BREDON-STYLE HOMOLOGY AND COHOMOLOGY FOR ALGEBRAIC STACKS

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Abstract. One of the main obstacles for proving Riemann-Roch for algebraic stacks is the lack of cohomology and homology theories that are closer to the K-theory and G-theory of algebraic stacks than the traditional cohomology and homology theories for algebraic stacks. In this paper we study in detail a family of cohomology and homology theories which we call *Bredon-style* theories that are of this type and in the spirit of the classical Bredon-cohomology and homology theories defined for the actions of compact topological groups on topological spaces. In a sequel to this paper, we establish Riemann-Roch theorems in this setting. We conclude with applications to *virtual fundamental classes* associated to dg-stacks: i.e. algebraic stacks (in general, of Artin type) provided with sheaves of commutative dgas, the main examples of which are moduli stacks of stable curves provided with a virtual structure sheaf associated to a perfect obstruction theory.

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1. **Introduction.**

Quotient stacks form a good class of algebraic stacks which are rather easily understood even without involving stack-theoretic terminology: these correspond to actions of smooth (affine) group-schemes on schemes. The traditional cohomology theories for studying such group-actions are the Borel-style equivariant cohomology theories. While quite easy to define and study, these theories, however, are only rather crude invariants of the group-action. Even for schemes with finite cohomological dimension, these cohomology theories need not vanish in infinitely many degrees. Moreover, the module structure of these cohomology theories over the representation ring of the group, factors through the completion of the representation ring at the augmentation ideal.

Finer equivariant cohomology theories have so far been defined only in the purely topological setting, namely for actions of compact groups on suitably nice topological spaces. These were originally introduced in [Br] and studied extensively by topologists (see [LMS] for example). One of the main difficulties in considering an analogous theory for the algebraic setting, for quotient stacks and in general for other algebraic stacks, was the lack of a suitable site or Grothendieck topology; this was rectified in our earlier work, [J-5]. (See [J-5] section 2 for a general discussion of Bredon style theories from a sheaf-cohomology point of view.)

Amplifying on the techniques developed there, we define and study in detail in this paper, cohomology and homology theories for algebraic stacks generalizing simultaneously Bredon style equivariant cohomology for group actions and Bloch-Ogus style theories for schemes and algebraic spaces. The following theorem summarizes some of the main properties of the Bredon-style cohomology and homology theories we define.

Throughout this theorem, we will assume that whenever a coarse moduli space is assumed to exist, it exists as a *quasi-projective scheme*. Moreover, we will assume that, in the equivariant case, provided with the action of a smooth group scheme, it is G-quasi projective, i.e. it admits a G-equivariant locally closed immersion into a projective space on which the group G acts linearly. There are two *distinct* versions of Bredon style cohomology and homology considered here, *one in general and the second when a coarse moduli space exists.* The *first version*, which is defined in general, uses hypercohomology on the isovariant étale site of the stack. The *second version*, uses hypercohomology on the étale site of the coarse moduli space when it exists. The two are different in general, but agree when the stack is a gerbe over its coarse moduli space. $\Gamma(\bullet)$ and $\Gamma^h(\bullet)$ will denote complexes of sheaves on the big iso-variant étale site of algebraic stacks or the big étale site of algebraic spaces as in section 3. (Strictly speaking these complexes need not be contravariant for arbitrary maps, but for the sake of this introduction one may assume they are. See section 3 for more precise details.) (The iso-variant étale site of algebraic stacks is recalled below, in the second section, following [J-5] section 3.) The Bredon cohomology (homology), $H^s_{Br}(\mathcal{S}, \Gamma(t))$ ($H_s^{Br}(\mathcal{S}, \Gamma(t))$) is defined by first defining certain presheaves $K\Gamma(\bullet)$ and $K\Gamma^h(\bullet)$ using the complexes $\Gamma(\bullet)$ and $\Gamma^h(\bullet)$. These are presheaves on the iso-variant étale site of the given stack or on the étale site of its coarse-moduli space. The Bredon cohomology (homology) groups are defined to be the hypercohomology on the isovariant étale site of the stack or the étale site of its coarse moduli space with respect to these presheaves. We also consider local Bredon cohomology groups, which are defined in Definition 5.13 below.)

All algebraic stacks considered in this paper are dg-stacks in the sense of Definition 2.10 and the dg-structure sheaf on a stack $\mathcal{S}$ will usually be denoted $\mathcal{A}_{\mathcal{S}}$ or simply $\mathcal{A}$. One motivation for considering such dg-stacks is the possibility of deriving various formulae for the virtual fundamental classes from Riemann-Roch. Throughout the following theorem we will assume that $\mathcal{S}$ is a dg-stack provided with a dg-structure sheaf $\mathcal{A}_{\mathcal{S}}$. (See sections 2 and 3 for an explanation of the terminology. See also 1.0.2 for our conventions regarding coarse moduli spaces.)

Theorem 1.1. *(Existence of Bredon-style theories with 'good' properties)*

(i) Assume that $f : \mathcal{S}' \to \mathcal{S}$ *is an arbitrary map of algebraic stacks. Then* f^* *defines a map* $H^s_{Br}(\mathcal{S}, \Gamma(t)) \to H^s_{Br}(\mathcal{S}', \Gamma(t))$ *making Bredon style cohomology a contravariant functor* $(alg.stacks/S) \to$ *(graded rings). Both Bredon style cohomology and Bredon style local cohomology are provided with ring structures.*

(ii) If, in addition f *is proper, one obtains a map* $f_* : H_s^{Br}(\mathcal{S}'; \Gamma(t)) \to H_s^{Br}(\mathcal{S}; \Gamma(t))$ *making Bredon style homology a covariant functor for proper maps* $(alg.stacks) \to (abelian \quad groups)$.

(iii) $H_*^{Br}(\mathcal{S}; \Gamma(\bullet))$ *is a module over* $H^*_{Br}(\mathcal{S}; \Gamma(\bullet))$ *and the latter is a module over* $\pi_*(K(\mathcal{S}, \mathcal{A}_{\mathcal{S}}))$*. In case a coarse-moduli space exists as a quasi-projective scheme,* $H_*^{Br}(\mathcal{S}; \Gamma^h(\bullet))$ *is a module over the corresponding local cohomology ring as well.*

(vi) Projection formula. Let $f : \mathcal{S}' \to \mathcal{S}$ *denote a proper map of algebraic stacks. Now the following diagram commutes:*

$$\begin{array}{ccccc} H^*(\mathcal{S};\Gamma(s)) \otimes H_*(\mathcal{S}';\Gamma(t)) & \xrightarrow{f^*\otimes id} & H^*(\mathcal{S}';\Gamma(s)) \otimes H_*(\mathcal{S}';\Gamma(t)) & \longrightarrow & H_*(\mathcal{S}';\Gamma(t-s)) \\ \downarrow{\scriptstyle id\otimes f_*} & & & & \downarrow{\scriptstyle f_*} \\ H^*(\mathcal{S};\Gamma(s)) \otimes H_*(\mathcal{S};\Gamma(t)) & & \longrightarrow & & H_*(\mathcal{S};\Gamma(t-s)) \end{array}$$

The corresponding assertion also holds with cohomology replaced by local cohomology.

(v) In case the algebraic stack $\mathcal{S}$ is a separated algebraic space of finite type over the base scheme, one obtains an isomorphism $H^*_{Br}(\mathcal{S},\Gamma(\bullet)) \cong H^*_{et}(\mathcal{S},\Gamma(\bullet))$ where the right hand side is the étale hypercohomology of $\mathcal{S}$ defined with respect to the complex $\Gamma(\bullet)$. Under the same hypothesis, one obtains an isomorphism $H^{Br}_*(\mathcal{S},\Gamma(\bullet)) \cong H^{et}_*(\mathcal{S},\Gamma(\bullet)) \cong \mathbb{H}^*_{et}(\mathcal{S},\Gamma^h(\bullet))$. (The corresponding statements hold generically if the algebraic stack $\mathcal{S}$ is a separated Deligne-Mumford stack which generically is an algebraic space, i.e. if the stack $\mathcal{S}$ is an orbifold.)

(vi) There exists a multiplicative homomorphism $ch : \pi_*K(\mathcal{S},\mathcal{A}) \to H^*_{Br}(\mathcal{S};\Gamma(\bullet))$ called the Chern character

(vii) The Riemann-Roch transformation and the fundamental class. Assume that a coarse-moduli space $\mathfrak{M}$ exists as a quasi-projective scheme associated to the algebraic stack $\mathcal{S}$ and that the natural map $p : \mathcal{S} \to \mathfrak{M}$ is of finite cohomological dimension . (This hypothesis is always satisfied in characteristic 0 by Artin stacks with quasi-finite diagonal: see the discussion in 1.0.2 below.) In this case there exists a Riemann-Roch transformation:

$\tau : \pi_*G(\mathcal{S},\mathcal{A}) \to H^{Br}_*(\mathcal{S};\Gamma(\bullet))$

Moreover the Chern-character and τ are compatible in the usual sense:

i.e. $\tau(\alpha \circ \beta) = \tau(\alpha) \circ ch(\beta)$, where $\alpha \varepsilon \pi_0(G(\mathcal{S},\mathcal{A}_{\mathcal{S}}))$ and $\beta \varepsilon \pi_0(K(\mathcal{S},\mathcal{A}_{\mathcal{S}}))$.

(viii) Moreover, in this case there exists a fundamental class $[\mathcal{S}] \varepsilon H^{Br}_*(\mathcal{S},\Gamma^h(\bullet))$ such that cap-product with this class induces a map:

$\cap[\mathcal{S}] : H^*_{Br}(\mathcal{S},\Gamma(\bullet)) \to H^{Br}_*(\mathcal{S},\Gamma(\bullet))$.

The fundamental class $[\mathcal{S}]$ is defined to be the term of highest weight (and degree = twice the weight) in $\tau(\mathcal{A}_{\mathcal{S}})$. (Classes in $H^{Br}_n(\mathcal{S},\Gamma(t))$ have degree n and weight t.)

(ix) The corresponding assertion also holds for local cohomology. i.e. The following holds. Assume the hypothesis in (vii) and let $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ denote a closed immersion into a regular scheme. Now there exists a map $PL : H^*_{Br,\mathcal{S}}(\tilde{\mathfrak{M}};\Gamma(r)) \to H^{Br}_*(\mathcal{S};\Gamma(d-r))$ where d denotes the dimension of $\mathfrak{M}$ and where $H^*_{Br,\mathcal{S}}(\tilde{\mathfrak{M}};\Gamma(r))$ denotes Bredon-style local cohomology.

(x) Let $\mathcal{S}$ denote a non-dg stack and let $\pi : \mathcal{S} \times \mathbb{A}^1 \to \mathcal{S}$ denote the obvious map. Now $\pi^* : H^*_{Br}(\mathcal{S};\Gamma(\bullet)) \cong H^*_{Br}(\mathcal{S} \times \mathbb{A}^1;\Gamma(\bullet))$ provided the stack $\mathcal{S}$ is smooth in the general case (i.e. when cohomology is computed on the isovariant étale site) and provided both the stack and its coarse moduli space are smooth in the second case (i.e. when cohomology is computed on the étale site of the coarse moduli space). The corresponding assertion also holds for local cohomology in case the stack $\mathcal{S}$ alone is restricted to be smooth.

(xi) Assume that a coarse moduli space $\mathfrak{M}$ exists for the stack $\mathcal{S}$. Let $\bar{\mathcal{E}}$ denote a vector bundle on $\mathfrak{M}$ and let $\mathcal{E}$ denote its pull-back to the stack $\mathcal{S}$. Let $\mathbb{P}(\mathcal{E})$ be the associated projective space with the dg-structure sheaf $\pi^*(\mathcal{A})$ where $\mathcal{A}$ is the dg-structure sheaf on $\mathcal{S}$ and $\pi : \mathbb{P}(\mathcal{E}) \to \mathcal{S}$ is the obvious projection. Then:

$H^*_{Br}(\mathbb{P}(\mathcal{E}),\Gamma(\bullet)) \simeq \oplus_{i=0}^{i=n} H^*_{Br}(\mathcal{S},\Gamma(\bullet))$.

The induced map in Bredon homology $\pi_* : H^{Br}_*(\mathbb{P}(\mathcal{E}),\Gamma(\bullet)) \to H^{Br}_*(\mathcal{S},\Gamma(\bullet))$ factors as $H^{Br}_*(\mathbb{P}(\mathcal{E}),\Gamma(\bullet)) \to \oplus_{i=0}^{i=n} H^{Br}_*(\mathcal{S},\Gamma(\bullet)) \to H^{Br}_*(\mathcal{S},\Gamma(\bullet))$.

(xii) In case a smooth group scheme acts on the stack $\mathcal{S}$, the equivariant versions of the above results also hold.

We consider compatibility of the above theories with other cohomology- homology theories in the following theorem.

Theorem 1.2. *Assume the complexes* $\Gamma(\bullet)$ *and* $\Gamma^h(\bullet)$ *extend to the big smooth site of all algebraic stacks.*

(i) Then there exists a map ϕ_* *from* $H_*^{Br}(\mathcal{S}, \Gamma(\bullet))$ *to the hypercohomology of the stack computed on the smooth site, namely* $\mathbb{H}^*_{smt}(\mathcal{S}, \Gamma^h(\bullet))$. *This provides a fundamental class in* $H_*^{smt}(\mathcal{S}, \Gamma(\bullet)) = \mathbb{H}^*_{smt}(\mathcal{S}, \Gamma^h(\bullet))$ *for algebraic stacks satisfying the hypothesis in Theorem (1.1(ix)). This map is compatible with respect to proper push-forward associated to proper representable morphisms, for all algebraic stacks with coarse moduli spaces.*

(ii) There exists a map $\psi^* : H^*_{Br}(\mathcal{S}, \Gamma(\bullet)) \rightarrow H^*_{smt}(\mathcal{S}, \Gamma(\bullet))$. *The maps* ϕ_* *and* ψ^* *are compatible with respect to the obvious pairings between cohomology and cohomology. Moreover* $\psi^* \circ ch = Ch \circ \psi^*$ *where ch (Ch) denotes the Chern character in Bredon cohomology (smooth cohomology, respectively) at least when coarse-moduli spaces are assumed to exist.*

(iii) Let $\mathcal{S}$ *denote a Deligne-Mumford stack over an algebraically closed field k provided with the trivial action by a smooth group scheme G. Assume that the dg-structure sheaf* $\mathcal{A} = \mathcal{O}_{\mathcal{S}}$. *Then there exists a finer variant of Bredon homology denoted* $H_*^{Br,et}(I_{\mathcal{S}}/S, G, \Gamma^h(\bullet)) \underset{\mathbb{Q}}{\otimes} \mathbb{Q}(\mu_\infty)$ *that maps to* $H_*^{Br}(\mathcal{S}, G, \Gamma^h(\bullet)) \underset{\mathbb{Q}}{\otimes} \mathbb{Q}(\mu_\infty)$ *compatible with respect to push-forwards. (Here* $\mathbb{Q}(\mu_\infty)$ *is the* $\mathbb{Q}$*-algebra generated by the roots of unity in the field k: one may fix an imbedding of this into* $\mathbb{C}$.*) Moreover, the latter map is an isomorphism when the stack* $\mathcal{S}$ *is smooth and the orders of the stabilizer groups of the stack at every point is prime to the characteristic of k.*

Remarks 1.3. Suppose $\mathcal{S}$ is a separated Deligne-Mumford stack. Then the maps considered in (i) provide maps $H_*^{Br}(I_{\mathcal{S}}, \Gamma(\bullet)) \rightarrow \mathbb{H}^*_{smt}(I_{\mathcal{S}}, \Gamma^h(\bullet))$ and similarly $H^*_{Br}(I_{\mathcal{S}}, \Gamma(\bullet)) \rightarrow \mathbb{H}^*_{smt}(I_{\mathcal{S}}, \Gamma(\bullet))$. Recall the targets of these maps identify with the (finer) homology and cohomology of the stack $\mathcal{S}$ as defined in [To] or [CR].

Since the equivariant theory is important in some of the applications of our theory, for example to virtual fundamental classes, we will discuss our results often in the equivariant context.

Moreover, observe that one of the main difficulties with Riemann-Roch for algebraic stacks is the fact that G-theory for algebraic stacks is essentially a Bredon-style homology theory; and it does not behave well functorially with respect to other homology theories that are not of Bredon type. In fact, we show (in a sequel, see [J-6])) that the Riemann-Roch problem for algebraic stacks that admit coarse moduli spaces (observe this includes also some Artin stacks) can be solved fairly easily using the Bredon-style homology theories we study in this paper.

It is worthwhile making some comparison of our theories with those adopted by Toen in his work on Riemann-Roch for Deligne-Mumford stacks, making use of the work of Vistoli. (See [To], [Vi-1]. The theory defined in [CR] is essentially the same, but uses slightly different terminology). There the information about the action of the isotropy on cohomology is incorporated into the picture by considering cohomology theories associated not to the original stack, but instead its inertia stack. Therefore, at least modulo torsion, and for Deligne-Mumford stacks, the cohomology theories for stacks in [To] are the usual cohomology theories applied instead to the inertia stack or to its moduli space. Though this approach has the advantage of providing a quick definition of the improved cohomology theories and has found applications in mirror symmetry problems (for example, to define the stringy Hodge numbers) it runs into various difficulties as far as Riemann-Roch is concerned: for example, one needs to always restrict to (preferably smooth) Deligne-Mumford stacks and checking the Riemann-Roch square commutes for non-representable maps is difficult.

Our approach is fundamentally different from the above approach in the following manner: we define cohomology and homology theories where the information about the action of the isotropy is built into the cohomology or homology theory. In particular, these theories are not the the *usual* theories applied to special spaces, but fundamentally new theories that reduce to the usual theories when applied to schemes or algebraic spaces. By varying the complexes $\Gamma(\bullet)$ and $\Gamma^h(\bullet)$ we obtain different theories: for example, restricting to smooth stacks and letting $\Gamma(\bullet) = \Gamma^h(\bullet)$ be the motivic complexes, we obtain motivic cohomology for algebraic stacks finer than the one adopted in [J-2] and [J-3]. Moreover our definition is such that checking the commutativity of the Riemann-Roch square for non-representable proper maps is rather easy. In addition we are able to handle stacks that are not smooth and not necessarily Deligne-Mumford. Applications to virtual structure sheaves and virtual fundamental classes make it necessary that we work throughout in the general context of DG-stacks. Further more, our Riemann-Roch makes full use of the existing Riemann-Roch at the level of the moduli spaces. Moreover, as shown in (ii) of the last theorem, our theory admits a variant that is closely related to the theory of [To] and [CR]. (See [J-6] for further details on Riemann-Roch.)

We conclude with one application of our theories: we show that Kontsevich's conjectural formula for the virtual fundamental class is valid when we define virtual fundamental classes with values in smooth homology. (Several more applications exist, for example, to various formulae for virtual structure sheaves and virtual fundamental classes: these are discussed in [J-6] and [J-8].) Let $\mathcal{S}$ denote a Deligne-Mumford stack provided with a perfect obstruction theory $E^\bullet = E^{-1} \rightarrow E^0$. Observe that since we are considering smooth cohomology, there is no need to assume the existence of global resolutions for the perfect complexes E^i, $i = 0, -1$ to be able to define Chern classes: we define the Todd class $Td(P)$ for any perfect complex P on the stack $\mathcal{S}$ by the Todd polynomial in the Chern classes of P with values in $\mathbb{H}^*_{smt}(\mathcal{S}; \Gamma(\bullet)) \otimes \mathbb{Q}$. Since the stack $\mathcal{S}$ is Deligne-Mumford and we are considering smooth cohomology with rational coefficients, it follows readily (see (7.0.5 below)) that the Todd class $T(P)$ is a *unit* for any perfect complex P. Let $[\mathcal{S}]^{virt}_{Br}$ denote the fundamental class of the dg-stack $(\mathcal{S}, \mathcal{O}^{virt}_{\mathcal{S}})$ in Bredon-homology and let $[\mathcal{S}]^{virt} = \phi_*([\mathcal{S}]^{virt}_{Br})$ denote the image of the above class in smooth homology under the map to smooth homology considered in the last theorem. We define the *virtual Todd class* of the obstruction theory $E^\bullet$ as $Td(E_0).Td(E_1)^{-1}$ where $E_i = (E^i)^\vee$. *We also call this the Todd class of the virtual tangent bundle and denote it by* $Td(T\mathcal{S}^{virt})$. We define the Todd homomorphism:

$$\tau^{smt} : \pi_0(K(\mathcal{S}, \mathcal{O}^{virt}_{\mathcal{S}})) \rightarrow \mathbb{H}^{smt}_*(\mathcal{S}, \Gamma(\bullet)) \tag{1.0.1}$$

by $\tau^{smt}(\mathcal{F}) = (\phi_*(ch^{Br}(\mathcal{F}) \cap [\mathcal{S}]^{virt}_{Br})) \cap Td(T\mathcal{S}^{virt})$, where ch^{Br} denotes the Chern-character map into Bredon cohomology.

Theorem 1.4. *(Kontsevich's conjectural formula for the virtual fundamental class) Assume the above situation. Then the Todd class* $Td(T\mathcal{S}^{virt})$ *is invertible in* $H^*_{smt}(\mathcal{S}; \Gamma(\bullet)) \otimes \mathbb{Q}$ *and we obtain:*

$$[\mathcal{S}]^{virt} = \tau^{smt}(\mathcal{O}^{virt}_{\mathcal{S}}) \cap Td(T\mathcal{S}^{virt})^{-1}$$

Here is an outline of the layout of the paper. In section 2, we recall the main results on the isovariant étale site from [J-2] and also briefly discuss the rudiments of DG-stacks. (Full details can be found in [J-7].) All stacks we consider in this paper will be DG-stacks in the sense of 2.10 . The next three sections are devoted to a detailed study of the cohomology and homology theories we define: we call these Bredon-style theories since they incorporate many of the nice features of the equivariant theories of Bredon (for compact group actions : see [Br]). We define these by beginning with homology-cohomology theories already defined on algebraic spaces in the sense of Bloch and Ogus. (See [Bl-O].) These are axiomatized in section 3 and section 4 discusses several examples of such theories, for example, continuous étale cohomology, De Rham cohomology, cohomology based on Gersten complexes etc. Then we define several variants of Bredon style cohomology and homology theories in detail in section 5. This is followed by a detailed proof of theorems 1.1, 1.2 and 1.4.

We have devoted an appendix to take care of the necessary technical details for handling presheaves of ring and module spectra.

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We will adopt the following terminology throughout the paper.

1.0.2. **Basic frame work.** Let S denote a Noetherian separated scheme which will serve as the base scheme. All objects we consider will be *locally finitely presented over S, and locally Noetherian.* In particular, all objects we consider are locally quasi-compact. However, our main results are valid, for the most part, only for objects that are finitely presented over the base scheme S.

We will adopt the following conventions regarding moduli spaces. A *coarse moduli-space* for an algebraic stack $\mathcal{S}$ will be a *proper map* $p : \mathcal{S} \rightarrow \mathfrak{M}_{\mathcal{S}}$ (with $\mathfrak{M}_{\mathcal{S}}$ an algebraic space) which is a uniform categorical quotient and a uniform geometric quotient in the sense of [K-M] 1.1 Theorem. In particular, p is universal with respect to maps from $\mathcal{S}$ to algebraic spaces. (Note: this may be different from the notion adopted in [Vi-1].) It is shown in [K-M] that if the stack $\mathcal{S}$ is Deligne-Mumford, of finite type over k and the obvious map $I_{\mathcal{S}} \rightarrow \mathcal{S}$ is finite, then a coarse moduli space exists with all of the above properties, except the map p may not be proper (i.e. finite). However, if $\mathcal{S}$ is also separated over k, then the map p will also be proper (i.e. finite). To see this, observe (see [Vi-1]) that one may find an étale covering $\mathfrak{M}' \rightarrow \mathfrak{M}_{\mathcal{S}}$ so that the induced map $p' : \mathcal{S} \underset{\mathfrak{M}_{\mathcal{S}}}{\times} \mathfrak{M}' \rightarrow \mathfrak{M}'$ is finite. (In fact, one may assume that the stack $\mathcal{S} \underset{\mathfrak{M}_{\mathcal{S}}}{\times} \mathfrak{M}'$ is the quotient stack associated a finite group action.) Therefore, in this case p itself is finite and a coarse moduli space in our sense exists. Moreover, for purposes of defining the Riemann-Roch transformation, we will assume that p has *finite cohomological dimension.* (*We say that a map* $f : \mathcal{S}' \rightarrow \mathcal{S}$ *of* *algebraic stacks has finite cohomological dimension if there exists an integer $N >> 0$ so that $R^i f_*(M) = 0$ for all $i > N$ and all $\mathcal{O}_{\mathcal{S}'}$-modules.*) (Observe that this hypothesis is satisfied if the order of the residual gerbes are prime to the residue characteristics, for example in characteristic 0 for all Artin stacks with quasi-finite diagonal. Proposition 5.14(i) of [J-5] shows that in characteristic 0, generically one may assume the stack is a neutral gerbe. When the stack has quasi-finite diagonal, the stabilizer groups are finite.)

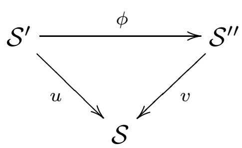
Given a presheaf of spectra P, $P_{\mathbb{Q}}$ will denote its localization at $\mathbb{Q}$. (Observe that then $\pi_*(P_{\mathbb{Q}}) = \pi_*(P) \otimes \mathbb{Q}$.)

2. The isovariant étale site of algebraic stacks and dg stacks: a quick review

Let $\mathcal{S}$ denote an algebraic stack finitely presented over the base scheme S. Now we define and study several new sites associated to stacks in this section. Recall the *inertia stack* $I_{\mathcal{S}}$ associated to $\mathcal{S}$ is defined by the fibered product $\mathcal{S} \underset{\Delta, \mathcal{S} \underset{S}{\times} \mathcal{S}, \Delta}{\times} \mathcal{S}$. Since $\Delta : \mathcal{S} \rightarrow \mathcal{S} \underset{S}{\times} \mathcal{S}$ is representable, so is the obvious induced map $I_{\mathcal{S}} \rightarrow \mathcal{S}$.

Definitions 2.1. (i) Let $f : \mathcal{S}' \rightarrow \mathcal{S}$ be a map of algebraic stacks. We say f is *isovariant* if the natural map $I_{\mathcal{S}'} \rightarrow I_{\mathcal{S}} \underset{\mathcal{S}}{\times} \mathcal{S}'$ is a 1−isomorphism, where $I_{\mathcal{S}'}$ ($I_{\mathcal{S}}$) denotes the inertia stack of $\mathcal{S}'$ ($\mathcal{S}$, respectively).

(ii) *The smooth and étale sites.* Given an algebraic stack $\mathcal{S}$, we let $\mathcal{S}_{smt}$ ($\mathcal{S}_{\underline{smt}}$) denote the site whose objects are smooth maps $u : \mathcal{S}' \rightarrow \mathcal{S}$ of algebraic stacks (smooth maps $u : U \rightarrow \mathcal{S}$ with U an algebraic space). Given two such objects $u : \mathcal{S}' \rightarrow \mathcal{S}$ and $v : \mathcal{S}'' \rightarrow \mathcal{S}$, a morphism $u \rightarrow v$ is a commutative triangle of stacks



(i.e. There is *given* a 2-isomorphism $\alpha : u \rightarrow v \circ \phi$.) The site $\mathcal{S}_{et}$ is the full sub-category of $\mathcal{S}_{smt}$ consisting of étale representable maps $u : \mathcal{S}' \rightarrow \mathcal{S}$, where $\mathcal{S}'$ is an algebraic stack. Finally, when $\mathcal{S}$ is a Deligne-Mumford stack, $\mathcal{S}_{\underline{et}}$ will denote the full sub-catgeory of $\mathcal{S}_{et}$ consisting of étale maps $u : U \rightarrow \mathcal{S}$ with U an algebraic space as objects.

(iii) *The isovariant étale and smooth sites.* If $\mathcal{S}$ is an algebraic stack, $\mathcal{S}_{iso.et}$ will denote the full sub-category of $\mathcal{S}_{et}$ consisting of (representable) maps $u : \mathcal{S}' \rightarrow \mathcal{S}$ that are also *isovariant*. $\mathcal{S}_{iso.smt}$ is defined similarly as a full sub-category of $\mathcal{S}_{smt}$. For the most part we will only consider the site $\mathcal{S}_{iso.et}$. (It follows from the lemma below that these indeed define pre-topologies (or sites) in the sense of Grothendieck.)

(iv) We will consider sheaves on any of the above sites with values in the category of abelian groups, or modules over a ring etc. If $\mathcal{C}$ is any one of the above sites, we will denote the corresponding category of sheaves on $\mathcal{C}$ by $Sh(\mathcal{C})$.

Lemma 2.2. *(See [J-5] section 3) (i) Isovariant maps are representable.*

(ii) Isovariant maps are stable by base-change and composition.

Example 2.3. (Quotient stacks). Let G denote a smooth group scheme acting on an algebraic space X. Now the objects of $X/G_{iso.et}$ may be identified with maps $u : U \rightarrow X$ where U is an algebraic space provided with a G-action so that u is étale and induces an isomorphism on the isotropy groups. Observe that any representable map $\mathcal{S}' \rightarrow X/G$ of algebraic stacks may identified with a G-equivariant map $u : U \rightarrow X$. The iso-variance forces isomorphism of the isotropy sub-groups.

Definition 2.4. An *algebraic groupoid* $\mathcal{X}$ consists of a pair (X_0, X_1) of algebraic spaces provided with the following data:

i) maps $s, t : X_1 \rightarrow X_0$ (s= the *source*, t = the *target*)

ii) a map $\mu : X_1 \underset{t, X_0, s}{\times} X_1 \rightarrow X_1$ which is associative in the obvious sense

iii) a map $e : X_0 \rightarrow X_1$ so that the composition $s \circ e = id_{X_0} = t \circ e$, a map $in : X_1 \rightarrow X_1$ so that, $in^2 = id_{X_1}$, $s \circ in = t$, $t \circ in = s$, $s \circ \mu = s \circ pr_2$ and $t \circ \mu = t \circ pr_1$. (Observe that, since $in^2 = id_{X_1}$, in must be an isomorphism.) Moreover

iv) $\mu \circ (id_{X_1} \times e) = \mu \circ (e \times id_{X_1}) = id_{X_1}$, $\mu \circ (in \times id) = e \circ s$ and $\mu \circ (id \times in) = e \circ t$.

2.1. **Classifying simplicial algebraic spaces.**

2.1.1. *Sites.* Let $\mathcal{S}$ denote an algebraic stack and let $x : X \to \mathcal{S}$ denote an atlas. Let $B_x\mathcal{S}$ denote the classifying simplicial algebraic space associated to x: i.e. $(B_x\mathcal{S})_n = (cosk_0^{\mathcal{S}} X)_n = X \underset{\mathcal{S}}{\times} X ... \underset{\mathcal{S}}{\times} X$. Now one defines the *small* smooth (étale) site of $B_x\mathcal{S}$ as in [Fr] p. 7. Recall each object in this site will be an object in the smooth (étale site) of some $(B_x\mathcal{S})_n$ for some n and a morphism between two such objects will be a map lying over some structure map of $B_x\mathcal{S}$. We will denote these sites by $B_x\mathcal{S}._{smt}$ ($B_x\mathcal{S}._{et}$, respectively). The corresponding *big* sites will be denoted $SMT(B_x\mathcal{S}.)$ ($ET(B_x\mathcal{S}.)$, respectively). Recall that an object in the corresponding big site consists of an object U in $SMT(B_x\mathcal{S}_n)$ ($ET(B_x\mathcal{S}_n)$) for some fixed integer n with morphisms between two such objects defined as morphisms lying over some structure map of the simplicial space $B_x\mathcal{S}$. Coverings are defined in the obvious manner and coincide in the small and the corresponding big sites.

2.1.2. *Topoi.* Given a site as above associated to a simplicial algebraic space $X_\bullet$ a sheaf F on $X_\bullet$ in the above site will be given by a collection $F = \{F_n|n\}$ of sheaves F_n on the corresponding site of X_n along with maps $\Phi_\alpha : \alpha^*(F_n) \to F_m$ for any structure map $\alpha : X_m \to X_n$. Moreover the maps $\{\Phi_\alpha|\alpha\}$ are required to satisfy an obvious compatibility condition. The category of all sheaves of sets on the small smooth site (the small étale site, the big smooth site, the big étale site) of $X_\bullet$ will be denoted $Sh_{sets}(X_{\bullet,smt})$ ($Sh_{sets}(X_{\bullet,et})$, $Sh_{sets}(SMT(X_\bullet))$, $Sh_{sets}(ET(X_\bullet))$, respectively). A sheaf $F = \{F_n|n\}$ on a simplicial space $X_\bullet$ has *descent* if the maps Φ_α are all isomorphisms. The category of sheaves with descent forms a *full* sub-category closed under extensions. This will be denoted $Sh^{des}_{sets}(X_{\bullet,smt})$, on the small smooth site for example.

2.1.3. The above discussion also applies to truncated simplicial algebraic spaces and in particular to algebraic groupoids. Given an algebraic groupoid $\mathcal{X}$, one defines the associated small (big) smooth and étale sites as the corresponding sites of the truncated simplicial space consisting of the X_0, X_1 and $X_2 = X_1 \underset{X_0}{\times} X_1$ along with the given structure maps between them. A sheaf on such a site will consist of a collection of sheaves $F = \{F_n|n = 0, 1, 2\}$, with F_i on X_i along with structure maps $\{\Phi_\alpha : \alpha^*(F_n) \to F_m|\alpha\}$ as above. The category of sheaves of sets on the small étale site of $\mathcal{X}$ will be denoted $Sh_{sets}(\mathcal{X}_{et})$, for example. The corresponding *full* sub-category of sheaves with descent will be denoted $Sh^{des}_{sets}(\mathcal{X}_{et})$.

Remark 2.5. All the above definitions apply to abelian sheaves or sheaves of R-modules, where R is a commutative ring. However, for the most part we will be concerned with the topoi of sheaves of sets. We will also consider mostly the étale sites.

We will presently recall several basic results on iso-variant étale sites from [J-5]. All stacks will be finitely presented over the base scheme.

Proposition 2.6. *Let $\mathcal{S}$ denote an algebraic stack, $x : X \to \mathcal{S}$ an atlas, $\mathcal{X}$ = the associated algebraic groupoid and $B_x\mathcal{S}$ = the associated classifying simplicial algebraic space.*

i) There exist maps $\bar{x} : (B_x\mathcal{S})_{et} \to \mathcal{S}_{et}$ and $\tilde{x} : \mathcal{X} = tr_2(B_x\mathcal{S})_{et} \to \mathcal{S}_{et}$ of sites

ii) One obtains an equivalence of categories:

$$Sh_{sets}(\mathcal{S}_{smt}) \simeq Sh^{des}_{sets}(\mathcal{X}_{smt}) \simeq Sh^{des}_{sets}(B_x\mathcal{S}_{smt}) \simeq Sh^{des}_{sets}(\mathcal{X}_{et}) \simeq Sh^{des}_{sets}(B_x\mathcal{S}_{et}).$$

(Here tr_2 denotes the truncation of the classifying simplicial algebraic space $B_x\mathcal{S}$ above degree 2.)

The following result should be taken as the key to understanding and working with the isovariant sites.

Theorem 2.7. *Assume that a coarse moduli space $\mathcal{M}$ exists (as an algebraic space) for the stack $\mathcal{S}$ and that $\mathcal{S}$ is a (faithfully flat) gerbe over $\mathcal{M}$. Now the functor $V \mapsto V \underset{\mathcal{M}}{\times} \mathcal{S}$, $\mathcal{M}_{et} \to \mathcal{S}_{iso.et}$ is an equivalence of sites. Therefore one obtains an equivalence of the following categories of sheaves:*

$Sh(\mathcal{S}_{iso.et})$ and $Sh(\mathcal{M}_{et})$ where the sheaves are either sheaves of sets or sheaves with values in any Abelian category.

Theorem 2.8. *Let $\mathcal{S}$ denote a quasi-compact algebraic stack with $x : X \to \mathcal{S}$ an atlas. Now there exists a finite filtration of $\mathcal{S}$*

$$\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_n = \mathcal{S} \tag{2.1.4}$$

by locally closed algebraic sub-stacks so that $Sh((\mathcal{S}_i - \mathcal{S}_{i-1})_{iso.et})$ is equivalent to the topos of sheaves on an algebraic space: here sheaves mean sheaves of sets or sheaves with values in any Abelian category. The iso-variant étale site has a conservative family of points and the points correspond to the geometric points of the coarse-moduli space of $\mathcal{S}_i - \mathcal{S}_{i-1}$ for all i.

Corollary 2.9. *(i) Let $\mathcal{S}$ denote an algebraic stack over S. (Recall by our hypotheses, this is required to be Noetherian.) If $\{F_\alpha|\alpha\}$ is a filtered direct system of presheaves of abelian groups or spectra on $\mathcal{S}_{iso.et}$, one obtains a natural quasi-isomorphism $\underset{\alpha}{colim}\mathbb{H}(\mathcal{S}_{iso.et}, F_\alpha) \simeq \mathbb{H}(\mathcal{S}_{iso.et}, \underset{\alpha}{colim}F_\alpha)$. (Here the hypercohomology is defined using the Godement resolution as in 8.0.8.)*

(ii) If $f : \mathcal{S}' \to \mathcal{S}$ is a map of algebraic stacks and $Rf_* = f_*\mathcal{G}$, with $\mathcal{G} = \underset{\Delta}{holim}\, G^\bullet$ computed on the iso-variant étale site of $\mathcal{S}'$, one obtains a similar quasi-isomorphism $\underset{\alpha}{colim} Rf_*F_\alpha \simeq Rf_*(\underset{\alpha}{colim}F_\alpha)$ for a filtered direct system of presheaves $\{F_\alpha|\alpha\}$. (Here $\mathcal{G}^\bullet$ denote the cosimplicial object defined by the Godement resolution - see 8.0.8.)

(iii) Finite cohomological dimension with respect to sheaves of $\mathbb{Q}$-vector spaces: under the above hypotheses, the site $\mathcal{S}_{iso.et}$ has finite cohomological dimension with respect to all sheaves of $\mathbb{Q}$-vector spaces.

Proof. The first two results are proved in [J-5]. The last follows readily from Theorems 2.7 and 2.8. □

2.2. DG-stacks.

Definition 2.10. A DG-stack is an algebraic stack $\mathcal{S}$ of Artin type which is also Noetherian provided with a sheaf of commutative dgas, $\mathcal{A}$, on $\mathcal{S}_{smt}$, so that $\mathcal{A}^i = 0$ for $i > 0$ or $i << 0$, $\mathcal{A}^0 = \mathcal{O}_{\mathcal{S}}$ and each $\mathcal{A}^i$ is a coherent $\mathcal{O}_{\mathcal{S}}$-module. (Observe that our hypotheses imply that $\mathcal{H}^*(\mathcal{A})$ is a sheaf of graded Noetherian rings.) (The need to consider such stacks should be clear from section 5 where we consider applications to virtual structure sheaves and virtual fundamental classes. See [J-7] for a comprehensive study of such stacks from a K-theory point of view.) For the purposes of this paper, we will define a DG-stack $(\mathcal{S}, \mathcal{A})$ to have property P if the associated underlying stack $\mathcal{S}$ has property P: for example, $(\mathcal{S}, \mathcal{A})$ is *smooth* if $\mathcal{S}$ is smooth. Often it is convenient to also include disjoint unions of such algebraic stacks into consideration.

2.2.1. *Morphisms of dg stacks.* A 1-morphism $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ of DG-stacks is a morphism of the underlying stacks $\mathcal{S}' \to \mathcal{S}$ together with a map $\mathcal{A} \to f_*(\mathcal{A}')$ compatible with the map $\mathcal{O}_{\mathcal{S}} \to f_*(\mathcal{O}_{\mathcal{S}'})$. Such a morphism will have property P if the associated underlying 1-morphism of algebraic stacks has property P. Clearly DG-stacks form a 2-category. If $(\mathcal{S}, \mathcal{A})$ and $(\mathcal{S}', \mathcal{A}')$ are two DG-stacks, one defines their *product* to be the product stack $\mathcal{S} \times \mathcal{S}'$ endowed with the sheaf of DGAs $\mathcal{A} \boxtimes \mathcal{A}'$. An *action* of a group scheme G on a DG-stack $(\mathcal{S}, \mathcal{A})$ will mean morphisms $\mu, pr_2 : (G \times \mathcal{S}, \mathcal{O}_G \boxtimes \mathcal{A}) \to (\mathcal{S}, \mathcal{A})$ and $e : (\mathcal{S}, \mathcal{A}) \to (G \times \mathcal{S}, \mathcal{O}_G \boxtimes \mathcal{A})$ satisfying the usual relations.

2.2.2. A left $\mathcal{A}$-module is a complex of sheaves M of $\mathcal{O}_{\mathcal{S}}$-modules, *bounded above* and so that M is a sheaf of left-modules over the sheaf of dgas $\mathcal{A}$. The category of all left $\mathcal{A}$-modules and morphisms will be denoted $Mod_l(\mathcal{S}, \mathcal{A})$. We define a map $f : M' \to M$ in $Mod_l(\mathcal{S}, \mathcal{A})$ to be a quasi-isomorphism if it is a quasi-isomorphism of $\mathcal{O}_{\mathcal{S}}$-modules: observe that this is equivalent to requiring that $\mathcal{H}^*(Cone(f)) = 0$ in $Mod(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$. This is in view of the fact that the mapping cone of the given map $f : M' \to M$ of $\mathcal{A}$-modules taken in the category of $\mathcal{O}_{\mathcal{S}}$-modules has an induced $\mathcal{A}$-module structure. A diagram $M' \overset{f}{\to} M \to M'' \to M[1]$ in $Mod_l(\mathcal{S}, \mathcal{A})$ is a *distinguished triangle* if there is a map $M'' \to Cone(f)$ in $Mod_l(\mathcal{S}, \mathcal{A})$ which is a quasi-isomorphism. Since we assume $\mathcal{A}$ is a sheaf of commutative dgas, there is an equivalence of categories between left and right modules; therefore, henceforth we will simply refer to $\mathcal{A}$-modules rather than left or right $\mathcal{A}$-modules. An $\mathcal{A}$-module M is *perfect* if the following holds: there exists a non-negative integer n and distinguished triangles $F_iM \to F_{i+1}M \to \mathcal{A} \overset{L}{\underset{\mathcal{O}_S}{\otimes}} P_{i+1} \to F_iM[1]$ in $Mod(\mathcal{S}, \mathcal{A})$, for all $0 \le i \le n-1$ and so that $F_0M \simeq \mathcal{A} \overset{L}{\underset{\mathcal{O}_S}{\otimes}} P_0$ with each P_i a perfect complex of $\mathcal{O}_{\mathcal{S}}$-modules and there is given a quasi-isomorphism $F_nM \to M$ of $\mathcal{A}$-modules. The morphisms between two such objects will be just morphisms of $\mathcal{A}$-modules. This category will be denoted $Perf(\mathcal{S}, \mathcal{A})$. M is coherent if $\mathcal{H}^*(M)$ is bounded and finitely generated as a sheaf of $\mathcal{H}^*(\mathcal{A})$-modules. Again morphisms between two such objects will be morphisms of $\mathcal{A}$-modules. This category will be denoted $Coh(\mathcal{S}, \mathcal{A})$. A left-$\mathcal{A}$-module M is *flat* if $M \underset{\mathcal{A}}{\otimes} - : Mod(\mathcal{S}, \mathcal{A}) \to Mod(\mathcal{S}, \mathcal{A})$ preserves quasi-isomorphisms.

Definition 2.11. The categories $Coh(\mathcal{S}, \mathcal{A})$ and $Perf(\mathcal{S}, \mathcal{A})$ along with quasi-isomorphisms as $\mathcal{A}$-modules form Waldhausen categories with fibrations and weak-equivalences. The *fibrations* are maps of $\mathcal{A}$-modules that are degree-wise surjections (i.e. surjections of $\mathcal{O}_{\mathcal{S}}$-modules) and the *weak-equivalences* are maps of $\mathcal{A}$-modules that are quasi-isomorphisms. We will let $Coh(\mathcal{S}, \mathcal{A})$ $(Perf(\mathcal{S}, \mathcal{A}))$ denote the above category with this Waldhausen structure. The K-theory (G-theory) spectra of $(\mathcal{S}, \mathcal{A})$ will be defined to be the K-theory of the Waldhausen category $Perf(\mathcal{S}, \mathcal{A})$ ($Coh(\mathcal{S}, \mathcal{A})$, respectively) and denoted $K(\mathcal{S}, \mathcal{A})$ ($G(\mathcal{S}, \mathcal{A})$, respectively). When $\mathcal{A} = \mathcal{O}_{\mathcal{S}}$, $K(\mathcal{S}, \mathcal{A})$ ($G(\mathcal{S}, \mathcal{A})$) will be denoted $K(\mathcal{S})$ ($G(\mathcal{S})$, respectively). Let $Perf_{fl}(\mathcal{S}, \mathcal{A})$ denote the full sub-category of $Perf(\mathcal{S}, \mathcal{A})$ consisting of flat $\mathcal{A}$-modules. This sub-category inherits a Waldhausen category structure from the one on $Perf(\mathcal{S}, \mathcal{A})$.

Proposition 2.12. *(i)If M is perfect, it is coherent.*

(ii) Let $M \varepsilon Perf(\mathcal{S}, \mathcal{A})$. Then there exists a flat $\mathcal{A}$-module $\tilde{M} \varepsilon Perf(\mathcal{S}, \mathcal{A})$ together with a quasi-isomorphism $\tilde{M} \rightarrow M$.

(iii) Let $M' \rightarrow M \rightarrow M'' \rightarrow M'[1]$ denote a distinguished triangle of $\mathcal{A}$-modules. Then if two of the modules M', M and M'' are coherent (perfect) $\mathcal{A}$-modules, then so is the third.

(iv) Let $\phi : (\mathcal{S}', \mathcal{A}') \rightarrow (\mathcal{S}, \mathcal{A})$ denote a map of dg-stacks. Then one obtains an induced functor $\phi^* : Perf_{fl}(\mathcal{S}, \mathcal{A}) \rightarrow Perf_{fl}(\mathcal{S}', \mathcal{A}')$ of Waldhausen catgeories with fibrations and weak-equivalences.

(v) Assume in addition to the situation in (iii) that $\mathcal{S}' = \mathcal{S}$ and that the given map $\phi : \mathcal{A}' \rightarrow \mathcal{A}$ is a quasi-isomorphism. Then $\phi_* : Perf(\mathcal{S}, \mathcal{A}) \rightarrow Perf(\mathcal{S}, \mathcal{A}')$ defines a functor of Waldhausen categories with fibrations and weak-equivalences. Moreover, the compositions $\phi_* \circ \phi^*$ and $\phi^* \circ \phi_*$ are naturally quasi-isomorphic to the identity.

Proof. In view of the results in Appendix B, one may replace the stack by the simplicial scheme $B_x\mathcal{S}$ where $x : X \rightarrow \mathcal{S}$ is an atlas and $B_x\mathcal{S}$ is the corresponding classifying simplicial space. To simplify the discussion, we will, however, pretend $B_x\mathcal{S}$ is just $\mathcal{S}$ itself.

(i) follows readily. Given any $M \varepsilon Mod(\mathcal{S}, \mathcal{A})$, one may find a flat $\mathcal{A}$-module $\tilde{M}$ together with a quasi-isomorphism $\tilde{M} \rightarrow M$: this follows readily since we are considering all $\mathcal{O}_{\mathcal{S}}$-modules and not just quasi-coherent $\mathcal{O}_{\mathcal{S}}$-modules. Given the $\mathcal{A}$-modules F_iM associated to M, one may define $F_i\tilde{M}$ by the canonical homotopy pull-back: $F_i\tilde{M} = F_iM \underset{M}{\overset{h}{\times}} \tilde{M}$ - see the definition of the latter in [T-T] (1.1.2.5). Since the obvious map $F_i\tilde{M} \rightarrow F_iM$ is a quasi-isomorphism, it follows that $\tilde{M} \varepsilon Perf(\mathcal{S}, \mathcal{A})$. This proves (ii).

To prove (iii), it suffices to show that if M' and M'' are coherent (perfect) then so is M. The coherence of M is clear and to see that M is perfect, one may proceed as follows. One may start with the $\{F_iM'' | i = 0, \cdots, n''\}$, $\{F_jM' | j = 0, \cdots, n'\}$ and define $F_{i+n'+1}M = M \underset{M''}{\overset{h}{\times}} F_iM''$. $F_{n'}M = M \underset{M''}{\overset{h}{\times}} 0 = M'$; now one may continue this by defining $F_jM = F_jM'$, $j = 0, \cdots, n'$. Therefore, it is clear that $M \varepsilon Perf(\mathcal{S}, \mathcal{A})$. This proves (iii).

(ii) shows how to define the functor ϕ^*. Since ϕ^* identifies with $L\phi^*$, it is clear it sends quasi-isomorphisms (distinguished triangles) of $\mathcal{A}$-modules to quasi-isomorphisms (distinguished triangles, respectively) of $\mathcal{A}'$-modules. Since ϕ^* is defined by tensor product, it clearly preserves surjections and hence fibrations.This proves (iv).

The obvious map $\mathcal{A}' \rightarrow \mathcal{A}$ defines the functor ϕ_* that sends an $\mathcal{A}$-module M to the same $\mathcal{O}_{\mathcal{S}}$-module M, but viewed as an $\mathcal{A}'$-module via the map $\mathcal{A}' \rightarrow \mathcal{A}$. Therefore the distinguished triangle $F_iM \rightarrow F_{i+1}M \rightarrow \mathcal{A} \underset{\mathcal{O}_S}{\overset{L}{\otimes}} P \rightarrow F_iM[1]$ is sent to the same distinguished triangle; since $\mathcal{A}' \rightarrow \mathcal{A}$ is a quasi-isomorphism, it follows that $\mathcal{A}' \underset{\mathcal{O}_S}{\overset{L}{\otimes}} P \rightarrow \mathcal{A} \underset{\mathcal{O}_S}{\overset{L}{\otimes}} P$ is also a quasi-isomorphism for any complex of $\mathcal{O}_{\mathcal{S}}$-modules P. Therefore, ϕ_* sends $Perf(\mathcal{S}, \mathcal{A})$ to $Perf(\mathcal{S}, \mathcal{A}')$ preserving quasi-isomorphisms and surjections which are the fibrations. Assuming the existence of functorial flat resolutions (which follows since the smooth sites of algebraic stacks locally of finite type are essentially small), one shows readily that the two compositions $\phi_* \circ \phi^*$ and $\phi^* \circ \phi_*$ are naturally quasi-isomorphic to the identity functors. □

Remarks 2.13. 1. Observe that the above K-theory spectra, $K(Perf(\mathcal{S}, \mathcal{O}_{\mathcal{S}}))$ and $K(Perf(\mathcal{S}, \mathcal{A}))$ are in fact E^{∞}-ring spectra and the obvious augmentation $\mathcal{O}_{\mathcal{S}} \rightarrow \mathcal{A}$ makes $K(\mathcal{S}, \mathcal{A})$ a $K(\mathcal{S})$-algebra. Given two modules M and N over $\mathcal{A}$, one may compute $\mathcal{H}^*(M \underset{\mathcal{A}}{\overset{L}{\otimes}} N)$ using the spectral sequence:

$$E_2^{s,t} = Tor_{s,t}^{\mathcal{H}^*(\mathcal{A})}(\mathcal{H}^*(M), \mathcal{H}^*(N)) \Rightarrow \mathcal{H}^*(M \underset{\mathcal{A}}{\overset{L}{\otimes}} N)$$

Since the above spectral sequence is strongly convergent, it follows that if M and N are coherent, so is $M \overset{L}{\underset{\mathcal{A}}{\otimes}} N$. It follows from this that $G(\mathcal{S}, \mathcal{A})$ is a module spectrum over $K(\mathcal{S}, \mathcal{A})$ as well.

2. Assume $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ is a *proper map* of DG-stacks so that $Rf_* : D_+(Mod(\mathcal{S}', \mathcal{O}_{\mathcal{S}'})) \to D_+(Mod(\mathcal{S}, \mathcal{O}_{\mathcal{S}}))$ has finite cohomological dimension. Now Rf_* induces a map $Rf_* : G(\mathcal{S}', \mathcal{A}') \to G(\mathcal{S}, \mathcal{A})$.

3. Assume that the dg-structure sheaf $\mathcal{A}$ is in fact the structure sheaf $\mathcal{O}$ and the stack $\mathcal{S}$ is *smooth*. Then it is shown in [J-3] (1.6.2) that the obvious map $K(\mathcal{S}) \to G(\mathcal{S})$ is a weak-equivalence.

Example 2.14. (Algebraic stacks provided with virtual structure sheaves) The basic example of a DG-stack that we consider will be an algebraic stack (typically of the form $\mathfrak{M}_{g,n}(X, \beta)$) provided with a *virtual structure sheaf* provided by a *perfect obstruction theory.* Here X is a projective variety, β is a one dimensional cycle and $\mathfrak{M}_{g,n}(X, \beta)$ denotes the stack of stable curves of genus g and n-markings associated to X. The virtual structure sheaf $\mathcal{O}^{virt}$ is the corresponding sheaf of dgas. Observe that if $M_{g,n}(X, \beta)$ denotes the corresponding coarse-moduli space, there are natural maps $K(M_{g,n}(X, \beta)) \to K(\mathfrak{M}_{g,n}(X, \beta)) \to K(\mathfrak{M}_{g,n}(X, \beta), \mathcal{O}^{virt})$ which are maps of E^∞-ring spectra.

Proposition 2.15. *Let $(\mathcal{S}, \mathcal{A})$ denote a DG-stack in the above sense and let $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ denote a map of dg-stacks.*

(i) An $\mathcal{A}$-module M is coherent in the above sense if and only if it is pseudo-coherent (i.e. locally on $\mathcal{S}_{smt}$ quasi-isomorphic to a bounded above complex of locally free sheaves of $\mathcal{O}_{\mathcal{S}}$-modules) with bounded coherent cohomology sheaves.

(ii) One has an induced map $f^* : K(\mathcal{S}, \mathcal{A}) \to K(\mathcal{S}', \mathcal{A}')$ and if f is proper and of finite cohomological dimension an induced map $f_* : G(\mathcal{S}', \mathcal{A}') \to G(\mathcal{S}, \mathcal{A})$ always.

(iii) If $\mathcal{H}^*(\mathcal{A}')$ is of finite tor dimension over $f^{-1}(\mathcal{H}^*(\mathcal{A})$, then one obtains an induced map $f^* : G(\mathcal{S}, \mathcal{A}) \to G(\mathcal{S}', \mathcal{A}')$.

(iv) If f_* sends $Perf(\mathcal{S}', \mathcal{A}')$ to $Perf(\mathcal{S}, \mathcal{A})$, then it induces a direct image map $f_* : K(\mathcal{S}', \mathcal{A}') \to K(\mathcal{S}, \mathcal{A})$.

Proof. In view of the hypotheses on $\mathcal{A}$, one may observe that if M is coherent as an $\mathcal{A}$-module, then the cohomology sheaves $\mathcal{H}^*(M)$ are bounded and coherent over the structure sheaf $\mathcal{O}_{\mathcal{S}}$. Therefore, if M is coherent as an $\mathcal{A}$-module, then M is pseudo-coherent as a complex of $\mathcal{O}_{\mathcal{S}}$-modules. Conversely suppose that M is an $\mathcal{A}$-module, so that, when viewed as a complex of $\mathcal{O}_{\mathcal{S}}$-modules, it is pseudo-coherent with bounded coherent cohomology sheaves. Now the cohomology sheaves $\mathcal{H}^*(M)$ are bounded and coherent $\mathcal{O}_{\mathcal{S}}$-modules. It follows that $\mathcal{H}^*(M)$ is finitely generated over $\mathcal{H}^*(\mathcal{A})$ and hence that M is coherent as an $\mathcal{A}$-module. This proves (i). The remaining statements are clear from the last proposition. ☐

Convention 2.16. Henceforth a stack will mean a DG-stack. DG-stacks whose associated underlying stack is of Deligne-Mumford type will be referred to as Deligne-Mumford DG-stacks.

Often we also need to include the action of an affine smooth group scheme, which may be defined as follows (see [J-2] section 5) for more details):

Definition 2.17. Let $\mathcal{S}$ denote an algebraic stack and let G denote an affine smooth group scheme over S. An action of G on $\mathcal{S}$ is given by the following:

representable maps $G \times \mathcal{S} \overset{\mu}{\to} \mathcal{S}$ and $G \times \mathcal{S} \overset{pr_2}{\to} \mathcal{S}$, along with a common section $s : \mathcal{S} \to G \times \mathcal{S}$ satisfying the usual relations when $G \times G \times \mathcal{S}$, $G \times \mathcal{S}$ and $\mathcal{S}$ are viewed as lax functors from schemes to sets.

An *action* of a group scheme G on a DG-stack $(\mathcal{S}, \mathcal{A})$ will mean morphisms $\mu, pr_2 : (G \times \mathcal{S}, \mathcal{O}_G \boxtimes \mathcal{A}) \to (\mathcal{S}, \mathcal{A})$ and $e : (\mathcal{S}, \mathcal{A}) \to (G \times \mathcal{S}, \mathcal{O}_G \boxtimes \mathcal{A})$ satisfying the relations as above.

Remark 2.18. It follows from the discussion in [J-5] Appendix that the quotient stack $[\mathcal{S}/G]$ exists in the above situation. Moreover, a G-equivariant quasi-coherent $\mathcal{O}_{\mathcal{S}}$-module identifies canonically with a quasi-coherent $\mathcal{O}_{[\mathcal{S}/G]}$-module. Therefore, there is no need to consider the equivariant situation separately: our general discussion of DG-stacks automatically incorporates a corresponding discussion for DG-stacks with G-action.

2.2.3. **K-theory and G-theory presheaves.** Assume that a coarse moduli space $\mathfrak{M}$ exists for the given stack $\mathcal{S}$. We let $p : \mathcal{S} \to \mathfrak{M}$ denote the obvious proper map. (In general, we will also let $p : \mathcal{S}_{smt} \to \mathcal{S}_{iso.et}$ denote the obvious map of sites.) In the presence of an action by a smooth affine group scheme G on $\mathcal{S}$ and (hence) on $\mathfrak{M}$,

$p^G : [\mathcal{S}/G] \rightarrow [\mathfrak{M}/G]$ will denote the induced map of quotient stacks. (In general, we will also let $p^G : [\mathcal{S}/G]_{smt} \rightarrow [\mathcal{S}/G]_{iso.et}$ denote the induced map of sites.)

In this situation, we let $\mathbf{K}(\quad)_{\mathfrak{M}}$ denote the presheaf of spectra defined on $\mathfrak{M}_{et}$ by $V \rightarrow K(V)$ = the K-theory spectrum of the Waldhausen category of vector bundles on V. ($\mathbf{G}(\quad)_{\mathfrak{M}}$ will denote the corresponding presheaf of spectra defined by the Waldhausen K-theory of the category of coherent sheaves.)

We let $\bar{\mathbf{K}}(\quad)_{\mathcal{S}}$ = the presheaf of spectra on the iso-variant étale site of $\mathcal{S}$ defined by $U \rightarrow \bar{\mathbf{K}}(U)$ = the K-theory spectrum of the Waldhausen category of vector bundles that are locally trivial on $U_{iso.et}$. $\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}}$ will denote the corresponding presheaf of spectra defined on the smooth site of $\mathcal{S}$ as in (2.11), where $\mathcal{A}$ is the given dg-structure sheaf. When the dg-structure sheaf $\mathcal{A} = \mathcal{O}_{\mathcal{S}}$, we will denote this simply by $\mathbf{K}(\quad)_{\mathcal{S}}$. In the presence of a group-scheme-action as above, we let $\mathbf{K}(\quad, G)_{\mathfrak{M}}$ ($\mathbf{K}(\quad, \mathcal{A}, G)_{\mathcal{S}} = \mathbf{K}_{[\mathcal{S}/G]}(\quad, \mathcal{A})$) denote the presheaf on $[\mathfrak{M}/G]_{iso.et}$ by $V \rightarrow \mathbf{K}(V, G)$ (the presheaf defined on $[\mathcal{S}/G]_{smt}$ by $U \rightarrow \mathbf{K}(U, \mathcal{A})$, respectively). (The subscripts in all of these will be omitted often if there is no cause for confusion.) Observe that the map $p^* : \mathbf{K}(\quad)_{\mathfrak{M}} \rightarrow \bar{\mathbf{K}}(\quad)_{\mathcal{S}}$ is a map of presheaves of ring-spectra.)

Proposition 2.19. *The functor p^* sending a vector bundle on $\mathfrak{M}$ to a vector bundle on the stack $\mathcal{S}$ induces a weak-equivalence $\mathbf{K}(V)_{\mathfrak{M}} \rightarrow \bar{\mathbf{K}}(V \underset{\mathfrak{M}}{\times} \mathcal{S})$ when the stack $\mathcal{S}$ is a gerbe over its coarse-moduli space $\mathfrak{M}$.*

Proof. Recall the functor $p^{-1} : \mathfrak{M}_{et} \rightarrow \mathcal{S}_{iso.et}$ sending $V \rightarrow V \underset{\mathfrak{M}}{\times} \mathcal{S}$ is an equivalence of sites: this follows from theorem (3.13) in [J-5]. □

3. Cohomology and homology theories for algebraic spaces

In order to define cohomology and homology theories on algebraic stacks the basic strategy adopted in this paper is the following: we begin with cohomology and homology theories defined on algebraic spaces in the setting of Bloch-Ogus (see [Bl-O]). We will assume these theories are defined by complexes of sheaves defined on the étale site of all algebraic spaces. (We cannot really say these are defined on the big étale site of algebraic spaces as they may not be contravariant for arbitrary maps.) By suitably modifying these using K-theoretic information, we are able to incorporate data about the isotropies at each point and therefore obtain cohomology and homology theories that are more suitable for algebraic stacks. If the complexes of abelian sheaves we start out with extend to the big isovariant étale site of algebraic stacks, we are able to define cohomology and homology theories for algebraic stacks in general using these; otherwise, we will only obtain cohomology and homology theories when the algebraic stacks have coarse moduli spaces.

Therefore, we begin this section by considering the key properties of these cohomology and homology theories on algebraic spaces we and recall the standard construction of higher Chern classes. We then consider the Chern character. In view of applications in later sections, we try to extend as much of the discussion as possible to the iso-variant étale site of algebraic stacks. In the next section, we provide a listing of standard examples of such theories.

Definition 3.1. (Basic hypotheses on cohomology-homology theories). Let S denote a base-scheme and let $(schemes/S)$ ($(alg.spaces/S)$, $(alg.stacks/S)$, respectively) denote the category of all locally Noetherian schemes over S (the category of all locally Noetherian algebraic spaces over S, the category of all locally Noetherian algebraic stacks over S, respectively). (We will provide the first two with the étale topology and the last with the iso-variant étale topology to make them into sites.) If G denotes a smooth S-group scheme, we let $(alg.stacks/S, G)$ denote the category of all locally Noetherian algebraic stacks over S provided with the action of G as defined earlier. We will denote any of these categories *generically* by $\mathcal{C}$: however, to distinguish between the two, the first (the second) will be referred to as the non-equivariant case (the equivariant case, respectively). A *duality theory* on the category $\mathcal{C}$ is given by a collection of complexes $\{\Gamma_Z(r)|r\}$ and $\{\Gamma^h_Z(r)|r\}$ for each object Z in the site so that the following axioms hold. $\Gamma_Z(r)$ ($\Gamma^h_Z(r)$) is a complex of abelian sheaves on Z_{et} ($Z_{iso.et}$) if Z is an algebraic space (Z is an algebraic stack, respectively). (The subscript Z will be often dropped.)

(i) Each $\Gamma_Z(r)$ ($\Gamma^h_Z(r)$) is required to be trivial in negative degrees (in positive degrees, respectively). Moreover, $\Gamma_Z(r)$ ($\Gamma^h_Z(r)$) is trivial for r outside of the interval $[0, \infty)$ (the interval $[-\infty, d]$, respectively) where $d = dim(Z)$. There exists pairings $\Gamma_Z(r) \overset{L}{\underset{\mathbb{Z}}{\otimes}} \Gamma_Z(s) \rightarrow \Gamma_Z(r+s)$, $\Gamma_Z(r) \overset{L}{\underset{\mathbb{Z}}{\otimes}} \Gamma^h_Z(s) \rightarrow \Gamma^h_Z(s-r)$ for each Z. These pairings are associative with unit (i.e. $\Gamma_Z(0)$ in degree 0 is a commutative ring with unit) and the first pairing is graded commutative.

Remark 3.2. In the next section it will be particularly convenient to first replace the complexes $\Gamma(\bullet)$ $(\Gamma^h(\bullet))$ by the presheaves of spectra $Sp(\Gamma(\bullet))$ $(Sp(\Gamma^h(\bullet))$, respectively). These are defined in the appendix.

(ii) If X is a scheme or an algebraic space over S, we let $H^i(X, \Gamma_X(r)) = \mathbb{H}^i_{et}(X, \Gamma_X(r))$ and $H_i(X, \Gamma_X(r)) = \mathbb{H}^{-i}_{et}(X, \Gamma^h_X(r))$. (The right hand sides are the étale hypercohomology groups.) Moreover, under the same hypotheses, if Y is a closed sub-scheme (algebraic sub-space) of X, we let $H^i_Y(X, \Gamma_X(r)) = \mathbb{H}^i_{et,Y}(X, \Gamma_X(r))$. In case X is an algebraic stack over S (with Y a closed algebraic sub-stack), we let $H^i(X, \Gamma_X(r)) = \mathbb{H}^i_{iso.et}(X, \Gamma_X(r))$ $(H_i(X, \Gamma_X(r)) = \mathbb{H}^{-i}_{iso.et}(X, \Gamma^h_X(r))$, $H^i_Y(X, \Gamma_X(r)) = \mathbb{H}^i_{iso.et,Y}(X, \Gamma_X(r))$, respectively). We will let, $\mathbb{H}(X, \Gamma_X(r))_{\mathbb{Q}}$ $(\mathbb{H}_Y(X, \Gamma_X(r))_{\mathbb{Q}}$, $\mathbb{H}(X, \Gamma^h_X(r))_{\mathbb{Q}})$ denote the corresponding hypercohomology objects tensored with $\mathbb{Q}$.

3.0.4. One of the basic hypotheses we require is that for each fixed integer r (n), $\mathbb{H}^n(X, \Gamma_X(r))_{\mathbb{Q}}$ $(\mathbb{H}^n_Y(X, \Gamma_X(r))_{\mathbb{Q}}$, $\mathbb{H}^n(X, \Gamma^h_X(r))_{\mathbb{Q}})$ vanish in all but a finite interval containing n (r, respectively) depending on X, if $dim(X) < \infty$, and the choice of the complexes $\{\Gamma(r), \Gamma^h(s) | r, s\}$. (This is true for most cohomology-homology theories we consider; for motivic cohomology and homology, this is also true modulo the Beilinson-Soulé vanishing conjecture.) The index denoting n in $\mathbb{H}^n(\quad, \Gamma(r))$ and $\mathbb{H}^n(\quad, \Gamma^h(r))$ will be called the *degree* while the index denoting r above will be called the *weight*.

(iii) For each fixed r and each map $f : Z' \to Z$ in the site $\mathcal{C}$, there is given a unique map $f^{-1}(\Gamma_Z(r)) \to \Gamma_{Z'}(r)$ so that these are compatible with compositions and flat base-change. So defined, cohomology (and cohomology with supports in a closed sub-scheme (algebraic sub-space, algebraic sub-stack) is contravariant. Homology is covariant for all proper maps (and contravariant for flat maps with constant relative dimension)).

Stated more precisely this means the following:

For each algebraic space or stack Z, we will let $\Gamma(r)_Z$ $(\Gamma^h(r)_Z)$ denote the restriction of $\Gamma(r)$ $(\Gamma^h(r))$ to the étale site (iso-variant étale site if Z is an algebraic stack, respectively) of Z. Given a map (a proper map) $f : X \to Y$ of algebraic spaces or stacks, (proper over the base scheme S, respectively), we will require that there is given a map $\Gamma_Y(r) \to Rf_*\Gamma_X(r)$ $(Rf_*\Gamma^h_X(s) \to \Gamma^h_Y(s)$, respectively) which is compatible with compositions. Similarly if $f : X \to Y$ is a flat map of constant relative dimension c, we assume that we are given a map $\Gamma^h_Y(r) \to Rf_*\Gamma^h_X(r+c)[dc]$, (where d is a positive integer, depending on the duality theory) which is compatible with compositions and with the direct image maps so that for a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

with g flat and f proper, the square

$$\begin{array}{ccc} H_*(X', \Gamma(\bullet)) & \xrightarrow{f'_*} & H_*(Y', \Gamma(\bullet)) \\ {\scriptstyle g'^*}\uparrow & & \uparrow{\scriptstyle g^*} \\ H_*(X, \Gamma(\bullet)) & \xrightarrow{f_*} & H_*(Y, \Gamma(\bullet)) \end{array}$$

commutes.

(iii)' Often we will also need to make the additional hypothesis that there exists a natural quasi-isomorphism $\Gamma^h_X(\bullet) \simeq Rf^!\Gamma^h_Y(\bullet)$ where $Rf^!$ is a right adjoint to Rf_* in the situation of (iii) with f proper. (This will be only in those situations where the right adjoint $Rf^!$ exists.)

(iv) *Localization sequence* Let $i : Y \to X$ denote a closed immersion of algebraic spaces with $j : U = X - Y \to X$ the corresponding open immersion. Now there exists a long exact sequence

$$... \to H_i(Y, \Gamma(j)) \to H_i(X, \Gamma(j)) \xrightarrow{j^*} H_i(U, \Gamma(j)) \to H_{i-1}(Y, \Gamma(j)) \to ...$$

so that for all proper maps $f : X \to X'$, there exists a map from the long exact sequence above to the corresponding long exact sequence for $(f(Y), X')$.

(v) *Homotopy invariance property.* For any X and $p : \mathbb{A}^1_X \to X$ the natural map, the induced map $p^* : H_i(X, \Gamma(r)) \to H_{i+d}(\mathbb{A}^1_X, \Gamma(r+1))$ is an isomorphism. (Here d is again a positive integer depending the complex $\Gamma(r)$.)

(vi) *Homology and cohomology of* $\mathbb{P}(\mathcal{E})$ where X is an algebraic space and $\mathcal{E}$ is a vector bundle on X. (Recall this means $\mathcal{E}$ is locally trivial on the étale topology of X.) In case X is an algebraic stack, let $\mathcal{E}$ denote a vector bundle on the stack $\mathcal{S}$ that is locally trivial on some iso-variant étale cover of $\mathcal{S}$. In this case there exists a canonical class $c_1(\mathcal{E}) \varepsilon H^d(X; \Gamma(1))$ so that if $\pi : \mathbb{P}(\mathcal{E}) \to X$ is the given map, the map π^* gives us isomorphisms $\Sigma_{i=0}^n \pi^*(\quad) \cap c_1(\mathcal{E})^i : \oplus_{i=0}^n \mathbb{H}_*(X; \Gamma(\bullet)) \to \mathbb{H}_*(\mathbb{P}(\mathcal{E}); \Gamma(\bullet))$ $(\Sigma_{i=0}^n \pi^*(\quad) \cup c_1(\mathcal{E})^i : \oplus_{i=0}^n \mathbb{H}^*(X; \Gamma(\bullet)) \to \mathbb{H}^*(\mathbb{P}(\mathcal{E}); \Gamma(\bullet))$.

(vii) *Projection formula.* Let $f : X \to X'$ be a proper map so that

$$\begin{array}{ccc} Y & \longrightarrow & X \\ {\scriptstyle f_Y}\downarrow & & \downarrow{\scriptstyle f_X} \\ Y' & \longrightarrow & X' \end{array}$$

is cartesian with $Y' \to X'$ a closed immersion. Now $f_*(\alpha) \cap z = f_*(\alpha \cap f^*(z))$, $\alpha \varepsilon H_i(X, \Gamma(r))$ and $z \varepsilon H^j_{Y'}(X', \Gamma(s))$ and the cap product pairing is the one induced by the second pairing in (i) on taking hypercohomology.

(viii) *Fundamental class, cohomological semi-purity, purity and Poincaré-Lefschetez duality.* If X is a quasi-projective scheme of pure dimension n, we require that there exist a fundamental class $[X] \varepsilon H_{dn}(X, \Gamma_X(n))$ which restricts to a fundamental class in $H_{dn}(U, \Gamma_U(n))$ for each U in the étale site of X. Moreover if $\tilde{i} : X \to \tilde{X}$ is a closed immersion of X into a smooth quasi-projective scheme $\tilde{X}$, there exists a pairing $H^i_X(\tilde{X}, \Gamma_X(r)) \otimes H_j(\tilde{X}, \Gamma_{\tilde{X}}(s)) \to H_{j-i}(X, \Gamma_X(s-r))$. This pairing defines an *isomorphism* when $[\tilde{X}] \varepsilon H_{dn}(\tilde{X}, \Gamma_{\tilde{X}}(n))$ is used: moreover, varying U over all neighborhoods of a point, we see that we obtain a quasi-isomorphism $Ri^! \Gamma_{\tilde{X}}(s)[dn] \to \Gamma^h_X(n-s)$. In particular (taking $\tilde{X} = X$) when X itself is smooth, we see that $\Gamma_X(s)[dn] \simeq \Gamma^h_X(n-s)$ where n is the dimension of X.

For a quasi-projective scheme X of pure dimension imbedded in $\tilde{X}$ as above so that the codimension is c, we see that the fundamental class of X corresponds to a class in $H^{dc}_X(\tilde{X}, \Gamma_X(c)) = H^{dc}(X, R\tilde{i}^! \Gamma_X(c))$ which defines a similar class on restriction to any U in the étale site of X. We call this the *Koszul-Thom class* and denote it by $[T]$. Observe that now we have the formula: $[T] \cap [\tilde{X}] = [X]$. Moreover taking cup-product with the class $[T]$ defines a map $\tilde{i}^*(\Gamma_{\tilde{X}}(r))[-dc] \to R\tilde{i}^!(\Gamma_{\tilde{X}}(r+c))$.

We will also require that cohomology satisfy a *cohomological semi-purity and purity* hypothesis as follows: if $i : X \to Y$ is a closed immersion (closed regular immersion) of pure codimension c, then $H^i_X(Y; \Gamma(\bullet)) = 0$ for all $i < d.c$ (and in addition, $H^{dc}_X(Y; \Gamma(c)) \neq 0$ and that X defines a class in $H^{dc}_X(Y; \Gamma(c))$, respectively).

(ix) *Higher Chern classes.* If the complexes $\Gamma(r)$ and $\Gamma^h(s)$ are defined on the big étale site of S-schemes, they clearly extend to the big étale site of simplicial schemes over S. We will assume these are not the l-adic complexes, but the complexes defining any one of the other theories in section 4. Let $\mathbf{K} = \mathbf{K}(\quad)_{\mathcal{S}}$ denote the K-theory presheaf of spectra on the big étale site of algebraic spaces : i.e. given an algebraic space $\mathcal{S}$, $\Gamma(\mathcal{S}, \mathbf{K}) = K(\mathcal{S})$. Let $\mathbf{K}_0$ denote the presheaf of fibrant simplicial sets forming the 0-th term of this presheaf of spectra. Now we assume there exist *universal Chern-classes* $C(i) \varepsilon \mathbb{H}^{di}(BGL_\bullet, \Gamma(di))$ where $BGL_\bullet$ denotes $\lim_{N \to \infty} BGL_N$. These universal Chern classes may be viewed as maps of simplicial presheaves $\mathbf{K}_0 \simeq \underline{\mathbb{Z}} \times \mathbb{Z}_\infty(BGL) \to Sp(\Gamma(i)[di])_0$ on the étale site of a given algebraic space $\mathcal{S}$ and define Chern classes $C(i)_n : \pi_n(K(\mathcal{S})) \to H^{di-n}(\mathcal{S}, \Gamma(di))$ for each $n \geq 0$ and each i - see [Bl-1], for example. (Here d is an integer depending on the given duality theory.)

Let $\tilde{C}h(i)$ denote the i-th Newton polynomial in the universal Chern classes $C(0)$,..., $C(i) \varepsilon H^*(BGL_\bullet, \Gamma(di))$.Now $Ch(i) = \tilde{C}h(i)/i!$ is the component of degree di of the Chern character Ch. Then $Ch(i)$ defines a map $\mathbf{K}_0 = \underline{\mathbb{Z}} \times \mathbb{Z}_\infty(BGL) \to \mathbb{H}(\quad; Sp(\Gamma(di)))_{\mathbb{Q}}$ on the étale site of a given algebraic space X and therefore induces a map $Ch(i)_n : \pi_n K(X)) \to H^{di-n}(X, \Gamma(di))_{\mathbb{Q}}$. One may obtain a delooping of this Chern-character as in section 5.

To consider the l-adic case, we simply observe that the discussion on the l-adic case as in 5.0.8 applies here as well.

As an immediate consequence of the above axioms we derive the following corollary.

Corollary 3.3. *Assume the situation in (vi). Now there exist quasi-isomorphisms* $R\pi_*(\Gamma^h(\bullet)_{|\mathbb{P}(\mathcal{E})}) \simeq \oplus_{i=0}^{i=n}\Gamma^h(\bullet)_{|X}$ *and* $R\pi_*(\Gamma(\bullet))_{|\mathbb{P}(\mathcal{E})} \simeq \oplus_{i=0}^{i=n}\Gamma(\bullet)_{|X}$ *where* $\mathcal{E}$ *denotes a rank* n *vector bundle on the algebraic space (stack)* X.

Proof. Both statements are clear on working locally on the appropriate site: in the case when X is an algebraic space (stack), one works locally on the étale site (iso-variant étale site, respectively) of X. □

4. The main sources of Bredon style cohomology-homology theories for algebraic stacks

In this section, we will consider typical examples of cohomology-homology theories on algebraic spaces that give rise to Bredon style cohomology and homology theories on algebraic stacks. The first is continuous l-adic étale cohomology and homology (for any prime l different from the residue characteristics) which, we show extends to define continuous l-adic cohomology and homology on the iso-variant étale site of algebraic stacks (with finite l-cohomological dimension). Therefore, continuous l-adic étale cohomology and homology extends to define Bredon style theories for all algebraic stacks (with finite l-cohomological dimension). The remaining cohomology and homology theories remain restricted to either algebraic spaces or quasi-projective schemes (often defined over a field) and therefore give rise to Bredon style theories only for algebraic stacks that have coarse moduli-spaces, for example those stacks that have a finite diagonal.

4.1. *Continuous étale Cohomology and Homology.* (See [Jan].) We prefer continuous étale cohomology as it is better behaved than étale cohomology. Given a complex of l-adic sheaves $K = \{K_{l^\nu}|\nu \geq 0\}$ on the étale site of a scheme or algebraic space X, we let $\mathbb{H}_{cont}(X, K) \otimes \mathbb{Q} = R(\lim_{\infty \leftarrow \nu} \circ \Gamma)(X_{et}, K_{l^\nu}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. This defines continuous étale cohomology: $H^i_{cont}(X, (r)) = H^i(\mathbb{H}_{cont}(X, \{\underline{\mathbb{Z}}/l^\nu(r)|\nu\}))$ where each $\underline{\mathbb{Z}}/l^\nu$ is the obvious constant sheaf and r denotes the obvious Tate twist. We define continuous étale homology as the continuous étale hypercohomology with respect to the dualizing complex $\mathbb{D}(r) = \{R\pi^!(\underline{\mathbb{Z}}/l^\nu(r))|\nu \geq 0\}$ i.e. $H_i^{cont}(X, (r)) \otimes \mathbb{Q} = H^{-i}(\mathbb{H}_{cont}(X_{et}, \mathbb{D}(r)))$. (Here $\pi : X \to S$ is the structure morphism.) Observe that $d = 2$ in this case.

Now we extend these to the isovariant étale site of algebraic stacks. Given a complex of l-adic sheaves $K = \{K_{l^\nu}|\nu \geq 0\}$ on the isovariant étale site of an algebraic stack $\mathcal{S}$, we let $\mathbb{H}_{cont}(\mathcal{S}, K)$ be defined exactly as in the case when $\mathcal{S}$ is an algebraic space. Observe that the functor $\lim_{\infty \leftarrow \nu}$ sends injectives to objects that are acyclic for Γ. Therefore, in case K satisfied the Mittag-Leffler condition, one may identify $\mathbb{H}_{cont}(\mathcal{S}, K)$ with $\lim_{\infty \leftarrow \nu} R\Gamma(\mathcal{S}_{iso.et}, K_\nu)$. Now $H^i_{cont}(\mathcal{S}, r)$ is defined exactly as in the case $\mathcal{S}$ is an algebraic space.

To be able to define homology in a similar manner, we will restrict to the category of algebraic stacks that are *proper* over the base scheme S. We will adopt the technique of compactly generated triangulated categories to first define a functor $f^!$ associated to any proper map $f : \mathcal{S}' \to \mathcal{S}$ of algebraic stacks. We begin by recalling the notion of *compact objects* from [N] p. 210. We let $D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$ denote the derived category of bounded below complexes of $\mathbb{Z}/l^\nu$-modules. An object $K \varepsilon D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$ is *compact* if for any collection $\{F_\alpha|\alpha\}$ of objects in $D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$

(4.1.1) $$Hom_{D_+(\mathcal{S}_{iso.et};\mathbb{Z}/l^\nu)}(K, \underset{\alpha}{\oplus} F_\alpha) \cong \underset{\alpha}{\oplus} Hom_{D_+(\mathcal{S}_{iso.et};\mathbb{Z}/l^\nu)}(K, F_\alpha)$$

Proposition 4.1. *(i) Every object of the form* $j^{\#}_{U!}j^*_U(\underline{\mathbb{Z}}/l^\nu[n])$ *for* $U \varepsilon \mathcal{S}_{iso.et}$ *and* n *an integer is compact. (ii) The category* $D_+(\mathcal{S}_{iso.et};\ \underline{\mathbb{Z}}/l^\nu)$ *is compactly generated by the above objects as* U *varies among a cofinal set of neighborhoods of all the points i.e. the above collection of objects is a small set* T *of compact objects in* $D_+(\mathcal{S}_{iso.et};\ \underline{\mathbb{Z}}/l^\nu)$*, closed under suspension (i.e. under the translation functor* $[1]$*), so that* $Hom_{D_+(\mathcal{S}_{iso.et};\mathbb{Z}/l^\nu)}(T, x) = 0$ *for all* T *implies* $x = 0$.

Proof. (i) Observe that $Hom_{D_+(\mathcal{S}_{iso.et};\underline{\mathbb{Z}}/l^\nu)}(j^{\#}_{U!}(j^*_U\underline{\mathbb{Z}}/l^\nu[n])),\ F) \cong Hom_{D_+(\mathcal{S}_{iso.et};\underline{\mathbb{Z}}/l^\nu)}(j^*_U(\underline{\mathbb{Z}}[n]),\ j^*_U F)$ $\cong Hom_{D_+(\mathcal{S}_{iso.et};\underline{\mathbb{Z}}/l^\nu)}(\underline{\mathbb{Z}}/l^\nu_{|U},\ j^*_{|U}(F)) \cong R\Gamma(U,\ F)$. Therefore, one now observes that

$$Hom_{D_+(\mathcal{S}_{iso.et};\underline{\mathbb{Z}}/l^\nu)}(j^{\#}_{U!}(j^*_U(\underline{\mathbb{Z}}/l^\nu[n])),\ \underset{\alpha}{\oplus} F_\alpha) \cong R\Gamma(U,\ \underset{\alpha}{\oplus} F_\alpha) \cong \underset{\alpha}{\oplus} R\Gamma(U,\ F_\alpha)$$

(Theorem 4.4 below shows that $R\Gamma$ commutes with filtered colimits.) This proves (i). Suppose $R\Gamma(U,\ F) = 0$ for all U that form a cofinal system of neighborhoods of all points in the site $\mathcal{S}_{iso.et}$. Now it follows immediately from the observation that one has *enough points* for the site $\mathcal{S}_{iso.et}$ that F is *acyclic* and therefore is isomorphic to 0 in the derived category $D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$. This proves (ii). □

Definition 4.2. (Compactly generated triangulated categories) Let **S** denote a triangulated category. Suppose all small co-products exist in **S**. Suppose also that there exists a small set of objects S of **S** so that

(i) for every $s\epsilon S$, $Hom_{\mathbf{S}}(s, -)$ commutes with co-products in the second argument and

(ii) if $y\epsilon\mathbf{S}$ is an object so that $Hom_{\mathbf{S}}(s, y) = 0$ for all $s\epsilon S$, then $y = 0$.

Such a triangulated category is said to be *compactly generated*. An object s in a triangulated category $\mathbf{S}$ is called *compact* if it satisfies the hypothesis (i) above.

Theorem 4.3. *(Neeman: see [N] Theorems 4.1 and 5.1) Let $\mathbf{S}$ denote a compactly generated triangulated category and let $F : \mathbf{S} \to \mathbf{T}$ denote a functor of triangulated categories. Suppose F has the following property:*

if $\{s_\lambda|\lambda\}$ is a small set of objects in $\mathbf{S}$, the co-product $\underset{\lambda}{\sqcup}F(s_\lambda)$ exists in $\mathbf{T}$ and the natural map $\underset{\lambda}{\sqcup}F(s_\lambda) \to F(\underset{\lambda}{\sqcup}s_\lambda)$ is an isomorphism.

Then F has a right adjoint G. Moreover the functor G preserves co-products (i.e. if $\{t_\alpha|\alpha\}$ is a small set of objects in $\mathbf{T}$ whose sum exists in $\mathbf{T}$, $G(\underset{\alpha}{\sqcup}t_\alpha) = \underset{\alpha}{\sqcup}G(t_\alpha)$) if for every s in a generating set S for $\mathbf{S}$, $F(s)$ is a compact object in $\mathbf{T}$.

We will apply the above theorem in the following manner. (Recall that we have restricted to algebraic stacks that are quasi-compact and quasi-separated. It follows that the iso-variant étale site of all the stacks we consider are *coherent* in the sense of [SGA] 4, Exposé VI. See Propositions 2.1, 2.2 and Corollary 4.7.) If $\mathcal{S}$ is such an algebraic stack, $D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$ will denote the derived category of bounded below complexes of sheaves of $\mathbb{Z}/l^\nu$-modules for some l different from the residue characteristics and $\nu \geq 1$.

Theorem 4.4. *Let $f : \mathcal{S}' \to \mathcal{S}$ denote a proper (not necessarily representable) map of algebraic stacks. Now f defines a right derived functor functor $Rf_* : D_+(\mathcal{S}'_{iso.et}; \mathbb{Z}/l^\nu) \to D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$ that commutes with filtered colimits and therefore with sums. Therefore, Rf_* has a right adjoint which we will denote by $Rf^!$.*

Moreover if f has finite l-cohomological dimension on the iso-variant étale sites, $Rf_*(j^{\#}_{U!}(j^*_U(\underline{\mathbb{Z}}/l^\nu[n])))$ is a compact object in $D_+(\mathcal{S}'_{iso.et}; \mathbb{Z}/l^\nu)$ for all objects $j_U : U \to \mathcal{S}$ in the site $\mathcal{S}_{iso.et}$ and all integers n and the functor $Rf^!$ preserves sums.

Proof. Evidently the derived categories $D_+(\mathcal{S}'_{iso.et}; \mathbb{Z}/l^\nu)$ and $D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$ are triangulated categories. Next we showed above in Proposition 4.1, that $\{j^{\#}_{U!}(j^*_U(\underline{\mathbb{Z}}/l^\nu[n]))|j_U : U \to \mathcal{S} \quad in \quad \mathcal{S}_{iso.et}, \quad n\epsilon\mathbb{Z}/l^\nu\}$ is a small set of compact objects that generate the category $D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$. Therefore, if Rf_* preserves sums, Theorem 4.3 shows it has an adjoint $Rf^!$. The functor Rf_* preserves all filtered colimits and finite sums, since the site is coherent: therefore it preserves all sums. The functor $Rf^!$ preserves sums, if $Rf_*(j^{\#}_{U!}(j^*_U(\underline{\mathbb{Z}}/l^\nu[n])))$ is a compact object in $D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$ for all objects $j_U : U \to \mathcal{S}'$ in the site $\mathcal{S}'_{iso.et}$ and all integers n. As shown in Theorem 2.8, one may filter the above stacks by locally closed algebraic sub-stacks $\{\mathcal{S}_i|i\}$ and $\{\mathcal{S}'_i|i\}$ so that the stacks $\mathcal{S}_{i+1} - \mathcal{S}_i$ and $\mathcal{S}'_{i+1} - \mathcal{S}'_i$ are gerbes over their coarse-moduli spaces and that there is an equivalence of the corresponding iso-variant étale sites with the étale sites of the corresponding coarse moduli spaces. Therefore, one may assume without loss of generality that the stacks under consideration are in fact algebraic spaces: it suffices to show that the functor $Rf_!$ sends the compact objects $j^{\#}_{U!}(j^*_U(\underline{\mathbb{Z}}/l^\nu[n]))$ to compact objects. The functor $Rf_!$ now corresponds to the derived direct image functor with compact supports of the induced map of the moduli spaces. Therefore it sends constructible sheaves to complexes with constructible bounded cohomology: now any bounded complex with constructible cohomology sheaves is a compact object in the derived category of sheaves of $\mathbb{Z}/l^\nu$-modules on the étale site of algebraic spaces. □

Definition 4.5. (i) Let $f : \mathcal{S} \to S$ denote the structure map of the stack $\mathcal{S}$. *Assume this is proper and that the base scheme S is Noetherian, regular and of dimension at most* 1. Now we define the dualizing complex on $D_+(\mathcal{S}_{iso.et}; \mathbb{Z}/l^\nu)$ by $D_{\mathcal{S},\nu}(s) = Rf^!(\underline{\mathbb{Z}}/l^\nu(s))$.

(ii) We define complexes $\Gamma_l(r)$ on $(alg.stacks/S)_{iso.et}$ by $\Gamma(U, \Gamma_l(r)) = R(\underset{\infty\leftarrow\nu}{\lim} \circ\Gamma)(U, \underline{\mathbb{Z}}/l^\nu(r))$ and $\Gamma^h(s)$ restricted to $\mathcal{S}_{iso.et}$ be defined by $\Gamma(U, \Gamma^h_l(s)) = R(\underset{\infty\leftarrow\nu}{\lim} \circ\Gamma)(U, D_{\mathcal{S},\nu}(s))$.

(iii) We define $H^{cont}_i(\mathcal{S}, (s)) = \mathbb{H}^{-i}(R(\underset{\infty\leftarrow\nu}{\lim} \circ\Gamma)(\mathcal{S}, D_{\mathcal{S},\nu}(s))$.

4.1.2. Observe that $H^i_{cont}(\mathcal{S}, (r)) = H^i(\mathbb{H}(\mathcal{S}_{iso.et}, \Gamma_l(r))$ and that $H^{cont}_i(\mathcal{S}, (s)) = H^{-i}(\mathbb{H}(\mathcal{S}_{iso.et}, \Gamma^h_l(s))$. (These follow from the observation that $\Gamma_l(r) = R\underset{\infty\leftarrow\nu}{\lim}\underline{\mathbb{Z}}/l^\nu$ and $\Gamma^h_l(s) = R\underset{\infty\leftarrow\nu}{\lim}D_{\mathcal{S},\nu}(s)$ and that these complexes of sheaves consist of injectives.)

4.1.3. *Basic hypothesis for isovariant étale cohomology.* We will assume throughout the paper that, whenever we consider continuous étale or isovariant étale cohomology and homology as above, this will be only for objects of finite l-cohomological dimension, where l is a prime different from the residue characteristics.

The remaining cohomology theories are defined only on algebraic spaces.

(ii) and (iii) *Variants of the Gersten complex.* Here $d = 1$. We may first of all define the complexes $\Gamma(r) = \pi_r(\mathbf{K}(\quad))$ for all r where $\mathbf{K}(\quad)$ denotes the presheaf of K-theory spectra on the big étale site of all algebraic spaces. Similarly we may define $\Gamma^h(s) = \pi_s(\mathbf{G}(\quad))$ for all s, where $\mathbf{G}(\quad)$ denotes the presheaf of G-theory spectra on the restricted big étale site of all algebraic spaces. (i.e. The site where the objects are all algebraic spaces, morphisms are only flat maps and coverings are étale coverings.)

We may also define complexes $\Gamma(r) = R^*(r)$ and $\Gamma^h(s) = R_*(s)$ of presheaves on the same restricted big étale site. For each integer p we define the presheaf $U \mapsto R^*(U, p)$ on the étale site of a stack $\mathcal{S}$ which is the complex :

(4.1.4) $$\underset{x \epsilon U^{(0)}}{\oplus} K_p(k(x)) \rightarrow \cdots \rightarrow \underset{x \epsilon U^{(i)}}{\oplus} K_{p-i}(k(x)) \rightarrow \cdots \rightarrow \underset{x \epsilon U^{(p)}}{\oplus} \mathbb{Z}$$

and the presheaf $U \mapsto R_*(U, p)$ which is the complex:

(4.1.5) $$\cdots \rightarrow \underset{x \epsilon U^{(i)}}{\oplus} K_{p+i}(k(x)) \rightarrow \cdots \rightarrow \underset{x \epsilon U^{(0)}}{\oplus} K_p(k(x))$$

(iv) *De Rham Cohomology and Homology.* (See [Hart].) Here $d = 2$ and we require that the base scheme S is the spectrum of a field of characteristic 0. If X is a smooth algebraic space, we let $\Gamma(q) = \Omega_X =$ the De Rham complex of $\mathcal{S}$ for all $q \geq 0$. We let $\Gamma^h(q) = \Gamma(q)$ in this case. We define $\Gamma(q)$ and $\Gamma^h(q)$ only if X admits a closed immersion into a smooth algebraic space $\tilde{X}$. The complexes $\Gamma^h(q)$ (for all $q \geq 0$) are defined as $Ri^!\Omega^{\cdot}_{\tilde{X}}$, where $i : X \rightarrow \tilde{X}$ is the closed immersion into a smooth algebraic space. The De Rham homology of X is defined as the hypercohomology with respect to this complex. The complex $\Gamma(q)$ is defined in this case as the formal completion of the complex $\Omega^{\bullet}_{\tilde{X}}$ along X. The De Rham cohomology of X is the hypercohomology with respect to this complex.

(v) *Motivic cohomology and the higher Chow groups of Bloch.* Here we assume the base scheme S is the spectrum of a field k. Strictly speaking the higher Chow groups form a homology theory, since they are covariant for all proper maps. They are also contravariant for flat maps and Bloch shows (see [Bl-1]) that they are in fact contravariant for arbitrary maps between smooth schemes. However, the cycle complex itself is not contravariantly functorial, whereas the motivic complex is in fact contravariantly functorial for arbitrary maps between smooth schemes. Therefore, we let $\Gamma(r) = \mathbb{Z}(r)=$ the codimension r motivic complex for all smooth schemes of finite type over k. (We do not define the complexes $\Gamma(\bullet)$ for non-smooth schemes.) We let $\Gamma^h(s)$ be defined by the *dimension s* higher cycle complex of Bloch. (See [FV] for possible extensions and variations.) In this case $d = 2$ once again.

5. Bredon style cohomology and homology: the different variations

In this section we define and study Bredon style cohomology and homology theories in detail. The Chern-character is crucial for this: therefore we begin by defining a Chern-character map for the K-theory of vector bundles that are locally trivial on the isovariant étale site of algebraic stacks.

Proposition 5.1. *Let $(S, \mathcal{O})$ denote a locally ringed site with $\mathcal{O}$ = a sheaf of commutative rings with 1. Assume the site S has enough points. Let $Mod_{l.fr}(S, \mathcal{O})$ denote the category of all locally free and finite rank sheaves of $\mathcal{O}$-modules. Let $C_b(Mod_{l.fr}(S, \mathcal{O}))$ denote the category of all bounded chain complexes of such sheaves of $\mathcal{O}$-modules.*

(i) For each U in the site S, $C_b(Mod_{l.fr}(U, \mathcal{O}_{|U}))$ has the structure of a complicial bi-Waldhausen category with cofibrations and weak-equivalences: the cofibrations are maps of complexes that are degree-wise split injective and weak-equivalences are maps that are quasi-isomorphisms. Now $U \mapsto K(C_b(Mod_{l.fr}(U, \mathcal{O}_{|U})))$ defines a presheaf of spaces on the site S (denoted $K(C_b(Mod_{l.fr}(\quad, \mathcal{O})))$).

(ii) Let $B_\bullet GL_n(\mathcal{O})$ denote the obvious simplicial presheaf on the site S and let $B_\bullet GL(\mathcal{O}) = \underset{n\rightarrow\infty}{\lim} B_\bullet GL_n(\mathcal{O})$. Then there exists a natural map $\underline{\mathbb{Z}} \times \mathbb{Z}_\infty(B_\bullet GL(\mathcal{O})) \rightarrow K(C_b(Mod_{l.fr}(\quad, \mathcal{O})))$ of presheaves of spaces which is a weak-equivalence stalk-wise. (Here $\mathbb{Z}_\infty$ denotes the Bousfield-Kan completion.)

Proof. The assertions in (i) are all clear from [T-T] section 1. The second assertion may be obtained from the following observations. The continuity property of the K-theory functor (see [Qu-1] section 2) and the observation that the Quillen K-theory agrees with the Waldhausen style K-theory shows (see [T-T] (1.11.2)) that the stalk of the presheaf $K(C_b(Mod_{l.fr}(\ \ , \mathcal{O})))$ at the point s may be identified with $K(C_b(Mod_{fr}(\mathcal{O}_s)))$. Now the telescope construction of Grayson (see [Gray]) provides the weak-equivalence in (ii). □

Remark 5.2. The main example to keep in mind is where the site is the isovariant étale site of an algebraic stack provided with the obvious structure sheaf. The last weak-equivalence enables us to produce higher Chern classes for vector bundles that are locally trivial on the isovariant étale site.

Finally it suffices to recall the definition of the functor GL for algebraic stacks. Evidently this is defined on the smooth site of a given stack; however, we may extend it to a presheaf on the big smooth site of all algebraic stacks as follows.

5.0.6. *The functor GL on $(alg.stacks)_{iso.et}$.* Recall that the structure sheaf $\mathcal{O}$ on an algebraic stack $\mathcal{S}$ may be defined as follows. Let $x : X \to \mathcal{S}$ denote a smooth surjective map from an algebraic space. Now

$$\Gamma(\mathcal{S}, \mathcal{O}) \qquad = ker(\Gamma(X, \mathcal{O}) \rightrightarrows \Gamma(X \underset{\mathcal{S}}{\times} X, \mathcal{O})$$

Now one defines the contravariant functor $GL_n(\mathcal{O})$ on the category $(alg.stacks/S)$ by $\mathcal{S}' \mapsto GL_n(\Gamma(\mathcal{S}', \mathcal{O}))$. Letting $GL_{n,S}$ also denote the functor represented by the group scheme $GL_{n,S}$ on the category $(alg.stacks/S)$, one obtains the natural isomorphism:

$$Hom_{alg.stacks/S}(\mathcal{S}', GL_{n,S}) \qquad = \Gamma(\mathcal{S}', GL_n)$$

$$= ker(\Gamma(X, GL_{n,\mathcal{S}}) \rightrightarrows \Gamma(X \underset{\mathcal{S}}{\times} X, GL_{n,\mathcal{S}}))$$

One may similarly define the functors $B_kGL_{n,S}$ for all $k \geq 0$ so that $Hom_{alg.stacks}(\mathcal{S}', B_kGL_{n,S}) = B_kGL_n(\Gamma(\mathcal{S}', \mathcal{O}))$. (We will often omit the base-scheme S and simply denote $B_kGL_{n,S}$ as B_kGL_n.)

Let $\{\Gamma(r)|r\}$ denote a collection of complexes of sheaves on the big site $(algebraic\quad spaces/S)_{e.t}$ so that they extend to a collection of presheaves on the big site $(algebraic\quad stacks/S)_{iso.et}$ as in Definition 3.1. In view of the results in the last corollary and the proposition, we may observe that one obtains the Chern-character

$$Ch_i : \pi_*\bar{\mathbf{K}}(\ \ ) \to \pi_*\mathbb{H}(\ \ , Sp(\Gamma(di)[di]))_{\mathbb{Q}} \tag{5.0.7}$$

as a map of presheaves on the site $(algebraic\quad stacks/S)_{iso.et}$. The above Chern character $Ch = \underset{i}{\Pi} Ch_i$ provides $\underset{i}{\Pi}\pi_*\mathbb{H}(\ \ , Sp(\Gamma(di)[di]))_{\mathbb{Q}}$ the structure of a presheaf of modules over $\pi_*\bar{K}(\ \ )_{\mathbb{Q}}$.

5.0.8. **The l-adic case.** We pause to consider the l-adic situation here. Let $\mathcal{S}$ denote a given algebraic stack. We let $\{\mathbb{Z}/l^\nu(r)|\nu\}$ denote the obvious inverse system of l-adic sheaves on $\mathcal{S}_{iso.et}$ (or on $(\mathfrak{M}_{et})$, if a coarse-moduli space $\mathfrak{M}$ for $\mathcal{S}$ exists). One now forms the associated inverse system of presheaves of spaces $\{Sp_0(\mathbb{Z}/l^\nu(r)|\nu\}$, where $Sp_0(\mathbb{Z}/l^\nu(r))$ denotes the 0-th term of the presheaf of spectra $Sp(\mathbb{Z}/l^\nu(r))$. Now one observes that the presheaf of spaces $\underset{\nu}{\text{holim}}\mathbb{H}(\ \ , Sp(\mathbb{Z}/l^\nu(r))_{\mathbb{Q}}$ defines the continuous l-adic cohomology on taking the homotopy groups. One observes that the same computations as in [Sch] now define the l-adic Chern character $Ch : \bar{K}(\ \ ) \to \underset{\nu}{\text{holim}}\mathbb{H}(\ \ , Sp(\mathbb{Z}/l^\nu(r))_{\mathbb{Q}}$ as a map of presheaves of spaces. We let $\mathbb{H}(\ \ , \Gamma_l(r))_{\mathbb{Q}} = \underset{\nu}{\text{holim}}\mathbb{H}(\ \ , Sp(\mathbb{Z}/l^\nu(r))_{\mathbb{Q}}$ and let Ch_i denote the i-th component of the above Chern-character.

Assume further that the algebraic stack $\mathcal{S}$ is of finite type over the base-scheme. We proceed to consider decompositions of $\pi_*(\mathbf{K}(\ \ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$ compatible with the Chern-character considered above. First observe that if X is any scheme, the Adams' operations ψ^k act on $\pi_*(\mathbf{K}(X))_{\mathbb{Q}}$ and are compatible with respect to pull-backs. Therefore, one obtains a decomposition of the presheaf $U \mapsto \pi_*(\mathbf{K}(U))_{\mathbb{Q}}$ into eigen-spaces for the action of the Adams' operations: we will denote the eigen-space on which ψ^k acts by k^n as $\pi_*(\mathbf{K}(\ \ ))_{\mathbb{Q}}(i)$. When a coarse-moduli space $\mathfrak{M}$ is assumed to exist (as before) for the stack $\mathcal{S}$, we therefore obtain a decomposition for each n:

$$\pi_n(\mathbf{K}(\ \ )_{\mathfrak{M}})_{\mathbb{Q}} = \sqcup_i \pi_n(\mathbf{K}(\ \ )_{\mathfrak{M}})_{\mathbb{Q}}(i) \tag{5.0.9}$$

To obtain a decomposition of the presheaf $\pi_*(\bar{\mathbf{K}}(\ \ )_{\mathcal{S}})_{\mathbb{Q}}$ compatible with the Chern character into a fixed cohomology theory we proceed as follows. We fix the cohomology theory and consider the Chern-character $Ch : \pi_*(\bar{\mathbf{K}}(\ \ )_{\mathcal{S}})_{\mathbb{Q}} \to \mathbb{H}^*_{iso.et}(\ \ , \Gamma(*))_{\mathbb{Q}}$. Now we take the inverse images of the graded components $\mathbb{H}^*_{iso.et}(\ \ , \Gamma(r))_{\mathbb{Q}}$ by the Chern-character to obtain a decomposition of the presheaf $\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}})_{\mathbb{Q}}$ compatible with the Chern-character. For each n, we denote this decomposition as:

(5.0.10) $$\pi_n(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}} = \sqcup_i \pi_n(\bar{\mathbf{K}}(\quad)_{\mathcal{S}})_{\mathbb{Q}}(i)$$

5.0.11. Ideally one would like to be able to obtain a decomposition of the presheaf $\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}})_{\mathbb{Q}}$ making use of Adams' operations. However, though one may use the results of [Gray] to define λ-operations on the K-theory of any exact category, it is still unknown if they define a λ-ring structure there, in general: see[J- 4] for partial results in this direction. Therefore, we resorted to the technique in (5.0.10). Given the above decompositions of $\pi_*(\mathbf{K}(\quad)_{\mathfrak{M}_{\mathbb{Q}}}$ and $\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}}$ one may define an induced decomposition on $\pi_*(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$ as follows.

Consider first the case when a coarse-moduli space $\mathfrak{M}$ is assumed to exist for the given algebraic stack $\mathcal{S}$. For each $i \geq 0$, let $\pi_*\mathbf{K}(\quad, A)_{\mathcal{S}_{\mathbb{Q}}}(i)$ be defined by the co-cartesian square:

(5.0.12) $$\begin{array}{ccc} \pi_*(\mathbf{K}(\quad)_{\mathfrak{M}})_{\mathbb{Q}}(i) & \longrightarrow & \pi_*(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}(i) \\ \uparrow & & \uparrow \\ \pi_*(\mathbf{K}(\quad)_{\mathfrak{M}})_{\mathbb{Q}} & \longrightarrow & \pi_*(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}} \end{array}$$

where $\pi_*(\mathbf{K}(\quad)_{\mathfrak{M}})_{\mathbb{Q}}(i)$ is the eigen-space with weight k^i for the action of ψ^k. Since $\pi_*(\mathbf{K}(\quad)_{\mathfrak{M}})_{\mathbb{Q}}(i)$ splits off $\pi_*(\mathbf{K}(\quad)_{\mathfrak{M}})_{\mathbb{Q}}$ and for each integer n, $\pi_n(\mathbf{K}(\quad)_{\mathfrak{M}})_{\mathbb{Q}} \cong \underset{i\geq 0}{\oplus} \pi_n(\mathbf{K}(\quad)_{\mathfrak{M}})_{\mathbb{Q}}(i)$, it follows that $\pi_*(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}(i)$ splits off $\pi_*(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$ and for each fixed integer n, $\pi_n(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}} \cong \underset{i\geq 0}{\oplus} \pi_n(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}(i)$.

In the case we use the decomposition of $\pi_*(\bar{\mathbf{K}}_{\mathcal{S}})_{\mathbb{Q}}$ as in (5.0.10) in the place of the decomposition of $\pi_*(\mathbf{K}(\quad)_{\mathfrak{M}})_{\mathbb{Q}}$, we obtain a similar decomposition of $\pi_*(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$. In the presence of an action by a smooth group scheme G on the stack, one may start with a decomposition of the presheaf $\pi_*(\bar{\mathbf{K}}(\quad)_{[\mathcal{S}/G]})_{\mathbb{Q}}$ instead of $\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}})_{\mathbb{Q}}$: this will be induced by an equivariant Chern-character map into $\mathbb{H}^*_{iso.et}(\quad, \Gamma(*))_{[\mathcal{S}/G]}$. In the case where the stack has a coarse moduli space $\mathfrak{M}$ with the action by a smooth group scheme G, one may also use a decomposition on $\pi_*(\mathbf{K}(\quad)_{[\mathfrak{M}/G]})_{\mathbb{Q}}$ in the place of the decomposition on $\pi_*(\mathbf{K}(\quad)_{\mathfrak{M}})_{\mathbb{Q}}$ to obtain such a decomposition. The decomposition of $\pi_*(\mathbf{K}(\quad)_{[\mathfrak{M}/G]})_{\mathbb{Q}}$ may be obtained from a decomposition of $\mathbb{H}^*_{[\mathfrak{M}/G]_{iso.et}}(\quad, \Gamma(*))_{\mathbb{Q}}$ making use of the Chern-character. The hypothesis (3.0.4) shows that all the hypercohomology considered above vanish in all but finitely many weights for a fixed degree. Observe, as a result that $\pi_*(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}})$ is a presheaf of *bi-graded* rings: the index denoting n in π_n will be called the *degree* while the index denoting the weight i in the decomposition considered above will be called the *weight*.

Definition 5.3. Let S denote a given base scheme (or algebraic space) and let $\{\Gamma(r)|r\}$, $\{\Gamma^h(s)|s\}$ denote a collection of complexes of sheaves as in Definition 3.1. Observe that the hyper-cohomology $\pi_*(\mathbb{H}(\quad, Sp(\Gamma(\bullet)) = \Pi_{n,r}\pi_n(\mathbb{H}(\quad, Sp(\Gamma(r))))$ and $\pi_*(\mathbb{H}(\quad, Sp(\Gamma^h(\bullet)))) = \Pi_{n,r}\pi_n(\mathbb{H}(\quad, Sp(\Gamma^h(r))))$are also pre-sheaves of bi-graded abelian groups. *The hypercohomology* $\mathbb{H}(\quad, Sp(\Gamma(\bullet)))$ *and* $\mathbb{H}(\quad, Sp(\Gamma(\bullet)))$ *denote hypercohomology computed on the iso-variant étale site of the stack* $\mathcal{S}$. In the non-equivariant situation, we define

$$K\Gamma^h_{\mathcal{S}}(\bullet) = \mathcal{RH}om_{\pi_*(\bar{\mathbf{K}}(\quad))_{\mathcal{S}_{\mathbb{Q}}}}(\pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})) \text{ and}$$

$$K\Gamma_{\mathcal{S}}(\bullet) = \pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}) \overset{L}{\underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}{\otimes}} \pi_*\mathbb{H}(\quad, Sp\Gamma(\bullet))_{\mathbb{Q}}.$$

(Recall from 2.2.3 that $p : \mathcal{S}_{smt} \to \mathcal{S}_{iso.et}$ is the obvious map of sites. Here π_* denotes the homotopy groups of the spectra considered above.)

5.0.13. Here we invoke the definition (8.0.11) to define $K\Gamma^h_{\mathcal{S}}(\bullet)$ with $\mathcal{A} = \pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})$, $M = \pi_*(p_*(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}))$, $N = \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})$, $m^* = \mathcal{H}om(\lambda_M, N)$ and $n_* = \mathcal{H}om(\mathcal{A} \otimes M, \lambda_N)$,

λ_M = the obvious map $\pi_*(p_*(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}})) \otimes \pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}}) \to \pi_*(p_*(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}))$ given by the obvious module structure and λ_N = the pairing $\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}}) \otimes \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}) \to \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})$ given by multiplication with $\pi_*(Ch)$. The presheaf $K\Gamma_{\mathcal{S}}(\bullet)$ is defined similarly making use of the definition (8.0.11). Now $K\Gamma_{\mathcal{S}}(\bullet)$ define presheaves of $\mathbb{Q}$-vector spaces on the site $\mathcal{S}_{iso.et}$. If $\mathcal{S}$ belongs to the category $(alg.stacks/S, G)$, (making use of Proposition (5.6) below) we let

$$K\Gamma^h_{[\mathcal{S}/G]}(\bullet) = \mathcal{RH}om_{\pi_*(\bar{\mathbf{K}}(\quad)_{[\mathcal{S}/G]_{\mathbb{Q}}})}(\pi_*(p^G_*\mathbf{K}(\quad, \mathcal{A}, G)_{\mathcal{S}_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h(\bullet))_{\mathbb{Q}}))) \text{ and}$$

$$K\Gamma_{[\mathcal{S}/G]}(\bullet) = \pi_*(p^G_*\mathbf{K}(\quad, \mathcal{A}, G)_{\mathcal{S}_{\mathbb{Q}}}) \overset{L}{\underset{\pi_*(\bar{\mathbf{K}}(\quad)_{[\mathcal{S}/G]_{\mathbb{Q}}})}{\otimes}} \pi_*\mathbb{H}(\quad, Sp\Gamma(\bullet))_{\mathbb{Q}}$$

which are both presheaves on $[\mathcal{S}/G]_{iso.et}$. (Recall again from (2.2.3) that $p^G : [\mathcal{S}/G]_{smt} \to [\mathcal{S}/G]_{iso.et}$ is the obvious map of sites.) We again make use of the pairings as in 5.0.13 above to define these presheaves of spectra. Observe that the presheaves $K\Gamma_{\mathcal{S}}(\bullet)$ and $K\Gamma^h_{\mathcal{S}}(\bullet)$ get an induced decomposition into bi-graded components, induced from the decompositions in (5.0.12) above, the corresponding decomposition of the presheaves $\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})$ and the decomposition of the hyper-cohomology $\pi_*(\mathbb{H}(\quad, Sp(\Gamma(\bullet)))) = \underset{n,t}{\oplus}\pi_n\mathbb{H}(\quad, Sp(\Gamma(t)))$, $\pi_*(\mathbb{H}(\quad, Sp(\Gamma^h(\bullet)))) = \underset{n,t}{\oplus}\pi_n\mathbb{H}(\quad, Sp(\Gamma^h(t)))$.

Definition 5.4. (**Bredon cohomology and homology for general algebraic stacks**.) Assume the above situation. Let $\mathcal{S}$ denote an algebraic stack. We let *the Bredon cohomology* spectrum be defined by

(i) $\mathbb{H}_{Br}(\mathcal{S}, \Gamma(\bullet)) = R\Gamma(\mathcal{S}; K\Gamma_{\mathcal{S}}(\bullet)) = R\Gamma(\mathcal{S}, \pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}} \overset{L}{\underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathbb{Q}})}{\otimes}} \pi_*\mathbb{H}(\quad, Sp(\Gamma(\bullet))_{\mathbb{Q}}))$

and *the Bredon homology* spectrum by

(ii) $\mathbb{H}^{Br}(\mathcal{S}, \Gamma^h(\bullet)) = R\Gamma(\mathcal{S}; K\Gamma^h_{\mathcal{S}}(\bullet)) = R\Gamma(\mathcal{S}, RHom_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}(\pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}), \pi_*\mathbb{H}(\quad, Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}))$

where the derived functors are taken on the iso-variant étale site. We let

(iii) $H^s_{Br}(\mathcal{S}, \Gamma(t)) = Gr_{-s,t}H^0(\mathbb{H}_{Br}(\mathcal{S}, \Gamma(\bullet))) = Gr_{-s,t}H^0(\Gamma(\mathcal{S}, \pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}} \overset{L}{\underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathbb{Q}})}{\otimes}} \pi_*\mathbb{H}(\quad, Sp(\Gamma(\bullet))_{\mathbb{Q}})))$

and

(iv) $H_s^{Br}(\mathcal{S}, \Gamma^h(t)) = Gr_{s,t}H^0(\mathbb{H}^{Br}(\mathcal{S}, \Gamma^h(\bullet)))$

$= Gr_{s,t}(H^0(\Gamma(\mathcal{S}, RHom_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}(\pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}), \pi_*\mathbb{H}(\quad, Sp(\Gamma^h(\bullet))))_{\mathbb{Q}}))).$

Here $Gr_{p,q}$ denotes the associated graded term with p denoting the degree and q denoting the weight. The derived functors are taken on the iso-variant étale site.

Remark 5.5. Alternatively, one could make the following definitions:

(iii)' $H^s_{Br}(\mathcal{S}, \Gamma(t)) = \underset{s=i-u}{\oplus} Gr_{u,t}H^i(\mathbb{H}_{Br}(\mathcal{S}, \Gamma(\bullet)))$

$= \underset{s=i-u}{\oplus} Gr_{u,t}H^i(R\Gamma(\mathcal{S}, \pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}} \overset{L}{\underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathbb{Q}})}{\otimes}} \pi_*\mathbb{H}(\quad, Sp(\Gamma(\bullet))_{\mathbb{Q}})))$ and

(iv)' $H_s^{Br}(\mathcal{S}, \Gamma^h(t)) = \underset{s=-i+u}{\Pi} Gr_{u,t}H^i(\mathbb{H}^{Br}(\mathcal{S}, \Gamma^h(\bullet)))$

$= \underset{s=-i+u}{\Pi} Gr_{u,t}(H^i(R\Gamma(\mathcal{S}, RHom_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}(\pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}), \pi_*\mathbb{H}(\quad, Sp(\Gamma^h(\bullet))))_{\mathbb{Q}})))$

Here $Gr_{p,q}$ has the same meaning as before. These alternate definitions are more appealing in that they also consider the higher cohomology groups. However, the theories defined by the alternate definitions will fail in general to satisfy the property (v) in Theorem 1.1. Apart from this, all the other properties in Theorem 1.1 are satisfied by these alternate theories.

5.0.14. As observed in 8.0.12, the induced filtration

$$\{F_{p,q}|p,q\} \text{ on } RHom_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}(\pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}))$$

has the property that the natural map

$$F_{p,q}RHom_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}(\pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}))$$

$$\to RHom_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}(\pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}))$$

is a split monomorphism. Therefore, one may define maps $H^0(\mathbb{H}^{Br}(\mathcal{S}, \Gamma^h(\bullet))) \to Gr_{p,q}(\mathbb{H}^{Br}(\mathcal{S}, \Gamma^h(\bullet)))$ for all p, q. i.e. One may define a map $\mathbb{H}^{Br}(\mathcal{S}, \Gamma^h(\bullet))) \to \underset{s,t}{\Pi}\mathbb{H}_s^{Br}(\mathcal{S}, \Gamma(t))$.

Proposition 5.6. *(See [J-1] Appendix.) Let $\mathcal{S}$ denote an algebraic stack with an action by a smooth group scheme G. Now there exists a quotient stack $[\mathcal{S}/G]$ which is an algebraic stack and a representable smooth map $\mathcal{S} \to \mathcal{S}/G$.*

Definition 5.7. In case $\mathcal{S}$ is an algebraic stack provided with the action by a smooth group scheme G, we will now define the Bredon style G-equivariant cohomology spectrum of $\mathcal{S}$ as $R\Gamma([\mathcal{S}/G]; K\Gamma_{[\mathcal{S}/G]}(\bullet))_{\mathbb{Q}}$. Similarly we may define the Bredon style G-equivariant homology spectrum of $\mathcal{S}$ as $R\Gamma([\mathcal{S}/G]; K\Gamma^h_{[\mathcal{S}/G]}(\bullet))_{\mathbb{Q}}$. Then we take the decomposition of the cohomology groups of the above and define bi-graded (Bredon-style equivariant) cohomology and homology as before.

5.1. **Cohomology-Homology theories when a coarse-moduli space exists:the main choice.**

In this case we may adopt the following alternate formulation of cohomology and homology theories.

Definition 5.8. (Bredon style cohomology and homology for algebraic stacks when a coarse moduli space exists) Let $\{\Gamma(r)|r\}$, $\{\Gamma^h(s)|s\}$ denote a collection of complexes of sheaves as in Definition 3.1 either on the the category $(alg.spaces/S)$ or on the the category $(alg.spaces/S, G)$. If $\mathcal{S}$ is an algebraic stack with coarse moduli space $\mathfrak{M}$ belonging to the former category (with $p : \mathcal{S} \to \mathfrak{M}$ denoting the obvious map), we define

$$K\Gamma^h_{\mathcal{S}}(\bullet) = \mathcal{RH}om_{\pi_*(\mathbf{K}(\quad)_{\mathfrak{M}_{\mathbb{Q}}})}(\pi_*(p_*(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}})), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})) \quad and \tag{5.1.1}$$

$$K\Gamma_{\mathcal{S}}(\bullet) = \pi_*(p_*(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}})) \underset{\pi_*(\mathbf{K}(\quad)_{\mathfrak{M}_{\mathbb{Q}}})}{\overset{L}{\otimes}} \pi_*(\mathbb{H}(\quad, Sp\Gamma(\bullet))_{\mathbb{Q}})$$

where the hyper-cohomology $\mathbb{H}(\quad, Sp(\Gamma(\bullet))$ and $\mathbb{H}(\quad, Sp(\Gamma^h(\bullet)))$ are *computed on the étale site of the coarse moduli space $\mathfrak{M}$ associated to the stack $\mathcal{S}$.* (Once again we make use of the pairings as in 5.0.13 to define these presheaves of spectra.)

Now we define the *Bredon cohomology* and *Bredon homology* spectra as follows:

(i) $\mathbb{H}_{Br}(\mathcal{S}, \Gamma(\bullet)) = \mathbb{R}\Gamma(\mathfrak{M}; K\Gamma_{\mathcal{S}}(\bullet)) = R\Gamma(\mathfrak{M}; \pi_*(p_*(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}})) \underset{\pi_*(\mathbf{K}(\quad)_{\mathfrak{M}_{\mathbb{Q}}})}{\overset{L}{\otimes}} \pi_*(\mathbb{H}(\quad, Sp(\Gamma(\bullet)))_{\mathbb{Q}}))$

(ii) $\mathbb{H}^{Br}(\mathcal{S}, \Gamma^h_{\mathcal{S}}(\bullet)) = \mathbb{R}\Gamma(\mathfrak{M}; K\Gamma^h_{\mathcal{S}}(\bullet)) = \Gamma(\mathfrak{M}; RHom_{\pi_*(\mathbf{K}(\quad)_{\mathfrak{M}_{\mathbb{Q}}})}(\pi_*(p_*(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}})), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})))$

Here the derived functors are taken on the étale site of $\mathfrak{M}$. Apart from this change the remaining definitions in Definition 5.4 carry over.

In this situation, one may define a map $\mathbb{H}^{Br}(\mathcal{S}, \Gamma^h(\bullet))) \to \prod_{s,t} \mathbb{H}^{Br}_s(\mathcal{S}, \Gamma(t))$ as before. Next suppose that $\mathfrak{M}$ is a coarse-moduli space for the stack $\mathcal{S}$ and the stack is provided with the action by a smooth group scheme G. We let $\mathbf{K}(\quad)_{[\mathfrak{M}/G]}$ denote the presheaf of K-theory spectra on $[\mathfrak{M}/G]_{iso.et}$ defined by sending $U \to \mathbf{K}(U)$ and let $p^G : [\mathcal{S}/G] \to [\mathfrak{M}/G]$ denote the obvious map. We let

$$\begin{aligned} K\Gamma^h_{[\mathcal{S}/G]}(\bullet) &= \mathcal{RH}om_{\pi_*(\mathbf{K}(\quad)_{[\mathfrak{M}/G]_{\mathbb{Q}}})}(\pi_*(p^G_*\mathbf{K}(\quad, \mathcal{A}, G)_{\mathcal{S}_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h(\bullet))_{\mathbb{Q}}))) \quad and \\ K\Gamma_{[\mathcal{S}/G]}(\bullet) &= \pi_*(p^G_*\mathbf{K}(\quad, \mathcal{A}, G)_{\mathcal{S}_{\mathbb{Q}}}) \underset{\pi_*(\mathbf{K}(\quad)_{[\mathfrak{M}/G]_{\mathbb{Q}}})}{\overset{L}{\otimes}} \pi_*(\mathbb{H}(\quad, Sp\Gamma(\bullet))_{\mathbb{Q}})) \end{aligned} \tag{5.1.2}$$

where $\mathbb{H}(\quad, Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}$ ($\mathbb{H}(\quad, Sp(\Gamma(\bullet)))_{\mathbb{Q}}$) is the presheaf $U \to \mathbb{H}_{iso.et}(U, Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}$ ($U \to \mathbb{H}_{iso.et}(U, Sp(\Gamma(\bullet)))_{\mathbb{Q}}$, respectively), $U \varepsilon [\mathfrak{M}/G]_{iso.et}$. The above are all presheaves on $[\mathfrak{M}/G]_{iso.et}$.

An alternate choice is to define the above presheaves as follows:

$$\begin{aligned} K\Gamma^h_{[\mathcal{S}/G]}(\bullet) &= \mathcal{RH}om_{\pi_*(\mathbf{K}(\quad, G)_{[\mathfrak{M}/G]_{\mathbb{Q}}})}(\pi_*(p^G_*\mathbf{K}(\quad, \mathcal{A}, G)_{\mathcal{S}_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, G, Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})) \quad and \\ RK\Gamma_{[\mathcal{S}/G]}(\bullet) &= \pi_*(p^G_*\mathbf{K}(\quad, \mathcal{A}, G)_{\mathcal{S}_{\mathbb{Q}}}) \underset{\pi_*(\mathbf{K}(\quad, G)_{[\mathfrak{M}/G]_{\mathbb{Q}}})}{\overset{L}{\otimes}} \pi_*(\mathbb{H}(\quad, G, Sp\Gamma(\bullet))_{\mathbb{Q}}) \end{aligned} \tag{5.1.3}$$

where $\mathbb{H}(\quad, G, Sp(\Gamma^h(\bullet)))$ ($\mathbb{H}(\quad, G, Sp(\Gamma(\bullet))))$ is the presheaf $U \to \mathbb{H}_{et}(EG \underset{G}{\times} U, Sp(\Gamma^h(\bullet)))$ ($U \to \mathbb{H}_{et}(EG \underset{G}{\times} U, Sp(\Gamma(\bullet))))$, $U \varepsilon [\mathfrak{M}/G]_{iso.et}$ respectively). The functor $U \mapsto \mathbf{K}_{[\mathfrak{M}/G]}(U, G)$ is the presheaf of spectra on $[\mathfrak{M}/G]_{iso.et}$ defined by $\mathbf{K}_{[\mathfrak{M}/G]}(U, G) = \mathbf{K}(Perf([U/G])$. (Strictly speaking, the last is not a Bredon style theory, but a combination of Bredon and Borel style theories.)

Remark 5.9. For $\mathbb{H}_{et}(EG\underset{G}{\times}U, Sp(\Gamma^h(\bullet)))$ to be defined, we will need to assume either that $\Gamma^h(\bullet)$ is a complex of sheaves on the big étale site of all algebraic spaces, or that the hyper-cohomology $\mathbb{H}_{et}(X, Sp(\Gamma^h(\bullet)))$ is contravariantly functorial in X for all smooth maps. In the former case, $\Gamma^h(\bullet)$ defines a complex on the étale site of the simplicial algebraic space $EG\underset{G}{\times}U$, so that $\mathbb{H}_{et}(EG\underset{G}{\times}U, Sp(\Gamma^h(\bullet)))$ is defined. In the latter case, one may adopt the technique in [J-2] section (3.6.4) to define this: i.e. one first takes $\{\mathbb{H}_{et}((EG\underset{G}{\times}U)_n, Sp(\Gamma^h(\bullet)))|n\}$. This forms a co-chain complex of abelian spectra; one applies a functor DN that produces a cosimplicial object from this and then takes its homotopy limit over Δ to define $\mathbb{H}_{et}(EG\underset{G}{\times}U, Sp(\Gamma^h(\bullet)))$.

Next we provide the following definition of G-equivariant cohomology and homology for an algebraic stack $\mathcal{S}$ with a coarse moduli space $\mathfrak{M}$ and provided with the action by a smooth group scheme G.

Definition 5.10. (i) $H^s_{Br}(\mathcal{S}, G; \Gamma(t)) = Gr_{-s,t}H^0(R\Gamma([\mathfrak{M}/G], K\Gamma_{[\mathcal{S}/G]}(\bullet)))$

(ii) $H_s^{Br}(\mathcal{S}, G; \Gamma^h(t)) = Gr_{s,t}H^0(R\Gamma([\mathfrak{M}/G], K\Gamma_{[\mathcal{S}/G]}(\bullet)))$

Here the derived functors are taken on the iso-variant étale site of $[\mathfrak{M}/G]$. Now the remaining definitions in Definition 5.4 carry over.

Remark 5.11. Next consider the case where the stack $\mathcal{S}$ is in fact an algebraic space. In this case, one may observe that the above definitions of equivariant cohomology and homology are indeed variants of Bredon style equivariant cohomology and homology originally considered for the action of compact groups on topological spaces.

In the case of Deligne-Mumford stacks, we will also define a variant of the above cohomology and homology theories replacing algebraic K-theory with étale K-theory which we define to be the hyper-cohomology on the étale site of the stack. Given a Deligne-Mumford stack $\mathcal{S}$ defined over an algebraically closed field k with the *trivial* action of a smooth group scheme G, we let $K_{et}(\mathcal{S}, G) = \mathbb{H}_{et}(\mathcal{S}, \mathbf{K}(\quad, G)_{\mathcal{S},\mathbb{Q}})$ and the presheaf $K_{et}(\quad, G)_{\mathcal{S}}$ be defined by $U \to K_{et}(U, G)$ with $U \varepsilon \mathcal{S}_{et}$. (Observe that since G acts trivially on $\mathcal{S}$, $\mathbf{K}(U, G) \simeq \mathbf{K}(U) \otimes R(G)$, where $R(G)$ is the representation ring of G at least if G is diagonalizable. This is because giving a G-action on a vector bundle is equivalent to giving a grading by the character group of G. Therefore, in this case, $K_{et}(\quad, G)_{\mathcal{S}} = (K_{et}(U)) \otimes R(G)$ as well.) This will be useful in relating the Bredon homology theories with those of the inertia stacks (for Deligne-Mumford stacks) in the sense of [CR], [Vi-2] and [To]. Let $\mathcal{S}$ denote such an algebraic stack with a *trivial* action by a smooth group scheme G. Let $I_{\mathcal{S}}$ denote the associated inertia stack. Let $p_0 : I_{\mathcal{S}} \to \mathcal{S}$ denote the obvious map.

Definition 5.12. (*Relative Étale form of cohomology and homology for inertia stacks*)

(i) $\mathbb{H}_{Br-et}(I_{\mathcal{S}}/\mathcal{S}, G, \Gamma(\bullet)) = \mathbb{R}\Gamma([\mathfrak{M}/G], \pi_*(p^G_* p_{0*}(K_{et}(\quad, G)_{I_{\mathcal{S}}\mathbb{Q}})) \underset{\pi_*(K(\quad)_{[\mathfrak{M}/G]_{\mathbb{Q}}})}{\overset{L}{\otimes}} \pi_*(\mathbb{H}(\quad, G, Sp(\Gamma(\bullet)))_{\mathbb{Q}}))$ and

(ii) $\mathbb{H}^{Br-et}(I_{\mathcal{S}}/\mathcal{S}, G, \Gamma(\bullet)) = \mathbb{R}\Gamma([\mathfrak{M}/G], \mathcal{RH}om_{\pi_*(\mathbf{K}(\quad)_{[\mathfrak{M}/G]_{\mathbb{Q}}})}(\pi_*(p^G_* p_{0*}(K_{et}(\quad, G)_{I_{\mathcal{S}}\mathbb{Q}})), \pi_*(\mathbb{H}(\quad, G, Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})))$

(iii) $\mathbb{H}^{Br-et}(S, G, \Gamma(\bullet)) = \mathbb{R}\Gamma([\mathfrak{M}/G], \mathcal{RH}om_{\pi_*(\mathbf{K}(\quad)_{[\mathfrak{M}/G]_{\mathbb{Q}}})}(\pi_*(p^G_*(K_{et}(\quad, G)_{\mathcal{S}\mathbb{Q}})), \pi_*(\mathbb{H}(\quad, G, Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})))$

(iv) One now takes the decomposition of the K-theory spectra as in (5.0.12) and making use of (8.0.12), takes H^0 and the associated graded terms as before to define the cohomology and homology groups $H^s_{Br-et}(I_{\mathcal{S}}/\mathcal{S}, G, \Gamma(t))$, $H_s^{Br-et}(I_{\mathcal{S}}/\mathcal{S}, G, \Gamma(t))$ and $H_s^{Br-et}(S, G, \Gamma^h(t))$.

We will next consider Bredon-style *local cohomology* for algebraic stacks. In this case we will always assume that the coarse moduli space is a scheme that admits a closed immersion into a regular scheme. In the equivariant case, for the action of a group scheme G, we will assume that such an equivariant closed immersion into a regular scheme exists onto which the G-action extends. Observe that if the coarse-moduli space is a quasi-projective scheme (a G-quasi-projective scheme in the sense of [J-1]), such a closed immersion always exists.

Definition 5.13. (Bredon-style local cohomology) Let $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ denote a closed immersion of the moduli-space into a regular scheme. Recall that the complexes $\{\Gamma(r)|r\}$ are defined on $(alg.spaces/S)$ and in particular on the étale site of $\tilde{\mathfrak{M}}$ as well. Therefore we may define the presheaf

$$\pi_*(i_*(p_*(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}\mathbb{Q}}))) \underset{\pi_*(i_*(\mathbf{K}(\quad)_{\mathfrak{M}\mathbb{Q}}))}{\overset{L}{\otimes}} \pi_*(\mathbb{H}(\quad, i_*Ri^!Sp(\Gamma(\bullet)))_{\mathbb{Q}}) \text{ on the étale site of } \tilde{\mathfrak{M}}.$$

(The presheaf $i_*\mathbf{K}(\quad)_{\mathfrak{M}}$ is the presheaf of spectra defined on $\tilde{\mathfrak{M}}_{et}$ by extension by zero of the presheaf $\mathbf{K}(\quad)_{\mathfrak{M}}$.) We will denote this presheaf by $i_*Ri^!(K\Gamma(\bullet))$ for convenience. Now we will let

$$\mathbb{H}^s_{Br,\mathcal{S}}(\tilde{\mathfrak{M}},\Gamma(t)) = Gr_{-s,t}H^0(\mathbb{R}\Gamma(\tilde{\mathfrak{M}}, i_*i^!\mathbf{K}\Gamma(\bullet)))$$

The derived functor is taken on the étale site of $\tilde{\mathfrak{M}}$. In the equivariant case, when $i:\mathfrak{M}\to\tilde{\mathfrak{M}}$ is a G-equivariant closed immersion into a regular G-scheme, one defines

$$i_*Ri^!K\Gamma(*,G) = \pi_*(i_*(p_*(\mathbf{K}(\quad,\mathcal{A},G)_{\mathcal{S}_{\mathbb{Q}}}))) \underset{\pi_*(i_*\mathbf{K}(\quad,G)_{[\mathfrak{M}/G]_{\mathbb{Q}}})}{\overset{L}{\otimes}} \pi_*(\mathbb{H}(\quad,G,i_*Ri^!Sp(\Gamma(\bullet)))_{\mathbb{Q}})$$

where $i_*\mathbf{K}(\quad,G)_{[\mathfrak{M}/G]}$ is the presheaf defined on the site $[\tilde{\mathfrak{M}}/G]_{iso.et}$ by $V\to\bar{\mathbf{K}}(V\underset{\tilde{\mathfrak{M}}}{\times}\mathfrak{M},G)$. We let

$$\mathbb{H}^s_{Br,\mathcal{S}}(\tilde{\mathfrak{M}},G,\Gamma(t)) = Gr_{-s,t}H^0(\Gamma([\tilde{\mathfrak{M}}/G], i_*Ri^!K\Gamma(*,G)))$$

The derived functor is taken on the iso-variant étale site of $[\tilde{\mathfrak{M}}/G]$.

Under the hypothesis in (iii)" in 3.1, one may show readily that this is independent of the chosen closed immersion.

Remarks 5.14. 1. Observe that the presheaves $K\Gamma(\bullet)$ and $K\Gamma^h(\bullet)$ defined above are replacements for the complexes $\Gamma(\bullet)$ and $\Gamma^h(\bullet)$ as in Definition 3.1 for algebraic stacks: these incorporate the action of the isotropy into the very definition. The main choice of cohomology and homology for algebraic stacks is as suitable hyper-cohomology with respect to these complexes. Moreover if the stack $\mathcal{S}$ happens to be an algebraic space, there is a natural weak-equivalence $K\Gamma_{\mathcal{S}}(\bullet)\simeq Sp(\Gamma_{\mathcal{S}}(\bullet))$. (This is proved in Theorem 1.1 (vi).) In case the stack is an *orbifold*, i.e. a Deligne-Mumford stack that is generically a scheme, we see that the same weak-equivalence holds generically. Similarly $\{K\Gamma^h_{\mathcal{S}}(\bullet)|\mathcal{S}\}$ extends $\Gamma^h(\bullet)$ to the category $(alg.stacks/S)$.

2. Observe that even in situations where a coarse moduli space exists, the iso-variant étale topology on the stack is, in general, finer than the étale topology of the moduli space. (They are equivalent only when the stack is a gerbe over the moduli space as proved in Theorem (2.7).) Therefore the definitions in 5.4 provide cohomology theories that are, in general, *distinct* from the ones in 5.8.

5.1.3. Finally in all of the above it is possible to replace π_* by just π_0: this will give Bredon-style cohomology and homology that are suitable for Riemann-Roch at the level of the Grothendieck groups without the hypothesis of quasi-projectivity on the moduli spaces.

6. Proofs of Theorems 1.1 and 1.2

We will adopt the following convention throughout:

Convention If $\mathcal{S}$ is an algebraic stack ($f:\mathcal{S}'\to\mathcal{S}$ is a map of algebraic stacks), we may assume that it satisfies (both $\mathcal{S}$ and $\mathcal{S}'$ satisfy, respectively) the hypothesis that it has a coarse-moduli space which exists as an algebraic space: in this case we will let $\{\Gamma^h(r)|r\}$ be any collection of complexes satisfying the general hypotheses in Definition 3.1. If the moduli spaces are not assumed to exist, we will need to assume the complexes $\{\Gamma^h(r)|r\}$ are defined as in Definition 4.5, for example, they define continuous l-adic étale cohomology and homology. We will provide proofs of statements in detail as much as possible for the version defined as hyper-cohomology on the isovariant étale site. The proofs of the corresponding statements for the versions defined using the moduli spaces follow along the same lines by appropriate modifications.

Basic observations. When the moduli spaces are assumed to be quasi-projective, one may observe the following: if $f:\mathcal{S}'\to\mathcal{S}$ is a map of algebraic stacks and $\bar{f}:\mathfrak{M}'\to\mathfrak{M}$ is the corresponding map of the moduli spaces, one may find regular schemes $\tilde{\mathfrak{M}}'$ containing $\mathfrak{M}'$ as a closed sub-scheme ($\tilde{\mathfrak{M}}$ containing $\mathfrak{M}$ as a closed sub-scheme, respectively) and a map $\tilde{f}:\tilde{\mathfrak{M}}'\to\tilde{\mathfrak{M}}$ extending $\bar{f}$. The map $\tilde{f}$ may be chosen to be proper if the original map f is. Moreover, in the equivariant case, we may do all this equivariantly.

The hypothesis (iii) of 3.1 which is assumed to hold on the big isovariant étale site shows that if $f:\mathcal{S}'\to\mathcal{S}$ is a map of algebraic stacks, there is an induced map $Sp(\Gamma_{\mathcal{S}}(r))\to Rf_*Sp(\Gamma_{\mathcal{S}'}(r))$ for all r. Next observe that f already induces maps $f^*:K(U,\mathcal{A})\to K(\mathcal{S}'\underset{\mathcal{S}}{\times}U,\mathcal{A}')$ (equivalently $f^*:\mathbf{K}_{\mathcal{S}}(\quad,\mathcal{A})\to Rf_*\mathbf{K}_{\mathcal{S}'}(\quad,\mathcal{A}'))$ and $f^*:\bar{K}(U)\to\bar{K}(\mathcal{S}'\underset{\mathcal{S}}{\times}U)$ (equivalently $f^*:\bar{\mathbf{K}}_{\mathcal{S}}(\quad)\to Rf_*\bar{\mathbf{K}}_{\mathcal{S}'}(\quad))$ of (symmetric) ring spectra, for $U\varepsilon\mathcal{S}_{iso.et}$. At any rate, on taking the associated presheaves of homotopy groups, one obtains a map of the presheaves of graded rings.

To see that the induced maps on cohomology and homology preserve the weights as stated, one needs to observe first that if $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ is a map of dg-stacks, the induced map on K-theory presheaves $f^* : \pi_*\Gamma(U, \mathbf{K}_{\mathcal{S}}(\quad, \mathcal{A})) \to \pi_*\Gamma(\mathcal{S}' \underset{\mathcal{S}}{\times} U, \mathbf{K}_{\mathcal{S}'}(\quad, \mathcal{A}'))$ preserves weights, $U \varepsilon \mathcal{S}_{iso.et}$; this in turn follows from the observation that the induced map $f^* : \pi_*\Gamma(U, \bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}}) \to \pi_*\Gamma(\mathcal{S}' \underset{\mathcal{S}}{\times} U, \bar{\mathbf{K}}(\quad)_{\mathcal{S}'_{\mathbb{Q}}})$ (or in the presence of moduli-spaces, the induced map $\bar{f}^* : \pi_*\Gamma(V, \mathbf{K}(\quad)_{\mathfrak{M}_{\mathbb{Q}}}) \to \pi_*\Gamma(\mathfrak{M}' \underset{\mathfrak{M}}{\times} V, \mathbf{K}(\quad)_{\mathfrak{M}'_{\mathbb{Q}}})$, $V \varepsilon \mathfrak{M}_{et}$) preserves weights. (See also 5.0.12.) In addition to this, one also needs to make use of the basic hypotheses on weights on the complexes $\{\Gamma(r)|r\}$ and $\{\Gamma^h(s)|s\}$ and how they behave as in section 3. Consequently we observe that the map f induces the following map of presheaves of graded rings:

$$\pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}) \underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}{\overset{L}{\otimes}} \pi_*(\mathbb{H}(\quad, Sp(\Gamma_{\mathcal{S}}(\bullet)))_{\mathbb{Q}})$$
$$\to Rf_*(\pi_*(p'_*\mathbf{K}(\quad, \mathcal{A}')_{\mathcal{S}'_{\mathbb{Q}}})) \underset{Rf_*(\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}'_{\mathbb{Q}}}))}{\overset{L}{\otimes}} \pi_*(Rf_*\mathbb{H}(\quad, Sp(\Gamma_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}})$$
$$\to Rf_*(\pi_*(p'_*(\mathbf{K}(\quad, \mathcal{A}')_{\mathcal{S}'_{\mathbb{Q}}})) \underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}'_{\mathbb{Q}}})}{\overset{L}{\otimes}} \pi_*(\mathbb{H}(\quad, Sp(\Gamma_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}}))$$

In view of the definition of the functors $\overset{L}{\otimes}$ and $\mathcal{RH}om$ as in the appendix, these observations suffice to prove the contravariance and the ring structure on Bredon style cohomology and these extend to local cohomology and also to the equivariant case.

Now we consider the covariance property for Bredon homology. It suffices to show that, if $f : \mathcal{S}' \to \mathcal{S}$ is a proper map of algebraic stacks, one obtains an induced map

$$Rf_*\mathcal{RH}om_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}'_{\mathbb{Q}}})}(\pi_*(p'_*K(\quad, \mathcal{A}')_{\mathcal{S}'_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}}))$$
$$\xrightarrow{f_*} \mathcal{RH}om_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}'_{\mathbb{Q}}})}(\pi_*(K(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}})).$$

By (8.0.14) in the appendix, this is adjoint to a map

$$Rf_*\mathcal{RH}om_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}'_{\mathbb{Q}}})}(\pi_*(p'_*K(\quad, \mathcal{A}')_{\mathcal{S}'_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}})) \underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}'_{\mathbb{Q}}})}{\overset{L}{\otimes}} (\pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}))$$
$$\to \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}})$$

This map may be obtained as follows. One first observes there are natural maps $\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}})_{\mathbb{Q}} \to Rf_*(\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}'})_{\mathbb{Q}})$ and $p_*(\pi_*(\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}) \to Rf_*p'_*(\pi_*(\mathbf{K}(\quad, \mathcal{A}')_{\mathcal{S}'})_{\mathbb{Q}})$ of sheaves of graded rings. Therefore, we obtain the following sequence of maps:

$$Rf_*\mathcal{RH}om_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}'_{\mathbb{Q}}})}(\pi_*(p_*K(\quad, \mathcal{A}')_{\mathcal{S}'_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}})) \underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}{\overset{L}{\otimes}} \pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}})$$
$$\to Rf_*\mathcal{RH}om_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}'_{\mathbb{Q}}})}(\pi_*(p'_*K(\quad, \mathcal{A}')_{\mathcal{S}'_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}})) \underset{Rf_*\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}'_{\mathbb{Q}}})}{\overset{L}{\otimes}} Rf_*\pi_*(p'_*\mathbf{K}(\quad, \mathcal{A}')_{\mathcal{S}'_{\mathbb{Q}}})$$
$$\to Rf_*(\mathcal{RH}om_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}'_{\mathbb{Q}}})}(\pi_*(p'_*K(\quad, \mathcal{A}')_{\mathcal{S}'_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}})) \underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}'_{\mathbb{Q}}})}{\overset{L}{\otimes}} \pi_*(p'_*\mathbf{K}(\quad, \mathcal{A}')_{\mathcal{S}'_{\mathbb{Q}}}))$$

Rf_* composed with the obvious evaluation map defines a map from the last expression to $Rf_*\pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}})$. Finally, the hypothesis in Definition 3.0.4(iii) shows there exists a natural map from the last term to $\pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}})$. (Such a map exists in general for all Artin stacks, only for continuous étale cohomology, while it exists for all the cohomology-homology theories we consider provided a coarse moduli space exists.) The composition of the above maps provides the required covariant functoriality of Bredon homology.

It will be important for later applications to observe that the composition of the above maps also factors as the following composition:

$$Rf_*\mathcal{RH}om_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}'_{\mathbb{Q}}})}(\pi_*(p'_*\mathbf{K}(\quad,\mathcal{A}')_{\mathcal{S}'_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}})) \tag{6.0.4}$$
$$\rightarrow \mathcal{RH}om_{Rf_*(\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}'_{\mathbb{Q}}}))}(Rf_*\pi_*(p'_*(\mathbf{K}(\quad,\mathcal{A}')_{\mathcal{S}'_{\mathbb{Q}}})), Rf_*(\pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}'}(\bullet)))_{\mathbb{Q}})))$$
$$\rightarrow \mathcal{RH}om_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}(\pi_*(p_*\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}))$$

where the last map is defined in a manner parallel to the definitions of the first map in the above paragraph.

The compatibility of the direct image maps in Bredon homology with the map $\pi_*(\mathbb{H}^{Br}(\mathcal{S}, \Gamma^h(\bullet)))_{\mathbb{Q}} \rightarrow \prod_t \mathbb{H}^{Br}_*(\mathcal{S}, \Gamma(t))_{\mathbb{Q}}$ as in 5.0.14 follows from the basic observations above. (For example: the inverse image maps on K-theory preserve the weight filtrations considered above.)

To prove the third property one needs to observe first that $\mathcal{RH}om_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}(\pi_*(\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}}))_{\mathbb{Q}}))$ has the structure of a presheaf of modules over $\pi_*(\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}) \overset{L}{\underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}{\otimes}} \pi_*(H(\quad, Sp(\Gamma_{\mathcal{S}}(\bullet)))_{\mathbb{Q}})$. By (8.0.14) in the appendix, it suffices to show that there exists a map:

$$\pi_*(\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}) \underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}{\otimes} \pi_*(\mathbb{H}(\quad, Sp(\Gamma_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}) \underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}{\otimes} \mathcal{RH}om_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}(\pi_*(\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}))$$
$$\underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}{\otimes} \pi_*(\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}_{\mathbb{Q}}})$$
$$\rightarrow \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}})$$

We obtain such a map by composing the pairing $\pi_*(\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}) \otimes \pi_*(\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}) \rightarrow \pi_*(\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}_{\mathbb{Q}}})$ with the evaluation map $\pi_*(\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}) \underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}{\otimes} \mathcal{RH}om_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}(\pi_*(\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}})) \rightarrow \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}})$ which provides the map:

$$\pi_*(\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}) \underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}{\otimes} \pi_*(\mathbb{H}(\quad, Sp(\Gamma_{\mathcal{S}}(\bullet)))_{\mathbb{Q}})$$
$$\underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}{\otimes} \mathcal{RH}om_{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}(\pi_*(\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}), \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}})) \underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}{\otimes} \pi_*(\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}_{\mathbb{Q}}})$$
$$\rightarrow \pi_*(\mathbb{H}(\quad, Sp(\Gamma_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}) \otimes \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}).$$

Now we compose with the pairing:

$$Sp(\Gamma_{\mathcal{S}}(\bullet)) \otimes Sp(\Gamma^h_{\mathcal{S}}(\bullet)) \rightarrow Sp(\Gamma^h_{\mathcal{S}}(\bullet))$$

to complete the required pairing. (One may readily verify the required associativity of the pairing.) The pairing between local cohomology and homology is defined similarly.

The projection formula in (iv) may be derived in the usual manner making use of the definitions of the inverse image in cohomology, direct image in homology and the pairings defined above between cohomology and homology.

Now we consider (v). Observe that in this case the presheaves $\bar{\mathbf{K}}(\quad)_{\mathcal{S}}$, $\mathbf{K}(\quad)_{\mathfrak{M}}$ and $\mathbf{K}(\quad)_{\mathcal{S}}$ are identical, so that the term $\pi_*(\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}) \overset{L}{\underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathcal{S}_{\mathbb{Q}}})}{\otimes}} \pi_*(\mathbb{H}(\quad, Sp(\Gamma_{\mathcal{S}}(\bullet)))_{\mathbb{Q}}) \simeq \pi_*(\mathbb{H}(\quad, Sp(\Gamma_{\mathcal{S}}(\bullet)))_{\mathbb{Q}})$. Therefore it follows that in this case, the above presheaf reduces to $\pi_*(\mathbb{H}(\quad, Sp(\Gamma_{\mathcal{S}}(\bullet)))_{\mathbb{Q}})$, thereby proving the assertion in (v) for cohomology. The reasoning for homology is similar. This completes the proof of property (v).

The statement in (x) follows from the homotopy property for K-theory for smooth objects (observe that in this case the K-theory identifies with G-theory: see remark (2.13)) and the hypothesis 3.1 (v). (In more detail: the presheaves $p_*(\mathbf{K}(\quad))$ and $K(\quad)_{\mathfrak{M}}$ on $\mathfrak{M}_{et}$ have the homotopy property when a moduli space $\mathfrak{M}$ is assumed to exist and it and the given stack are smooth. Now the presheaf $\pi_*(p_*(\mathbf{K}(\quad))) \overset{L}{\underset{\pi_*(\bar{\mathbf{K}}(\quad)_{\mathfrak{M}})}{\otimes}} \pi_*\mathbb{H}(\quad, Sp(\Gamma(\bullet))$ also inherits the homotopy property. This proves the homotopy property for cohomology and the case of local cohomology is similar.) In the equivariant case, we will assume that the group scheme G acts trivially on the factor $\mathbb{A}^1$.

Now we consider (xi). Let $\bar{\mathcal{E}}$ denote a vector bundle on the coarse moduli-space $\mathcal{M}$ and let $\mathcal{E}$ denote its pull-back to the stack $\mathcal{S}$. Assume that $\mathcal{E}$ and $\bar{\mathcal{E}}$ are of rank $= n$. Let $\phi : P(\mathcal{E}) \rightarrow \mathcal{S}$ and $\bar{\phi} : P(\bar{\mathcal{E}}) \rightarrow \mathfrak{M}$, $p : \mathcal{S} \rightarrow \mathfrak{M}$ and $p_0 : P(\mathcal{E}) \rightarrow P(\bar{\mathcal{E}})$ denote the obvious maps. Now the hypothesis 3.1(vi) shows that there exists a canonical class $c_1(\bar{\mathcal{E}}) \varepsilon H^d_{et}(P(\bar{\mathcal{E}}); Sp\Gamma(1))$ and that the map $\Sigma_{i=0}^n \pi^*(\quad) \cap c_1(\bar{\mathcal{E}})^i$ induces a quasi-isomorphism:

$$R\bar{\phi}_*(Sp\Gamma^h(\bullet)) \simeq \oplus_{i=0}^n Sp(\Gamma^h(\bullet)) \tag{6.0.5}$$

where the derived functors are computed on the appropriate étale sites. One also obtains a quasi-isomorphism $\mathbb{H}_{et}(P(\bar{\mathcal{E}}); Sp\Gamma^h(\bullet)) \simeq \sqcup_{i=0}^n \mathbb{H}_{et}(\mathfrak{M}; Sp\Gamma^h(\bullet))$ and similarly for $\Gamma(\bullet)$ in the place of $\Gamma^h(\bullet)$.

We compute the K-theory of a projective space bundle over a dg-stack in [J-7] section 3. There it is shown that the usual formula holds, i.e. the following result holds.

Proposition 6.1. *(K-theory of projective space bundles over dg-stacks) The maps $\phi^*(\quad) \otimes \mathcal{O}_{\mathbb{P}}(-i) : \mathbf{K}(\mathcal{S}, \mathcal{A}) \rightarrow K(\mathbb{P}(\mathcal{E}), \pi^*(\mathcal{A}))$ induce a weak-equivalence: $\Pi_{i=0}^{r-1}\mathbf{K}(\mathcal{S}, \mathcal{A}) \rightarrow K(\mathbb{P}(\mathcal{E}), \pi^*(\mathcal{A}))$.*

Clearly one also has the weak-equivalence: $\mathbf{K}(P(\bar{\mathcal{E}})) \simeq \Pi_{i=0}^{r-1}\mathbf{K}(\mathfrak{M}).[\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-i)]$.

Moreover the corresponding assertion holds when $\mathfrak{M}$ is replaced by an object $U \varepsilon \mathfrak{M}_{et}$ and when the stack $\mathcal{S}$ is replaced by the pull-back $\mathcal{S} \underset{\mathfrak{M}}{\times} U$. Therefore, one obtains the weak-equivalence:

$$K(P(\mathcal{E}), \phi^*(\mathcal{A})) \simeq K(P(\bar{\mathcal{E}})) \overset{L}{\underset{K(\mathfrak{M})}{\otimes}} K(\mathcal{S}, \mathcal{A})$$

and the quasi-isomorphism of presheaves:

$$\pi_*(p_*\phi_*(\mathbf{K}(\quad, \phi^*(\mathcal{A})))_{P(\mathcal{E})}) = \pi_*(\bar{\phi}_* p_{0*}\mathbf{K}(\quad, \phi^*(\mathcal{A}))_{P(\mathcal{E})}) \simeq \pi_*(\bar{\phi}_*\mathbf{K}(\quad)_{P(\bar{\mathcal{E}})}) \overset{L}{\underset{\pi_*(\mathbf{K}(\quad)_{\mathfrak{M}})}{\otimes}} \pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}})$$

and similarly $R\bar{\phi}_*\mathbb{H}_{et}(\quad, Sp\Gamma^h(\bullet)) \simeq \sqcup_{i=0}^n \mathbb{H}_{et}(\quad, Sp\Gamma^h(\bullet))$. Therefore,

$$\mathcal{RH}om_{\pi_*(\bar{\phi}_*\mathbf{K}(\quad)_{P(\bar{\mathcal{E}})})}(\pi_*(p_*\phi_*\mathbf{K}(\quad, \pi^*(\mathcal{A}))_{P(\mathcal{E})}), \pi_*(R\bar{\phi}_*\mathbb{H}_{et}(\quad; Sp\Gamma^h(\bullet))))$$

$$\simeq \mathcal{RH}om_{\pi_*(\mathbf{K}(\quad)_{\mathfrak{M}})}(\pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}}), \pi_*(R\bar{\phi}_*\mathbb{H}_{et}(\quad); Sp\Gamma^h(\bullet))))$$

$$\simeq \sqcup_{i=0}^n \mathcal{RH}om_{\pi_*(\mathbf{K}(\quad)_{\mathfrak{M}})}(\pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}}), \pi_*(\mathbb{H}_{et}(\quad; Sp\Gamma^h(\bullet))))$$

Now the property (xi) for Bredon cohomology follows immediately from its definition. The corresponding assertion there on homology now follows from the observation in (6.0.4) and the definition of Bredon homology. (Take p' $(p,\ f)$ in (6.0.4) to be p_0 $(p,\ \bar{\phi}$, respectively).)

The rest of the discussion will be devoted to defining the Chern-character map, the Riemann-Roch transformations and in establishing their properties.

We begin by *defining the Chern-character*. In general, when the stack $\mathcal{S}$ is provided with the action of a smooth affine group scheme G, we let $Ch : \pi_*(K(\mathcal{S}, G)) \rightarrow \mathbb{H}_{Br,*}(\mathcal{S}, \Gamma(\bullet))$ be defined by the natural map
(6.0.6)

$$\pi_*(K([\mathcal{S}/G], \mathcal{A})) \overset{id \otimes 1}{\rightarrow} H^0(\Gamma([\mathcal{S}/G], K\Gamma(\bullet))) = H^0(\Gamma([\mathcal{S}/G], \pi_*(\mathbf{K}_{[\mathcal{S}/G]}(\quad, \mathcal{A}))_{\mathbb{Q}} \overset{L}{\underset{\pi_*(\bar{\mathbf{K}}_{[\mathcal{S}/G]}(\quad))_{\mathbb{Q}}}{\otimes}} \pi_*(\mathbb{H}_{iso.et}(\quad, \Gamma(\bullet))_{\mathbb{Q}})$$

One may also define *a local Chern-character* as follows in case a moduli-space exists as a quasi-projective scheme. We assume the situation of Definition 5.13. Let $i : \mathfrak{M} \rightarrow \tilde{\mathfrak{M}}$ denote the G-equivariant closed immersion into a smooth G-scheme. Recall $i_* R i^! K\Gamma(*, G) = i_*(\pi_*(p_*(\mathbf{K}(\quad, \mathcal{A}, G)_{\mathcal{S}}))_{\mathbb{Q}}) \overset{L}{\underset{i_*\pi_*(\mathbf{K}(\quad)_{[\mathfrak{M}/G]})_{\mathbb{Q}}}{\otimes}} \pi_*(\mathbb{H}(\quad, i_* R i^! Sp(\Gamma(\bullet)))_{\mathbb{Q}})$ where $i_*\mathbf{K}(\quad)_{[\mathfrak{M}/G]}$ is the presheaf defined on the site $[\tilde{\mathfrak{M}}/G]_{iso.et}$ by $V \rightarrow \bar{\mathbf{K}}(V \underset{\tilde{\mathfrak{M}}}{\times} \mathfrak{M})$. Therefore, there is a natural augmentation:

$$Ch_{\mathcal{S}|\tilde{\mathfrak{M}}} : \pi_*(\mathbf{K}(\mathcal{S}, \mathcal{A}, G)) \overset{id \otimes 1}{\rightarrow} H^0(\Gamma([\tilde{\mathfrak{M}}/G], i_* R i^! K\Gamma(*, G))) \tag{6.0.7}$$

Replacing $\tilde{\mathfrak{M}}$ everywhere by $\mathfrak{M}$ defines similarly an augmentation

$$Ch : \pi_*(\mathbf{K}(\mathcal{S}, \mathcal{A}, G)) \overset{id \otimes 1}{\rightarrow} H^0(\Gamma([\mathfrak{M}/G], K\Gamma(*, G))) \tag{6.0.8}$$

Definition 6.2. (Chern character and local Chern character) We define the Chern-character to be given by the map in (6.0.6) in general and by (6.0.8) when a coarse moduli space exists satisfying our hypotheses. The local Chern character with respect to a closed immersion $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ of the moduli-space into a smooth scheme is defined to be (6.0.7).

Remark 6.3. To understand these Chern characters, one needs to observe that they define operations on the Bredon-homology groups. In this sense they are *operational Chern classes* (in the same spirit as the Chern classes considered in [Ful]). When viewed as operations on Bredon-homology we see from the properties below that they have all the expected properties.

Next we proceed to define the Riemann-Roch transformations. Our technique is to define these using the techniques developed in the appendix A. This has the advantage that the Riemann-Roch transformation is defined also for singular stacks and also makes intrinsic use of the Riemann-Roch transformation at the level of the moduli spaces. Throughout the rest of this section we will assume that a moduli-space exists as a quasi-projective scheme and that it is G-quasi-projective when provided with the action of a smooth group-scheme G. Observe that any Artin stack with finite diagonal for which the coarse-moduli space exists as a quasi-projective scheme satisfies these hypotheses.

6.1. Let $\mathcal{S}$ denote an algebraic stack with $p : \mathcal{S} \to \mathfrak{M}$ the obvious proper map to its moduli-space. We will fix a closed immersion $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ with the latter *smooth* for the rest of the discussion. Observe that this is possible in view of the hypothesis that the moduli space $\mathfrak{M}$ is quasi-projective. In the equivariant case, we will assume that the moduli-space $\mathfrak{M}$ is G-quasi-projective, so that one can assume the group action G extends to $\tilde{\mathfrak{M}}$ and that the closed immersion i is G-equivariant. Given any presheaf P on $\mathfrak{M}_{et}$ we will consider its extension by zero, i_*P, on $\tilde{\mathfrak{M}}$. We will apply this extension functor to all the presheaves considered above.

Now we begin with the following key identification:

(6.1.1)

$$\mathcal{RH}om_{i_*(\pi_*(K(\quad)_{\mathfrak{M}}))_{\mathbb{Q}}}(i_*\pi_*(p_*G(\quad,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}, R\mathcal{H}om_{i_*(\pi_*(K(\quad)_{\mathfrak{M}}))_{\mathbb{Q}}}(i_*\pi_*(p_*K(\quad,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}, \pi_*(\mathbb{H}(\quad, i_*Sp(\Gamma^h(\bullet))))_{\mathbb{Q}}))$$

$$\simeq \mathcal{RH}om_{i_*(\pi_*(K(\quad)_{\mathfrak{M}}))_{\mathbb{Q}}}(i_*\pi_*(p_*G(\quad,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}} \underset{i_*(\pi_*(K(\quad)_{\mathfrak{M}}))_{\mathbb{Q}}}{\overset{L}{\otimes}} i_*\pi_*(p_*K(\quad,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}, \pi_*(\mathbb{H}(\quad, i_*Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}))$$

Since $i_*\pi_*(p_*G(\quad,\mathcal{A})_{\mathcal{S}})$ and $i_*\pi_*(p_*K(\quad,\mathcal{A})_{\mathcal{S}})$ are modules over $i_*\pi_*(K(\quad)_{\mathfrak{M}})$, the term

$$i_*\pi_*(p_*G(\quad,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}} \underset{i_*(\pi_*(K(\quad)_{\mathfrak{M}}))_{\mathbb{Q}}}{\overset{L}{\otimes}} i_*\pi_*(K(\quad,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$$

is defined as before. The above identification follows readily from the definitions of the terms and (8.0.14) (in the appendix). This enables us to define one step of the Riemann-Roch transformation as the following transformation.

Definition 6.4. We define the map of (complexes of) presheaves on $\tilde{\mathfrak{M}}_{et}$:

$$\tau'_{\mathcal{S},\tilde{\mathfrak{M}}} : i_*\pi_*(p_*G(\quad,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}} \to R\mathcal{H}om_{i_*(\pi_*(K(\quad)_{\mathfrak{M}}))_{\mathbb{Q}}}(i_*\pi_*(p_*\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}, \pi_*(\mathbb{H}(\quad, i_*Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}))$$

$$= i_*(R\mathcal{H}om_{\pi_*(K(\quad)_{\mathfrak{M}})_{\mathbb{Q}}}(\pi_*(p_*\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}, \pi_*(\mathbb{H}(\quad, Sp(\Gamma^h(\bullet))_{\mathbb{Q}}))))$$

as the map corresponding under the adjunction in (6.1.1) to the following map:

$$i_*\pi_*(p_*G(\quad,\mathcal{A})_{\mathcal{S}}) \underset{i_*(\pi_*(K(\quad)_{\mathfrak{M}}))}{\overset{L}{\otimes}} i_*\pi_*(p_*\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}}) \to i_*\pi_*(p_*G(\quad,\mathcal{A})_{\mathcal{S}}) \overset{p_*}{\to} i_*\pi_*((G(\quad)_{\mathfrak{M}})) \overset{\tau_{\mathfrak{M}}}{\to} \pi_*(\mathbb{H}(\quad, i_*Sp(\Gamma^h(\bullet))_{\mathbb{Q}}))$$

where the first map is given by the module structure of $G(\quad,\mathcal{A})_{\mathcal{S}}$ over $K(\quad,\mathcal{A})_{\mathcal{S}}$, the second is the push-forward by p and the third map is the Riemann-Roch transformation defined on $\tilde{\mathfrak{M}}$. (Recall from our hypotheses that p is of finite cohomological dimension.) In fact one may write the map $\tau_{\mathfrak{M}}$ as the composition of the following maps. (Given a presheaf P on $\tilde{\mathfrak{M}}_{et}$, we define $i^!P$ to be the homotopy fiber of the map $P \to j_*(P)$, where j is the complementary open immersion, complimentary to i):

$$i_*(\pi_*G(\quad)_{\mathfrak{M}})_{\mathbb{Q}} \simeq \pi_*(\text{homotopy fiber}(K(\quad)_{\tilde{\mathfrak{M}}})_{\mathbb{Q}} \to K(\quad)_{\tilde{\mathfrak{M}}-\mathfrak{M}})_{\mathbb{Q}}$$

$$\overset{ch}{\to} \pi_*(i_*i^!\mathbb{H}_{et}(\quad, Sp(\Gamma(\bullet)))_{\mathbb{Q}})^{\cap Td_{\tilde{\mathfrak{M}}|\mathfrak{M}} \circ \cap[\tilde{\mathfrak{M}}]} \!\!\to \pi_*(\mathbb{H}_{et}(\quad, i_*Sp(\Gamma^h(\bullet))_{\mathbb{Q}}))$$

where ch denotes the local Chern-Character that now presheafifies on $\tilde{\mathfrak{M}}_{et}$, $Td_{\tilde{\mathfrak{M}}}$ denotes the Todd-class of $\tilde{\mathfrak{M}}$ that localizes on $\tilde{\mathfrak{M}}_{et}$ and $[\tilde{\mathfrak{M}}]$ denotes the fundamental class of $\tilde{\mathfrak{M}}$ which also localizes on $\tilde{\mathfrak{M}}_{et}$.

Remark 6.5. 1. Observe that the map $\tau'_{\mathcal{S},\tilde{\mathfrak{M}}}$ admits the following alternate description. We will denote $i_*(G(\quad)_{\mathfrak{M}})$ as $K(\quad)_{\tilde{\mathfrak{M}}|\mathfrak{M}}$. Now the composition of the maps

$$i_*\pi_*(p_*G(\quad,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}} \overset{L}{\underset{i_*(\pi_*(K(\quad)_{\mathfrak{M}}))_{\mathbb{Q}}}{\otimes}} i_*\pi_*(p_*\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$$

$$\overset{i_*p_*}{\to} i_*\pi_*(p_*G(\quad,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}} \overset{p_*}{\to} i_*\pi_*((G(\quad)_{\mathfrak{M}}))_{\mathbb{Q}} = \pi_*(K(\quad)_{\tilde{\mathfrak{M}}|\mathfrak{M}})_{\mathbb{Q}}$$

defines a map

$$i_*\pi_*(p_*G(\quad,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}} \to R\mathcal{H}om_{i_*(\pi_*(K(\quad)_{\mathfrak{M}}))_{\mathbb{Q}}}(i_*\pi_*(p_*\mathbf{K}(\quad,\mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}, \pi_*(K(\quad)_{\tilde{\mathfrak{M}}|\mathfrak{M}})_{\mathbb{Q}}) \tag{6.1.2}$$

Next we compose this with the Riemann-Roch transformation defined as the composition of the local Chern character map followed by taking cup product with the Todd class of $\tilde{\mathfrak{M}}$ and then cap product with the fundamental class of $\tilde{\mathfrak{M}}$, i.e. we take $R\mathcal{H}om_{i_*(\pi_*(K(\quad)_{\mathfrak{M}}))}(\quad, i_*\tau_{\tilde{\mathfrak{M}}|\mathfrak{M}})$ where $i_*\tau_{\tilde{\mathfrak{M}}|\mathfrak{M}}$ is the map

$$i_*\tau_{\tilde{\mathfrak{M}}|\mathfrak{M}}:\pi_*(K(\quad)_{\tilde{\mathfrak{M}}|\mathfrak{M}})_{\mathbb{Q}} \overset{ch}{\to} i_*\pi_*(i^!\mathbb{H}_{et}(\quad,Sp(\Gamma(\bullet)))_{\mathbb{Q}})^{\cap Td_{\tilde{\mathfrak{M}}|\mathfrak{M}}\circ\cap[\tilde{\mathfrak{M}}]} \to \pi_*(\mathbb{H}_{et}(\quad,i_*Sp(\Gamma^h(\bullet))_{\mathbb{Q}})) \tag{6.1.3}$$

2. In order to verify that the adjunction as in (6.1.1) applies (see Proposition 8.0.14 in the appendix), one needs to use the following observations. First the projection formula implies the map i_*p_* is a map of module-spectra over $i_*K(\quad)_{\mathfrak{M}}$. Therefore it suffices to show the two maps $K_V(\tilde{V})\otimes K(V) \to K_V(\tilde{V}) \simeq G(V) \overset{\tau}{\to} \mathbb{H}_V(\tilde{V}, \underline{Sp}\Gamma(\bullet))$ and $K_V(\tilde{V})\otimes K(V) \overset{\tau\otimes ch}{\to} \mathbb{H}_V(\tilde{V}, Sp\Gamma(\bullet))\otimes\mathbb{H}(V, Sp\Gamma(\bullet)) \to H_V(\tilde{V}, Sp\Gamma(\bullet))$ of presheaves as $\tilde{V}$ varies over the étale site of $\tilde{\mathfrak{M}}$ with $V = \tilde{V}\underset{\tilde{\mathfrak{M}}}{\times}\mathfrak{M}$ are homotopic as maps of presheaves. Since both involve multiplication by the Todd class $Td_{\tilde{\mathfrak{M}}|\mathfrak{M}}$ and the fundamental class of $\tilde{\mathfrak{M}}$, this reduces to the multiplicative property of the Chern character.

Next assume, in addition to the above, that a smooth group scheme G acts on the algebraic stack $\mathcal{S}$ as in 2.17. Under the assumption that $\mathfrak{M}$ is G-quasi-projective, this action extends to an action on $\tilde{\mathfrak{M}}$. We will now replace $\tilde{\mathfrak{M}}_{et}$ by $[\tilde{\mathfrak{M}}/G]_{iso.et}$ and the category $Presh(\tilde{\mathfrak{M}}_{et})$ by $Presh([\tilde{\mathfrak{M}}/G]_{iso.et})$. The map p will be the obvious map of sites : $[\mathcal{S}/G]_{iso.et} \to [\tilde{\mathfrak{M}}/G]_{iso.et}$. With this change, all the basic definitions carry over.

Definition 6.6. Now we will define

$$\tau'_{[\mathcal{S}/G],[\tilde{\mathfrak{M}}/G]}:\pi_*(p_*G(\quad,G)_{\mathcal{S}})_{\mathbb{Q}} \to R\mathcal{H}om_{i_*\pi_*(K(\quad,G)_{\mathfrak{M}})_{\mathbb{Q}}}(i_*\pi_*(p_*\mathbf{K}(\quad,G)_{\mathcal{S}})_{\mathbb{Q}}, \pi_*(\mathbb{H}_{et}(\quad,G,i_*Sp(\Gamma^h(\bullet)))))$$

as the map corresponding under the adjunction in (6.1.1) to the following map of presheaves on $[\tilde{\mathfrak{M}}/G]_{iso.et}$:

$$i_*\pi_*(p_*G(\quad,G)_{\mathcal{S}})_{\mathbb{Q}} \overset{L}{\underset{i_*\pi_*(K(\quad,G)_{\mathfrak{M}})_{\mathbb{Q}}}{\otimes}} i_*\pi_*(p_*\mathbf{K}(\quad,G)_{\mathcal{S}})_{\mathbb{Q}} \to i_*\pi_*(p_*G(\quad,G)_{\mathcal{S}})_{\mathbb{Q}} \overset{p_*}{\to} i_*\pi_*(G(\quad,G)_{\mathfrak{M}})_{\mathbb{Q}}$$

$$\overset{\tau_{\mathfrak{M}}}{\to} \pi_*(\mathbb{H}_{et}(\quad,G,i_*Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})$$

Definition 6.7. (The Riemann-Roch and Lefschetz-Riemann-Roch transformations) Now we pre-compose $\tau'_{\mathcal{S},\tilde{\mathfrak{M}}}$ with the obvious augmentation $\pi_*(G(\mathcal{S})) \to H^0(\mathbb{H}_{et}(\mathfrak{M},\pi_*(p_*G(\quad)_{\mathcal{S}}))_{\mathbb{Q}})$ to define the Riemann-Roch transformation

$$\tau_{\mathcal{S},\tilde{\mathfrak{M}}}:\pi_*(G(\mathcal{S})) \to H^{Br}_*(\mathcal{S},\Gamma^h(\bullet)).$$

Similarly, in the equivariant case, we pre-compose $\tau'_{[\mathcal{S}/G],[\tilde{\mathfrak{M}}/G]}$ with the obvious augmentation $\pi_*(G(\mathcal{S},G)) \to H^0(\mathbb{H}_{iso.et}([\mathfrak{M}/G],\pi_*(p_*G(\quad,G)_{\mathcal{S}}))_{\mathbb{Q}})$ to obtain the Lefschetz-Riemann-Roch transformation

$$\tau_{[\mathcal{S}/G],[\tilde{\mathfrak{M}}/G]}:\pi_*(G(\mathcal{S},G)) \to H^{Br}_*(\mathcal{S},G,\Gamma^h(\bullet))$$

In view of the following proposition, we will denote $\tau_{\mathcal{S},\tilde{\mathfrak{M}}}$ ($\tau_{[\mathcal{S}/G],[\tilde{\mathfrak{M}}/G]}$) simply by $\tau_{\mathcal{S}}$ ($\tau_{[\mathcal{S}/G]}$, respectively).

Observe that the above transformations are defined using the closed immersion of $\mathfrak{M}$ into $\tilde{\mathfrak{M}}$. The following proposition shows that this is independent of such a closed immersion and depends only on $\mathfrak{M}$.

Proposition 6.8. *Assume the situation of 6.1. Let $\bar{i}: \tilde{\mathfrak{M}} \rightarrow \hat{\mathfrak{M}}$ denote a closed immersion into another smooth scheme; in the equivariant case we will assume that the action of G extends to $\hat{\mathfrak{M}}$ making the map $\bar{i}$ equivariant. Now the square*

$$\begin{array}{ccc} \pi_*(G(\mathcal{S}, G)) & \xrightarrow{\tau_{[\mathcal{S}/G],[\tilde{\mathfrak{M}}/G]}} & H^{Br}_*(\mathcal{S}, G, Sp(\Gamma^h(\bullet))) \\ \downarrow{id} & & \downarrow{id} \\ \pi_*(G(\mathcal{S}, G)) & \xrightarrow{\tau_{[\mathcal{S}/G],[\hat{\mathfrak{M}}/G]}} & H^{Br}_*(\mathcal{S}, G, Sp(\Gamma^h(\bullet))) \end{array}$$

commutes.

Proof. For simplicity we will only consider the case where the group G is trivial. From the alternate formulation of the maps $\tau_{[\mathcal{S}/G],[\tilde{\mathfrak{M}}/G]}$ and $\tau_{[\mathcal{S}/G],[\hat{\mathfrak{M}}/G]}$ as in (6.1.2) and (6.1.3) it suffices to check the commutativity of the following diagram of presheaves:

$$\begin{array}{ccc} \mathcal{RH}om_{\bar{i}_*(i_*(\pi_*(K(\quad)_{\mathfrak{M}})))_{\mathbb{Q}}}(X, \bar{i}_*\pi_*(K(\quad)_{\tilde{\mathfrak{M}}|\mathfrak{M}})_{\mathbb{Q}}) & \xrightarrow{\mathcal{RH}om(\quad, \bar{i}_* i_* \tau_{\tilde{\mathfrak{M}}})} & \mathcal{RH}om_{\bar{i}_*(i_*(\pi_*(K(\quad)_{\mathfrak{M}})))_{\mathbb{Q}}}(X, \pi_*(\mathbb{H}(\quad, \bar{i}_*(i_* Sp(\Gamma^h(\bullet))))_{\mathbb{Q}})) \\ \downarrow & & \downarrow{\bar{i}_*} \\ \mathcal{RH}om_{\bar{i}_*(i_*(\pi_*(K(\quad)_{\mathfrak{M}})))_{\mathbb{Q}}}(X, K(\quad)_{\hat{\mathfrak{M}}|\mathfrak{M}_{\mathbb{Q}}}) & \xrightarrow{\mathcal{RH}om(\quad, \bar{i}_* \tau_{\hat{\mathfrak{M}}})} & \mathcal{RH}om_{\bar{i}_*(i_*(\pi_*(K(\quad)_{\mathfrak{M}})))_{\mathbb{Q}}}(X, \pi_*(\mathbb{H}(\quad, \bar{i}_* Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})) \end{array}$$

where $X = \bar{i}_*(i_*\pi_*(p_*\mathbf{K}(\quad, \mathcal{A})_{\mathcal{S}}))_{\mathbb{Q}}$ and $K(\quad)_{\tilde{\mathfrak{M}}|\mathfrak{M}_{\mathbb{Q}}}$ is the presheaf on $\tilde{\mathfrak{M}}_{et}$ defined by $\tilde{U} \rightarrow K_{\mathfrak{M} \underset{\tilde{\mathfrak{M}}}{\times} \tilde{U}}(\tilde{U})_{\mathbb{Q}}$. This identifies with $i_*G(\quad)_{\mathfrak{M}_{\mathbb{Q}}}$. The presheaf $\bar{K}(\quad)_{\hat{\mathfrak{M}}|\mathfrak{M}}$ is defined similarly and identifies with $\bar{i}_* i_* G(\quad)_{\mathfrak{M}}$. The commutativity of the above square follows from the homotopy commutativity of the square of maps:

$$\begin{array}{ccc} \bar{i}_*K(\quad)_{\tilde{\mathfrak{M}}|\mathfrak{M}} & \xrightarrow{\bar{i}_* i_* \tau_{\tilde{\mathfrak{M}}}} & \mathbb{H}(\quad, \bar{i}_*(i_* Sp(\Gamma^h(\bullet))))_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ K(\quad)_{\hat{\mathfrak{M}}|\mathfrak{M}} & \xrightarrow{\bar{i}_* \tau_{\hat{\mathfrak{M}}}} & \mathbb{H}(\quad, \bar{i}_* Sp(\Gamma^h(\bullet)))_{\mathbb{Q}} \end{array}$$

This may be proved by deformation to the normal cone of $\tilde{\mathfrak{M}}$ in $\hat{\mathfrak{M}}$. The key observation to note here is that the deformation to the normal cone is canonical on the étale site of $\hat{\mathfrak{M}}$. Therefore, $ch(\quad) \circ \bar{i}_* = \bar{i}_*(ch(\quad).Td(N)^{-1})$ where N is the normal bundle associated to the closed immersion $\bar{i}$ and where ch denotes the local Chern character. The $\bar{i}_*$ on the right is the Gysin map in local cohomology which is given by cup product with the Koszul-Thom class, T, of the normal bundle N associated to the closed immersion $\bar{i}$. Since the classes $Td(N)$, and T localize on $\tilde{\mathfrak{M}}_{et}$, we have equality of the above maps as maps of presheaves. Finally observe that the Todd classes of both $\tilde{\mathfrak{M}}$ and $\hat{\mathfrak{M}}$ also localize on $\hat{\mathfrak{M}}_{et}$ and $Td_{\tilde{\mathfrak{M}}} = Td(N)^{-1}.Td_{\hat{\mathfrak{M}}|\tilde{\mathfrak{M}}}$ (where the last term is the restriction of$Td_{\hat{\mathfrak{M}}}$ to $\tilde{\mathfrak{M}}$), $[T] \cap [\hat{M}] = [\tilde{M}]$. These prove the proposition. □

Now we proceed to define *fundamental classes.* For simplicity we will restrict to the situation where the integer d as in definition (3.1)(viii) in section 3.1 is 2. Observe from the Proposition (6.10) below that $H^{Br}_*(\mathcal{S}, \Gamma(m)) = \pi_*(\mathbb{H}^{Br}(\mathcal{S}, \Gamma(m))) = 0$ for all $m > n$ if $\mathcal{S}$ is an algebraic stack of dimension n for which a coarse moduli space exists. Therefore, in general we define the fundamental class to be *the nonzero term in $\tau_{\mathcal{S}}(\mathcal{A})$ of the highest weight* k and degree $=2k$, where $\mathcal{A}$ is the given dg-structure sheaf of the dg-stack $\mathcal{S}$. (More detailed definition when the dg-structure sheaf $\mathcal{A}$ is obtained from a perfect obstruction theory is considered in [J-8]. There it is shown that the integer k coincides with the *virtual dimension* of the dg-stack $(\mathcal{S}, \mathcal{A})$.)

Next we will consider the case when the dg-structure sheaf $\mathcal{A}$ is just the usual structure sheaf $\mathcal{O}_{\mathcal{S}}$. Now it suffices to show that $H^{Br}_*(\mathcal{S}, \Gamma(n)) \neq 0$ where n is the dimension of the stack $\mathcal{S}$. Assuming this we let

(6.1.4) $$[\mathcal{S}] = \text{the term of degree } 2n \text{ and weight } n \text{ in } \tau_{\mathcal{S}}(\mathcal{O}_{\mathcal{S}})$$

Next we show that $H^{Br}_*(\mathcal{S}, \Gamma(n)) \neq 0$ under the assumption that the map $p: \mathcal{S} \rightarrow \mathfrak{M}$ is *finite.* First observe that $\mathbb{H}^{-2n}_{et}(\mathfrak{M}, \Gamma^h(n))$ has a *fundamental class*, $[\mathfrak{M}]$ by our hypothesis. Moreover, by the relationship between the Riemann-Roch transformation for the moduli space $\mathfrak{M}$ and its fundamental class $\tau_{\mathfrak{M}}(\mathcal{O}_{\mathfrak{M}}) = [\mathfrak{M}]$ + lower dimensional terms. (See [Ful] Theorem 18.3, (5).) From the definition of the Riemann-Roch transformation for the stack $(\mathcal{S}, \mathcal{O})$ (see 6.4 above), observe that $\tau_{\mathcal{S}}(\mathcal{O}_{\mathcal{S}})$ identifies with the morphism: $\mathcal{E} \mapsto \tau_{\mathfrak{M}}(p_*(\mathcal{E}))$, $\mathcal{E} \epsilon \pi_k(G(\mathcal{S}))$, $k \geq 0$ The map p is finite by assumption, and it induces a map $p_* : H_*^{Br}(\mathcal{S}, \Gamma(\bullet)) \rightarrow H_*^{Br}(\mathfrak{M}, \Gamma(\bullet)) = \mathbb{H}_{et}^*(\mathfrak{M}; \Gamma^h(\bullet))$. One may verify that $p_*(\tau_{\mathcal{S}}(\mathcal{O}_{\mathcal{S}}))$ identifies with the map $\mathcal{E} \mapsto \tau_{\mathfrak{M}}(p_* p^*(\mathcal{E})) = ch(\mathcal{E}).\tau_{\mathfrak{M}}(p_* p^*(\mathcal{O}_{\mathfrak{M}}))$, $\mathcal{E} \epsilon \pi_k(K(\mathfrak{M}))$, $k \geq 0$. Therefore, $p_*(\tau_{\mathcal{S}}(\mathcal{O}_{\mathcal{S}})_{2n}(n)) \epsilon \mathbb{H}_{et}^*(\mathfrak{M}; \Gamma^h(\bullet))$ is nothing other than a multiple of $\tau_{\mathfrak{M}}(\mathcal{O}_{\mathfrak{M}})_{2n}(n)$ by the degree of the projection map $p : \mathcal{S} \rightarrow \mathfrak{M}$. (First one shows this generically on the moduli space $\mathfrak{M}$ and then one uses cohomological semi-purity (see hypothesis (viii) in section 3) to conclude in general.) This shows $H_{2n}^{Br}(\mathcal{S}, \Gamma(n)) \neq 0$.

Remark 6.9. An entirely similar argument may be used to define a fundamental class in the equivariant situation, with values in the equivariant homology defined earlier. We skip the details.

The compatibility of the Chern-character and the Riemann-Roch transformation follows readily in view of the pairing established between Bredon cohomology and Bredon homology.

The map PL from local Bredon cohomology to Bredon homology is obtained by using the pairing established in (iii) and taking the product with the fundamental class in homology defined above. This completes the proof of theorem 1.1.

Proposition 6.10. *Let $\mathcal{S}$ denote an algebraic stack of dimension n for which a coarse moduli space exists. Then $H_*^{Br}(\mathcal{S}; \Gamma(m)) = 0$ for $m > n$.*

Proof. Recall $H_*^{Br}(\mathcal{S}; \Gamma_{\mathfrak{M}}^h(m)) = Gr_m H^0(\Gamma(\mathfrak{M}; \mathcal{RH}om_{\pi_*(\mathbf{K}(\ \ )_{\mathfrak{M},\mathbb{Q}})}(\pi_*(p_*\mathbf{K}(\ \ , \mathcal{A})_{\mathcal{S},\mathbb{Q}}), \pi_*(\mathbb{H}_{et}(\ \ ; Sp(\Gamma_{\mathfrak{M}}^h(\bullet)))_{\mathbb{Q}}))))$, where one decomposes $\pi_*(\mathbf{K}(\ \ )_{\mathfrak{M}})_{\mathbb{Q}}$ using Adams operations and $\pi_*(\mathbf{K}(\ \ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$ is decomposed correspondingly as in (5.0.12). Finally one takes the pieces of total weight m, coming from the graded terms of weight k in $\pi_*(\mathbf{K}(\ \ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$ and weight $m+k$ in $\Gamma^h(\bullet)$. By our hypothesis, the moduli space of $\mathcal{S}$ has dimension n and therefore, $\Gamma_{\mathfrak{M}}^h(m) = 0$ for for all $m > n$: see Definition 3.1(i). Since there are no terms of *negative* weight in $\pi_*(\mathbf{K}(\ \ , \mathcal{A})_{\mathcal{S}})_{\mathbb{Q}}$, it follows that the highest weighted terms appearing in $H_{Br}^*(\mathcal{S}; \Gamma^h(\bullet))$ are with weight $m = n$. □

Proof of Theorem (1.2).

(i) The map from Bredon homology to homology computed on the smooth site may be obtained as follows. Sending a vector bundle that is locally trivial on the isovariant étale site of an algebraic stack $\mathcal{S}$ to the same vector bundle, but now viewed as a vector bundle on the stack and then pulled back to a perfect complex of $\mathcal{A}_{\mathcal{S}}$-modules (i.e. tensored with $\mathcal{A}_{\mathcal{S}}$), defines a natural map of presheaves of spectra $\bar{\mathbf{K}}(\ \ ) \rightarrow \mathbf{K}(\ \ , \mathcal{A})_{\mathcal{S}}$. Moreover the natural map of sites, $\epsilon : \mathcal{S}_{smt} \rightarrow \mathcal{S}_{iso.et}$ defines a map of presheaves, $\mathbb{H}_{iso.et}(\ \ , \Gamma^h(\bullet)) \rightarrow \mathbb{H}_{smt}(\ \ , \epsilon^*(\Gamma^h(\bullet)))$. Therefore, one obtains a map $\phi_* : H_{Br}^*(\mathcal{S}, \Gamma(\bullet)) \rightarrow \mathbb{H}_{smt}^*(\mathcal{S}, \epsilon^*(\Gamma^h(\bullet)))$, sending a map $\pi_*(\mathbf{K}(\ \ , \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}}) \rightarrow \pi_*(\mathbb{H}_{iso.et}(\ \ , \Gamma^h(\bullet))_{\mathbb{Q}})$ of presheaves of $\pi_*(\bar{\mathbf{K}}(\ \ )_{\mathbb{Q}})$-modules to the map obtained by pre-composing with the map $\pi_*(\bar{\mathbf{K}}(\ \ )_{\mathbb{Q}}) \rightarrow \pi_*(\mathbf{K}(\ \ , \mathcal{A})_{\mathcal{S}_{\mathbb{Q}}})$ and composing at the end with the map $\mathbb{H}_{iso.et}(\ \ , \Gamma^h(\bullet))_{\mathbb{Q}} \rightarrow \mathbb{H}_{smt}(\ \ , \epsilon^*(\Gamma^h(\bullet)))_{\mathbb{Q}}$. When a coarse-moduli space exists one may replace $\bar{\mathbf{K}}(\ \ )_{\mathcal{S}}$ $(\mathbb{H}_{iso.et}(\ \ , \Gamma(\bullet)))$ with $\mathbf{K}(\ \ )_{\mathfrak{M}}$ $(\mathbb{H}_{et}(\ \ , \Gamma^h(\bullet))$ computed on the étale site of the coarse moduli space, respectively).

We will next consider the compatibility of this map with proper push-forward for representable proper maps in the case of l-adic cohomology when l is different from the residue characteristics and when moduli spaces are assumed to exist. Let $\mathcal{S}$ denote an algebraic stack with coarse moduli space $\mathfrak{M}$ and let $\bar{p} : \mathfrak{M}' \rightarrow \mathfrak{M}$ denote a *proper* map of algebraic spaces. Let $\mathcal{S}' = \mathfrak{M}' \underset{\mathfrak{M}}{\times} \mathcal{S}$ and let $p : \mathcal{S}' \rightarrow \mathcal{S}$ denote the induced proper map of algebraic stacks. In this case we replace $\mathbb{H}_{iso.et}(\ \ , \Gamma^h(\bullet))$ by $\mathbb{H}_{et}(\ \ , \Gamma^h(\bullet))$ (computed on the étale site of the coarse-moduli spaces). In view of the definition of the direct-image maps in Bredon homology, the compatibility of the map $\mathbb{H}_{et}(\ \ , \Gamma^h(\bullet)) \rightarrow \mathbb{H}_{smt}(\ \ , \epsilon^*(\Gamma^h(\bullet)))$ (with the first computed on the moduli space and the second on the stack) with pushforward by p_* follows readily by proper-base change in étale cohomology. In general, if $p_1 : \mathcal{S}_1 \rightarrow \mathcal{S}$ is a proper representable map of algebraic stacks and $\mathfrak{M}_1 = \mathfrak{M}'$ is the coarse-moduli space of $\mathcal{S}_1$, one may factor the map p_1 through p defined above and complete the proof. This proves the statements in (i) Theorem 1.2.

(ii) The map from Bredon cohomology to smooth cohomology (denoted ψ^*) may be obtained as follows: recall $H_{Br}^s(\mathcal{S}, \Gamma(t)) = Gr_{s,t}(\pi_*(K(\mathcal{S}))_{\mathbb{Q}} \overset{L}{\underset{\pi_* K(\mathfrak{M})_{\mathbb{Q}}}{\otimes}} \mathbb{H}_{et}^*(\mathfrak{M}, \Gamma(\bullet))_{\mathbb{Q}})$. Clearly this maps to $Gr_{s,t}(\pi_*(K(\mathcal{S}))_{\mathbb{Q}} \overset{L}{\underset{\pi_* K(\mathcal{S})_{\mathbb{Q}}}{\otimes}} \mathbb{H}_{smt}^*(\mathcal{S}, \epsilon^*\Gamma(\bullet))_{\mathbb{Q}})$ $\cong \mathbb{H}_{smt}^*(\mathcal{S}, \epsilon^*\Gamma(\bullet))_{\mathbb{Q}}$. This defines ψ^*. The compatibility of the maps ϕ_* and ψ^* with respect to the pairings between cohomology and homology reduces to the commutativity of the square:

$$\begin{array}{ccc} H^*_{Br}(\mathcal{S},\Gamma(\bullet)) \otimes H^{Br}_*(\mathcal{S},\Gamma(\bullet)) & \longrightarrow & H^{Br}_*(\mathcal{S},\Gamma(\bullet)) \\ \Big\downarrow{\scriptstyle \psi^*\otimes\phi_*} & & \Big\downarrow{\scriptstyle \phi_*} \\ H^*_{smt}(\mathcal{S},\Gamma(\bullet)) \otimes H^{smt}_*(\mathcal{S},\Gamma(\bullet)) & \longrightarrow & H^{smt}_*(\mathcal{S},\Gamma(\bullet)) \end{array}$$

To see this let $\mathcal{E}\varepsilon\pi_*(K(\mathcal{S}))_{\mathbb{Q}}$, $\alpha\varepsilon H^*_{iso.et}(\mathcal{S},\Gamma(\bullet))$ and $f\varepsilon Hom_{\pi_*(\bar{K}(\quad)_{\mathcal{S}})_{\mathbb{Q}}}(\pi_*(K(\quad)_{\mathcal{S}})_{\mathbb{Q}}, H^*_{iso.et}(\quad,\Gamma^h(\bullet))_{\mathcal{S}\mathbb{Q}}$. Then using the identification of $H^{smt}_*(\mathcal{S},\Gamma(\bullet))$ with $Hom_{\pi_*(\bar{K}(\quad)_{\mathcal{S}})_{\mathbb{Q}}}(\pi_*(\bar{K}(\quad)_{\mathcal{S}})_{\mathbb{Q}}, \mathbb{H}^*_{smt}(\quad,\epsilon^*(\Gamma^h(\bullet))_{\mathbb{Q}})$, composition of the maps in the top row and the right column will define the map $\mathcal{F}\mapsto\phi_*(f)(\epsilon^*(\mathcal{F})\circ\mathcal{E})\circ\epsilon^*(\alpha)$ where $\mathcal{F}\varepsilon\pi_*(\bar{K}(\quad)_{\mathcal{S}})_{\mathbb{Q}}$ and $\circ$ denotes the obvious pairings. The map in the left column sends $\mathcal{E}\otimes\alpha\otimes f$ to $Ch(\mathcal{E})\otimes\epsilon^*(\alpha)\otimes\phi_*(f)\varepsilon H^*_{smt}(\mathcal{S},\Gamma(\bullet))\otimes H^{smt}_*(\mathcal{S},\Gamma(\bullet))$. Once again using the identification of $H^{smt}_*(\mathcal{S},\Gamma(\bullet))$ with $Hom_{\pi_*(\bar{K}(\quad)_{\mathcal{S}})_{\mathbb{Q}}}(\pi_*(\bar{K}(\quad)_{\mathcal{S}})_{\mathbb{Q}}, \mathbb{H}$ the composition of the maps in the left column and the bottom row is seen to be the same map as given by the composition of the top row and the right column. (Observe that $\phi_*(f)(\epsilon^*(\mathcal{F})\circ\mathcal{E}) = \phi_*(f)(\mathcal{F})\circ Ch(\mathcal{E})$.)

These prove all but the last statement in (ii). To see this, observe that the module structure of $H^*_{smt}(\mathcal{S},\epsilon^*(\Gamma(\bullet)))_{\mathbb{Q}}$ over $\pi_*(K(\mathcal{S}))_{\mathbb{Q}}$ is given by the Chern character. Clearly this Chern character is compatible with the Chern character on the moduli space under pull-back by the map ϵ^* where $\epsilon:\mathcal{S}\to\mathfrak{M}$ will denote the obvious map. This proves the last statement in (ii).

We define the required finer variant of Bredon cohomology and homology in definition (5.12). Assume the hypotheses of Theorem 1.2. Clearly there is an obvious morphism $\pi_*(K(\quad,G)_{\mathcal{S}})_{\mathbb{Q}}\underset{\mathbb{Q}}{\otimes}\mathbb{Q}(\mu_\infty)\to\pi_*(K(\quad,G)_{I_{\mathcal{S}}})_{\mathbb{Q}}\underset{\mathbb{Q}}{\otimes}\mathbb{Q}(\mu_\infty)\to\pi_*(K_{et}(\quad,G)_{I_{\mathcal{S}}})_{\mathbb{Q}}\underset{\mathbb{Q}}{\otimes}\mathbb{Q}(\mu_\infty)$. Since the action of the inertia stack on vector bundles on $\mathcal{S}$ is diagonalizable, one may break up the last term into a sum of terms indexed by the characters of the inertia stack. This way one obtains a map from the last term into $\pi_*(K_{et}(\quad,G)_{I_{\mathcal{S}}})_{\mathbb{Q}}\underset{\mathbb{Q}}{\otimes}\mathbb{Q}(\mu_\infty)$. That this composite map is an isomorphism was shown in [To] and [Vi-1]: strictly speaking they only consider the case where the group G is trivial. That the isomorphism in [To] and [Vi-1] extend to this case was observed in [J-5]. Therefore, the second statement in Theorem 1.2 is an immediate consequence of the following isomorphisms:

$$R\Gamma([\mathfrak{M}/G],\pi_*(p^G_*p_{0*}((K_{et}(\quad,G)_{I_{\mathcal{S}}\mathbb{Q}})))\underset{\pi_*(\bar{\mathbf{K}}(\quad)_{[\mathfrak{M}/G]_{\mathbb{Q}}})}{\overset{L}{\otimes}}\pi_*(\mathbb{H}(\quad,Sp(\Gamma(\bullet)))_{\mathbb{Q}}))\underset{\mathbb{Q}}{\otimes}\mathbb{Q}(\mu_\infty)$$

$$\simeq R\Gamma([\mathfrak{M}/G],p^G_*p_{0*}(\pi_*(K_{et}(\quad,G)_{I_{\mathcal{S}}})_{\mathbb{Q}})\underset{\mathbb{Q}}{\otimes}\mathbb{Q}(\mu_\infty)\underset{\pi_*(\bar{\mathbf{K}}(\quad)_{[\mathfrak{M}/G]_{\mathbb{Q}}})\underset{\mathbb{Q}}{\otimes}\mathbb{Q}(\mu_\infty)}{\overset{L}{\otimes}}\pi_*(\mathbb{H}(\quad,Sp(\Gamma(\bullet)))_{\mathbb{Q}})\underset{\mathbb{Q}}{\otimes}\mathbb{Q}(\mu_\infty))\text{ and}$$

$$R\Gamma([\mathfrak{M}/G],\mathcal{RH}om_{\pi_*(\bar{\mathbf{K}}(\quad)_{[\mathfrak{M}/G]_{\mathbb{Q}}})}(\pi_*(p^G_*p_{0*}(K_{et}(\quad,G)_{I_{\mathcal{S}}\mathbb{Q}})),\pi_*(\mathbb{H}(\quad,G,Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}))\underset{\mathbb{Q}}{\otimes})\mathbb{Q}(\mu_\infty)$$

$$\simeq R\Gamma([\mathfrak{M}/G],\mathcal{RH}om_{\pi_*(\bar{\mathbf{K}}(\quad)_{[\mathfrak{M}/G]_{\mathbb{Q}}})}(\pi_*(p^G_*p_{0*}(K_{et}(\quad,G)_{I_{\mathcal{S}}\mathbb{Q}})),\pi_*(\mathbb{H}(\quad,G,Sp(\Gamma^h(\bullet)))_{\mathbb{Q}}))\underset{\mathbb{Q}}{\otimes}\mathbb{Q}(\mu_\infty))$$

$$\simeq R\Gamma([\mathfrak{M}/G],\mathcal{RH}om_{\pi_*(\bar{\mathbf{K}}(\quad)_{[\mathfrak{M}/G]})_{\mathbb{Q}}\underset{\mathbb{Q}}{\otimes}\mathbb{Q}(\mu_\infty)}(\pi_*(p^G_*p_{0*}(K_{et}(\quad,G)_{I_{\mathcal{S}}})_{\mathbb{Q}})\underset{\mathbb{Q}}{\otimes}\mathbb{Q}(\mu_\infty),\pi_*(\mathbb{H}(\quad,G,Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})\underset{\mathbb{Q}}{\otimes}\mathbb{Q}(\mu_\infty)))$$

(The first $\simeq$ is a consequence of the fact that $\mathbb{Q}(\mu_\infty)$ is flat over $\mathbb{Q}$.) This concludes the proof of Theorem (1.2).

Remark 6.11. It may be worthwhile observing that the composite map from $\pi_*(K(\mathcal{S}))$ to $H^{smt}_*(\mathcal{S},\Gamma(\bullet))_{\mathbb{Q}}$ is different from the more familiar one which factors through $\pi_*(K(B_x\mathcal{S}))$. The latter is defined by a Riemann-Roch transformation defined in $\pi_*K(B_x\mathcal{S})$.

7. Applications to virtual fundamental classes

7.0.5. **Proof of Theorem 1.4.** First observe in view of our hypotheses that if $\mathcal{S}$ is a Deligne-Mumford stack of finite type, for each fixed weight r, $H^i_{smt}(\mathcal{S},\Gamma(r))\otimes\mathbb{Q}=0$ for all but finitely many i. Therefore, the Chern classes for any perfect complex on the stack $\mathcal{S}$ with values in this cohomology that lie in positive degrees are all nilpotent; now the (usual) formula for the Todd class of any perfect complex with values in this smooth cohomology shows the Todd class of any perfect complex is invertible. Next observe that if we let $\mathcal{F}=\mathcal{O}^{virt}_{\mathcal{S}}$, then its Chern character $ch^{Br}(\mathcal{O}^{virt}_{\mathcal{S}})=1$ in $H^*_{Br}(\mathcal{S},\Gamma(\bullet))\otimes\mathbb{Q}$: see the remark below. Therefore, $\tau^{smt}(\mathcal{O}^{virt}_{\mathcal{S}})=\phi_*([\mathcal{S}]^{virt}_{Br})\cap Td(T\mathcal{S}^{virt})=[\mathcal{S}]^{virt}\cap Td(T\mathcal{S}^{virt})$. Since the Todd-class, $Td(T\mathcal{S}^{virt})$ is invertible, one multiplies by its inverse to obtain the required identification. This completes the proof of Kontsevich's formula.

Remark 7.1. *It is worthwhile pointing out that, in the proof of (7.0.5), it is important to consider the K-theory of the dg-stack, and not the stack with its usual structure sheaf* $\mathcal{O}_{\mathcal{S}}$. *It is only because we used the K-theory of the dg-stack both for the source of the Todd-homomorphism* τ^{smt} *and also in the definition of Bredon-style homology that we are able to obtain* $ch(\mathcal{O}_{\mathcal{S}}^{virt}) = 1$ *in* $H^*_{Br}(\mathcal{S}, \Gamma(\bullet))$. We called the formula for the virtual fundamental class in Theorem 1.4 Kontsevich's conjectural formula, since Kontsevich stated in [Kont], a formula like this could be used to define the virtual fundamental classes. However, the definition of the virtual fundamental classes by others, (for example, Behrend and Fantechi) used a different approach, namely using Gysin maps and cycle maps as in Fulton's intersection theory. The above result shows the virtual fundamental class defined using the Bredon homology as in Theorem 1.1(viii) agrees with the conjectured formula of Kontsevich. We show, in [J-8], Theorem 2.5, as a consequence of the Riemann-Roch theorem that the virtual fundamental class defined using Bredon homology as in Theorem 1.1(viii) agrees with the virtual fundamental class defined cycle theoretically by others.

8. **Appendix: A**

8.0.6. Throughout this section $\mathcal{S}$ will denote a site satisfying the following hypotheses.

8.0.7. In the language of [SGA]4 Exposé IV, there exists a conservative family of points on $\mathcal{S}$. Recall this means the following. Let $(sets)$ denote the category of sets. Now there exists a set $\bar{\mathcal{S}}$ with a map $p : (sets)^{\bar{\mathcal{S}}} \to \mathcal{S}$ so that the map $F \to p_* \circ U \circ a \circ p^*(F)$ is injective for all Abelian sheaves F on $\mathcal{S}$. (Equivalently, if $i_{\bar{s}} : (sets) \to \mathfrak{S}$ denotes the map of sites corresponding to a point $\bar{s}$ of $\bar{\mathcal{S}}$, an Abelian sheaf F on $\mathcal{S}$ is trivial if and only if $i^*_{\bar{s}}F = 0$ for all $\bar{s}\varepsilon\bar{\mathcal{S}}$.) Here $(sets)^{\bar{\mathcal{S}}}$ denotes the product of the category $(sets)$ indexed by $\bar{\mathcal{S}}$. a is the functor sending a presheaf to the associated sheaf and U is the forgetful functor sending a sheaf to the underlying presheaf. We will also assume that the corresponding functor $p^{-1} : \mathcal{S} \to (sets)^{\bar{\mathcal{S}}}$ commutes with fibered products. Given a presheaf $P\varepsilon Mod(\underline{S})$, we let $G^\bullet P : P...GP...G^2P...G^nP...$ denote the obvious cosimplicial object in $Mod(\underline{S})$, where $G = p_* \circ U \circ a \circ p^*$. Now we let

$$\mathcal{G}P = \underset{\Delta}{\text{holim}}\{G^nP|n\} \tag{8.0.8}$$

where $\underset{\Delta}{\text{holim}}\{G^nP|n\}$ denotes the homotopy inverse limit: see [J-2] section 6.

We will further assume that $\mathcal{S}$ is *essentially small* and for every object U in $\mathcal{S}$ the category of coverings of U in $\mathcal{S}$ is also *essentially small.*

8.0.9. If X is an object in the site $\mathcal{S}$, we will let $\mathcal{S}/X$ denote the category whose objects are maps $u : U \to X$ in $\mathcal{S}$ and where a morphism $\alpha : u \to v$ (with $v : V \to X$ in $\mathcal{S}$) is a commutative triangle

$$\begin{array}{ccc} U & \longrightarrow & V \\ & {\scriptstyle u}\searrow \quad \swarrow{\scriptstyle v} & \\ & X & \end{array}$$

We will further assume that the site $\mathcal{S}$ has a terminal object which will be denoted X (i.e. $\mathcal{S}/X = \mathcal{S}$) and that the category $\mathcal{S}$ is closed under finite inverse limits.

8.0.10. $Prsh(\mathcal{S})$ will denote the category of presheaves of abelian groups. An algebra in $Prsh(\mathcal{S})$ will mean an object which has the additional structure of a presheaf of bi-graded (commutative) algebras. Given such an algebra $\mathcal{A}$ in $Prsh(\mathcal{S})$, $Mod(\mathcal{S}, \mathcal{A})$ will denote the sub-category of presheaves that are presheaves of modules over $\mathcal{A}$. Observe that $Prsh(\mathcal{S})$ has a tensor structure defined by the tensor product of two presheaves. It also has an internal Hom which we denote by $\mathcal{H}om$. Given an algebra $\mathcal{A}$ in $Prsh(\mathcal{S})$, $M\varepsilon Mod(\mathcal{S}, \mathcal{A})$ and $N\varepsilon Mod(\mathcal{S}, \mathcal{A})$, $M\underset{\mathcal{A}}{\otimes}N$ is defined as the co-equalizer:

$$Coeq(\ M \otimes \mathcal{A} \otimes N \underset{n}{\overset{m}{\rightrightarrows}} M \otimes N\ ) \tag{8.0.11}$$

where $m : M \otimes \mathcal{A} \otimes N \to M \otimes N$ ($n : M \otimes \mathcal{A} \otimes N \to M \otimes N$) is the map $m = \lambda_M \otimes id_N$, with $\lambda_M : M \otimes \mathcal{A} \to M$ the module structure on M ($n = id_M \otimes \lambda_N$, with $\lambda_N : \mathcal{A} \otimes N \to N$ the module structure on N, respectively). Let

$\mathcal{H}om$ denote the internal hom in the category $Prsh$: this exists as a right adjoint to $\otimes$ since the category $Prsh$ has a small generating set. If $M \varepsilon Mod(\mathcal{S}, \mathcal{A})$ and $N \varepsilon Mod(\mathcal{S}, \mathcal{A})$, we also define:

$$\mathcal{H}om_{\mathcal{A},l}(M, N) = Equalizer(\ \mathcal{H}om(\mathcal{A}, N) \underset{n_*}{\overset{m*}{\rightrightarrows}} \mathcal{H}om(\mathcal{A} \otimes M, N)\ ) \tag{8.0.11}$$

where $m^* = \mathcal{H}om(\lambda_M, N)$ and $n_* = \mathcal{H}om(\mathcal{A} \otimes M, \lambda_N)$.

In case M and N in $Mod(\mathcal{S}, \mathcal{A})$ are also bi-graded, so that the module structures are compatible with the grading (i.e. $\mathcal{A}_{i,j} \otimes M_{i',j'}$ maps to $M_{i+i',j+j'}$ and similarly for N), one may observe readily that $M \underset{\mathcal{A}}{\otimes} N$ has an induced bi-grading.

8.0.12. One may filter M (N) by $F_k M = \underset{i \le k}{\oplus} M(i)$ and $F_k N = \underset{i \le k}{\oplus} N(i)$ so that the above definitions apply to define a filtration on $\mathcal{H}om_{\mathcal{A}}(M, N)$. However, in this case one may define an induced filtration $\{F_k \mathcal{H}om_{\mathcal{A}}(M, N)|k\}$ by $F_k \mathcal{H}om_{\mathcal{A}}(M, N) = \{f : M \to N | f(F_i M) \subseteq F_{i+k} N\}$. By projecting onto the summands in N, one may see readily that the natural maps $F_k \mathcal{H}om_{\mathcal{A}}(M, N) \to F_{k+1} \mathcal{H}om_{\mathcal{A}}(M, N)$ and $F_k \mathcal{H}om_{\mathcal{A}}(M, N) \to \mathcal{H}om_{\mathcal{A}}(M, N)$ are split monomorphisms.

The category $Mod(\mathcal{S}, \mathcal{A})$ has enough injectives which enable us to define $\mathcal{RH}om_{\mathcal{A}}(M, N)$ if M, $N \varepsilon Mod(\mathcal{S}, \mathcal{A})$. Then the above conclusions on $\mathcal{H}om_{\mathcal{A}}(M, N)$ extend to $\mathcal{RH}om_{\mathcal{A}}(M, N)$.

One may also easily define functorial *flat* resolutions of any $M \varepsilon Mod(\mathcal{S}, \mathcal{A})$ making use of the hypothesis that our site $\mathcal{S}$ is essentially small. Moreover, if M is a bi-graded object one may find a resolution by presheaves of bigraded flat modules over $\mathcal{A}$. This shows one may define $M \overset{L}{\underset{\mathcal{A}}{\otimes}} N$ in the obvious manner and that it gets an induced bi-grading if M and N are presheaves of bigraded $\mathcal{A}$-modules.

8.0.13. In case $\mathcal{A}$ and $\mathcal{B}$ are algebras in $Prsh$ and $M \varepsilon Mod(\mathcal{S}, \mathcal{A})$, $N \varepsilon Mod(\mathcal{S}, \mathcal{B})$ and P is a presheaf of $(\mathcal{A}, \mathcal{B})$bi-modules, then one obtains the usual adjunction:

$$\mathcal{RH}om_{\mathcal{A}}(M, \mathcal{RH}om_{\mathcal{B}}(P, N)) \cong \mathcal{RH}om_{\mathcal{B}}(M \overset{L}{\underset{\mathcal{A}}{\otimes}} P, N) \tag{8.0.14}$$

8.1. **From co-chain complexes to symmetric spectra.** We begin by recalling the functor

$$Sp : (abelian \quad groups) \to (symmetric \quad spectra) \tag{8.1.1}$$

from [HSS] Example 1.2.5. Let S^1 denote the simplicial 1-sphere which is obtained by identifying the boundary of $\Delta[1]$ to a point. We let $S^n = \overset{n}{\wedge} S^1$ = the n-sphere. If A is an abelian group we let $Sp'(A) = \{Sp'(A)_n | n\}$ denote the spectrum defined by $Sp'(A)_n = A \otimes (S^n)$ = the simplicial group given in degree k by the the sum of A indexed by the non-degenerate k-simplices of S^n_k and with the base-point identified to the zero element. The symmetric group Σ_n acts on $Sp'(A)_n$ in the obvious way by permuting the n-factors of S^1. If $A^\bullet$ is a co-chain complex (trivial in negative degrees), we let $DN(A^\bullet)$ denote the cosimplicial abelian group obtained in the usual manner. Now we apply the functor Sp' to $DN(A^\bullet)$ to obtain a cosimplicial object of symmetric spectra. The $\underset{\Delta}{\text{holim}}$ of the resulting object will define a symmetric spectrum we denote by $Sp'(A^\bullet)$. So defined Sp' now extends to a functor

$$Sp' : (co-chain \quad complexes \quad trivial \quad in \quad neg. \quad degrees) \to (symmetric \quad spectra) \tag{8.1.2}$$

This functor sends short-exact sequences of co-chain complexes to fibration sequences and quasi-isomorphisms to weak-equivalences.

Next assume that $A^\bullet$ is a co-chain complex that is trivial in degrees lower than N. Now we let $Sp'_N(A^\bullet) = (S^N) \wedge Sp'(A^\bullet[-N])$. One may verify that this extends the functor Sp' to all co-chain complexes that are trivial in degrees lower than N and having similar properties. Finally we skip the verification that there exists a natural weak-equivalence $Sp'_N(A^\bullet[-N]) \to S^k \wedge Sp'(A^\bullet[-N-k]) = Sp'_{N+k}(A^\bullet[-N-k])$. We let $Sp(A^\bullet) = \lim_{N \to \infty} Sp'_N(A^\bullet[-N])$. It follows in straightforward manner that this defines a functor

$$Sp : (co-chain \quad complexes) \to (symmetric \quad spectra) \tag{8.1.3}$$

and that this functor sends short exact sequences (quasi-isomorphisms) of complexes to fibration sequences (weak-equivalences) of spectra.

Lemma 8.1. *(i) If A is an abelian sheaf and $A[-k]$ denotes the co-chain complex of abelian sheaves concentrated in degree k, $\pi_i(Sp(A[-k])) = 0$ unless $i = -k$ and $\pi_{-k}(Sp(A[-k])) \cong A$.*

(ii) If $K^\bullet$ is a complex of abelian sheaves bounded below, then there exists an integer $N >> 0$ so that $\pi_i(Sp(K^\bullet)) = 0$ if $i > N$.

Proof. Assume the situation in (ii). Now there exists a spectral sequence:

$$E_2^{s,t} = H^s(\{\pi_t(Sp(K^k))|k\}) = H^s(\{\pi_t(Sp(K^k[0]))|k\}) \Rightarrow \pi_{-s+t}(Sp(K^\bullet))$$

Now let $K^\bullet = A[-k]$. In this case, the spectral sequence degenerates since $E_2^{s,t} = 0$ unless $t = 0$ and $s = k$. Therefore one computes $\pi_i(Sp(A[-k])) \cong A$ if $i = -k$ and trivial otherwise. This proves (i).

We may assume without loss of generality that $K^i = 0$ if $i < 0$. Now (i) shows that $\pi_i(Sp(K^k[0])) = 0$ unless $i = 0$. Therefore the $E_2^{s,t} = 0$ unless $t = 0$ and $H^s(\{\pi_0(Sp(K^k[0]))|k\}) = \pi_{-s}(Sp(K^\bullet))$. Since this is trivial for $s < 0$, it follows that $\pi_i(Sp(K^\bullet)) = 0$ unless $i \leq 0$. This proves (ii). □

Lemma 8.2. *(i) In case $\Gamma(\bullet) = \underset{r}{\oplus}\Gamma(r)$ is a differential graded algebra, $Sp(\Gamma(\bullet)) = \underset{r}{\Pi}Sp(\Gamma(r))$ is a ring object in the category of symmetric spectra.*

(ii) If $\Gamma^h(\bullet) = \underset{r}{\Pi}\Gamma^h(r)$ is a left (right, bi) differential graded module over $\Gamma(\bullet)$, then $Sp(\Gamma'(\bullet))$ is a left (right, bi) module spectrum over the ring spectrum $Sp(\Gamma(\bullet))$.

(iii) Let $\mathcal{S}$ denote a site with enough points. Then $\mathbb{H}(X, Sp(K^\bullet)) \cong Sp(\mathbb{H}(X, K^\bullet))$ where X denotes an object in the site $\mathcal{S}$ and $K^\bullet$ is a chain complex of abelian sheaves.

Proof. Ring objects in the category of co-chain complexes of abelian groups may be identified with differential graded algebras. Now it suffices to show the functor Sp sends ring objects to ring objects, which may be checked readily. This proves the first assertion and the second one may be checked similarly. Now we consider the last property. Since the site has enough points, one may use Godement resolution to compute the hypercohomology. Now the last statement follows from the fact the the homotopy inverse limits involved in the definition of hypercohomology commute with the homotopy inverse limit involved in the definition of the functor *sp*. □

9. Appendix B: K-theory and G-theory of algebraic stacks :replacement for the smooth site

We have added this section mainly because of the issues with the smooth site site of an algebraic stack that have come to light recently. We will essentially invoke the detailed paper of Martin Olsson (see [Ol]) where these issues are dealt with at length and consider only those results that are relevant for the K-theory and G-theory of algebraic stacks.

Let $\mathcal{S}$ denote a Noetherian algebraic stack defined over a Noetherian base scheme S, let $x : X \to \mathcal{S}$ denote a presentation and let $B_x\mathcal{S}$ denote the corresponding classifying simplicial algebraic space. Following [Ol] we will adopt the following terminology: $B_x\mathcal{S}^+$ will denote the associated *semi-simplicial algebraic space* obtained by forgetting the degeneracies. The étale site of $B_x\mathcal{S}^+$ is defined as usual: objects are objects U in the étale site of $(B_x\mathcal{S})_n$ for some n and if V is in the étale site of $(B_x\mathcal{S})_m$, a morphism $U \to V$ is an (étale) morphism lying over some structure map $(B_x\mathcal{S})_n^+ \to (B_x\mathcal{S})_m^+$. A sheaf F on $B_x\mathcal{S}_{et}^+$ consists of a collection of sheaves $\{F_n|n\}$, with F_n a sheaf on the étale site of $(B_x\mathcal{S})_n$ along with a compatible collection of morphisms $\{\alpha^*(F_n) \to F_m\}$ for each structure map $\alpha : B_x\mathcal{S}_m^+ \to B_x\mathcal{S}_n^+$. We say that a sheaf F on $B_x\mathcal{S}_{et}$ or $B_x\mathcal{S}_{et}^+$ has *descent* if all the above structure maps $\alpha^*(F_n) \to F_m$ are isomorphisms. Clearly there is a restriction functor $res : Sh(B_x\mathcal{S}_{et}) \to Sh(B_x\mathcal{S}_{et}^+)$ where Sh denotes the category of sheaves.

Let $Mod(\mathcal{S}, \mathcal{O})$ ($Qcoh(\mathcal{S}, \mathcal{O})$) denote the category of all $\mathcal{O}_\mathcal{S}$-modules (all quasi-coherent $\mathcal{O}_\mathcal{S}$-modules, respectively). Similarly let $Mod(B_x\mathcal{S}_{et}, \mathcal{O})$ and $Mod(B_x\mathcal{S}_{et}^+, \mathcal{O})$ denote the category of all $\mathcal{O}$-modules on $B_x\mathcal{S}_{et}$ ($B_x\mathcal{S}_{et}^+$, respectively); let $Qcoh(B_x\mathcal{S}_{et}, \mathcal{O})$ and $Qcoh(B_x\mathcal{S}_{et}^+, \mathcal{O})$ denote the corresponding categories of quasi-coherent $\mathcal{O}$-modules. If $\mathbf{A}$ denotes any of the abelian categories above, we will let $D^b(\mathbf{A})$ denote the corresponding bounded derived category. $D_{des}^b(Mod(B_x\mathcal{S}_{et}^+, \mathcal{O})$ will denote the full sub-category of complexes whose cohomology sheaves have descent, i.e. are cartesian sheaves as in [Ol].

Theorem 9.1. *The obvious inclusion functor $D^b(QCoh(\mathcal{S}, \mathcal{O})) \to D^b(Mod(\mathcal{S}, \mathcal{O}))$ is an equivalence of categories.*

Proof. Observe that one obtains a commutative diagram of derived categories:

$$\begin{array}{ccccc}
D^b(Mod(\mathcal{S},\mathcal{O})) & \longrightarrow & D^b_{des}(Mod(B_x\mathcal{S}^+_{et},\mathcal{O})) & \xleftarrow{res} & D^b_{des}(Mod(B_x\mathcal{S}_{et},\mathcal{O})) \\
\uparrow & & \uparrow & & \uparrow \\
D^b(Qcoh(\mathcal{S},\mathcal{O})) & \longrightarrow & D^b_{des}(Qcoh(B_x\mathcal{S}^+_{et},\mathcal{O})) & \xleftarrow{res} & D^b_{des}(Qcoh(B_x\mathcal{S}_{et}\mathcal{O}))
\end{array}$$

One observes that the maps in the top and bottom rows are all equivalences of catgeories as in [Ol] sections 2 and 5. Therefore it suffices to show that the right vertical map is an equivalence. This follows using the quasi-coherator defined in [T-T]. □

Let $(\mathcal{S}, \mathcal{A})$ denote a dg-stack as in section 2. Then $\mathcal{A}$ defines, by pull-back a sheaf of dgas on $B_x\mathcal{S}^+_{et}$ and on $B_x\mathcal{S}_{et}$ where $x : X \rightarrow \mathcal{S}$ is an atlas and $B_x\mathcal{S}$ is the corresponding simplicial classifying space. A sheaf of $\mathcal{A}$-modules M is *coherent* if the cohomology sheaves $\mathcal{H}^*(M)$ are sheaves of finitely generated $\mathcal{H}^*(\mathcal{A})$-modules. The category of all coherent $\mathcal{A}$-modules on $\mathcal{S}$ ($B_x\mathcal{S}_{et}$, $B_x\mathcal{S}^+_{et}$) will be denoted $Coh(\mathcal{S}, \mathcal{A})$ ($Coh(B_x\mathcal{S}_{et}, \mathcal{A})$, $Coh(B_x\mathcal{S}^+_{et}, \mathcal{A})$, respectively). Similarly one defines the category of *perfect complexes* on $\mathcal{S}$ ($B_x\mathcal{S}_{et}$, $B_x\mathcal{S}^+_{et}$) : see section 2. These will be denoted $Perf(\mathcal{S}, \mathcal{A})$, $Perf(B_x\mathcal{S}^+_{et}, \mathcal{A})$ and $Perf(B_x\mathcal{S}_{et}, \mathcal{A})$, respectively . Observe that these are all Waldhausen categories with fibrations and weak-equivalences where the fibrations are degree-wise surjections and the weak-equivalences are maps of $\mathcal{A}$-modules that are quasi-isomorphisms.

Proposition 9.2. *Let $(\mathcal{S}, \mathcal{A})$ denote a dg-stack as in section 2. Then the following hold:*

(i) The obvious functors $Coh(S, \mathcal{A})) \rightarrow Coh_{des}(B_x\mathcal{S}^+_{et}, \mathcal{A}) \leftarrow Coh_{des}(B_x\mathcal{S}_{et}, \mathcal{A})$ induce weak-equivalences of Waldhausen K-theories where the Waldhausen structure is as above. (The subscript des denotes the full-subcategory of complexes whose cohomology sheaves have descent.)

(ii) The obvious functors $Perf(\mathcal{S}, \mathcal{A}) \rightarrow Perf_{des}(B_x\mathcal{S}^+_{et}, \mathcal{A}) \leftarrow Perf_{des}(B_x\mathcal{S}_{et}, \mathcal{A})$ induce weak-equivalences of Waldhausen K-theories. (Again the subscript des denotes the full-subcategory of complexes whose cohomology sheaves have descent.)

Proof. In view of the equivalence of topoi on the smooth and étale sites we will work with the smooth site on $B_x\mathcal{S}^+$. Let $\bar{x}^* : Mod(\mathcal{S}_{smt}, \mathcal{O}) \rightarrow Mod(B_x\mathcal{S}^+_{smt}, \mathcal{O})$ denote the obvious inverse image functor $M \mapsto \{x^*_n(M)|n\}$. Here $x_n : B_x\mathcal{S}_n \rightarrow \mathcal{S}$ denotes the map induced by x. Let $\bar{x}_* : Mod(B_x\mathcal{S}^+_{smt}, \mathcal{O}) \rightarrow Mod(\mathcal{S}_{smt}, \mathcal{O})$ denote the functor sending $F = \{F_n|n\}$ to $ker(\delta^0 - \delta^1 : x_{0*}(F_0) \rightarrow x_{1*}(F_1))$. One observes that the compositions $R\bar{x}_* \circ \bar{x}^*$ is naturally quasi-isomorphic to the identity. Similary the composition $\bar{x}^* \circ R\bar{x}_*$ restricted to $D^b_{des}(Mod(B_x\mathcal{S}^+_{smt}, \mathcal{O}))$ is naturally quasi-isomorphic to the identity. Moreover, both the functors $\bar{x}^*$ and $R\bar{x}_*$ preserve the structure of being $\mathcal{A}$-modules. The functor $\bar{x}^*$ clearly preserves the structure of Waldhausen categories with fibrations and weak-equivalences. Moreover the above statements show $\bar{x}^*(f)$ is a quasi-isomorphism if and only if f is. Therefore the weak-equivalence of the K-theory spectra of $Coh(\mathcal{S}, \mathcal{A})$ and $Coh(B_x\mathcal{S}^+_{et}, \mathcal{A})$ follows. One observes that the restriction functor $Coh(B_x\mathcal{S}_{et}, \mathcal{A}) \rightarrow Coh(B_x\mathcal{S}^+_{et}, \mathcal{A})$ is fully-faithful at the level of the associated derived categories. Since the functor $\bar{x}^* : Coh(\mathcal{S}, \mathcal{A}) \rightarrow Coh(B_x\mathcal{S}^+_{et}, \mathcal{A})$ factors through $Coh(B_x\mathcal{S}_{et}, \mathcal{A})$ it follows that the functor res is full at the level of the associated derived categories. This proves the functor $res : Coh(B_x\mathcal{S}_{et}, \mathcal{A}) \rightarrow Coh(B_x\mathcal{S}^+_{et}, \mathcal{A})$ induces an equivalence of the associated derived categories proving it induces a weak-equivalence of the corresponding Waldhausen K-theory spectra. readily.

To see (ii) it suffices now to observe that both the functors $\bar{x}^*$ and $R\bar{x}_*$ preserve the property of being a perfect $\mathcal{A}$-module. In view of the definition of this in section 2, we reduce to proving the above functors preserve the property of being a perfect $\mathcal{O}$-module. This is clear for $\bar{x}^*$. Next suppose $P \varepsilon Perf(B_x\mathcal{S}^+_{smt}, \mathcal{O})$ and has cohomology sheaves with descent. Then $P = \bar{x}^*(Q)$ for some $Q \varepsilon Mod(\mathcal{S}_{smt}, \mathcal{O})$. Since P is perfect, Q is perfect as a complex of $\mathcal{O}$-modules on $\mathcal{S}_{smt}$. Now observe that $R\bar{x}_*(P) = Q$. Hence $R\bar{x}_*(P)$ is perfect. □

Remark 9.3. It should be easy to see from the last proposition that the K-theory of *dg*-stacks as in section 2 is contravariantly functorial and that the G-theory is *dg*-stacks is also contravariantly functorial for flat maps.

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