, the sheaf W_{Zar} is strictly homotopy invariant.

The proof of the theorem is based on certain known results concerning Witt groups:

(1) for each field extension k'/k one has

$$W(k') = W(\mathbf{A}_{k'}^1),$$

(2) for a point $x \in X$ and the local scheme $U = \text{Spec}(\mathcal{O}_{X,x})$ the Gersten-Witt complex

$$0 \rightarrow W(\mathcal{O}_{X,x}) \rightarrow W(k(X)) \xrightarrow{res} \coprod_{y \in U^{(1)}} W(k(y)) \xrightarrow{res} \coprod_{z \in U^{(2)}} W(k(z)) \xrightarrow{res} \dots$$

is exact by [B],

(3) for each field extension k'/k and each open $U \subset \mathbf{A}_{k'}^1$ there is an exact sequence of the Witt groups (see [L, Thm. 3.1])

$$0 \rightarrow W(U) \rightarrow W(k'(t)) \xrightarrow{\sum res_{\pi_x}} \coprod_{x \in U} W(k'(x)) \rightarrow 0.$$

The final version of this preprint was written during my visit to the Institute of Advanced Study which provided excellent conditions to finish this work.

1 Voevodsky trick

Suppose the ground field k is PERFECT. Let $\mathcal{F}$ be a homotopy invariant Nisnevich sheaf on the big Nisnevich site $\mathcal{S}m_{Nis}$. Assume furthermore that for an integer $j > 0$ the functors $H_{Nis}^i(-, \mathcal{F})$ are homotopy invariant provided that $i < j$. Finally assume that for each field extension k'/k one has $H_{Nis}^i(\mathbf{A}_{k'}^1, \mathcal{F}_{Nis}) = 0$ for $i > 0$. The following result is extracted from [V, Section 4.6].

1.0.2 Theorem (Voevodsky trick). *Suppose for each essentially smooth local scheme U and each essentially smooth divisor D on U the map*

$$H_{Nis}^j(U \times \mathbf{A}^1, \mathcal{F}) \rightarrow H_{Nis}^j((U - D) \times \mathbf{A}^1, \mathcal{F})$$

is injective. Then for each (essentially) k -smooth X

$$\text{Ker} [H_{Nis}^j(X \times \mathbf{A}^1, \mathcal{F}) \xrightarrow{i_0^*} H_{Nis}^j(X, \mathcal{F})] = 0.$$

The proof requires auxiliary results.

1.0.3 Lemma. *The Leray spectral sequence for the projection $p : X \times \mathbf{A}^1 \rightarrow X$ gives rise to a short exact sequence*

$$\{0\} \rightarrow H_{Nis}^j(X, R^0 p_*(\mathcal{F})) \xrightarrow{\alpha} H_{Nis}^j(X \times \mathbf{A}^1, \mathcal{F}) \xrightarrow{\beta} H_{Nis}^0(X, R^j p_*(\mathcal{F})).$$

and moreover $R^0 p_*(\mathcal{F}) = \mathcal{F}$ and $\alpha = p^*$.

1.0.4 Notation. Set $\bar{H}_{Nis}^j(X \times \mathbf{A}^1, \mathcal{F}) = \text{Ker} [H_{Nis}^j(X \times \mathbf{A}^1, \mathcal{F}) \xrightarrow{i_0^*} H_{Nis}^j(X, \mathcal{F})]$.

1.0.5 Lemma. *The map $\beta|_{\bar{H}} : \bar{H}_{Nis}^j(X \times \mathbf{A}^1, \mathcal{F}) \rightarrow H_{Nis}^0(X, R^j p_*(\mathcal{F}))$ is injective.*

This follows from Lemma 1.0.3.

1.0.6 Lemma. *Under the hypotheses of Theorem 1.0.2 for each essentially smooth closed $Z \subset X$ the map*

$$\bar{H}_{Nis}^j(X \times \mathbf{A}^1, \mathcal{F}) \rightarrow \bar{H}_{Nis}^j((X - Z) \times \mathbf{A}^1, \mathcal{F})$$

is injective.

To prove this Lemma consider the commutative diagram

$$\begin{array}{ccc} \bar{H}_{Nis}^j(X \times \mathbf{A}^1, \mathcal{F}) & \xrightarrow{\beta} & H_{Nis}^0(X, R^j p_*(\mathcal{F})) \\ \downarrow & & \downarrow \\ \bar{H}_{Nis}^j((X - Z) \times \mathbf{A}^1, \mathcal{F}) & \xrightarrow{\beta} & H_{Nis}^0(X - Z, R^j p_*(\mathcal{F})). \end{array}$$

The horizontal arrows are both injective by Lemma 1.0.5. For each $x \in X$ and $U = \text{Spec}(\mathcal{O}_{X,x})$ there exists an essentially smooth divisor D in U containing $Z \cap U$. Moreover the map $H_{Nis}^j(U \times \mathbf{A}^1, \mathcal{F}) \rightarrow H_{Nis}^j((U - D) \times \mathbf{A}^1, \mathcal{F})$ is injective. Thus for the open inclusion $j : X - Z \hookrightarrow X$ the adjunction sheaf morphism $R^j p_*(\mathcal{F}) \rightarrow j_* j^* R^j p_*(\mathcal{F})$ is injective. Taking the global sections we get the injectivity of the right hand side vertical arrow in the last diagram. The Lemma follows.

1.0.7 Lemma. *Under the hypotheses of Theorem 1.0.2 for each closed $Z \subset X$ the map*

$$\bar{H}_{Nis}^j(X \times \mathbf{A}^1, \mathcal{F}) \rightarrow \bar{H}_{Nis}^j((X - Z) \times \mathbf{A}^1, \mathcal{F})$$

is injective.

To proof this Lemma take an element $a \in \bar{H}_{Nis}^j(X \times \mathbf{A}^1, \mathcal{F})$ with $a|_{(X-Z) \times \mathbf{A}^1} = 0$ and consider the singular locus $\text{Sing}(Z)$ of Z . It is a proper closed subset of Z because the field k is perfect. Applying Lemma 1.0.6 to the pair $(X^{(1)}, Z^{(1)}) := (X - \text{Sing}(Z), Z - \text{Sing}(Z))$ and the element $a^{(1)} = a|_{X^{(1)}}$ we conclude that $0 = a|_{X - \text{Sing}(Z)} = a^{(1)}$. Replace Z by $\text{Sing}(Z)$ and $\text{Sing}(Z)$ by $\text{Sing}(\text{Sing}(Z))$. Since k is perfect $\text{Sing}(\text{Sing}(Z))$ is a proper closed subset of $\text{Sing}(Z)$. Applying Lemma 1.0.6 to the pair

$$(X - \text{Sing}(\text{Sing}(Z)), \text{Sing}(Z) - \text{Sing}(\text{Sing}(Z)))$$

we conclude that $a|_{X - \text{Sing}(\text{Sing}(Z))} = 0$. Repeating the process several times we get $a = 0$.

Proof of Theorem 1.0.2. Let $a \in \bar{H}_{Nis}^j(X \times \mathbf{A}^1, \mathcal{F})$. Since $H_{Nis}^j(\mathbf{A}_{k(X)}^1, \mathcal{F}) = 0$ there exists a closed Z in X such that a vanishes on $(X - Z) \times \mathbf{A}^1$. Lemma 1.0.7 completes the proof of the Theorem. □

2 Auxiliary results

In this section we prove some auxiliary results.

2.0.8 Lemma. *For an essentially k -smooth scheme X one has*

$$W_{Zar}(X) = \bigcap_{x \in X^{(1)}} W(\mathcal{O}_{X,x}) \subseteq W(k(X)),$$

where the intersection is taken over all codimension one points x in X .

Proof. For an essentially k -smooth variety X and a point $x \in X$ the map $W(\mathcal{O}_{X,x}) \rightarrow W(k(X))$ is injective [Oj] (it follows also from the result (2) listed in the Introduction). Thus $W_{Zar}(X) = \bigcap_{x \in X} W(\mathcal{O}_{X,x}) \subseteq W(k(X))$. The last intersection coincides with the one

$\bigcap_{x \in X^{(1)}} W(\mathcal{O}_{X,x})$ by the purity theorem [OP, Thm. A] (it follows also from the result (2) listed in the Introduction). □

2.0.9 Corollary. *For an essentially k -smooth variety X and an open imbedding $U \subset X$ the pull-back map*

$$W_{Zar}(X) \hookrightarrow W_{Zar}(U)$$

is injective.

2.0.10 Lemma. *For each field extension k'/k the restriction of the pre-sheaf W to the affine line $\mathbf{A}_{k'}^1$ is already a Zarisky sheaf and even a Nisnevich sheaf.*

Proof. For any open $U \subseteq \mathbf{A}_{k'}^1$ one has an exact sequence of the Witt groups (see (3))

$$0 \rightarrow W(U) \rightarrow W(k'(t)) \xrightarrow{\sum res_{\pi_x}} \coprod_{x \in U} W(k'(x)) \rightarrow 0.$$

This sequence and Lemma 2.0.8 shows that $W(U) = W_{Zar}(U)$. By Lemma 3.0.13 $W_{Zar}(U) = W_{Nis}(U)$. The Lemma follows. □

2.0.11 Corollary. *The following holds*

- (1) $W(\mathbf{A}_{k'}^1) = W_{Zar}(\mathbf{A}_{k'}^1) = W_{Nis}(\mathbf{A}_{k'}^1)$,
- (2) $H_{Zar}^i(\mathbf{A}_{k'}^1, W_{Zar}) = H_{Nis}^i(\mathbf{A}_{k'}^1, W_{Nis}) = 0$ for all $i > 0$.

Proof. The first assertion is clear. To prove the second one consider the sequence of Zarisky sheaves on $\mathbf{A}_{k'}^1$

$$0 \rightarrow W_{Zar} \rightarrow W(k'(t)) \xrightarrow{\sum res_{\pi_x}} \coprod_{x \in U} i_{x,*} W(k'(x)) \rightarrow 0.$$

on $\mathbf{A}_{k'}^1$ (the middle term here is considered as a constant sheaf). It is exact by the property (3) from the Introduction. So it can be considered as a flabby resolution of the sheaf W_{Zar} on $\mathbf{A}_{k'}^1$. Since the sequence of the global sections on $\mathbf{A}_{k'}^1$ is exact too the Zarisky cohomology $H_{Zar}^i(\mathbf{A}_{k'}^1, W_{Zar})$ vanishes for $i > 0$. The same arguments works for the Nisnevich sheaf W_{Nis} .

□

2.0.12 Lemma. *For a k -smooth scheme X and a closed subset $Z \subset X$ let $\mathcal{H}_Z^i(-, W_{Nis})$ be the Nisnevich sheaf associated with the pre-sheaf $U \mapsto H_{Z \cap U}^i(U, W_{Nis})$. For each smooth principal divisor $i : D \hookrightarrow X$ one has*

- (1) *the sheaves $\mathcal{H}_D^1(X, W_{Nis})$ and $i_*(W_{Nis})$ on X are isomorphic,*
- (2) *$\mathcal{H}_D^i(X, W_{Nis}) = 0$ for $i \neq 1$.*

Proof. This Lemma is the most difficult one in this Section. One can give a self-contained proof based on the trace method used in the proof of [OP, Thm. B]. However now we give a proof by a reference to [BGPW, Proof of Thm. 4.4].

Let $GW_C(X)$ and $GW_C(X, D)$ be the Gersten-Witt complex of X and of X with the supports on D respectively [BGPW, Def. 3.1]. A choice of the equation defining the divisor D identifies the Gersten-Witt complexes $GW_C(X, D)$ and $GW_C(D)[1]$ by [BGPW, Lemma 3.3]. Thus $H^i(GW_C(X, D)) = H^{i-1}(GW_C(D))$ for all $i \geq 0$. The Gersten-Witt complex $GW_C(X)$ (respectively $GW_C(D)$) is the complex of the global sections of a flabby resolution of the Nisnevich sheaf W_{Nis} on X (respectively on D) [BGPW, Proof of Lemma 4.2]. Thus

$$H_D^i(X, W_{Nis}) = H^i(GW_C(X, D)) = H^{i-1}(GW_C(D)) = H^{i-1}(D, W_{Nis}).$$

Thus one has $H_D^i(X, W_{Nis}) = H^{i-1}(D, W_{Nis})$. In the case of essentially k -smooth local Henselian X and $i \neq 1$ the group $H^{i-1}(D, W_{Nis})$ vanishes by [B, Thm ??] (use the property (2) from the Introduction). This proves the second assertion of the Lemma. The first one follows from the functoriality of the isomorphisms $H_{D \cap U}^1(U, W_{Nis}) = H^0(D \cap U, W_{Nis}) = H^0(U, i_* W_{Nis})$, where U runs over Zarisky open subsets of X .

□

3 Proof of the Main theorem

3.0.13 Lemma. *The Zarisky sheaf W_{Zar} coincides with the Nisnevich sheaf W_{Nis} .*

Proof. It suffices to prove that for an elementary Nisnevich square (see [MV, Lemma ??])

$$\begin{array}{ccc} \tilde{X} & \xleftarrow{\tilde{j}} & \tilde{U} \\ p \downarrow & & \downarrow q \\ X & \xleftarrow{j} & U \end{array}$$

the sequence

$$0 \rightarrow W_{Zar}(X) \xrightarrow{p^*+j^*} W_{Zar}(\tilde{X}) \oplus W_{Zar}(U) \xrightarrow{q^*-j^*} W_{Zar}(\tilde{U}) \quad (1)$$

is exact. Recall that the square is elementary if it is Cartesian with an open inclusion j and an etale morphism p such that p induces an isomorphism of the reduced closed subschemes $\tilde{X} - \tilde{U}$ and $X - U$.

The maps j^* and $\tilde{j}^*$ are injective by Corollary 2.0.9. So to prove the exactness of the sequence (1) it suffices to check that for a pair $(\tilde{a}, b) \in W_{Zar}(\tilde{X}) \oplus W_{Zar}(U)$ with $\tilde{j}^*(\tilde{a}) = q^*(b)$ there exist an $a \in W_{Zar}(X)$ such that $p^*(a) = \tilde{a}$ and $j^*(a) = b$.

Claim. The element $b \in W_{Zar}(U) \subset W(k(X))$ belongs to the subgroup $W_{Zar}(X)$. Given the Claim set $a = b \in W_{Zar}(X)$. Since $\tilde{j}^*$ is injective this a is the required element. It remains to check that the Claim.

To do that it suffices to check that for any codimension one point $x \in X$ the element b belongs to $W(\mathcal{O}_{X,x})$ (see 2.0.8). This is the case if $x \in U$. So we may assume that $x \in Z \cap X$ and consider the cartesian square

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}_{\tilde{X},x}) & \xleftarrow{\tilde{j}} & \mathrm{Spec}(k(\tilde{X})) \\ p \downarrow & & \downarrow q \\ \mathrm{Spec}(\mathcal{O}_{X,x}) & \xleftarrow{j} & \mathrm{Spec}(k(X)) \end{array}$$

which is the pull-back of the square above by means of the morphism $\mathrm{Spec}(\mathcal{O}_{X,x}) \rightarrow X$. The last square gives rise to a diagram of the Witt groups.

$$\begin{array}{ccccccc} 0 & \longrightarrow & W(\mathcal{O}_{\tilde{X},x}) & \xrightarrow{\tilde{j}^*} & W(k(\tilde{X})) & \xrightarrow{res_{\tilde{\pi}}} & W(k(x)) \longrightarrow 0 \\ & & p^* \uparrow & & q^* \uparrow & & id \uparrow \\ 0 & \longrightarrow & W(\mathcal{O}_{X,x}) & \xrightarrow{j^*} & W(k(X)) & \xrightarrow{res_{\pi}} & W(k(x)) \longrightarrow 0 \end{array}$$

The rows in the diagram are short exact sequences. The residue maps are defined by local parameters in the discrete valuation ring $\mathcal{O}_{X,x}$ and $\mathcal{O}_{\tilde{X},x}$. Choosing a local parameter $\pi \in \mathcal{O}_{X,x}$ we get a local parameter $\tilde{\pi} = p^*(\pi) \in \mathcal{O}_{\tilde{X},x}$. This choice of local parameters makes the last diagram commutative. Since $q^*(b) = \tilde{j}^*(\tilde{a})$ one has $res_{\tilde{\pi}}(q^*(b)) = 0$. Thus $res_{\pi}(b) = 0$ and b belongs to $W(\mathcal{O}_{X,x})$. The Claim follows. Whence the Lemma. $\square$

Proof of the Main Theorem. The outline of our proof follows [V, Proof of Thm. 4.27]. Check firstly that the Nisnevich sheaf W_{Nis} is *homotopy invariant*. For a smooth X consider the diagram

$$\begin{array}{ccccc} W_{Nis}(X \times \mathbf{A}^1) & \xrightarrow{J^*} & W_{Nis}(\mathbf{A}_{k(X)}^1) & \xleftarrow{can} & W(\mathbf{A}_{k(X)}^1) \\ i_{0,X}^* \downarrow & & i_{0,k(X)}^* \downarrow & & \downarrow i_{0,k(X)}^* \\ W_{Nis}(X) & \xrightarrow{j^*} & W_{Nis}(k(X)) & \xleftarrow{can} & W(k(X)) \end{array}$$

with the pull-back mappings J^* and j^* . The maps "can" are isomorphisms by Lemma 2.0.10 and the right hand side $i_{0,k(X)}^*$ is an isomorphism by Corollary 2.0.11. Thus the middle arrow $i_{0,k(X)}^*$ is an isomorphism too. The map J^* is injective by Corollary 2.0.9. Thus the map $i_{0,X}^*$ is injective too. Obviously $i_{0,X}^*$ is surjective. Thus it is an isomorphism and $p^* : W_{Nis}(X) \rightarrow W_{Nis}(X \times \mathbf{A}^1)$ is an isomorphism too. The homotopy invariance follows.

Now prove the homotopy invariance of $H_{Nis}^1(-, W_{Nis})$. The Leray spectral sequence for the projection $p : X \times \mathbf{A}^1 \rightarrow X$ gives rise to a short exact sequence of the sheaves

$$0 \rightarrow H_{Nis}^1(X, R^0 p_*(W_{Nis})) \xrightarrow{\alpha} H_{Nis}^1(X \times \mathbf{A}^1, W_{Nis}) \xrightarrow{\beta} H_{Nis}^0(X, R^1 p_*(W_{Nis})).$$

By the homotopy invariance of the sheaf W_{Nis} the group $H_{Nis}^1(X, R^0 p_*(W_{Nis}))$ coincides with the one $H_{Nis}^1(X, W_{Nis})$ and $\alpha = p^*$. To show that p^* is an isomorphism it suffices to check that the Nisnevich sheaf $R^1 p_*(W_{Nis})$ vanishes. For that it remains to show that for a Henselian essentially smooth local scheme X one has $H_{Nis}^1(X \times \mathbf{A}^1, W_{Nis}) = 0$.

Since $H_{Nis}^1(\mathbf{A}_{k(X)}^1, W_{Nis}) = 0$ (see 2.0.11) it suffices to check that the map

$$H_{Nis}^1(X \times \mathbf{A}^1, W_{Nis}) \rightarrow H_{Nis}^1(\mathbf{A}_{k(X)}^1, W_{Nis})$$

is injective. By the Voevodsky trick 1.0.2 it suffices to check that for a smooth divisor $i : D \hookrightarrow X$ the map

$$H_{Nis}^1(X \times \mathbf{A}^1, W_{Nis}) \rightarrow H_{Nis}^1((X - D) \times \mathbf{A}^1, W_{Nis})$$

is injective. For that it suffices to verify that the boundary map

$$H_{Nis}^1((X - D) \times \mathbf{A}^1, W_{Nis}) \xrightarrow{\partial_{\mathbf{A}^1}} H_{D \times \mathbf{A}^1}^1(X \times \mathbf{A}^1, W_{Nis})$$

from the localization sequence is surjective. To prove this we need auxiliary lemmas.

3.0.14 Lemma. *For any essentially k -smooth scheme S and its closed subscheme Z the map*

$$\gamma : H_Z^1(S, W_{Nis}) \rightarrow H^0(S, \mathcal{H}_Z^1(-, W_{Nis}))$$

is an isomorphism, where $\mathcal{H}_Z^1(-, W_{Nis})$ is the Nisnevich sheaf associated with the pre-sheaf $U \mapsto H_{Z \cap U}^1(U, W_{Nis})$.

In fact, the map γ fits in the exact sequence

$$0 \rightarrow H^1(S, \mathcal{H}_Z^0(W_{Nis})) \rightarrow H_Z^1(S, W_{Nis}) \xrightarrow{\gamma} H^0(S, \mathcal{H}_Z^1(W_{Nis})) \rightarrow H^2(S, \mathcal{H}_Z^0(W_{Nis}))$$

and the sheaf $\mathcal{H}_Z^0(W_{Nis}) := \mathcal{H}_Z^0(-, W_{Nis})$ vanishes by 2.0.9.

3.0.15 Lemma. *The map $p^* : H^0(X, \mathcal{H}_D^1(W_{Nis})) \rightarrow H^0(X \times \mathbf{A}^1, \mathcal{H}_{D \times \mathbf{A}^1}^1(W_{Nis}))$ is an isomorphism.*

Let $i : D \hookrightarrow X$ be the closed imbedding. Lemma 2.0.12, the commutative diagram

$$\begin{array}{ccccc} W_{Nis}(D \times \mathbf{A}^1) & \xrightarrow{\cong} & H^0(X \times \mathbf{A}^1, (i \times id)_* W_{Nis}) & \xrightarrow{\cong} & H^0(X \times \mathbf{A}^1, \mathcal{H}_{D \times \mathbf{A}^1}^1(W_{Nis})) \\ p^* \uparrow & & p^* \uparrow & & p^* \uparrow \\ W_{Nis}(D) & \xrightarrow{\cong} & H^0(X, i_* W_{Nis}) & \xrightarrow{\cong} & H^0(X, \mathcal{H}_D^1(W_{Nis})), \end{array}$$

and the homotopy invariance of the sheaf W_{Nis} prove the lemma.

3.0.16 Lemma. *For the Henselian local scheme X the boundary map $\partial : W_{Nis}(X - D) \xrightarrow{\partial} H_D^1(X, W_{Nis})$ is surjective.*

In fact, the next term in the localization sequence is $H_{Nis}^1(X, W_{Nis})$. It vanishes, because X is local for the Nisnevich topology.

To complete the proof of the homotopy invariance of $H_{Nis}^1(-, W_{Nis})$ consider the commutative diagram

$$\begin{array}{ccccc} W_{Nis}((X - D) \times \mathbf{A}^1) & \xrightarrow{\partial_{\mathbf{A}^1}} & H_{D \times \mathbf{A}^1}^1(X \times \mathbf{A}^1, W_{Nis}) & \xrightarrow{\gamma_{\mathbf{A}^1}} & H^0(X \times \mathbf{A}^1, \mathcal{H}_{D \times \mathbf{A}^1}^1(W_{Nis})) \\ p^* \uparrow & & p^* \uparrow & & \uparrow p^* \\ W_{Nis}(X - D) & \xrightarrow{\partial} & H_D^1(X, W_{Nis}) & \xrightarrow{\gamma} & H^0(X, \mathcal{H}_D^1(W_{Nis})). \end{array}$$

The maps γ and $\gamma_{\mathbf{A}^1}$ are isomorphisms by Lemma 3.0.14. The right hand side map p^* is an isomorphism by Lemma 3.0.15. Thus the middle arrow p^* is an isomorphism too. Since ∂ is a surjection the map $\partial_{\mathbf{A}^1}$ is surjective too. The homotopy invariance of $H_{Nis}^1(-, W_{Nis})$ is proved.

Now prove the homotopy invariance of $H_{Nis}^i(-, W_{Nis})$ for $i > 1$. We proceed induction by i . So assuming that the homotopy invariance holds for all $i < j$ prove it for $i = j$. The Leray spectral sequence for the projection p and the inductive hypotheses give the following short exact sequence

$$\{0\} \rightarrow H_{Nis}^j(X, R^0 p_*(W_{Nis})) \xrightarrow{\alpha} H_{Nis}^j(X \times \mathbf{A}^1, W_{Nis}) \xrightarrow{\beta} H_{Nis}^0(X, R^j p_*(W_{Nis})).$$

By the homotopy invariance of the sheaf W_{Nis} the group $H_{Nis}^j(X, R^0 p_*(W_{Nis}))$ coincides with the one $H_{Nis}^j(X, W_{Nis})$ and $\alpha = p^*$. To show that p^* is an isomorphism it suffices to check that the Nisnevich sheaf $R^j p_*(W_{Nis})$ vanishes. For that it remains to show that for a Henselian essentially smooth scheme X one has $H_{Nis}^j(X \times \mathbf{A}^1, W_{Nis}) = 0$.

Since $H_{Nis}^j(\mathbf{A}_{k(X)}^1, W_{Nis}) = 0$ (see 2.0.11) it suffices to check that the map

$$H_{Nis}^j(X \times \mathbf{A}^1, W_{Nis}) \rightarrow H_{Nis}^j(\mathbf{A}_{k(X)}^1, W_{Nis})$$

is injective. By the Voevodsky trick 1.0.2 it suffices to check that for an essentially smooth divisor $i : D \hookrightarrow X$ on an essentially smooth Henselian scheme X the map

$$H_{Nis}^j(X \times \mathbf{A}^1, W_{Nis}) \rightarrow H_{Nis}^j((X - D) \times \mathbf{A}^1, W_{Nis})$$

is injective. The localization sequence shows that it remains to check the vanishing of the group $H_{D \times \mathbf{A}^1}^j(X \times \mathbf{A}^1, W_{Nis})$ in this case. The spectral sequence

$$E_{p,q}^2 = H^p(X \times \mathbf{A}^1, \mathcal{H}_{D \times \mathbf{A}^1}^q(-, W_{Nis})) \Rightarrow H_{D \times \mathbf{A}^1}^{p+q}(X \times \mathbf{A}^1, W_{Nis})$$

shows that the group $H_{D \times \mathbf{A}^1}^j(X \times \mathbf{A}^1, W_{Nis})$ vanishes if the groups $H^p(X \times \mathbf{A}^1, \mathcal{H}_{D \times \mathbf{A}^1}^q(W_{Nis}))$ vanishes for all pairs (p, q) with $p + q = j$. The sheaves $\mathcal{H}_{D \times \mathbf{A}^1}^q(W_{Nis})$ vanish for $q \neq 1$ by 2.0.12. The sheaf $\mathcal{H}_{D \times \mathbf{A}^1}^1(W_{Nis})$ is isomorphic by Lemma 2.0.12 to the sheaf $(i \times id)_* W_{Nis}$ on $X \times \mathbf{A}^1$. Thus one has a chain of isomorphisms

$$H^{j-1}(X \times \mathbf{A}^1, \mathcal{H}_{D \times \mathbf{A}^1}^1(W_{Nis})) \cong H^{j-1}(X \times \mathbf{A}^1, (i \times id)_* W_{Nis}) = H^{j-1}(D \times \mathbf{A}^1, W_{Nis})$$

By the inductive hypothesis $H^{j-1}(D \times \mathbf{A}^1, W_{Nis}) = H^{j-1}(D, W_{Nis})$ and the last group vanishes because D is local Henselian and $j - 1 > 0$. Thus for local essentially smooth Henselian X the group $H_{D \times \mathbf{A}^1}^j(X \times \mathbf{A}^1, W_{Nis})$ vanishes. The inductive step is checked. The homotopy invariance is proved.

It remains to prove that $H_{Zar}^i(X, W_{Zar}) = H_{Nis}^i(X, W_{Nis})$ for any $i > 0$. The Leray spectral sequence for the morphism of the sites $\mathcal{S}m_{Nis} \rightarrow \mathcal{S}m_{Zar}$ and Lemma 2.0.8 shows that it suffices to prove the following

Claim. For a k -smooth scheme X its point x and the local scheme $U = \text{Spec } \mathcal{O}_{X,x}$ one has $H_{Nis}^i(U, W_{Nis}) = 0$ for $i > 0$.

To prove this Claim note that $H_{Nis}^i(\text{Spec } k(X), W_{Nis}) = 0$ for $i > 0$. It remains to prove the injectivity of the pull-back map $\eta^* : H_{Nis}^i(U, W_{Nis}) \rightarrow H_{Nis}^i(\text{Spec } k(X), W_{Nis})$ induced by the generic point $\eta : \text{Spec } k(X) \rightarrow U$. To do that note that the functor $\mathcal{S}m \rightarrow \mathcal{A}b$ taking a k -smooth variety Y to $H_{Nis}^i(Y, W_{Nis})$ satisfies the axioms P1, P2 and P3 of the paper [C-TO]. In fact, it commutes with direct limits of finitely generated k -algebras, it satisfies the homotopy invariance as we already proved and it satisfies the gluing property P3 because it is Nisnevich cohomology functor. Now by [C-TO, Th. ??] the map η^* is injective. Whence the Claim. Theorem 0.0.1 is proved. □

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