Slice filtration on motives and the Hodge conjecture

Annette Huber, July 2003/February 2005

Introduction

The slice filtration on Voevodsky's triangulated category of motives is defined by effectivity conditions. It is constructed and studied in [HK]. An analogous filtration on the homotopy category was introduced by Voevodsky. In this note we try to get a - conjectural - picture of the properties of the slice filtration by systematic use of the realization functor to the derived category of Hodge structures. A key ingredient is Grothendieck's Generalized Hodge Conjecture about the analogous filtration on pure Grothendieck motives.

This approach is successful, even if the answers are not what we had hoped for originally. An old example of Griffith's allows to deduce - using deep but standard conjectures - the following:

1. The slice filtration does not respect the subcategory of geometrical motives.
2. The slice filtration does not commute with the weight filtration.
3. The induced slice filtration functors on the (conjectural) abelian category of mixed motives are left exact but not exact.

Moreover, the induced filtration on the (conjectural) abelian category of pure motives agrees with the coniveau filtration. As a byproduct, Grothendieck's Generalized Hodge Conjecture is generalized to triangulated motives. The generalization is implied by the same set of conjectures.

J. Ayoub recently communicated a non-conditional argument for property 1. to me. This may be read as a confirmation for the conjectural picture we have of the theory of motives.

The note was written in context of the joint project with B. Kahn on the slice filtration and its properties, see [HK]. I would like to thank him heartily for many interesting discussions. Several people helped me in my hunt for a good example. I am indebted to H. Esnault, B. Herzog, U. Jannsen, B. Moonen, A. Mukherjee and C. Voisin. It is a pleasure to thank them.

1 Definition of the slice filtration

We review the construction of the slice filtration as constructed in [HK]. For the purpose of the present article it suffices to work over the field of complex numbers $\mathbb{C}$. Most results extend to all fields of characteristic zero. We restrict to a $\mathbb{Q}$-rational theory.

Let $D\mathcal{M}_{\mathrm{gm}} = D\mathcal{M}_{\mathrm{gm}}(\mathbb{C}) \otimes \mathbb{Q}$ be Voevodsky's category of geometrical motives ([V] section 2.1), $D\mathcal{M}^{\mathrm{eff}}_{\mathrm{gm}} = D\mathcal{M}^{\mathrm{eff}}_{\mathrm{gm}}(\mathbb{C}) \otimes \mathbb{Q}$ the full subcategory of effective motives ([V] Definition 2.1.1). Let $D\mathcal{M}^{\mathrm{eff}}_{-}$ the category of bounded above complexes of Nisnevich sheaves with transfers which have homotopy invariant cohomology, i.e., Voevodsky's category of motivic complexes ([V] section 3.1). Let $D\mathcal{M}_{-}$ be the category obtained from $D\mathcal{M}^{\mathrm{eff}}$ by formally inverting the Tate object. Quasi-invertibility (e.g. [HK] Prop. A.1) can be used to show that there is a natural full embedding

$$\iota : D\mathcal{M}^{\mathrm{eff}}_{-} \to D\mathcal{M}_{-} \ .$$

In all, there is a commutative diagram of full embeddings

$$\begin{tikzcd}
D\mathcal{M}^{\mathrm{eff}}_{\mathrm{gm}} \arrow[r] \arrow[d] & D\mathcal{M}_{\mathrm{gm}} \arrow[d] \\
D\mathcal{M}^{\mathrm{eff}}_{-} \arrow[r] & D\mathcal{M}_{-}
\end{tikzcd}$$

Lemma 1.1 ([HK], Prop. 1.1). *ι has a right adjoint τ, i.e., $\tau : D\mathcal{M}_{-} \to D\mathcal{M}^{\mathrm{eff}}_{-}$ and a natural transformation $\tau \to \mathrm{id}$ s.t.*

$$\mathrm{Hom}_{D\mathcal{M}_{-}}(\iota N, M) = \mathrm{Hom}_{D\mathcal{M}^{\mathrm{eff}}_{-}}(N, \tau(M))$$

for all $N \in D\mathcal{M}^{\mathrm{eff}}_{-}$, $M \in D\mathcal{M}_{-}$.

Proof. Let M, N as in the lemma. By definition, $M(m)$ is effective for m big enough. We put

$$\tau(M) := \lim_{m\to\infty} \underline{\mathrm{Hom}}_{D\mathcal{M}^{\mathrm{eff}}_{-}}(\mathbb{Q}(m), M(m))$$

where $\underline{\mathrm{Hom}}$ is internal Hom in $D\mathcal{M}^{\mathrm{eff}}$ ([V] 3.2). By quasi-invertibility of $\mathbb{Z}(1)$, the limit stabilizes: if $M = \tilde{M}(-m)$ with $\tilde{M} \in D\mathcal{M}^{\mathrm{eff}}_{-}$, $m \geq 0$, then

$$\tau(M) = \underline{\mathrm{Hom}}_{D\mathcal{M}^{\mathrm{eff}}_{-}}(\mathbb{Q}(m), \tilde{M}) \ .$$

The universal property is easy to check. □

For $n \in \mathbb{Z}$ let $D\mathcal{M}^{\geq n}_{-} = D\mathcal{M}^{\mathrm{eff}}_{-}(n)$. There is a sequence of functors

$$\nu^{\geq n} : D\mathcal{M}_{-} \to D\mathcal{M}^{\geq n}_{-}$$

right adjoint to the embedding. Explicitly:

$$\nu^{\geq n}(M) = \tau(M(-n))\,(n) \ .$$

Definition 1.2 ([HK] (1.1)). *The sequence of transformations*

$$\nu^{\geq n} \to \nu^{\geq n-1} \to \cdots \to \mathrm{id}$$

is called the slice filtration.

The same type of filtration is also considered by Voevodsky in terms of homotopy theory of schemes.

Let $D\mathcal{M}_-^{\leq n}$ be the category of motives on which $\nu^{\geq n+1}$ vanishes. Quite formally one deduces the existence of a sequence of functors

$$\nu_{\leq n} : D\mathcal{M} \to D\mathcal{M}_-^{\leq n}$$

sitting in natural distinguished triangles

$$\nu^{\geq n} \to \mathrm{id} \to \nu_{\leq n-1} \to \nu^{\geq n}[1] \ .$$

Example: Let $M = \mathbb{Q}(n)$. Then

$$\tau(M) = \begin{cases} \underline{\mathrm{Hom}}(\mathbb{Q}, \mathbb{Q}(n)) = \mathbb{Q}(n) & n \geq 0, \\ \underline{\mathrm{Hom}}(\mathbb{Q}(-n), \mathbb{Q}) = 0 & n < 0. \end{cases}$$

Lemma 1.3 ([HK] Prop. 1.7). *Let $M = M^c(X)$ where X is a variety of dimension at most d. Then*

$$\nu^{\geq n} M = \begin{cases} M & n \leq 0 \\ 0 & n > d \end{cases}$$

Proof. The first part only says that M is effective. The second assertion follows from duality and the known facts on motivic cohomology with values in $\mathbb{Q}$. □

This means that the slice filtration is finite, separated and exhaustive on geometrical motives. However, we do not know:

Question 1.4. *If $M \in D\mathcal{M}_{\mathrm{gm}}$, is it true that all $\nu^{\geq n} M$ are geometric?*

Contrary to my original hope, I expect the answer to be no, see 5.3 below for an argument relying on conjectures. Ayoub has recently communicated a non-conditional argument with the same conclusion.

2 Slice filtration on mixed Hodge structures

In order to understand what the slice filtration means let us first consider the toy model of mixed Hodge structures. We denote $\mathcal{H}$ the category of mixed polarizable $\mathbb{Q}$-Hodge structures. A Hodge structure is called *effective* if its non-zero Hodge numbers are concentrated in the first quadrant. The category of effective Hodge structure is denoted $\mathcal{H}^{\mathrm{eff}}$. Note that this category is stable under subquotients and extensions. A mixed Hodge structure is effective if and only if its simple subquotients are effective.

Lemma 2.1. *The inclusion $\iota : \mathcal{H}^{\mathrm{eff}} \to \mathcal{H}$ has a left adjoint τ, i.e., $\tau : \mathcal{H} \to \mathcal{H}^{\mathrm{eff}}$ and natural transformation* id $\to \tau$ *s.t.*

$$\mathrm{Hom}_{\mathcal{H}}(N, \iota M) = \mathrm{Hom}_{\mathcal{H}^{\mathrm{eff}}}(\tau N, M)$$

for all $N \in \mathcal{H}$, $M \in \mathcal{H}^{\mathrm{eff}}$.

Proof. This is just linear algebra. Let H be an object of $\mathcal{H}$. Then we define τH as the biggest quotient of H which is effective. We have to verify existence of this biggest quotient. Consider the set of all effective quotients of H ordered by the natural projections. This is an Artinian category. Suppose $p_1 : H \to H^1$ and $p_2 : H \to H^2$ are two effective quotients. Let K^i be the kernel of p_i and $K = K^1 \cap K^2$. We consider $H \to H/K$. It dominates H^1 and H^2. As subobject of $H^1 \oplus H^2$ the quotient H/K is effective. Hence the category of effective quotients has a unique maximal object. It is functorial. It is easy to check the universal property. □

Remark: We have switched from right adjoints in motives to left adjoints in Hodge structures. This corresponds to the fact that the Hodge realization functor is contravariant.

Let $\mathcal{H}^{\geq n} = \mathcal{H}^{\mathrm{eff}}(-n)$. There is a sequence of functors

$$\nu^{\geq n} : \mathcal{H} \to \mathcal{H}^{\geq n}$$

left adjoint to the embedding. Explicitly:

$$\nu^{\geq n}(H) = \tau(H(n))\,(-n)\ .$$

Definition 2.2. *The sequence of transformations*

$$\mathrm{id} \to \cdots \to \nu^{\geq n-1} \to \nu^{\geq n}$$

is called the slice cofiltration*.*

If $\tau H = 0$, this does *not* mean that H has Hodge numbers outside the first quadrant. It is easy to write down a simple, pure Hodge structure of weight 0 with Hodge type $(-1,1),(0,0),(1,-1)$. This Hodge structure has no effective quotient! This effect only occurs with $\mathbb{Q}$-Hodge structures. When working with $\mathbb{R}$-Hodge structures, simple objects have Hodge type $(p,q),(q,p)$ only. The slice functors become exact.

Lemma 2.3. *The functors $\nu^{\geq n}$ are right exact but not exact on $\mathcal{H}$.*

Proof. It suffices to consider τ. Rightexactness follows from the definition as biggest effective quotient. Assume now that τ is exact. Let H be a simple polarizable Hodge structure of positive weight which is *not* effective and $H^\vee$ its dual. Note that $H^\vee$ is not effective either. Let E be a non-trivial extension

$$0 \to \mathbb{Q}(0) \to E \to H \to 0\ .$$

They are classified by

$$\mathrm{Ext}^1_{\mathcal{H}}(H,\mathbb{Q}(0)) = \mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Q}(0),H^\vee) = \mathrm{Coker}(H^\vee_{\mathbb{Q}} \oplus F^0H^\vee_{\mathbb{C}} \to H^\vee_{\mathbb{C}}) \neq 0\ .$$

In fact, this is an infinite dimensional $\mathbb{Q}$-vector space because $H^\vee$ is not effective. Hence E exists. We apply τ to the sequence and get

$$0 \to \mathbb{Q}(0) \to \tau E \to 0 \to 0$$

because $\mathbb{Q}(0)$ is effective and H is not effective but simple. The isomorphism $\mathbb{Q}(0) \to \tau E$ together with the projection $E \to \tau E$ splits the original sequence. □

3 Hodge conjecture

Recall ([H1] 2.3.5, [H2]) that there is a *Hodge realization functor*

$$\underline{R}_{\mathcal{H}} : D\mathcal{M}_{\mathrm{gm}} \longrightarrow D^b(\mathcal{H}) \ .$$

We write $\underline{H}_{\mathcal{H}}(M) = \bigoplus H^i(\underline{R}_{\mathcal{H}}(M)) \in \mathcal{H}$.

If X is a smooth variety, then by construction $\underline{H}_{\mathcal{H}}(M(X))$ is singular cohomology $H^*(X(\mathbb{C}), \mathbb{Q})$ of the complex manifold $X(\mathbb{C})$ with the Hodge structure defined by Deligne [D1] Theorem 3.2.5 (iii).

If M is effective, then $\underline{H}_{\mathcal{H}}(M)$ is also effective. What about the converse? This is the set-up of the generalized Hodge conjecture.

Let $\mathcal{M}$ be Grothendieck's category of pure motives up to homological equivalence, see e.g. [S] 1.4.

Conjecture 3.1 (Hodge). *The functor* $\underline{H}_{\mathcal{H}} : \mathcal{M} \to \mathcal{H}$ *is fully faithful.*

In more down to earth terms this says something about (p,p)-cycles. There was also a more general conjecture by Hodge for (p,q)-cycles. It was "false for trivial reasons" as Grothendieck pointed out. The corrected version is:

Conjecture 3.2 (Grothendieck [G]). *The Hodge conjecture holds and a pure motive* $M \in \mathcal{M}$ *is effective if and only if* $\underline{H}_{\mathcal{H}}(M)$ *is effective.*

This usually goes by the name of generalized Hodge conjecture. I propose to extend the conjecture to $D\mathcal{M}_{\mathrm{gm}}$.

Conjecture 3.3 (GHC for triangulated motives). *An object* $M \in D\mathcal{M}_{\mathrm{gm}}$ *is effective if and only if its Hodge realization is effective.*

Why should this be true?

Today's standard conjectures

- GHC for pure Grothendieck motives up to homological equivalence (3.2).
- $D\mathcal{M}_{\mathrm{gm}}$ admits a t-structure τ^{mot}. Its heart $\mathcal{MM}$ (mixed motives) contains $\mathcal{M}$ as full subcategory. For each object of $D\mathcal{M}_{\mathrm{gm}}$ the filtration induced by the truncation functors $\tau^{\mathrm{mot}}_{\leq n}$ is finite, separated and exhaustive.
- There are weight filtration functors $W_{\leq n}$ on $D\mathcal{M}_{\mathrm{gm}}$ which commute with the t-structure and such that the pure objects in $\mathcal{MM}$ are in $\mathcal{M}$. For each object of $D\mathcal{M}_{\mathrm{gm}}$ the filtration induced by the truncation functors $W_{\leq n}$ is finite, separated and exhaustive.
- $\underline{H}_{\mathcal{H}}$ is compatible with t-structure and weights.

The cohomological functor of the motivic t-structure τ^{mot} is denoted H^i. Note that the Hodge realization is contravariant. This implies that $\underline{H}_{\mathcal{H}}(H^i(X)) = \underline{H}_{\mathcal{H}}^{-i}(X)$. We normalize the weight filtration such that

$$\underline{H}_{\mathcal{H}}(W_{\leq n}M) = \underline{H}_{\mathcal{H}}(M)/W_{-(n+1)}\underline{H}_{\mathcal{H}}(M) \ ,$$

i.e., a pure motive of weight n is mapped to a pure Hodge structure of weight $-n$. If X is a smooth proper variety the conjectures imply that $H^i(X)$ is pure of weight i.

Proposition 3.4. *We assume the above set of conjectures. Then:*

1. $\underline{H}_{\mathcal{H}}$ *is conservative on* $D\mathcal{M}_{\mathrm{gm}}$*, i.e., if* $\underline{H}_{\mathcal{H}}(M) = 0$ *then* $M = 0$.
2. *A pure Grothendieck motive is effective in* $\mathcal{M}$ *if and only if it is effective in* $D\mathcal{M}_{\mathrm{gm}}$.
3. $M \in D\mathcal{M}_{\mathrm{gm}}$ *is effective if and only if all* $H^i(M)$ *are effective in* $\mathcal{MM}$ *and if and only if all* $\mathrm{Gr}_j H^i(M)$ *are effective in* $\mathcal{M}$.
4. *GHC holds for triangulated motives, i.e., conjecture 3.3 is true.*

Proof. By today's standard conjectures, the H^i and Gr_j^W are conservative and commute with $\underline{H}_{\mathcal{H}}$. This reduces the question to pure Grothendieck motives. In this case it is the faithfulness part of the Hodge conjecture.

Suppose M is an effective object of $\mathcal{M}$. By the Hodge conjecture, $\mathcal{M}$ is a semi-simple category. Without restriction M is pure of weight i. By definition this means that it is direct summand of $H^i(X)$ for a smooth projective variety X. By the Hodge conjecture, the $H^i(X)$ satisfy hard Lefschetz. By a general argument of Deligne (see [D2]) this implies that in $D\mathcal{M}_{\mathrm{gm}}$

$$M(X) = \bigoplus H^i(X)[-i] \ .$$

Hence M is a direct summand of $M(X)[i]$. As $D\mathcal{M}_{\mathrm{gm}}^{\mathrm{eff}}$ is pseudo-abelian, this implies that M is also effective viewed as object of $D\mathcal{M}_{\mathrm{gm}}$. Conversely, if M is in $\mathcal{M} \cap D\mathcal{M}_{\mathrm{gm}}^{\mathrm{eff}}$, then its Hodge realization is effective. By GHC for pure motives, M is an object of M^{eff}.

Note that $\underline{H}_{\mathcal{H}}(D\mathcal{M}_{\mathrm{gm}}^{\mathrm{eff}}) \subset \mathcal{H}^{\mathrm{eff}}$ and that $D\mathcal{M}_{\mathrm{gm}}^{\mathrm{eff}}$ is stable under triangles: if two vertices of a triangle are effective, then so is the third. Now let M be a triangulated motive such that $\underline{H}_{\mathcal{H}}(M) = \bigoplus \underline{H}_{\mathcal{H}}(H^i(M))$ is effective. Hence all $H^{-i}(\mathrm{Gr}_j^W M)$ are effective in $\mathcal{M}$ by the GHC for $\mathcal{M}$. By the considerations above this implies that all $H^i(\mathrm{Gr}_j^W M)$ are effective in $D\mathcal{M}_{\mathrm{gm}}$. M is successive extension of effective objects, hence effective.

This implies the non-trivial part of GHC for triangulated motives. The remaining statements follows for GHC for $D\mathcal{M}_{\mathrm{gm}}$. ☐

Lemma 3.5. *If the motivic t-structure and the weight filtration exist on* $D\mathcal{M}_{\mathrm{gm}}$*, they extend to a t-structure and weight filtration on* $D\mathcal{M}_{-}$.

Proof. We use two facts on $D\mathcal{M}_-$.

- Every object is third vertex in a distinguished triangle where the other two are (infinite) direct sums of geometrical motives.
- For direct sums of geometrical motives

$$\mathrm{Hom}_{D\mathcal{M}_-}(\bigoplus_{i\in I} M_i, \bigoplus_{j\in J} N_j) = \prod_{i\in I}\bigoplus_{j\in J} \mathrm{Hom}_{D\mathcal{M}_{\mathrm{gm}}}(M_i, N_j).$$

Consider the smallest full subcategory $\tau^{\mathrm{mot}}_{\leq n} D\mathcal{M}_-$ of $D\mathcal{M}_-$ which contains $\tau^{\mathrm{mot}}_{\leq n} M$ for all geometrical motives M, is closed under direct sums and such that if $M_1 \to M_2 \to M_3$ is a distinguished triangle with M_1, M_3 in $\tau^{\mathrm{mot}}_{\leq n} D\mathcal{M}_-$, then M_2 in $\tau^{\mathrm{mot}}_{\leq n} D\mathcal{M}_-$. Dually we define $\tau^{\mathrm{mot}}_{\geq n+1} D\mathcal{M}_-$. We claim that this is a t-structure. The vanishing of morphisms and behaviour under shifts follows from the above facts. We also need to check that for all $M \in D\mathcal{M}_-$ there are $\tau^{\mathrm{mot}}_{\leq n} M \in \tau^{\mathrm{mot}}_{\leq n} D\mathcal{M}_-$ and $\tau^{\mathrm{mot}}_{\geq n+1} M \in \tau^{\mathrm{mot}}_{\geq n} D\mathcal{M}_-$ such that there is a distinguished triangle

$$\tau^{\mathrm{mot}}_{\leq n} M \to M \to \tau^{\mathrm{mot}}_{\geq n+1} M\ .$$

This holds for geometrical motives and extends to direct sums of geometrical motives. Let $M_1 \to M_2 \to M_3$ be a distinguished triangle and assume that truncation is defined on M_2 and M_3. Let $\tilde{H}^n(M_3) = \mathrm{Im}\,(H^n(M_2) \to H^n(M_3))$. A modified truncation of M_3 is defined by the diagram of triangles

$$\begin{tikzcd}
\tau^{\mathrm{mot}}_{\leq n-1} M_3 \arrow[r] & \tau^{\mathrm{mot}}_{\leq n} M_3 \arrow[r] & H^n(M_3)[-n] \\
\tau^{\mathrm{mot}}_{\leq n-1} M_3 \arrow[r] \arrow[u] & \tau'_{\leq n} M_3 \arrow[r] \arrow[u] & \tilde{H}^n(M_3)[-n] \arrow[u]
\end{tikzcd}$$

By construction the map $\tau^{\mathrm{mot}}_{\leq n} M_2 \to \tau^{\mathrm{mot}}_{\leq n} M_3$ factors through $\tau'_{\leq n} M_3$. We define $\tau^{\mathrm{mot}}_{\leq n} M_1$ by the distinguished triangle

$$\tau^{\mathrm{mot}}_{\leq n} M_1 \to \tau^{\mathrm{mot}}_{\leq n} M_2 \to \tau'_{\leq n} M_3\ .$$

Define $\tau'_{\geq n+1} M_3$ and $\tau^{\mathrm{mot}}_{\geq n+1} M_1$ as third vertices in the distinguished triangles

$$\tau'_{\leq n} M_3 \to M_3 \to \tau'_{\geq n+1} M_3$$
$$\tau^{\mathrm{mot}}_{\leq n} M_1 \to M_1 \to \tau^{\mathrm{mot}}_{\geq n+1} M_1$$

We automatically get a distinguished triangle

$$\tau^{\mathrm{mot}}_{\geq n+1} M_1 \to \tau^{\mathrm{mot}}_{\geq n+1} M_2 \to \tau'_{\geq n+1} M_3\ .$$

It is easy to see that $\tau^{\mathrm{mot}}_{\leq n} M_1 \in \tau^{\mathrm{mot}}_{\leq n} D\mathcal{M}_-$ and $\tau^{\mathrm{mot}}_{\geq n+1} M_1 \in \tau^{\mathrm{mot}}_{\geq n+1} D\mathcal{M}_-$.

We skip the arguments for the weight filtration, which are simpler. □

We now can use GHC in order to get a conjectural understanding of the slice filtration. $\nu^{\geq n}$ on $D\mathcal{M}_{\mathrm{gm}}$ defines $H^0\nu^{\geq n}$ on $\mathcal{M}\mathcal{M}$.

Proposition 3.6. *Assume again today's set of standard conjectures. $M \in \mathcal{MM}$ is effective if and only if $H^0\nu^{\geq n}M = M$. The functors $H^0\nu^{\geq n}$ and $H^0\nu_{\leq n}$ are left exact on $\mathcal{MM}$. $H^0\nu^{\geq n}$ is right adjoint to the inclusion $\iota : \mathcal{MM}^{\mathrm{eff}} \to \mathcal{MM}$.*

One should think of $\nu^{\geq 0}$ as the derived functor of $H^0\nu^{\geq 0}$.

Proof. It suffices to consider $n = 0$. Let $M \in \mathcal{MM}$. M effective means $M = \nu^{\geq 0}M$, in particular $H^i(\nu^{\geq 0}M) = 0$ for $i \neq 0$. Conversely, assume $M = H^0(\nu^{\geq 0}M)$. Clearly $\nu^{\geq 0}M$ is effective and hence also its H^0.

For left exactness let $M \in \mathcal{MM}$ be a mixed motive. Consider the distinguished triangle

$$\nu^{\geq 0}M \to M \to \nu_{>0}M$$

We first want to show that $H^i(\nu_{<0}M)$ is effective for $i \neq 0$. Consider the long exact sequence with respect to H^i. It yields isomorphisms

$$H^i\nu_{<0}M \to H^{i+1}\nu^{\geq 0}M$$

for $i \neq -1, 0$. The object on the right is effective hence so is the object on the left. The same sequence yields

$$0 \to H^{-1}\nu_{<0}M \to H^0\nu^{\geq 0}M \to M \to H^0\nu_{<0}M \to H^1\nu^{\geq 0}M \to 0$$

It remains to show that subobjects of effective motives in $\mathcal{MM}$ are effective: this is true by GHC for triangulated motives because the dual is true in $\mathcal{H}$. Hence by adjunction

$$\mathrm{Hom}(\tau^{\mathrm{mot}}_{<0}\nu_{<0}M, \nu_{<0}M) = \mathrm{Hom}(\tau^{\mathrm{mot}}_{<0}\nu_{<0}M, \nu_{\geq 0}\nu_{<0}M) = 0\ ,$$

i.e., $\tau^{\mathrm{mot}}_{<0}M = 0$. In particular, $H^0\nu_{<0}$ is left exact. By the above isomorphisms this also implies that $H^0\nu^{\geq 0}$ is left exact.

Let $M \in \mathcal{MM}$, $N \in \mathcal{MM}^{\mathrm{eff}}$. Then

$$\begin{aligned}\mathrm{Hom}_{\mathcal{MM}}(N, M) &= \mathrm{Hom}_{D\mathcal{M}_{\mathrm{gm}}}(N, M) = \mathrm{Hom}_{D\mathcal{M}^{\mathrm{eff}}_{-}}(N, \nu^{\geq 0}M)\\ &= \mathrm{Hom}_{D\mathcal{M}^{\mathrm{eff}}_{-}}(N, H^0\nu^{\geq 0}M) = \mathrm{Hom}_{\mathcal{MM}}(N, H^0\nu^{\geq 0}M)\end{aligned}$$

The crucial third equality holds because N and $\nu^{\geq 0}M$ have cohomology concentrated in degree 0 and in non-negative degrees respectively. □

Remark: Recall that the t-structure on $D\mathcal{M}_{\mathrm{gm}}$ extends to a t-structure on $D\mathcal{M}_{-}$. The proposition implies that $\nu^{\geq n}$ is also left exact on $D\mathcal{M}_{-}$.

Question 3.7. *Does the slice filtration commute with the weight filtration?*

I think that the answer is no, see 5.3 below for an argument relying on conjectures.

4 Coniveau filtration

In this section we concentrate on pure motives. As a left exact functor, $H^0\nu^{\geq n}$ respects the category of pure motives $\mathcal{M}$. In fact, on a simple pure motive it is either the zero or the full object. We are going to review Grothendieck's coniveau filtration. Note that we have to reverse all arrows because we use covariant motives whereas his setting was contravariant.

Definition 4.1 ([G]). *Let X be a smooth proper variety. The* coniveau filtration *on $M = H^{-i}(X)$ is defined as*

$$F'^p M = M/\operatorname{Im}\left(\bigoplus H^{-i}(U)\right)$$

where the sum runs over all open subvarieties $U \subset X$ such that $T = X \smallsetminus U$ has codimension at least p.

Alternatively, $F'^p M$ can be described as the smallest quotient of M such that all Gysin morphisms $H^{-i}(X) \to H^{-i+2q}(\tilde{T})(q)$ for all $\tilde{T} \to X$ with $\tilde{T}$ smooth, projective, $\dim X - \dim \tilde{T} = q \geq p$ factor through $F'^p M$.

Proposition 4.2. *Assume today's standard conjectures. Let X be smooth and proper, $M = H^{-i}(X)$. Then the composition*

$$H^0\nu^{\geq p} M \to M \to F'^p M$$

is an isomorphism. The slice filtration provides a splitting of the coniveau filtration.

Proof. The key observation is that the $H^{-i+2q}(\tilde{T})(q)$ of the alternative description are in $D\mathcal{M}^{\geq q} \subset D\mathcal{M}^{\geq p}$. Hence a simple constituent of M which is not in $D\mathcal{M}^{\geq p}$ is also mapped to zero in $F'^p M$. Conversely, let $M' \subset M$ be a simple direct summand which is in $D\mathcal{M}^{\geq p}$. It is direct summand of some $H^{-i+2p}(Y)(p)$. with Y smooth and projective. The projection $M \to M'$ is induced by a morphism of motives $\phi : H^{-i}(X) \to H^{-i+2p}(Y)(p)$. We assume that pure motives in $\mathcal{MM}$ are Grothendieck motives, hence this morphism is represented by a cycle T in $X \times Y$ with $\dim X - \dim T = p$. Let $\tilde{T}$ be a desingularization of T. Then ϕ is the composition $H^{-i}(X) \to H^{-i+2p}(\tilde{T})(p) \to H^{-i+2p}(Y)(p)$. As M' is a direct summand of $H^{-i+2p}(Y)(p)$, it is also a direct summand of $H^{-i+2p}(\tilde{T})(p)$. This implies that M' does not vanish in $F'^p M$ either. $\square$

GHC for pure motives can be formulated as saying that $H^0\nu^{\geq p}$ commutes with the Hodge realization functor.

Question 4.3. *Does $\nu^{\geq n}$ on $D\mathcal{M}_-$ commute with the Hodge realization?*

In order for this question to make sense, we first have to extend the Hodge realization to a functor on $D\mathcal{M}_-$. It will have values in $D^+(\text{Pro}-\mathcal{H})$ where $\text{Pro}-\mathcal{H}$ is the pro-category of Hodge structures. The question can be reduced to the case of $M \in \mathcal{M}$. However, I do not have a guess for the answer.

5 The counterexample

Let X be a smooth projective variety and Z a cohomologically trivial cycle of codimension 2. By the Abel-Jacobi map it induces an extension of mixed Hodge structures

$$0 \to \underline{H}^3_{\mathcal{H}}(X) \to H_Z \to \mathbb{Q}(-2) \to 0\ .$$

Lemma 5.1. *Let X be a generic quintic in $\mathbb{P}^4$, $H = \underline{H}^3_{\mathcal{H}}(X)$. Then H is a simple Hodge structure of weight 3 with $H^{3,0} \neq 0$. The image of the Abel-Jacobi map in $\mathrm{Ext}^1(\mathbb{Q}(-2), \underline{H}^3_{\mathcal{H}}(X))$ is not finite dimensional.*

Proof. Quintics in $\mathbb{P}^4$ are simply connected Calabi-Yau threefolds and very well studied. In particular, their H^3 is primitive and has Hodge type

$$(3,0), (2,1), (1,2), (0,3)\ .$$

By [PS] Corollary 18 it is simple for a generic X. The Abel-Jacobi map on homologically trivial cycles was studied by Griffiths and Clemens in this example. By [C] Theorem 6 its image in $\mathrm{Ext}^1(\mathbb{Q}(-2), \underline{H}^3_{\mathcal{H}}(X))$ is not finite dimensional. □

I would like to thank C. Voisin and B. Moonen for pointing these arguments out to me.

Corollary 5.2. *Assume today's standard conjectures. Let X be as in the lemma, $M = H^{-3}(X)^\vee(2)$. Then $H^0\nu^{\geq 0}M = 0$ and $\mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Q}(0), M)$ is infinite dimensional.*

Proof. $M = H^3(X)^\vee(2)$ is simple because its Hodge realization is simple. Moreover, its Hodge realizations is of type $(-1,2), (0,1), (1,0), (2,-1)$, i.e., M is not effective. Hence it does not have any effective subobjects. By duality and functoriality

$$\mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Q}(0), M) \cong \mathrm{Ext}^1_{\mathcal{MM}}(H^{-3}(X), \mathbb{Q}(2)) \to \mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Q}(-2), \underline{H}^3_{\mathcal{H}}(X)).$$

The Abel-Jacobi map factors through this map. By the Lemma the dimension of the Ext-group has to be infinite. □

Proposition 5.3 (see Questions 1.4 and 3.7). *Assume today's standard conjectures. Then $H^0\nu^{\geq n}$ is not exact, the functors $\nu^{\geq n}$ do not respect geometrical motives and do not commute with the weight filtration.*

Proof. It suffices to consider $n = 0$. Let M be as in the corollary. Consider a non-trivial extension

$$0 \to M \to E \to \mathbb{Q}(0) \to 0\ .$$

The long exact sequence for $H^i\nu^{\geq 0}$ starts

$$0 \to 0 \to H^0\nu^{\geq 0}E \to \mathbb{Q}(0) \to H^1\nu^{\geq 0}M \to\ \ .$$

If $H^0\nu^{\geq 0}$ was exact, then $H^0\nu^{\geq 0}E \cong \mathbb{Q}(0)$. If $\nu^{\geq 0}$ commuted with the weight filtration, then $H^1\nu^{\geq 0}M$ would be pure of weight 1. The boundary map would vanish and again $H^0\nu^{\geq 0}E \cong \mathbb{Q}(0)$. In both cases this isomorphism together with the inclusion $H^0\nu^{\geq 0}E \to E$ would split the sequence and we would have a contradiction.

Now assume that $\nu^{\geq 0}M$ is geometric. We have

$$\mathrm{Hom}_{D\mathcal{M}_{\mathrm{gm}}}(\mathbb{Q}, M[1]) = \mathrm{Hom}_{D\mathcal{M}^{\mathrm{eff}}_{-}}(\mathbb{Q}, \nu^{\geq 0}M[1]) \ .$$

As M is simple and non-effective, $H^0\nu^{\geq 0}M = 0$. By assumption $H^1\nu^{\geq 0}M$ is geometric, hence

$$\mathrm{Hom}_{D\mathcal{M}_{\mathrm{gm}}}(\mathbb{Q}, M[1]) = \mathrm{Hom}_{D\mathcal{M}^{\mathrm{eff}}_{\mathrm{gm}}}(\mathbb{Q}, H^1\nu^{\geq 0}M) \ .$$

For pure motives, we have

$$\mathrm{Hom}_{\mathcal{M}}(\mathbb{Q}, N) = \mathrm{Hom}_{\mathcal{H}}(\mathbb{Q}, N)$$

by the Hodge conjecture, in particular this is a finite dimensional vector space. Hence this is also true for mixed motives. This contradicts the infinite dimensionality established in corollary 5.2. $\square$

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Annette Huber
Math. Institut, Augustusplatz 10/11, Universität Leipzig, 04109 Leipzig
huber@mathematik.uni-leipzig.de