NORM VARIETIES

ANDREI SUSLIN AND SEVA JOUKHOVITSKI

ABSTRACT. For given symbol in the n-th Milnor K-group modulo prime l we construct a splitting variety with several properties. This variety is l-generic, meaning that it is generic with respect to splitting fields having no finite extensions of degree prime to l. The degree of its top Milnor class is not divisible by l^2, and a certain motivic cohomology group of this variety consists of units. The existence of such varieties is needed in Voevodsky's part of the proof of the Bloch-Kato conjecture. In the course of the proof we also establish Markus Rost's degree formula.

CONTENTS

0. INTRODUCTION

The purpose of this paper is to present a part of Markus Rost's work on Norm Varieties. The primary goal is to prove the following result, formulated in [MC/l, theorem 6.3], that is necessary to complete the inductive step in the proof of the Bloch–Kato conjecture.

0.1. **Theorem** (M. Rost). *Let l be a prime number and k be a field of characteristic zero that contains a primitive l-th root of unity. For any non-trivial n-symbol $\{a_1, \dots, a_n\} \in K_n^M(k)/l$ there exists a splitting variety X such that*

1) X is a $\nu_{\leqslant n-1}$ variety

2) the sequence

$$H_{-1,-1}(X \times X) \xrightarrow{(p_1)_* - (p_2)_*} H_{-1,-1}(X) \longrightarrow H_{-1,-1}(k) = k^*$$

is exact.

2000 *Mathematics Subject Classification.* Primary 19E15. Secondary 14C25, 14F42, 19D45.
Key words and phrases. Splitting variety, Bloch-Kato conjecture, chain lemma.
First author acknowledges support by the NSF under the agreement No. DMS-0100586.
Second author acknowledges support by the NSF under the agreement No. DMS-0111298.

An interested reader may find out more about both the history and the strategy of the proof of the conjecture in the introduction to the paper [MC/2]. A diagram illustrating various implications of results in the motivic cohomology that are used in the inductive step appears in the introduction to the notes [EMS].

Observe that to get the proof of Bloch–Kato conjecture in general it suffices to prove the main theorem above for any prime number $l > 2$ and any base field satisfying conditions of the theorem. Moreover, the base field may be assumed to be l-special, see definition 1.11 below. While the restriction on the characteristic is not essential for many steps of the construction, we impose it for the sake of simplicity. Thus we freely use the resolution of singularities technique by Hironaka [H]. We don't assume that l is odd because it gives no simplification.

The present proof of the theorem in weight n uses theorem A.1 which in turn follows from the Bloch–Kato conjecture in weights $\leqslant n-1$. Thus, rather then being a completely independent statement it is a part of inductive step

Bloch–Kato conjecture in weights $\leqslant n-1$	$\Longrightarrow$	Bloch–Kato conjecture in weight n.

Here is the outline of the argument. In the first section we give all the necessary definitions and introduce the so-called group of reduced 0-dimensional $\mathcal{K}_1$-cycles on a smooth scheme to replace the $(-1,-1)$-homology of the main theorem. We discuss a number of properties of these groups and formulate theorem 1.21 which is the central result of these notes. In short, it states that splitting varieties of a special type exist, and that any such variety satisfies the claim of the main theorem. At the end of section one we show that theorem 1.21 implies theorem 0.1 if the base field is l-special.

In section two we describe an inductive construction of l-generic splitting varieties for a symbol. These varieties are constructed from symmetric powers and are, in fact, exactly the ones we want to produce. Toward the end we show that theorem 1.21 implies theorem 0.1 for a base field that is not necessary l-special.

The next two sections deal with pseudo-Galois (i.e. 'Galois almost everywhere') coverings. In section three we define the η invariant of such coverings and show that it satisfies an appropriate degree formula. In section four it is shown, by means of introduction of b-classes, that knowing the η invariant of the l-th (Cartesian) power of a variety over its l-th cyclic power is essentially the same as knowing whether it is a $\nu_{\leqslant n-1}$ variety, as defined in 1.20.

Finally in section five we use Markus Rost's *chain lemma* to show that the variety in question is indeed $\nu_{\leqslant n-1}$, and also to prove the *multiplication principle* for reduced 0-dimensional $\mathcal{K}_1$-cycles. In turn, the multiplication principle together with the *norm principle* (see [RH]) obviously imply the remaining claim of the theorem 1.21 and hence the main theorem as well.

In the appendix we prove an auxiliary result that is crucial for our construction of l-generic splitting varieties but otherwise is independent from the rest of the paper.

Vigilant readers will surely notice that the expression "Norm Varieties" does not appear anywhere in the text. Let us point out that the term *norm variety* was coined to describe a variety given by an equation $N(x) = a$ where N is any norm map, while x and a are whatever circumstances dictate. Hence both the splitting varieties constructed from the symmetric powers in section two and the varieties $S(\sqrt[\ell]{\alpha})$ of section five deserve that name. Which ones the paper is entitled after is anybody's guess.

This article is based on a series of lectures given by the first author during the fall term of 2004 at the Institute for Advanced Study. We would like to thank the following people and organizations: the Institute for providing a perfect environment for our work, the National Science Foundation for its support, Chuck Weibel for the editorial assistance, and the referee for through review and useful suggestions.

1. Reduced 0-dimensional $\mathcal{K}_1$-cycles

We would like to transform the statement of the main theorem 0.1 into one about algebraic cycles.

Let X be a smooth, irreducible, projective variety of dimension d. Recall that the $\mathcal{K}$-cohomology groups of X are those of the Gersten complex

$$K^M_{d+1}(k(X)) \longrightarrow \coprod_{\operatorname{codim} x=1} K^M_d(k(x)) \longrightarrow \cdots \longrightarrow \coprod_{\operatorname{codim} x=d} K^M_1(k(x)).$$

The last cohomology group is

$$H^d(X,\mathcal{K}_{d+1}) = \operatorname{coker}\left(\coprod_{\dim \bar{y}=1} K^M_2(k(y)) \xrightarrow{\partial} \coprod_{x\in X \text{ closed}} K^M_1(k(x))\right).$$

Finally recall that the latter group may also be denoted $A_0(X,\mathcal{K}_1)$ and, when written in this form is called the group of 0-dimensional $\mathcal{K}_1$-cycles.

The connection between the motivic homology $H_{-1,-1}(X)$ and $A_0(X,\mathcal{K}_1)$ is a direct consequence of standard relations.

1.1. **Lemma.** *Let X be a smooth, irreducible, projective variety of dimension d. Then $H_{-1,-1}(X) = A_0(X,\mathcal{K}_1)$.*

Proof. Using duality (see [TC]) and usual isomorphism between motivic and $\mathcal{K}$-cohomology, as well as the above remarks we compute:

$$\begin{aligned}
H_{-1,-1}(X) &= \operatorname{Hom}_{DM_-}(\boldsymbol{Z}(-1)[-1], M(X)) && \text{(definition)}\\
&= \operatorname{Hom}_{DM_-}(\boldsymbol{Z}(-1)[-1], \mathcal{H}\mathrm{om}(M(X), \boldsymbol{Z}(d)[2d])) && \text{(duality)}\\
&= \operatorname{Hom}_{DM_-}(\boldsymbol{Z}(-1)[-1]\otimes M(X), \boldsymbol{Z}(d)[2d]) &&\\
&= \operatorname{Hom}_{DM_-}(M(X), \boldsymbol{Z}(d+1)[2d+1]) &&\\
&= H^{2d+1}(X, \boldsymbol{Z}(d+1)) && \text{(definition)}\\
&= H^d(X,\mathcal{K}_{d+1}) && \text{[MC/2, lemma 4.11]}\\
&= A_0(X,\mathcal{K}_1) && \text{(definition)}
\end{aligned}$$

with $\mathcal{H}\mathrm{om}$ being the internal Hom-object in the category DM_-.

Note that a proper morphism $f : X \to Y$ induces a map of Gersten complexes, hence a map $f_* : A_0(X,\mathcal{K}_1) \to A_0(Y,\mathcal{K}_1)$. Consequently the groups $A_0(-,\mathcal{K}_1)$ are covariant, in particular, with respect to morphisms of projective varieties. Moreover the map f_* is compatible with the corresponding map of $(-1,-1)$ homology groups.

1.2. **Notation.** For a smooth, irreducible, projective variety X let $\bar{A}_0(X,\mathcal{K}_1)$ denote the group of *reduced* 0-dimensional $\mathcal{K}_1$-cycles, i.e.,

$$\bar{A}_0(X,\mathcal{K}_1) := \operatorname{coker}\Big(A_0(X\times X,\mathcal{K}_1) \xrightarrow{(p_1)_* - (p_2)_*} A_0(X,\mathcal{K}_1)\Big).$$

Finally let us point out that the map $N : A_0(X,\mathcal{K}_1) \to A_0(\operatorname{Spec} k,\mathcal{K}_1) = k^\times$ induced by the structure map is the sum of norm maps of Milnor K-groups, and that it obviously factors through $\bar{A}_0$. Now we can make a trivial but very important observation.

1.3. **Remark.** A projective variety X verifies the second requirement of the main theorem 0.1 if and only if the norm map $N : \bar{A}_0(X,\mathcal{K}_1) \to k^\times$ is injective.

Observe that the group $\bar{A}_0(X,\mathcal{K}_1)$ is generated by elements of the form $[x,\mu]$, where $x \in X$ is a closed point, $\mu \in k(x)^\times$. Such an element may be thought of either as the image of μ under the canonical map $k(x)^\times = \bar{A}_0(\operatorname{Spec} k(x),\mathcal{K}_1) \to \bar{A}_0(X,\mathcal{K}_1)$ corresponding to embedding of x into X or simply as μ placed at x.

Let L/k be a field extension. A morphism $\phi : \operatorname{Spec} L \to X$ is determined by a point x of X and a field embedding $k(x) \hookrightarrow L$ over k. We will refer to such ϕ as *an L-valued point of X*. If L/k is a finite extension then x must be a closed point of X. For such a point the map ϕ_* defined above admits a very explicit description. It is induced by the norm map

$$\begin{array}{ccc} L^\times & \xrightarrow{N_{L/k(x)}} & k(x)^\times \\ \| & & \downarrow {\scriptstyle\text{canonical map}} \\ A_0(\operatorname{Spec} L,\mathcal{K}_1) & \xrightarrow{\phi_*} & A_0(X,\mathcal{K}_1). \end{array}$$

This allows us to give the following description of $\bar{A}_0(X,\mathcal{K}_1)$.

1.4. **Lemma.** *$\bar{A}_0(X,\mathcal{K}_1)$ is obtained from $A_0(X,\mathcal{K}_1)$ by factoring out all relations of the form $\phi_*(\lambda) - \psi_*(\lambda)$ where L is any finite extension of k, $\lambda \in L^\times$, and $\phi,\psi : \operatorname{Spec} L \to X$ are any two L-valued points.*

Proof. Any two morphisms $\phi,\psi : \operatorname{Spec} L \to X$ determine the product morphism $(\phi,\psi) : \operatorname{Spec} L \to X\times X$. For any $\lambda \in L^\times$ therefore $\phi_*(\lambda) - \psi_*(\lambda) = ((p_1)_* - (p_2)_*)(\phi,\psi)_*(\lambda)$ vanishes in $\bar{A}_0(X,\mathcal{K}_1)$. Conversely every element in the image of $(p_1)_* - (p_2)_*$ must be a sum of terms of that form.

Making the right choice of L we obtain

1.5. **Corollary.** *1) Assume that $x, x' \in X$ are closed points such that there exists an isomorphism $\sigma : k(x) \xrightarrow{\sim} k(x')$. Then for every $\lambda \in k(x)^\times$, $[x,\lambda] = [x',\sigma(\lambda)]$ in $\bar{A}_0(X,\mathcal{K}_1)$.*
2) Assume that $x, x' \in X$ are closed points such that there exists a field embedding $k(x') \hookrightarrow k(x)$. Then for every $\lambda \in k(x)^\times$, $[x,\lambda] = [x', N_{k(x)/k(x')}(\lambda)]$ in $\bar{A}_0(X,\mathcal{K}_1)$.

1.6. **Corollary.** *If X has a k-rational point x_0 then $N : \bar{A}_0(X,\mathcal{K}_1) \xrightarrow{\sim} k^\times$ is an isomorphism.*

Proof. The morphism $X \to \operatorname{Spec} k$ induces the map $N : \bar{A}_0(X,\mathcal{K}_1) \to k^\times$ that sends $[x,\mu]$ to $N_{k(x)/k}(\mu)$. It has right inverse that maps $\mu \in k^\times$ to $[x_0,\mu]$. It is enough to show that the latter is surjective. Let $x \in X$ be any closed point. Then according to corollary 1.5 for each $\mu \in k(x)^\times$ we get $[x, \mu] = [x_0, N_{k(x)/k}(\mu)]$ in $\bar{A}_0(X, \mathcal{K}_1)$.

1.7. **Corollary.** *If X has a closed point of degree n then both the kernel and the cokernel of $N : \bar{A}_0(X, \mathcal{K}_1) \longrightarrow k^\times$ are annihilated by n.*

Proof. Let $x \in X$ be a point with $[k(x) : k] = n$. After extension of scalars to $k(x)$ the map in question becomes an isomorphism. The usual transfer argument completes the proof.

Now we recall the notion of a generic splitting variety. Let $\{\underline{a}\} = \{a_1, \dots, a_n\} \in K_n^M(k)/l$ be a non-zero symbol. Any extension L of k such that $\{\underline{a}\} = 0$ in $K_n^M(L))/l$ is called *a splitting field of* $\{\underline{a}\}$.

1.8. **Definition.** A smooth variety X is *a splitting variety* for a non-zero symbol $\{\underline{a}\} \in K_n^M(k)/l$ if

1) $\{\underline{a}\} = 0$ in $K_n^M(k(X))/l$.

It is a *generic splitting variety* if in addition

2) for any splitting field L of $\{\underline{a}\}$ there is an L-valued point $\operatorname{Spec} L \to X$ over k.

1.9. **Remark.** Observe that for any $x \in X$ the map $K_n^M(k)/l \to K_n^M(k(x))/l$ factors through a (non-canonical) specialization map $K_n^M(k(X))/l \to K_n^M(k(x))/l$. Therefore if X is a splitting variety for $\{\underline{a}\}$ then $\{\underline{a}\} = 0$ in $K_n^M(k(x))/l$ for every point x of X.

Unfortunately, generic splitting varieties are only known to exist for $n \leqslant 3$ and also for arbitrary n provided $l = 2$. Observe however that if L'/L is a finite extension of degree prime to l and L' splits $\{\underline{a}\}$ then by the usual transfer argument L splits $\{\underline{a}\}$ as well. Therefore we can, without much loss, relax the definition as follows.

1.10. **Definition.** A smooth variety X is *an l-generic splitting variety* for a non-zero symbol $\{\underline{a}\} \in K_n^M(k)/l$ if

1) X is a splitting variety for $\{\underline{a}\}$ and

2) for any splitting field L of $\{\underline{a}\}$ there is a finite extension L'/L of degree prime to l and an L'-valued point $\operatorname{Spec} L' \to X$ over k.

It is convenient to have another description of generic splitting varieties.

1.11. **Definition.** A field F is called *l-special* provided F has no finite extensions of degree prime to l or equivalently if $Gal(F_{alg}/F)$ is a pro-l-group.

1.12. **Remark.** Let L be a field of characteristic other then l. Choose an algebraic closure L_{alg} of L. Let $G := Gal(L_{alg}/L)$ and G_l be a Sylow l-subgroup of the profinite group G. Let $\tilde{L} := L_{alg}^{G_l}$ be the subfield fixed by G_l. Then $\tilde{L}$ is l-special. At the same time the degree of every finite subextension L'/L is prime to l. Such field $\tilde{L}$ is called *a maximal extension of L of degree prime to l.* (If *char* $L = l$ then $\tilde{L} = L_{sep}$ has the required property.)

Using the notion of l-special field one can define l-generic splitting varieties as follows.

1.10′ **Definition.** A variety X is an *l-generic splitting variety* for a non-zero symbol $\{\underline{a}\} \in K_n^M(k)/l$ if

1) X is a splitting variety for $\{\underline{a}\}$ and

2) every l-special splitting field F of $\{\underline{a}\}$ has an F-valued point $\operatorname{Spec} F \to X$ over k.

Note that definitions 1.10 and 1.10′ are equivalent. Indeed if X is an l-generic splitting variety according to 1.10 and L is l-special then L'/L must be a trivial extension and any L'-valued point is an L-valued point. Conversely let X be an l-generic splitting variety according to 1.10′ and L be a splitting field for $\{\underline{a}\}$. Let $\tilde{L}$ be a maximal extension of L of degree prime to l. Since $\{\underline{a}\}$ vanishes over L it does so over $\tilde{L}$ hence X has an $\tilde{L}$-valued point. It is supported in a point x of X. Since $k(x)$ is finitely generated over k there exists some finite subextension L'/L such that field embedding $k(x) \hookrightarrow \tilde{L}$ factors through L'. Thus, by construction, X has an L'-valued point supported in x and L'/L is finite of degree prime to l.

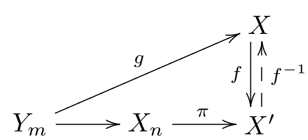
Everywhere in these notes all splitting varieties are always assumed to be smooth and projective.

1.13. **Lemma.** *Let $f : X \to X'$ be a birational morphism of projective varieties. Then for each point x' in the smooth locus of X' there exists $x \in X$ such that $f(x) = x'$ and $k(x) = k(x')$.*

Proof. Assume first that X' is smooth and $f : X' \to X$ is a blow-up with a smooth center. In this special case the claim holds for obvious reasons.

In the general case consider the inverse rational map $f^{-1} : X' - \to X$. Using the resolution of singularities one can find a tower of blow-ups $X_n \to \cdots \to X_1 \to X'$ such that X_n is smooth and $\pi : X_n \to X'$ is an isomorphism over the smooth locus of X'. In particular the fiber of π over x' consists of a single point x'' with the same residue field.

Then again using the resolution of singularities for the morphism $f^{-1} \circ \pi$ one constructs a tower of blow-ups with smooth centers $Y_m \to \cdots \to Y_1 \to X_n$ such that $f^{-1} \circ \pi : X_n = X' - \to X$ lifts to a morphism $g : Y_m \longrightarrow X$.



According to the special case one can further lift $x'' \in X_n$ to $x''' \in Y_m$ with the same residue field. Setting $x := g(x''')$ we observe that $f(x) = x'$ and moreover that $k(x') = k(x''') \supseteq k(x) \supseteq k(x')$. Hence the residue fields $k(x)$ and $k(x')$ are the same.

1.14. **Notation.** For a variety X set $FE_X := \{F/k : X \text{ has an } F\text{-valued point}\}$. ($FE$ stands for 'field extension'.)

Using a tower of blow-ups as in the previous lemma one can readily prove the following.

1.15. **Lemma.** *Assume that $X - \to X'$ is a rational map of smooth projective varieties. Then $FE_X \subseteq FE_{X'}$.*

1.16. **Remark.** According to the lemma the property of being a generic splitting variety for a given symbol is a birational invariant.

1.17. **Notation.** For a variety X set $FE^l_X := \{F/k : F \text{ is } l\text{-special and } X \text{ has an } F\text{-valued point}\}$.

We will repeatedly use the following technical statement.

1.18. **Lemma.** *Let $f : X \to X'$ be a dominant morphism of degree prime to l of projective varieties of the same dimension. Let L' be any field and $\psi : \operatorname{Spec} L' \to X'$ be a morphism supported in the smooth locus of X'. Then ψ may be lifted to a morphism $\phi : \operatorname{Spec} L \to X$ so that L/L' is a finite extension of degree prime to l.*

Proof. According to the Raynaud-Gruson platification theorem [RG] there exists a blow-up $p : B_{Z'}X' \to X'$, not necessarily with a smooth center, such that the proper pull-back $p^! f$ of f is flat. Since $p^! f$ is flat proper, and generically finite (because so is f), it is flat and finite.

Let $\tilde{X}'$ be a variety resolving the singularities of $B_{Z'}X'$ and let $\tilde{X}$ be the pull-back of the corresponding square.

$$\begin{array}{ccccccc}
\operatorname{Spec} L & \overset{\tilde{\phi}}{\dashrightarrow} & \tilde{X} & \longrightarrow & p^! X & \longrightarrow & X \\
\vdots & & \downarrow \tilde{f} & \square & \downarrow p^! f & & \downarrow f \\
\operatorname{Spec} L' & \underset{\tilde{\psi}}{\dashrightarrow} & \tilde{X}' & \longrightarrow & B_{Z'}X' & \xrightarrow[p]{} & X'
\end{array}$$

Since $\tilde{X}' \to X'$ is a birational morphism, lemma 1.13 allows to lift the morphism $\psi : \operatorname{Spec} L' \to X'$ to $\tilde{\psi} : \operatorname{Spec} L' \to \tilde{X}'$. Consider the fiber of $\tilde{f}$ over the L'-valued point $\tilde{\psi}$. It is a finite scheme of degree prime to l over $\operatorname{Spec} L'$ and hence has a closed point also of degree prime to l over $\operatorname{Spec} L'$. This point provides a morphism $\tilde{\phi} : \operatorname{Spec} L \to \tilde{X}$ that lifts $\tilde{\psi}$ with L/L' being a finite extension of degree prime to l. Composing $\tilde{\phi}$ with the other two morphisms in the top row of the diagram we get the required lifting of ψ.

1.19. **Corollary.** *Assume that $X - \to X'$ is a dominant rational map of smooth projective varieties of the same dimension and of degree prime to l. Then $FE^l_X = FE^l_{X'}$.*

Proof. According to lemma 1.15 we may replace X by any birationally equivalent smooth projective variety. Thus by resolution of singularities we may assume that $f : X \to X'$ is a morphism. Then the inclusion $FE^l_X \subseteq FE^l_{X'}$ is obvious.

To prove the opposite inclusion consider an l-special field L' such that X' has an L'-valued point. By lemma 1.18 X has an L-valued point for an appropriate extension L/L'. Since L' is l-special this extension is in fact trivial, that is, X also has an L'-valued point.

To conclude this section we recall definition of ν_n- and $\nu_{\leqslant n}$-varieties and state the core theorem 1.21, the proof of which will occupy the remainder of the paper. We then show that theorem 1.21 implies the main theorem 0.1 in those cases we are mostly interested in.

1.20. **Definition** ([MC/2] after lemma 3.1, [MC/l] chapter 4 and definition 6.2)**.** Let X be a smooth variety of dimension d over a field k. Consider the total Chern class $c = 1 + c_1 + \cdots + c_d : K_0(X) \to CH^*(X)$. One may formally write $c = (1 - x_1) \ldots (1 - x_d)$ with $\deg x_i = 1$ and then define the m-th Milnor's class

$$s_m := (-x_1)^m + \cdots + (-x_d)^m : K_0(X) \to CH^*(X).$$

In short, s_m is the m-th Newton symmetric polynomial in Chern roots. It follows that s_m is additive and that for a line bundle L one has $s_m(L) = (c_1(L))^m$.

Assume further that X is a smooth projective variety of dimension $d = l^n - 1$. It is known that in this case the number $\deg_k s_d(X) = \deg_k s_d(T_X)$ is a multiple of l. X is said to be *a ν_n-variety* provided

$$\deg_k s_d(X) \neq 0 \mod l^2\mathbf{Z}.$$

X is said to be *a $\nu_{\leqslant n}$-variety* if X is a ν_n-variety and for each $1 \leqslant i \leqslant n-1$ there exist a ν_i-variety X_i and a morphism $X_i \to X$.

1.21. **Theorem** (M. Rost). *Let $n \geqslant 2$ and $0 \neq \{\underline{a}\} = \{a_1, \dots, a_n\} \in K_n^M(k)/l$. Then:*

1) there exists a geometrically irreducible projective l-generic splitting variety for $\{\underline{a}\}$ of dimension $l^{n-1} - 1$.

Assume further that the field k is l-special. If X is a projective l-generic splitting variety for $\{\underline{a}\}$ of dimension $l^{n-1} - 1$ then:

2) X is a geometrically irreducible ν_{n-1}-variety;

3) each element of $\bar{A}_0(X, \mathcal{K}_1)$ is of the form $[x, \lambda]$, where $x \in X$ is a closed point of degree $[k(x) : k] = l$ and $\lambda \in k(x)^\times$.

1.22. **Corollary.** *Let k be l-special and X be a projective l-generic splitting variety for $\{\underline{a}\}$ of dimension $l^{n-1} - 1$. Then X is a $\nu_{\leqslant n-1}$-variety.*

Proof. We have to show that for every $1 \leqslant i < n-1$ there is a ν_i-variety equipped with a morphism to X. Consider the non-zero symbol $\{a_1, \dots, a_{i+1}\} \in K_{i+1}^M(k)/l$. By the theorem it has an l-generic splitting variety X_i of dimension $l^i - 1$. Its function field $k(X_i)$ splits $\{a_1, \dots, a_{i+1}\}$ and hence splits $\{a_1, \dots, a_n\}$. Therefore there exists a finite extension $F/k(X_i)$ of degree prime to l and an F-valued point $\operatorname{Spec} F \to X$. Choosing a model for F and resolving the singularities of the corresponding rational map, we get a smooth projective variety X'_i of the same dimension as X_i and a pair of morphisms

$$X_i \xleftarrow{\ f\ } X'_i \xrightarrow{\ g\ } X$$

so that f is dominant of degree prime to l. By corollary 1.19 $FE^l_{X_i} = FE^l_{X'_i}$. Consequently X'_i is another l-generic splitting variety for an $(i+1)$-symbol and because $\dim X'_i = \dim X_i = l^i - 1$ it is ν_i by the theorem. Thus we have constructed a morphism g from a ν_i-variety to X.

1.23. **Remark.** For $i = 0$ the same argument applied to $X_0 = \operatorname{Spec} k(\sqrt[l]{a_1})$ shows that X has an F-valued point where $F/k(\sqrt[l]{a_1})$ is a finite extension of degree prime to l. However since k is l-special this extension can only be the trivial one, that is X must have a $k(\sqrt[l]{a_1})$-valued point.

1.24. **Corollary.** *Let k be l-special and X be a projective l-generic splitting variety for $\{\underline{a}\}$ of dimension $l^{n-1} - 1$. Then the norm map $N : \bar{A}_0(X, \mathcal{K}_1) \to k^\times$ is injective.*

Proof. Consider $[x, \lambda] \in \ker N$ with $[k(x) : k] = l$. Let σ be a generator of $Gal(k(x)/k) \cong \mathbf{Z}/l$. By Hilbert Theorem 90 $N_{k(x)/k}(\lambda) = 1$ implies $\lambda = \mu^{1-\sigma}$ for some $\mu \in k(x)^\times$. Thus $[x, \lambda] = [x, \mu] - [x, \sigma(\mu)] = 0$ by corollary 1.5.

1.25. **Remark.** Evidently corollaries 1.22 and 1.24 along with remark 1.3 allow us to conclude that theorem 1.21 implies the main theorem 0.1 for any l-special base field.

2. Symmetric powers

In order to prove the existence clause of theorem 1.21 we use the following construction suggested by V. Voevodsky. It is based on the notion of symmetric powers that we briefly recall below.

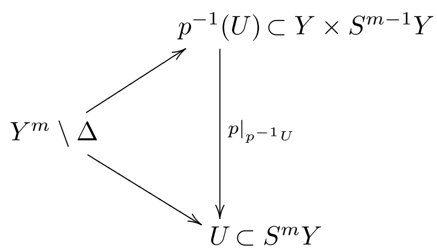
Let Y be a quasi-projective variety. The symmetric group Σ_m acts on the m-fold product Y^m and we let S^mY (or $Symm^mY$) denote the quotient variety Y^m/Σ_m.

For every normal and irreducible scheme T one can identify the set of morphisms $\mathrm{Hom}(T, S^mY)$ with the set of all effective cycles $Z \subset Y \times T$ such that each component of Z is finite surjective over T and that the degree of Z over T is m.

Assume that Y is smooth and geometrically irreducible hence S^mY is geometrically irreducible and normal. The identity morphism $id : S^mY \to S^mY$ then corresponds to the *incidence cycle* $Z \subset Y \times S^mY$. In fact Z is a closed subscheme equal to the image of the closed embedding $Y \times S^{m-1}Y \to Y \times S^mY$ mapping (y, z) to $(y, z + y)$.

Let $p : Y \times S^{m-1}Y \to Y \times S^mY \to S^mY$ be the composition of the above morphism with the projection onto the second factor. It is finite surjective of degree m.

Consider the largest open subscheme $Y^m \setminus \Delta$ of Y^m on which Σ_m acts freely. (Δ denotes the union of all the 'diagonals' in Y^m.) Set $U := (Y^m \setminus \Delta)/\Sigma_m \subset S^mY$. From the diagram



where both slant arrows are Galois étale coverings, we see that $p|_{p^{-1}U}$ is a finite étale map of degree m and that U is smooth.

Note that $p_*(\mathcal{O}_{Y\times S^{m-1}Y})$ is a coherent $\mathcal{O}_{S^mY}$-algebra and that the sheaf $\mathcal{A} := p_*(\mathcal{O}_{Y\times S^{m-1}Y})|_U$ is a locally free $\mathcal{O}_U$-algebra of rank m. Let $V := \mathrm{Spec}\,(S^*\mathcal{A}^\#)$ be the m-dimensional vector bundle over U corresponding to $\mathcal{A}$. (Here $\mathcal{A}^\#$ denotes the dual sheaf and S^* denotes its symmetric algebra.)

Since $\mathcal{A}$ is a locally free algebra, there is a well-defined norm function $N : \mathcal{A} \to \mathcal{O}_U$. Moreover locally N is a homogenous polynomial function of degree m, that is, $N \in S^m(\mathcal{A}^\#)$.

We will construct l-generic splitting varieties by induction. The case of $n = 2$ is well-known; one can choose a splitting variety to be the Severi-Brauer variety of a cyclic algebra associated to the symbol $\{a_1, a_2\}$.

From now on we have to assume that $char\ k \neq l$. We further assume that $n > 2$, that Y in the preceding construction is a smooth projective geometrically irreducible l-generic splitting variety for $\{a_1, \ldots, a_{n-1}\}$ of dimension $l^{n-2} - 1$, and that $m = l$.

Let $W \subset V$ be the hypersurface defined by the equation $N - a_n = 0$.

2.1. **Lemma.** *W is smooth over U (and hence smooth) and geometrically irreducible.*

Proof. Notice that every homogeneous polynomial $P(x_1, \ldots, x_k)$ of arbitrary degree m satisfies

$$\sum_{i=1}^{k} x_i \frac{\partial P}{\partial x_i} = mP(x_1, \ldots, x_k).$$

Since N is locally a form of degree l and $a_n \neq 0$, the first claim follows from the Jacobian criterion.

To prove the second we may first replace k by its algebraic closure. Assume that W is not irreducible. Hence there exists a point $u \in U$ such that the homogenous polynomial $N_u - a_n$ with coefficients in $\mathcal{O}_u$ is reducible. Then algebraic lemma 2.2 below would imply that $N_u = M^m$ is a power of a non-trivial linear form $M : \mathcal{A}_u \to \mathcal{O}_u$. Therefore the degeneracy locus of N_u would be $\ker M$, a non-zero proper $\mathcal{O}_u$-submodule of $\mathcal{A}_u$. However this is not possible. If the fiber $p^{-1}(u)$ consists of a single point $\hat{u}$ then the algebra $A_u = \mathcal{O}_{\hat{u}}$ has no zero-divisors and the degeneracy locus of N_u is trivial. If the fiber $p^{-1}(u)$ consists of a several point $\hat{u}_1, \ldots, \hat{u}_k$ then $A_u = \mathcal{O}_{\hat{u}_1} \oplus \cdots \oplus \mathcal{O}_{\hat{u}_k}$ and the degeneracy locus of N_u is $\{(\lambda_1, \ldots, \lambda_k)$ such that at least one of λ_i is zero$\}$, that is not a submodule.

2.2. **Lemma.** *Let N be a form of prime degree m in n variables with coefficients in a UFD B/k and let $a \neq 0$ in k. The following conditions are equivalent.*

1) *The polynomial $N - a \in B[X_1, \ldots, X_n]$ is irreducible;*
2) *The polynomial $N - aT^m \in B[T, X_1, \ldots, X_n]$ is irreducible;*
3) *N does not equal aM^m for any linear form M.*

Proof. The equivalence of the first two conditions is obvious. The last two are equivalent thanks to the Gauss lemma applied to $B[X_1, \ldots, X_n]$ and its fraction field.

By the resolution of singularities we can embed W as an open subvariety into a smooth, projective, and geometrically irreducible variety X. Note that $\dim X = \dim W = \dim V - 1 = \dim U + l - 1 = l \dim Y + l - 1 = l(l^{n-2} - 1) + l - 1 = l^{n-1} - 1$ just as required. In order to prove the existence part of theorem 1.21, it remains to show that X is indeed an l-generic splitting variety for $\{\underline{a}\}$. This will be done in several steps.

2.3. **Lemma.** *Let F/k be any field extension such that $W(F) \neq \emptyset$. Then the symbol $\{a_1, \ldots, a_n\} = 0$ in $K_n^M(F)$.*

Proof. To specify an F-valued point x of W one may specify the underlying F-valued point $\tilde{x}$ of U and a rational point $\hat{x}$ in the fiber $V_{\tilde{x}}$ such that $N(\hat{x}) - a_n = 0$. (Note that $V_{\tilde{x}} \cong \boldsymbol{A}^l_F$.) The point $\tilde{x}$ corresponds to a cycle $x_1 + \ldots + x_k$ on Y_F such that $\sum [F(x_i) : F] = l$. Then the point $\hat{x}$ has 'coordinates' $(\lambda_1, \ldots, \lambda_k) \in F(x_1) \times \cdots \times F(x_k)$ and by assumption $a_n = N(\hat{x}) = \prod N_{F(x_i)/F}(\lambda_i)$.

By construction, Y has an $F(x_i)$-valued point for each $1 \leqslant i \leqslant k$ and it follows that $\{a_1, \ldots, a_{n-1}\} = 0$ in $K_{n-1}^M(F(x_i))/l$. Thus in $K_n^M(F))/l$

$$\{a_1, \ldots, a_n\} = \sum N_{F(x_i)/F}(\{a_1, \ldots, a_{n-1}, \lambda_i\}) = 0.$$

Applying the previous lemma to $F = k(W) = k(X)$ we conclude that X is a splitting variety for $\{\underline{a}\}$.

2.4. **Proposition** (V. Voevodsky)**.** *Assume that the Bloch–Kato conjecture holds in weight* $(n-1)$*. Let* $\{a_1, \ldots, a_{n-1}\} \in K^M_{n-1}(k)/l$ *be any non-zero symbol. Assume that* k *is* l*-special and* Y *is a* $\nu_{\leqslant n-2}$ *splitting variety for the symbol* $\{a_1, \ldots, a_{n-1}\}$*. Then*

$$\{a \text{ such that } \{a_1, \ldots, a_{n-1}, a\} = 0 \text{ in } K^M_n(k)/l\} = (k^\times)^l N(\bar{A}_0(Y, \mathcal{K}_1)).$$

Proof. In fact this a paraphrase of theorem A.1 in appendix.

Finally we are able to show that for every l-special field F that splits $\{\underline{a}\}$ X has an F-valued rational point. Two cases are possible.

First case. F does not split $\{a_1, \ldots, a_{n-1}\}$. Then by proposition 2.4 applied to Y_F we get $a_n \in (F^\times)^l N(\bar{A}_0(Y_F, \mathcal{K}_1))$. Hence by theorem 1.21 part 3, which applies to Y_F by the inductive assumption, there exists $y \in Y_F$ such that $[F(y) : F] = l$ and $\lambda \in F(y)^\times$ so that $a_n = a^l N_{F(y)/F}(\lambda) = N_{F(y)/F}(a\lambda)$. This data determines an F-valued point of the hypersurface W thus one of X.

Second case. (This argument is due to V. Voevodsky.) F splits $\{a_1, \ldots, a_{n-1}\}$ hence Y_F has a rational point. By lemma 2.5 below Y_F has l distinct rational points $y_1, \ldots, y_l$ that determine an F-rational point $y = y_1 + \ldots + y_l$ of U_F. This point along with $(1, \ldots, 1, a_n) \in V_y$ determines an F-rational point in the fiber W_y. Hence W_F has a rational point. Once again this data determines an F-valued point of W thus one of X.

2.5. **Lemma.** *Let* F *be* l*-special, and* Y *be a smooth projective variety over* F *of dimension at least 1. If* $Y(F) \neq \emptyset$ *then* $Y(F)$ *is infinite.*

Proof. We may assume that Y/F is a curve. Let $y_1, \ldots, y_k$ be distinct rational points on Y. We need to exhibit one more point. Consider a divisor $\sum_1^k n_i y_i$ such that $n_i > 0$, $\sum n_i > 2g - 2$, and $(\sum n_i, l) = 1$. By the Riemann-Roch theorem we can find a rational function f such that $(f)_\infty = \sum_1^k n_i y_i$. Let $(f)_0 = \sum m_j z_j$. Note that all z_j are different from all y_i and that $\sum m_j [F(z_j) : F] = \sum n_i$ is prime to l. Hence for at least one j the degree $[F(z_j) : F]$ is prime to l. Since F has no finite extensions of degree prime to l, z_j is another rational point.

Finally we will show (again using parts 2 and 3 of Rost's theorem 1.21) that the construction described above produces a variety satisfying the claim of the main theorem 0.1 for a field k that is not necessary l-special.

Let k be any field of characteristic zero and $\{\underline{a}\} = \{a_1, \ldots, a_n\} \in K^M_n(k)/l$ be any non-zero n-symbol. The case $n = 2$ is well-known and we assume that $n > 2$. Let k' denote a maximal extension of k of degree prime to l.

Let X_1 be the Severi-Brauer variety of $\{a_1, a_2\}$. Let X_i for $2 \leqslant i \leqslant n-1$ be consecutively constructed from one another by means of the procedure described above. We already know, among other properties, that X_i is a splitting variety for $\{a_1, \ldots, a_{i+1}\}$ for each i.

2.6. **Proposition.** X_{n-1} *is a* $\nu_{\leqslant n-1}$ *variety.*

Recall that variety X of dimension $d = l^m - 1$ is ν_m if $\deg_k s_d(X)$ is a multiple of l but not of l^2. Consequently the property to be ν_m depends on the base field.

2.7. **Lemma.** *Let X be a smooth projective variety over F.*

1) Let F'/F be any field extension such that $X_{F'}$ is irreducible. Then X is ν_m over F if and only if $X_{F'}$ is ν_m over F'.

2) Let F/F'' be a finite extension of degree prime to l. Then X is ν_m over F if and only if X is ν_m over F''.

Proof. Obviously all the varieties under consideration have the same dimension. Moreover $\deg_{F'} s_d(X_{F'}) = \deg_F s_d(X)$ proves the first claim, while $\deg_{F''} s_d(X) = [F : F''] \deg_F s_d(X)$ proves the second one.

Note that the construction of splitting varieties given above is stable with respect to an extension of scalars. In particular $(X_i)_{k'}$ are splitting varieties for the non-zero symbols $\{a_1, \ldots, a_{i+1}\} \in K^M_{i+1}(k')/l$. Since k' itself is l-special each $(X_i)_{k'}$ is ν_i over k'. First part of the above lemma implies that each X_i is ν_i over k.

We proceed as in corollary 1.22 and find ν_i varieties X'_i over k' that fit into the diagrams

$$\begin{array}{ccc} & X'_i & \\ {}^{f_i}\swarrow & & \searrow^{g_i} \\ (X_i)_{k'} & & (X_{n-1})_{k'}. \end{array}$$

All these diagrams must be defined over some finite subextension k''/k, that is, they could be obtained by an extension of scalars from k'' to k' from

$$\begin{array}{ccc} & X''_i & \\ {}^{f'_i}\swarrow & & \searrow^{g'_i} \\ (X_i)_{k''} & & (X_{n-1})_{k''}. \end{array}$$

In particular each $X'_i = (X''_i)_{k'}$. By the first part of the preceding lemma X''_i is ν_i over k''. Since degree of k''/k is prime to l second part of the lemma shows that X''_i is ν_i over k as well. Composing g'_i with the projection $(X_{n-1})_{k''} \to X_{n-1}$ we get the required map $X''_i \to X_{n-1}$ from a ν_i-variety to X_{n-1} for each $1 \leqslant i < n-1$. Since X_{n-1} itself is a ν_{n-1}-variety we conclude that X_{n-1} is $\nu_{\leqslant n-1}$.

2.8. **Remark.** It was noted by A. Vishik that using the Landweber-Novikov operations in algebraic cobordisms one can prove that every ν_{n-1}-variety is in fact a $\nu_{\leqslant n-1}$-variety. That fact makes the above argument unnecessary.

2.9. **Proposition.** *The norm map $N : \bar{A}_0(X_{n-1}, \mathcal{K}_1) \to k^\times$ is injective.*

Proof. Set $E := k(\sqrt[l]{a_1})$. Since E splits $\{a_1, a_2\}$ and *char* $k = 0$ the Severi-Brauer variety X_1 has infinitely many E-rational points. The argument preceding 2.5 shows that each X_i has infinitely many E-rational points. Thus by corollary 1.7 the kernel of N is annihilated by $[E : k] = l$. On the other hand $\ker N$ vanishes after extension to k' hence the orders of all its elements are prime to l. Thus $\ker N = 1$.

3. Rost's degree formula

To prove the second claim of theorem 1.21 we will develop a version of the degree formula invented by Markus Rost. With that goal in mind we begin by defining the notion of degree for zero-cycles on an open subscheme relative to the ambient projective variety.

Let S/k be any projective (not necessarily smooth) variety of dimension d. The degree homomorphism $\deg_S : CH_0(S) \to \boldsymbol{Z}$ is nothing but the proper push-forward π_* induced by the structure morphism $\pi : S \to \operatorname{Spec} k$. Let $I(S) := \deg_S CH_0(S)$ denote the subgroup of $\boldsymbol{Z}$ generated by the degrees of the closed points of S.

For a proper morphism $i : S_0 \to S$ of projective varieties there is the usual commutative diagram of push-forwards:

$$\begin{array}{ccc} CH_0(S_0) & \xrightarrow{i_*} & CH_0(S) \\ & {\scriptstyle \deg_{S_0}} \searrow \quad \swarrow {\scriptstyle \deg_S} & \\ & Z. & \end{array}$$

Let S/k be a projective variety, $S_0 \subset S$ a closed subscheme, $U = S \setminus S_0$ the complementary open subscheme, and denote the inclusion morphisms by i and j, respectively. From the diagram

$$\begin{array}{ccccccc} CH_0(S_0) & \xrightarrow{i_*} & CH_0(S) & \xrightarrow{j^*} & CH_0(U) & \longrightarrow & 0 \\ \downarrow {\scriptstyle \deg_{S_0}} & & \downarrow {\scriptstyle \deg_S} & & \vdots {\scriptstyle \deg_U} & & \\ I(S_0) & \longrightarrow & \boldsymbol{Z} & \longrightarrow & \boldsymbol{Z}/I(S_0) & & \end{array}$$

we get a homomorphism $\deg_U : CH_0(U) \to \boldsymbol{Z}/I(S_0)$.

The following lemma summarizes the basic properties of the homomorphism $\deg_U$.

3.1. **Lemma.**
1) Let $S \supset S_0 \supset S_0'$ be a projective variety and two of its closed subschemes, let $U := S \setminus S_0$ and $U' := S \setminus S_0'$. Then
$I(S_0') \subseteq I(S_0)$ *and*
$\forall Z \in CH_0(U') \quad \deg_{U'}(Z) \equiv \deg_U(Z|_U) \mod I(S_0)$.
2) Let $f : S \to S'$ be a morphism of projective varieties, and $S_0' \subset S'$ be a closed subscheme. Set $S_0 := f^{-1}(S_0')$, $U' := S' \setminus S_0'$, and $U := S \setminus S_0$. Then $I(S_0) \subseteq I(S_0')$ and the diagram

$$\begin{array}{ccc} CH_0(U) & \xrightarrow{(f|_U)_*} & CH_0(U') \\ \downarrow {\scriptstyle \deg_U} & & \downarrow {\scriptstyle \deg_{U'}} \\ \boldsymbol{Z}/I(S_0) & \longrightarrow & \boldsymbol{Z}/I(S_0') \end{array}$$

commutes.
3) Let $f : S \to S'$ be a morphism of projective varieties of the same dimension. Let $S_0 \subset S$, $S_0' \subset S'$ be closed subschemes, and set $U := S \setminus S_0$, $U' := S' \setminus S_0'$. Assume that $f^{-1}(S_0') \subseteq S_0$ and hence $f(U) \subseteq U'$. Finally assume that U' is smooth. Then for every cycle $Z \in CH_0 U'$

$$\deg_U\left((f|_U)^*(Z)\right) \equiv (\deg f)\, \deg_{U'}(Z) \mod I(S_0) + I(S_0').$$

Proof. (Sketch) For 1) note that $S_0' \subset S_0$ implies that $U \subset U'$, and $I(S_0') \subseteq I(S_0)$, so all the claims make sense and follow from the definition. Similarly, 2) follows from the commutative diagram

$$\begin{array}{ccccccc} CH_0(S_0) & \longrightarrow & CH_0(S) & \longrightarrow & CH_0(U) & \longrightarrow & 0 \\ \big\downarrow{\scriptstyle (f|_{S_0})_*} & & \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle (f|_U)_*} & & \\ CH_0(S_0') & \longrightarrow & CH_0(S') & \longrightarrow & CH_0(U') & \longrightarrow & 0. \end{array}$$

For 3) observe, first of all, that for $Z \in CH_0(U')$

$$(f|_U)^*(Z) = (f|_{f^{-1}(U')})^*(Z)|_U.$$

Hence by 1)

$$\deg_U (f|_U)^*(Z) \equiv \deg_{f^{-1}(U')}(f|_{f^{-1}(U')})^*(Z) \mod I(S_0).$$

Thus replacing U by $f^{-1}(U')$ we may assume that $U = f^{-1}(U')$. Now $f|_U : U \to U'$ is proper and the projection formula yields:

$$(f|_U)_*((f|_U)^*(Z)) = (\deg f)Z \in CH_0(U').$$

Finally according to 2) we get

$$\begin{aligned}(\deg f)\deg_{U'}(Z) &\equiv \deg_{U'}(f|_U)_*((f|_U)^*(Z)) \\ &\equiv \deg_U((f|_U)^*(Z)) \qquad \mod I(S_0) + I(S_0').\end{aligned}$$

Next we construct an invariant for pseudo-Galois coverings.

3.2. **Definition.** Let $p : X \to S$ be a finite surjective morphism of integral schemes. Let G be a finite group acting on X over S. The covering p is called *pseudo-Galois* provided that $k(X)/k(S)$ is a Galois field extension and the natural map $G \to Gal(k(X)/k(S))$ is an isomorphism.

3.3. **Remark.** Under the conditions of the definition there is an induced birational morphism $\bar{p} : X_G \to S$. If in addition S is normal then $\bar{p}$ is an isomorphism by Zariski's Main Theorem.

3.4. **Remark.** It is well known and easy to check that every diagram of the form

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ S' & \longrightarrow & S \end{array}$$

where the vertical morphisms are Galois coverings with the same group G and the top horizontal morphism is G-equivariant is in fact Cartesian.

3.5. **Notation.** Let $S_{unr} \subseteq S$ be the open subscheme over which the morphism p is étale, and let $S_{ram} := S \setminus S_{unr}$ be the closed ramification subscheme.

To simplify matters we will only consider pseudo-Galois coverings with $G = \boldsymbol{Z}/l$. We assume that $char\, k \neq l$ and that k contains a primitive l-th root of unity. Furthermore we choose an identification $\mu_l = \boldsymbol{Z}/l$.

The Kummer sequence $1 \longrightarrow \mu_l \longrightarrow \mathbb{G}_m \xrightarrow{l} \mathbb{G}_m \longrightarrow 1$ induces an epimorphism $H^1_{et}(S_{unr}, \mu_l) \twoheadrightarrow {}_l Pic(S_{unr})$.

Finally starting with p we get an étale Galois covering $p^{-1}(S_{unr}) \longrightarrow S_{unr}$, the corresponding element in $H^1_{et}(S_{unr}, \boldsymbol{Z}/l) = H^1_{et}(S_{unr}, \mu_l)$, its image in ${}_l Pic(S_{unr})$, and thus an invertible sheaf $L(X/S)$ on S_{unr}.

3.6. **Definition.** Assume that $p : X \to S$ is a pseudo-Galois covering with group $G = \boldsymbol{Z}/l$, that S is projective, and the assumptions made above hold. Assume further that there exists a closed subscheme $S_{bad} \subset S$ such that:

(a) $I(S_{bad}) \subseteq l\boldsymbol{Z}$;
(b) $S_{good} := S \setminus S_{bad}$ is smooth;
(c) over S_{good} the morphism p is étale.

This data determines an invertible sheaf $L(X/S) \in {}_l Pic(S_{good})$ and a zero-cycle $Z(X/S)$ defined as $c_1(L(X/S))^{\dim S} \in CH_0(S_{good})$. Finally we define the η-invariant of the covering p to be:

$$\eta(X/S) := \deg_{S_{good}}(c_1(L(X/S))^{\dim S}) \in \boldsymbol{Z}/l.$$

3.7. **Remark.** Note that $\eta(X/S)$ does not depend on the choice of a closed subscheme S_{bad}. If $\tilde{S}_{bad}$ is another such subscheme one could compute η using $\tilde{S}_{bad} \cup S_{bad}$ and, according to lemma 3.1(1), get the same result. At the same time $\eta(X/S)$ depends on the choice of a primitive root of unity $\zeta \in \mu_l$. Once ζ is replaced by ζ^s, $L(X/S)$ gets replaced by $L(X/S)^{\otimes s}$, and $\eta(X/S)$ by $(s^{\dim S})\eta(X/S)$. This will not cause any difficulties as long as the same choice is maintained throughout.

3.8. **Theorem** (Markus Rost's Degree Formula)**.** *Assume that k is a field of characteristic zero, that X/S and X'/S' are two pseudo-Galois coverings with the same Galois group $G = \boldsymbol{Z}/l$, that both S and S' are projective of the same dimension d, and that $\eta(X/S)$ and $\eta(X'/S')$ are defined. Then for any G-equivariant rational map $g : X - \to X'$*

$$\eta(X/S) = (\deg g)\, \eta(X'/S') \in \boldsymbol{Z}/l.$$

Proof. Note that g induces a morphism from a neighborhood of the generic point of S to S'. Hence there is a unique rational map $f : S - \to S'$ compatible with g and clearly $\deg f = \deg g$.

3.9. **Lemma.** *Let X/S and X'/S' be pseudo-Galois coverings with the same group G fitting into an equivariant diagram of morphisms and rational maps*

$$\begin{array}{ccc} X & \overset{g}{\dashrightarrow} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ S & \overset{f}{\dashrightarrow} & S'. \end{array}$$

1) Assume that g is everywhere defined and that S is normal. Then f is everywhere defined.
2) Assume that f is everywhere defined and that X is normal. Then g is everywhere defined.

Proof. 1) The morphism g induces a morphism $\bar{g} : X_G \to X'_G$. Since S is normal, p induces an isomorphism $\tilde{p} : X_G \to S$. Hence the birational map f may be defined everywhere by $\tilde{p}' \circ \bar{g} \circ (\tilde{p})^{-1}$, where $\tilde{p}' : X'_G \to S'$ is induced by p'.

2) Let $\tilde{X}$ be the normalization of S' in $k(X')$. Since X'/S' is finite there is a morphism $\rho : \tilde{X} \to X'$ over S'. Because X is normal g induces a morphism $\tau : X \to \tilde{X}$. Evidently the morphism $\tau \circ \rho$ represents the rational map g.

First we prove the special case of the theorem. Suppose that g is everywhere defined and that S is normal. By the lemma f is everywhere defined and we get the diagram of morphisms

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ S & \xrightarrow{f} & S'. \end{array}$$

Replacing S_{bad} by $S_{bad} \cup f^{-1}(S'_{bad})$ if necessary we may assume that $S_{bad} \supseteq f^{-1}(S'_{bad})$. In the equivariant diagram

$$\begin{array}{ccc} p^{-1}(S_{good}) & \xrightarrow{g|_{p^{-1}(S_{good})}} & (p')^{-1}(S'_{good}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ S_{good} & \xrightarrow{f|_{S_{good}}} & S'_{good} \end{array}$$

both vertical arrows are étale coverings with the same Galois group G. According to remark 3.4, the left one is the pull-back of the right one and the diagram is Cartesian. In particular $L(X/S) = (f|_{S_{good}})^* L(X'/S')$. (Recall that both are in ${}_l Pic(S_{good})$.) Therefore $Z(X/S) = (f|_{S_{good}})^*(Z(X'/S'))$ in $CH_0(S_{good})$. Finally part (3) of lemma 3.1 completes the proof of this special case.

Now to the general case.

3.10. **Lemma.** *Assume that X/S is any pseudo-Galois covering with Galois group $G = \boldsymbol{Z}/l$. Let $\phi : \tilde{S} \to S$ be a birational morphism. Let $\tilde{X}$ be the normalization of $\tilde{S}$ in the finite field extension $k(X) \supset k(S) = k(\tilde{S})$.*
1) There exists a unique morphism $\psi : \tilde{X} \to X$ that completes the diagram:

$$\begin{array}{ccc} \operatorname{Spec} k(\tilde{X}) & = & \operatorname{Spec} k(X) \\ \downarrow & & \downarrow \\ \tilde{X} & \xrightarrow{\psi} & X \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{S} & \xrightarrow{\phi} & S. \end{array}$$

Moreover ψ is G-equivariant and $\tilde{X}/\tilde{S}$ is a pseudo-Galois covering with group G.
2)Assume in addition that S and $\tilde{S}$ are projective, $\tilde{S}$ is smooth and $\eta(X/S)$ is defined. Then $\eta(\tilde{X}/\tilde{S})$ is defined as well and $\eta(X/S) = \eta(\tilde{X}/\tilde{S}) \in \boldsymbol{Z}/l$.

Proof. The first claim is trivial by construction.

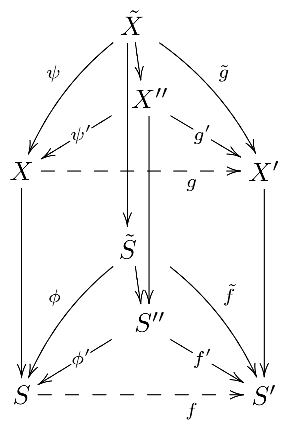
To prove the second one we need to check the conditions (a), (b), and (c) of definition 3.6. Set $\tilde{S}_{bad} := \phi^{-1}(S_{bad})$ hence $\tilde{S}_{good} = \phi^{-1}(S_{good})$. Since $I(\tilde{S}_{bad}) \subseteq I(S_{bad}) \subseteq l\boldsymbol{Z}$ condition (a) holds. Since $\tilde{S}$ is smooth so is $\tilde{S}_{good}$ hence (b) holds. For (c) observe that $\tilde{S}_{good}$ is smooth and hence normal, thus $\tilde{S}_{good} = (\tilde{p}^{-1}(\tilde{S}_{good}))_G$. In the following G-equivariant diagram the right vertical arrow is an étale Galois covering. Moreover since the action of G on $p^{-1}(S_{good})$ is free the action of G on

$\tilde{p}^{-1}(\tilde{S}_{good})$ is free as well.

$$\begin{array}{ccc} \tilde{p}^{-1}(\tilde{S}_{good}) & \xrightarrow{\psi} & p^{-1}(S_{good}) \\ \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} \\ \tilde{S}_{good} & \xrightarrow{\phi} & S_{good} \end{array}$$

Hence $\tilde{p} : \tilde{p}^{-1}(\tilde{S}_{good}) \to (\tilde{p}^{-1}(\tilde{S}_{good}))_G = \tilde{S}_{good}$ is also an étale Galois covering. Finally the equality $\eta(X/S) = \eta(\tilde{X}/\tilde{S})$ in $\boldsymbol{Z}/l$ follows from the special case of the theorem.

Now let X, S, X', S', g be as in the statement of the theorem. Set X'' to be the closure of the graph of g in $X \times X'$. Two projections induce the birational morphism $\psi' : X'' \to X$ and the morphism $g' : X'' \to X'$, so that $g' = g \circ \psi'$ as rational maps. Moreover G acts on X'', and ψ', g' are equivariant. Set $S'' := (X'')_G$, and choose a birational morphism $\tilde{S} \to S''$ with $\tilde{S}$ smooth. Let $\tilde{X}$ be the normalization of $\tilde{S}$ in $k(X) \supset k(S) = k(\tilde{S})$ as in the lemma. We get the diagram



where ϕ is the obvious composition, ψ is the morphism that comes from the lemma, the morphism $\tilde{X} \to X''$ also comes from the lemma, $\tilde{g}$ is another composition, and the remaining morphisms are the obvious ones.

Applying the lemma to $\tilde{X}/\tilde{S}$ and X/S we conclude that $\eta(\tilde{X}/\tilde{S})$ is defined and that

$$\eta(\tilde{X}/\tilde{S}) = \eta(X/S).$$

The coverings $\tilde{X}/\tilde{S}$ and X'/S' meet the conditions for the special case, so

$$\eta(\tilde{X}/\tilde{S}) = \deg \tilde{g}\, \eta(X'/S').$$

Since $\deg \tilde{g}$ equals $\deg g$ these two relations complete the proof of theorem 3.8.

Now we compute the η-invariant for coverings of a special type.

3.11. **Definition.** Let S/k be an arbitrary scheme, L an invertible sheaf of $\mathcal{O}_S$-modules and $\alpha \in \Gamma(S, L^{\otimes l})$ a global section. (Recall that we assume k to contain an l-th primitive root of unity.) Let $\boldsymbol{A}(L) := \operatorname{Spec}(S^*(L^{\#}))$ denote the line bundle corresponding to L. The sheaf $L_{\boldsymbol{A}(L)}$ has a canonical section T corresponding to the diagonal embedding $\boldsymbol{A}(L) \xrightarrow{\Delta} \boldsymbol{A}(L) \times_k \boldsymbol{A}(L) = \boldsymbol{A}(L_{\boldsymbol{A}(L)})$. Finally let $S(\sqrt[l]{\alpha})$

be the effective Cartier divisor in $\boldsymbol{A}(L)$ defined by the global section $T^{\otimes l} - \alpha \in \Gamma(\boldsymbol{A}(L), L^{\otimes l}_{\boldsymbol{A}(L)})$.

If L is trivial over some open affine $U \subset S$, then α determines a regular function a on U. Hence over U the scheme $S(\sqrt[l]{\alpha})$ is given by the equation $T^l - a = 0$ in $U \times \boldsymbol{A}^1$. In particular $S(\sqrt[l]{\alpha}) \to S$ is flat and finite of degree l.

3.12. **Lemma.** *Assume that S is smooth and irreducible and that $\alpha \notin \Gamma(S, L)^{\otimes l}$. Then $\phi : S(\sqrt[l]{\alpha}) \to S$ is a pseudo-Galois covering with group $G = \boldsymbol{Z}/l$.*

Proof. First we verify that $S(\sqrt[l]{\alpha})$ is integral. This may be checked locally. Over an affine U as above $S(\sqrt[l]{\alpha})$ coincides with $\operatorname{Spec} A[T]/T^l - a$ where $A = k[U]$. $\operatorname{Spec} A[T]/T^l - a$ is integral if and only if $T^l - a \in A[T]$ is irreducible. However if the latter polynomial is reducible then $a = b^l$ for $b \in k(U) = k(S)$ hence $\alpha = \beta^{\otimes l}$ for a rational section β of L. Noting that $l(\beta) = (\alpha)$ and so β has no poles we conclude that β is regular, and $\alpha \in \Gamma(S, L)^{\otimes l}$, a contradiction.

$\mathbb{G}_m$ acts naturally on $\boldsymbol{A}(L)$ and so does $\mu_l \subset \mathbb{G}_m$. As is evident from the local description $S(\sqrt[l]{\alpha})$ is μ_l-invariant and moreover

$$\mu_l \xrightarrow{\sim} Gal(k(U(\sqrt[l]{a}))/k(U)) = Gal(k(S(\sqrt[l]{\alpha}))/k(S)).$$

The identification $\mu_l = \boldsymbol{Z}/l$ completes the proof.

Evidently the covering ϕ is unramified away from the vanishing subscheme of $V(\alpha)$ of α. Thus the following corollary is almost straightforward.

3.13. **Corollary.** *Assume that S is smooth, projective, and irreducible, and that $\alpha \notin \Gamma(S, L)^{\otimes l}$. Assume further that $I(V(\alpha)) \subseteq l\boldsymbol{Z}$. Then $\eta(S(\sqrt[l]{\alpha})/S)$ is defined and equals* $\deg(-c_1 L)^{\dim S} \mod l\boldsymbol{Z}$.

Proof. Since ϕ is étale over $S_{good} := S \setminus V(\alpha)$ then η is defined. Since the invertible sheaf corresponding to this covering is the dual sheaf $L^\vee$ then $\eta(S(\sqrt[l]{\alpha})/S) = \deg(c_1 L^\vee)^{\dim S} = \deg(-c_1 L)^{\dim S}$.

4. Computations with b-classes

4.1. **Definition.** Let X/k be a smooth geometrically irreducible projective variety. The group $G = \boldsymbol{Z}/l$ acts on the irreducible variety X^l by cyclic permutations of factors. We call the factor variety $C^l(X) := (X^l)_G$ *the l-th cyclic power of X.*

4.2. **Remark.** Note that $C^l(X)$ is a normal projective variety and that the projection $p : X^l \to C^l(X)$ is a pseudo-Galois covering with group G. Let $\Delta : X \to X^l$ be the diagonal embedding of the fixed-point subscheme and let Δ_X be its image. Then $X^l \setminus \Delta_X \to C^l(X) \setminus p(\Delta_X)$ is an étale Galois covering with group G. In particular $C^l \setminus p(\Delta_X)$ is smooth. We thus set $C^l(X)_{bad} := p(\Delta_X)$.

4.3. **Lemma.** *The composite $X \overset{\Delta}{\hookrightarrow} X^l \xrightarrow{p} C^l(X)$ is a closed embedding identifying X with $p(\Delta_X)$.*

Proof. The statement is local with respect to X. For $X = \operatorname{Spec} A$ we need to show that

$$(A^{\otimes l})^G \longrightarrow A^{\otimes l} \xrightarrow{mult} A$$

is surjective. This is so because for every $a \in A$ the composition $(A^{\otimes l})^G \longrightarrow A$ maps $(1/l)\sum_1^l (1 \otimes \cdots \otimes a \otimes \cdots \otimes 1)$ to a. (Recall that *char* $k = 0$.)

It follows that $C^l(X)_{bad}$ is isomorphic to X. Hence $\eta_l(X) := \eta(X^l/C^l(X))$ is defined if and only if $I(X) \subseteq l\boldsymbol{Z}$.

When $l = 2$, the invariant η_2 may be computed via the following result.

4.4. **Theorem** (Rost)**.** *Let X/k be a smooth geometrically irreducible projective variety of dimension d. Then $\deg(c_d(-T_X)) \in 2\boldsymbol{Z}$. If in addition $I(X) \subseteq 2\boldsymbol{Z}$ then*

$$\eta_2(X) = \frac{\deg(c_d(-T_X))}{2} \mod 2\boldsymbol{Z}.$$

Proof. See Merkurjev's notes on the degree formula [M].

We will be mostly interested in the case $l > 2$. Let $c = 1 + c_1 + \cdots + c_d : K_0(X) \to CH^*(X)$, where $d = \dim X$, be the total Chern class. As in definition 1.20 we formally write $c = (1 - x_1) \dots (1 - x_d)$ with $\deg x_i = 1$.

4.5. **Definition.** *The total b-class of X* is a map $b = b^{(l)} : K_0(X) \to CH^*(X)$ defined as $b = b^{(l)} := (1 - x_1^{l-1}) \dots (1 - x_d^{l-1}) =: 1 + b_1 + \dots + b_l + \dots$

4.6. **Remark.** Note that the operation b is multiplicative, that is, $b(V \oplus V') = b(V)b(V')$ and that $b(L) = 1 - (-c_1(L))^{l-1}$ for a line bundle L. By the splitting principle, these two properties completely determine b. Also note that $b_i = 0$ unless $l - 1 | i$ and that $b_i = c_i$ for $l = 2$.

4.7. **Theorem** (Rost)**.** *Let X/k be a smooth geometrically irreducible projective variety of dimension d. Let l be a fixed prime. Then $\deg(b_d(-T_X)) \in l\boldsymbol{Z}$. If in addition $I(X) \subseteq l\boldsymbol{Z}$ then*

$$\eta_l(X) = \frac{\deg(b_d(-T_X))}{l} \mod l\boldsymbol{Z}.$$

In particular, $\eta_l(X) = 0 \mod l\boldsymbol{Z}$ if d is not a multiple of $l - 1$.

Proof. See Rost's 'Notes on Degree Formula' on the web page [CL].

4.8. **Proposition.** *Assume that $d = l^n - 1$ and that the conditions of the theorem hold. Then*

$$\deg(b_d(-T_X)) = \deg(s_d(T_X)) \mod l^2\boldsymbol{Z}.$$

In particular, if η_l is defined then

$$\eta_l(X) = \frac{\deg(s_d(T_X))}{l} \mod l\boldsymbol{Z}.$$

Proof. It is enough to prove the statement over the algebraic closure of k. Essentially one may either assume that $k = \mathbb{C}$ and use the topological complex cobordism theory or use the algebraic cobordism theory of Morel and Levine [LM].

Let $\Omega = MU_*$ be the Lazard ring of bordism classes. We need to show that the following map is the zero map.

$$\begin{array}{ccc} \Omega_d & \longrightarrow & \boldsymbol{Z}/l \\ [X] & \longmapsto & \dfrac{\deg(b_d(-T_X))}{l} - \dfrac{\deg(s_d(T_X))}{l}. \end{array}$$

After localization at the prime l the component Ω_d is additively generated by decomposable elements and the class of any hypersurface of degree l in $\boldsymbol{P}^{d+1}$.

Case of a decomposable $[X]$. Suppose $X = X_1 \times X_2$ with $d_1 := \dim X_1, d_2 := \dim X_2 < d$. Because $s_d(Y) = 0$ whenever $\dim Y < d$ we get

$$s_d(T_X) = s_d(T_{X_1\times X_2}) = s_d(p_1^*(T_{X_1}) \oplus p_2^*(T_{X_2})) = p_1^*(s_d(T_{X_1})) + p_2^*(s_d(T_{X_2})) = 0$$

Since the total b-class is multiplicative and commutes with pull-backs

$$b_d(-T_X) = b_d(-T_{X_1\times X_2}) = \sum_{i+j=d} p_1^*(b_i(-T_{X_1}))p_2^*(b_j(-T_{X_2}))$$
$$= p_1^*(b_{d_1}(-T_{X_1}))p_2^*(b_{d_2}(-T_{X_2}))$$

because all other terms vanish for dimensional reasons. Recall however that by theorem 4.7 each factor is a multiple of l hence the product vanishes modulo l^2. We see that $[X_1 \times X_2]$ indeed maps to zero.

Case of a hypersurface. Let $X \subset \boldsymbol{P}^{d+1}$ be a hypersurface of degree l. The ideal sheaf I of X is isomorphic to $\mathcal{O}_{\boldsymbol{P}^{d+1}}(-l)$ so the normal sheaf $N = (I|_X)^\vee$ is isomorphic to $\mathcal{O}_{\boldsymbol{P}^{d+1}}(l)|_X$. From the two standard exact sequences

$$0 \longrightarrow \mathcal{O}_{\boldsymbol{P}^{d+1}} \longrightarrow \mathcal{O}_{\boldsymbol{P}^{d+1}}(1)^{d+2} \longrightarrow T_{\boldsymbol{P}^{d+1}} \longrightarrow 0$$

$$0 \longrightarrow T_X \longrightarrow T_{\boldsymbol{P}^{d+1}}|_X \longrightarrow N \longrightarrow 0$$

we conclude that $[-T_X] = V|_X \in K_0(X)$ where

$$V := [\mathcal{O}_{\boldsymbol{P}^{d+1}}(l)] - (d+2)[\mathcal{O}_{\boldsymbol{P}^{d+1}}(1)] + [\mathcal{O}_{\boldsymbol{P}^{d+1}}] \in K_0(\boldsymbol{P}^{d+1}).$$

Therefore $s_d(T_X) = -s_d(-T_X) = -s_d(V)\,.\,X$ and $b_d(-T_X) = b_d(V)\,.\,X$.

Let $H = c_1(\mathcal{O}_{\boldsymbol{P}^{d+1}}(1))$ be the class of a hypersurface in $CH^1(\boldsymbol{P}^{d+1})$. Then (since $d = l^n - 1$) we get $s_d(V) = (lH)^d - (l^n+1)H^d = (l^d - l^n - 1)H^d$ so $s_d(-T_X) = (l^d - l^n - 1)(H^d\,.\,X)$. Hence $\deg(s_d(T_X)) = -(l^d - l^n - 1)\deg(H^d\,.\,X) = (1 - l^d + l^n)l$ and we conclude that

$$\frac{\deg(s_d(T_X))}{l} = 1 \mod l\boldsymbol{Z}.$$

Next we observe that $b_d(V)\,.\,X$ is the degree zero component of $b(V)\,.\,X$. Since $\deg(H^d\,.\,X) = l$ it follows from remark 4.6 that $\deg(b_d(-T_X))/l$ is the coefficient of H^d in the power series expansion of $b(V) = (1 - (-lH)^{l-1})/(1 - (-H)^{l-1})^{l^n+1}$ and we only need to know it modulo l.

Notice that $(-1)^{l-1} = 1 \mod l\boldsymbol{Z}$ for any prime l. Therefore we have the following sequence of identities in $\boldsymbol{Z}/l[[t]]$.

$$\begin{aligned}
\frac{(1-(-lt)^{l-1})}{(1-(-t)^{l-1})^{l^n+1}} &= \frac{1}{(1-t^{l-1})^{l^n+1}} \\
&= \frac{1}{(1-t^{l-1})^{l^n}(1-t^{l-1})} \\
&= \frac{1}{(1-t^{l^n(l-1)})(1-t^{l-1})} \\
&= (1 + t^{l^n(l-1)} + t^{2l^n(l-1)} + \dots)(1 + t^{l-1} + t^{2(l-1)} + \dots).
\end{aligned}$$

Since $l^n(l-1) > l^n - 1 = d$ the t^d term of the product above has coefficient 1 and we conclude that

$$\frac{\deg(b_d(-T_X))}{l} = 1 \mod l\boldsymbol{Z}$$

as well. This completes the proof of proposition 4.8.

5. The Chain Lemma

Let J be an invertible sheaf on X. A non-zero l-form $\gamma : J^{\otimes l} \to \mathcal{O}_X$ may be viewed as an element of $\Gamma(X, J^{\otimes(-l)})$. Let $U \subset X \setminus V(\gamma)$ be an open subscheme trivializing J and let $u \in \Gamma(U, J)$ be a non-vanishing section. Then $\gamma = au^{\otimes(-l)}$ for an appropriate $a \in \Gamma(U, \mathcal{O}_X^\times)$. Since a is well defined up to an l-th power the form γ gives rise to a well-defined element $\gamma_U \in \Gamma(U, \mathcal{O}_X^\times)/\Gamma(U, \mathcal{O}_X^\times)^l$.

Choose $x \in X \setminus V(\gamma)$. The above construction applied to neighborhoods of x provides an element $\gamma_x \in \mathcal{O}_x^\times/(\mathcal{O}_x^\times)^l$. Let $\gamma(x) \in k(x)^\times/(k(x)^\times)^l$ denote the corresponding element. When x is chosen to be the generic point it defines an element $\gamma(X) \in k(X)^\times/(k(X)^\times)^l$ assigned to the form γ. By abuse of notation we will write just γ instead of $\gamma(X)$ since no confusion will occur.

Let $J_1, \dots, J_n$ be invertible sheaves equipped with non-zero l-forms $\gamma_1, \dots, \gamma_n$ respectively. We can assign the symbol $\{\gamma_1, \dots, \gamma_n\} \in K_n^M(k(X))/l$ to this collection of sheaves and forms.

5.1. **Theorem** (Rost's Chain Lemma). *Let $\{a_1, \dots, a_n\} \in K_n^M(k)/l$ be a non-trivial n-symbol. Then there exists a smooth projective cellular variety S/k and a collection of invertible sheaves $J = J_1, J_1' \dots, J_{n-1}, J_{n-1}'$ equipped with non-zero l-forms $\gamma = \gamma_1, \gamma_1', \dots, \gamma_{n-1}, \gamma_{n-1}'$ respectively, satisfying the following conditions.*

(1) $\dim S = l(l^{n-1} - 1) = l^n - l$;
(2) $\{a_1, \dots, a_n\} = \{a_1, \dots, a_{n-2}, \gamma_{n-1}, \gamma_{n-1}'\} \in K_n^M(k(S))/l$,
$\{a_1, \dots, a_{i-1}, \gamma_i\} = \{a_1, \dots, a_{i-2}, \gamma_{i-1}, \gamma_{i-1}'\} \in K_i^M(k(S))/l$ *for* $2 \leqslant i \leqslant n-1$,
and in particular $\{a_1, \dots, a_n\} = \{\gamma, \gamma_1', \dots, \gamma_{n-1}'\} \in K_n^M(k(S))/l$;
(3) $\gamma \notin \Gamma(S, J)^{\otimes(-l)}$, *as is evident from (2);*
(4) *for any $s \in V(\gamma_i)$ or $V(\gamma_i')$ the field $k(s)$ splits $\{a_1, \dots, a_n\}$;*
(5) $I(V(\gamma_i)), I(V(\gamma_i')) \subseteq l\boldsymbol{Z}$ *for all i, as follows from (4);*
(6) $\deg(c_1(J)^{\dim S})$ *is relatively prime to l.*

Proof. See Markus Rost's 'Notes on Degree Formula' web page [CL] and also [RH].

Here is our first application of the Chain Lemma.

5.2. **Proposition.** *Let X be an geometrically irreducible l-generic splitting variety for a non-zero symbol $\{\underline{a}\} = \{a_1, \dots, a_n\} \in K_n^M(k)/l$ of dimension $d = l^{n-1} - 1$. Then X is a ν_{n-1}-variety.*

Proof. We adopt all the notation in the statement of the Chain Lemma. By construction $k(S)(\sqrt[l]{\gamma})$ splits $\{\underline{a}\}$. Let F_∞ be a maximal extension of $k(S)$ of degree prime to l. Then $F_\infty(\sqrt[l]{\gamma})$ is l-special and also splits $\{\underline{a}\}$. Hence there exists a morphism $\operatorname{Spec} F_\infty(\sqrt[l]{\gamma}) \to X$ over k. Since X is of finite type this morphism may be factored through $\operatorname{Spec} F(\sqrt[l]{\gamma}) \to X$ for a certain finite subextension $k(S) \subset F \subset F_\infty$. Starting with the embedding $k(S) \subset F$ we choose a model for F and then resolve singularities to obtain a smooth projective variety $\tilde{S}$ equipped with a dominant morphism $h : \tilde{S} \longrightarrow S$ of degree prime to l, and a rational map $\phi : \tilde{S}(\sqrt[l]{\gamma}) - \to X$. (Since $k(\tilde{S}(\sqrt[l]{\gamma})) = k(\tilde{S})(\sqrt[l]{\gamma}) = F(\sqrt[l]{\gamma})$.)

Let σ be a generator of $G = \boldsymbol{Z}/l = \mu_l$. By construction we get an equivariant diagram of pseudo-Galois coverings

$$\begin{array}{ccc} \tilde{S}(\sqrt[l]{\gamma}) & \overset{(\phi,\phi\sigma,\ldots,\phi\sigma^{l-1})}{\dashrightarrow} & X^l \\ \downarrow & & \downarrow \\ \tilde{S} & \dashrightarrow & C^l(X) \end{array}$$

with the bottom map induced by the top one. Note that $\dim S = \dim C^l(X)$. We will apply Rost's degree formula to this diagram.

First observe that by the Chain Lemma the form γ is not an l-th power and $I(V(\gamma)) \subseteq l\boldsymbol{Z}$. Hence $\eta(\tilde{S}(\sqrt[l]{\gamma})/\tilde{S})$ is defined and

$$\begin{aligned} \eta(\tilde{S}(\sqrt[l]{\gamma})/\tilde{S}) &= \deg(c_1(h^*(J))^{\dim \tilde{S}}) \\ &= \deg(h^*(c_1(J))^{\dim S}) \\ &= \deg(h_*(h^*(c_1(J))^{\dim S})) \\ &= \deg h \deg(c_1(J)^{\dim S}). \end{aligned}$$

Note that both factors are prime to l, by construction and by the Chain Lemma respectively.

Next recall that by theorem 4.7 and proposition 4.8,

$$\eta(X^l/C^l(X)) = \frac{\deg s_d(T_X)}{l} \mod l\boldsymbol{Z}.$$

Finally by the Degree Formula

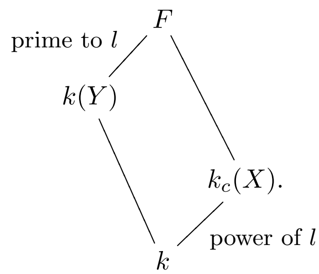
$$\deg h \deg(c_1(J)^{\dim S}) = \deg g \frac{\deg s_d(T_X)}{l} \mod l\boldsymbol{Z}$$

where $g := (\phi, \phi\sigma, \ldots, \phi\sigma^{l-1})$. We readily conclude that neither of the factors on the right is a multiple of l. In particular $\deg s_d(X) = \deg s_d(T_X) \neq 0 \mod l^2\boldsymbol{Z}$. Thus X is ν_{n-1}.

5.3. **Remark.** For any variety X/k let $k_c(X)$ denote *the field of constants of X*, that is, the algebraic closure of k in $k(X)$. It is well known, and easy to verify, that X is geometrically irreducible if and only if $k_c(X) = k$. Also note that a rational map $X \dashrightarrow Y$ induces an embedding $k_c(Y) \hookrightarrow k_c(X)$.

5.4. **Proposition.** *Assume that k is an l-special field. Then every l-generic splitting variety X/k for a symbol $\{\underline{a}\}$ is geometrically irreducible.*

Proof. Let Y be a geometrically irreducible l-generic splitting variety for $\{\underline{a}\}$ that exists according the first part of 1.21. Then there exists an extension $F/k(Y)$ of degree prime to l and a point $\operatorname{Spec} F \to X$. The fields involved form a diagram of embeddings



Since Y is geometrically irreducible $k(Y) \otimes_k k_c(X)$ is a subfield of F. Thus a degree count shows that $k_c(X) = k$ and X is geometrically irreducible.

The second claim of theorem 1.21 now follows from the above two propositions.

The second application of the Chain Lemma will be concerned with the so-called *multiplication principle*.

Consider the variety S of the Chain Lemma. Let $s \in S \setminus \bigcup_1^{n-1}(V(\gamma_i) \cup V(\gamma_i'))$ be a rational point. Specialization of $\{\gamma(S), \gamma_1'(S), \dots, \gamma_{n-1}'(S)\}$ from $K_n^M(k(S))/l$ to $K_n^M(k(s))/l$ amounts to evaluation. Hence

$$\{a_1, \dots, a_n\} = \{\gamma(s), \gamma_1'(s), \dots, \gamma_{n-1}'(s)\} \text{ in } K_n^M(k)/l.$$

In particular $k(\sqrt[l]{\gamma(s)})$ splits the symbol $\{\underline{a}\}$. (It may be shown that specialization to a rational point of S provides a universal way to rewrite the symbol.)

5.5. **Theorem.** *Let k be l-special. Let E/k be a cyclic extension of degree l splitting $\{\underline{a}\}$. Then there is a rational point $s \in S$ such that $k(\sqrt[l]{\gamma(s)}) = E$.*

Proof. Recall that there is a dominant morphism of smooth projective varieties $\tilde{S} \to S$, of degree prime to l, along with an equivariant diagram of pseudo-Galois coverings

$$\begin{array}{ccc} \tilde{S}(\sqrt[l]{\gamma}) & \overset{g=(\phi,\phi\sigma,\dots,\phi\sigma^{l-1})}{\dashrightarrow} & X^l \\ \downarrow & & \downarrow \\ \tilde{S} & \dashrightarrow & C^l(X). \end{array}$$

Moreover $\deg g$ is prime to l and *a fortiori* g is dominant. Using a resolution of singularities we can find a birational morphism $\hat{S} \to \tilde{S}$ from a smooth projective variety such that the composition $\hat{S} \to \tilde{S} - \to C^l(X)$ is everywhere defined. Then the previous diagram induces the following one:

$$\begin{array}{ccc} \hat{S}(\sqrt[l]{\gamma})_{\text{norm}} & \xrightarrow{\hat{g}} & X^l \\ \downarrow & & \downarrow \\ \hat{S} & \xrightarrow{\hat{f}} & C^l(X). \end{array}$$

This diagram is also equivariant and consists of pseudo-Galois coverings. Indeed, normalization does not change these properties. The bottom map is everywhere defined by construction, and so is the top one by lemma 3.9.

On the other hand, since E is also l-special and splits $\{\underline{a}\}$ there is an E-valued point $\psi : \operatorname{Spec} E \to X$ that gives rise to the diagram

$$\begin{array}{ccc} \operatorname{Spec} E & \xrightarrow{(\psi,\psi\sigma,\ldots,\psi\sigma^{l-1})} & X^l \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & C^l(X). \end{array}$$

Let the rational point $z \in C^l(X)$ be the image of the bottom map. Since the diagonal of X^l has no rational points, z belongs to both the smooth locus of $C^l(X)$ and the unramified locus of $X^l \to C^l(X)$. Hence the diagram is Cartesian and the fiber over z consists of a single E-rational point of X^l.

Since $\hat{f}$ is a dominant morphism of degree prime to l and z is smooth, using lemma 1.18 we can lift z to a rational point $\hat{s} \in \hat{S}$. Note that $V(\gamma)$ has no rational points, and that $\tilde{S}(\sqrt[l]{\gamma}) \to \hat{S}$ is unramified away from $V(\gamma)$. Therefore the fiber over $\hat{s}$ is the same in both $\hat{S}(\sqrt[l]{\gamma})$ and $\hat{S}(\sqrt[l]{\gamma})_{\mathrm{norm}}$. Moreover since the diagram in question is locally Cartesian near z, this fiber is a single point with residue field $k(\sqrt[l]{\gamma(\hat{s})}) = E$. Let s be the image of $\hat{s}$ under the projection $\hat{S} \to S$. Since both $\hat{s}$ and s are rational we conclude that $k(\sqrt[l]{\gamma(s)}) = k(\sqrt[l]{\gamma(\hat{s})}) = E$.

As an easy corollary we get following statement, also referred to as *the Chain Lemma.*

5.6. **Theorem.** *Let k be l-special and let $E_1, \ldots, E_n$ be a sequence of cyclic splitting fields of degree l for a non-trivial symbol $\{a_1, \ldots, a_n\} \in K_n^M(k)/l$. Then there exist $a'_1, \ldots, a'_n \in k^\times$ such that $\{a_1, \ldots, a_n\} = \{a'_1, \ldots, a'_n\}$ and that E_i splits $\{a'_1, \ldots, a'_i\}$ for each $1 \leqslant i \leqslant n$.*

Proof. Using induction on j we will show that we can rewrite the given symbol so that the last condition holds for $1 \leqslant i \leqslant j$.

The case $j = 1$ is settled by the previous theorem.

Induction step from $j-1$ to j. Applying the assumption to $E_2, \ldots, E_j$ we may rewrite the symbol so that E_2 splits $\{a_1\}$, $\ldots$, E_j splits $\{a_1, \ldots, a_{j-1}\}$. By the previous theorem we may find a rational point $s \in S$ such that $E_1 = k(\sqrt[l]{\gamma(s)})$. We set $a'_1 := \gamma(s)$, $a'_2 := \gamma'_1(s)$, $\ldots$, $a'_n := \gamma'_{n-1}(s)$. Then E_1 splits $\{a'_1\}$ and for each $1 < i \leqslant j$ the field E_j splits $\{a_1, \ldots, a_{j-1}\}$ hence also splits $\{a_1, \ldots, a_{j-1}, \gamma'_j(s)\} = \{\gamma(s), \gamma'_1(s), \ldots, \gamma_{j-2}(s), \gamma'_{j-1}(s)\} = \{a'_1, \ldots, a'_j\}$.

Observe that the third part of theorem 1.21 would be an immediate corollary of the following two statements. (Recall that we assume k to be l-special and X to be an $l^{n-1}-1$ dimensional l-generic splitting variety for an n-symbol $\{\underline{a}\}$.)

5.7. **Proposition** (Multiplication Principle)**.** *Let $[x, \lambda], [x', \lambda'] \in \bar{A}_0(X, \mathcal{K}_1)$ be such that $[k(x) : k] = [k(x') : k] = l$. Then there exist $x'' \in X$, $\lambda'' \in k(x'')^\times$ such that $[k(x'') : k] = l$ and $[x, \lambda] + [x', \lambda'] = [x'', \lambda'']$.*

5.8. **Proposition** (Norm Principle)**.** *Let $[x, \lambda] \in \bar{A}_0(X, \mathcal{K}_1)$ be such that $[k(x) : k] = l^m$, where $m > 1$. Then there exist $x_i \in X$, $\lambda_i \in k(x_i)^\times$ such that $[k(x_i) : k] < [k(x) : k]$ for all i and $[x, \lambda] = \sum_i [x_i, \lambda_i]$.*

Below we will give a proof of the multiplication principle. The proof of the norm principle will appear in [RH].

Using theorem 5.6 we may rewrite $\{\underline{a}\} = \{a'_1, a'_2, \dots\}$ so that $k(x)$ splits $\{a'_1\}$ and $k(x')$ splits $\{a'_1, a'_2\}$. Let $D := \binom{a'_1, a'_2}{k}$ be the cyclic division algebra and let $Y := SB(D)$ be its Severi-Brauer variety.

The following two facts in one form or another are well established in the folklore, so we only sketch their proofs.

5.9. **Lemma.** *The multiplication principle holds for Y.*

Proof. Let $[y, \lambda]$ in $\bar{A}_0(Y, \mathcal{K}_1)$ be such that $[k(y) : k] = l$. Then $k(y)$ may be identified with a maximal subfield of D and moreover $N([y, \lambda]) = \operatorname{Nrd}(\lambda) \in k^\times$. Recall that according to [MS] the map

$$N : A_0(Y, \mathcal{K}_1) \xrightarrow{\sim} \operatorname{Nrd}(D^\times) \subseteq k^\times$$

is an isomorphism. (Thus in this special case $\bar{A}_0(Y, \mathcal{K}_1) = A_0(Y, \mathcal{K}_1)$.)

Let $[y', \lambda']$ be the other summand. Form $\lambda\lambda'$ in $D^\times$ and choose $y'' \in Y$ such that $\lambda\lambda' \in k(y'')$. Since $N([y, \lambda])N([y', \lambda']) = \operatorname{Nrd}(\lambda\lambda') = N([y'', \lambda\lambda'])$ and N is an isomorphism, we conclude that

$$[y, \lambda] + [y', \lambda'] = [y'', \lambda\lambda'].$$

5.10. **Lemma.** *Let $f : \tilde{Z} \to Z$ be a dominant morphism of smooth projective varieties of degree relatively prime to l. Then $f_* : \bar{A}_0(\tilde{Z}, \mathcal{K}_1) \to \bar{A}_0(Z, \mathcal{K}_1)$ is an isomorphism.*

Proof. Recall that the base field k is assumed to be l-special. Hence for each generator $[z, \lambda]$ of $\bar{A}_0(Z, \mathcal{K}_1)$ one can find, according to 1.18, a point $\tilde{z} \in \tilde{Z}$ that maps to z so that $k(\tilde{z}) = k(z)$. Thus $f_*([\tilde{z}, \lambda]) = [z, \lambda]$ and we conclude that f_* is surjective.

To prove injectivity we first show that the composition $f^* f_*$ coincides with multiplication by $\deg f$.

Choose any generator $[\tilde{z}, \lambda]$ of $\bar{A}_0(\tilde{Z}, \mathcal{K}_1)$. Let $z = f(\tilde{z})$. As above, one can find $\tilde{z}'$ in the fiber over z having residue field $k(\tilde{z}') = k(z)$. According to corollary 1.5 we get $[\tilde{z}, \lambda] = [\tilde{z}', N_{k(\tilde{z})/k(z)}(\lambda)]$. Thus replacing one by the other we may assume that $\tilde{z}$ and $z = f(\tilde{z})$ have isomorphic residue fields. Consider any open $U \subset Z$ over which f is finite. One can show that $\bar{A}_0(Z, \mathcal{K}_1)$ and $\bar{A}_0(\tilde{Z}, \mathcal{K}_1)$ are generated by points from U and $\tilde{U} := f^{-1}(U)$ respectively. Hence we may assume that $\tilde{z} \in \tilde{U}$. In this case the fiber of f over z is finite. Assume that this fiber consists of points $\tilde{z}_1 = \tilde{z}, \dots, \tilde{z}_k$, counted with multiplicities. Explicit computation shows that:

$$\begin{aligned} f^* f_*([\tilde{z}, \lambda]) = f^*([z, \lambda]) &= \sum_1^k [\tilde{z}_i, \lambda] = \sum_1^k [\tilde{z}, N_{k(\tilde{z}_i)/k(z)}(\lambda)] \\ &= \sum_1^k [\tilde{z}, \lambda^{[k(\tilde{z}_i):k(z)]}] = \Big(\sum_1^k [k(\tilde{z}_i) : k(z)]\Big)[\tilde{z}, \lambda] = (\deg f)[\tilde{z}, \lambda]. \end{aligned}$$

In particular we conclude that $\ker f_*$ is annihilated by $\deg f$. On the other hand, the diagram

$$\begin{array}{ccc} \bar{A}_0(\tilde{Z}, \mathcal{K}_1) & \xrightarrow{f_*} & \bar{A}_0(Z, \mathcal{K}_1) \\ {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle N} \\ k^\times & \xrightarrow{=} & k^\times \end{array}$$

along with corollary 1.7 demonstrates that $\ker f_* \subseteq \ker N$ is annihilated by the degree of any closed point, that is, by some power of l. Since $(deg f, l) = 1$ we conclude that $\ker f = 0$, i.e., f_* is injective as well.

The rest of the proof of the multiplication principle for the generic splitting variety X goes as follows. Recall that $Y = SB(D)$ and note that $k(Y)$ splits $\{\underline{a}\}$. Therefore we can construct a smooth projective variety $\tilde{Y}$ along with a dominant morphism $p : \tilde{Y} \to Y$ of degree relatively prime to l such that there exists a morphism $\pi : \tilde{Y} \to X$.

Let $y, y' \in Y$ be such that $k(y) \simeq k(x)$, $k(y') \simeq k(x')$. According to 5.9 one can find another point $y'' \in Y$ of degree l and $\lambda'' \in k(y'')^\times$ such that $[y, \lambda] + [y', \lambda'] = [y'', \lambda'']$ in $\bar{A}_0(Y, \mathcal{K}_1)$. Points y, y', y'' may be lifted as in the proof of 5.10 to $\tilde{y}, \tilde{y}', \tilde{y}'' \in \tilde{Y}$ with the same residue fields and moreover $[\tilde{y}, \lambda] + [\tilde{y}', \lambda'] = [\tilde{y}'', \lambda'']$ in $\bar{A}_0(\tilde{Y}, \mathcal{K}_1)$. Pushing $\tilde{y}, \tilde{y}', \tilde{y}''$ down to X we finally obtain points z, z', z'' such that $k(z) \simeq k(\tilde{y}) \simeq k(y) \simeq k(x)$, similarly $k(z') \simeq k(y')$, and $k(z'') \simeq k(y'')$, along with the relation

$$[x, \lambda] + [x', \lambda'] = [z, \lambda] + [z', \lambda'] = [z'', \lambda'']$$

as required.

Appendix A. Cokernel of the norm map

In this appendix we outline a proof of the following theorem of Voevodsky, which is crucial for the construction of l-generic splitting varieties. We use here the machinery developed by Voevodsky in [MC/l], borrowing also some ideas from [MC/2] and [OVV].

In what follows we assume that the Bloch–Kato conjecture has already been established in weights $\leqslant n$. In particular we assume, as a part of the induction process, that for each non-trivial symbol in K_n^M there exists a ν_{n-1} splitting variety.

A.1. **Theorem.** *Assume that the base field k is l-special and that $\{a_1, \dots a_n\} \in K_n^M/l$ is a non-trivial symbol. Let X be a smooth ν_{n-1} splitting variety for the symbol $\{a_1, \dots a_n\}$. Then the following sequence is exact.*

$$A_0(X, \mathcal{K}_1) \xrightarrow{N} k^\times \xrightarrow{\{a_1, \dots a_n\}\cup} K_{n+1}^M(k)/l$$

Proof. Our approach is to perform a series of reductions. We will be freely using the techniques of [MC/l, sections 5, 6], sometimes without direct reference.

To begin note that the composition of the above two maps is trivial by projection formula. It thus suffices to prove exactness for the following sequence.

$$A_0(X, \mathcal{K}_1) \xrightarrow{N} k^\times \xrightarrow{\{a_1, \dots a_n\}\cup} H_{et}^{n+1}(k, \mu_l^{\otimes n+1})$$

Note further that X is an l-generic splitting variety for $\{a_1, \dots a_n\}$ according to theorem 7.3 of [MC/l]. (Otherwise one can make it a condition of the theorem to prove.) Since $k(\sqrt[l]{a_1})$ splits the symbol and is l-special we conclude that X has points of degree l and in particular that $N(A_0(X, \mathcal{K}_1)) \supseteq (k^\times)^l$. This shows that it will be enough to establish the exactness of the sequence

$$A_0(X, \mathcal{K}_1)/l \xrightarrow{N} k^\times/(k^\times)^l \xrightarrow{\{a_1, \dots a_n\}\cup} H_{et}^{n+1}(k, \mu_l^{\otimes n+1}).$$

Set $d := \dim X = l^{n-1} - 1$, and write $\tau : \boldsymbol{Z}(d)[2d] \to M(X)$ for the fundamental class of X. It is well-known that after the identification $A_0(X, \mathcal{K}_1) = H^{2d+1,d+1}(X)$

the homomorphism N is identified with the pull-back via the fundamental class

$$(\tau)^* : H^{2d+1,d+1}(X) \longrightarrow H^{2d+1,d+1}(\boldsymbol{Z}(d)[2d]) = H^{1,1}(k) = k^\times.$$

Denote further by $\mathcal{X} := \check{C}(X)$ the Čech simplicial scheme of X, see e.g., [MC/2, appendix B]. The validity of the Bloch–Kato conjecture in weights $\leqslant n$ provides for the following computation of some of the motivic cohomology groups of $\mathcal{X}$:

(1) $H^{p,q}(\mathcal{X}, \boldsymbol{Z}/l) = H^{p,q}(k, \boldsymbol{Z}/l)$ for $p \leqslant q \leqslant n$

(2) $H^{p,p-1}(\mathcal{X}, \boldsymbol{Z}/l) = \ker\left(H^p_{et}(k, \mu_l^{\otimes(p-1)}) \longrightarrow H^p_{et}(k(X), \mu_l^{\otimes(p-1)})\right)$ for $p \leqslant n+1$.

We fix once and for all a primitive l-th root of unity in k that allows us to identify all the étale sheaves $\mu_l^{\otimes i}$. Let $\delta \in H^n_{et}(k, \mu_l^{\otimes(n-1)})$ denote the cohomology class corresponding to $\{a_1, \dots a_n\} \in H^n_{et}(k, \mu_l^{\otimes n})$ under this identification. According to (2) above we see that $\delta \in H^{n,n-1}(\mathcal{X}, \boldsymbol{Z}/l)$. Since $H^{n+1,n}(\mathcal{X}, \boldsymbol{Z}/l) \hookrightarrow H^{n+1}_{et}(k, \mu_l^{\otimes n})$ we finally conclude that it suffices to establish the exactness of the sequence

$$H^{2d+1,d+1}(X, \boldsymbol{Z}/l) \xrightarrow{(\tau_{\mathcal{X}})^*} H^{1,1}(\mathcal{X}, \boldsymbol{Z}/l) \xrightarrow{\cup\delta} H^{n+1,n}(\mathcal{X}, \boldsymbol{Z}/l),$$

where $\tau_{\mathcal{X}} : M(\mathcal{X})(d)[2d] \to \boldsymbol{Z}(d)[2d] \xrightarrow{\tau} M(X)$ is the relative fundamental class.

To analyze the image of $(\tau_{\mathcal{X}})^*$ we recall the construction presented in [MC/l, section 6]. We set

$$\mu := \tilde{Q}_0 Q_1 \dots Q_{n-1}(\delta) \in H^{2b+1,b}(\mathcal{X}, \boldsymbol{Z}_{(l)}) = \mathrm{Hom}_{DM_-(k)}(M(\mathcal{X}), M(\mathcal{X})(b)[2b+1])$$

where $b = (l^{n-1}-1)/(l-1)$, Q_i are the Milnor operations, and $\tilde{Q}_0$ is the integral Bockstein homomorphism.

We also set $M := \mathit{cone}(\mu)[-1]$ so that we get a distinguished triangle

$$M(\mathcal{X})(b)[2b] \xrightarrow{x} M \xrightarrow{y} M(\mathcal{X}) \xrightarrow{\mu} M(\mathcal{X})(b)[2b+1]$$

and set further $M_i := S^i(M)$. Voevodsky constructs in [MC/l, section 5] the canonical distinguished triangles

$$M_{i-1}(b)[2b] \xrightarrow{v} M_i \xrightarrow{S^i y} M(\mathcal{X}) \xrightarrow{s} M_{i-1}(b)[2b+1]$$

and

$$M(\mathcal{X})(bi)[2bi] \xrightarrow{S^i x} M_i \xrightarrow{u} M_{i-1} \xrightarrow{r} M(\mathcal{X})(bi)[2bi+1].$$

The motive $M := M_{l-1}$ is of special interest. It is called *the generalized Rost motive.* Voevodsky proves in [MC/l, section 5] that there exists $\lambda : M(X) \to M_{l-1} = M$ which makes the following diagram commutative

$$\begin{array}{ccc} M(X) & \xrightarrow{\lambda} & M_{l-1} \\ \downarrow{\scriptstyle \pi_{\mathcal{X}}} & & \downarrow{\scriptstyle S^{l-1}y} \\ M(\mathcal{X}) & = & M(\mathcal{X}) \end{array}$$

and also that the diagram

$$\begin{array}{ccc} M(\mathcal{X})(d)[2d] & = & M(\mathcal{X})(d)[2d] \\ \downarrow{\scriptstyle \tau_{\mathcal{X}}} & & \downarrow{\scriptstyle S^{l-1}x} \\ M(X) & \xrightarrow{\lambda} & M_{l-1} \end{array}$$

commutes up to a scalar from $\boldsymbol{Z}^*_{(l)}$. Finally he also proves that λ is a split epimorphism so that M is a direct summand in $M(X)$ in the category of motives with $\boldsymbol{Z}_{(l)}$-coefficients.

From the second of the above diagrams we conclude immediately that the image of $\tau^*_{\mathcal{X}} : H^{2d+1,d+1}(X, \boldsymbol{Z}/l) \to H^{1,1}(\mathcal{X}, Z/l)$ contains the image of $(S^{l-1}x)^* : H^{2d+1,d+1}(M, \boldsymbol{Z}/l) \to H^{1,1}(\mathcal{X}, Z/l)$. Furthermore the exact sequence of motivic cohomology corresponding to the distinguished triangle

$$M(\mathcal{X})(d)[2d] \xrightarrow{S^{l-1}x} M_{l-1} \xrightarrow{u} M_{l-2} \xrightarrow{r} M(\mathcal{X})(d)[2d+1]$$

shows that

$$\operatorname{im}((S^{l-1}x)^*) = \ker\left(H^{1,1}(\mathcal{X}, \boldsymbol{Z}/l) \xrightarrow{r^*} H^{2d+2,d+1}(M_{l-2}, \boldsymbol{Z}/l)\right).$$

Next consider the exact cohomology sequence corresponding to the distinguished triangle

$$M_{l-2}(b)[2b] \xrightarrow{v} M_{l-1} \xrightarrow{S^{l-1}y} M(\mathcal{X}) \xrightarrow{s} M_{l-2}(b)[2b+1].$$

namely

$$H^{2d+2+2b,d+b+1}(M_{l-1}, Z/l) \longrightarrow H^{2d+2,d+1}(M_{l-2}, Z/l) \xrightarrow{s^*}$$
$$\xrightarrow{s^*} H^{2d+2+2b+1,d+b+1}(\mathcal{X}, Z/l) \longrightarrow H^{2d+2+2b+1,d+b+1}(M_{l-1}, Z/l).$$

Since M_{l-1} is a direct summand in $M(X)$ we conclude that the cohomology group $H^{2(d+b+1),d+b+1}(M_{l-1}, Z/l)$ is a direct summand in $H^{2(d+b+1),d+b+1}(X, Z/l) = CH^{d+b+1}(X)/l = 0$. (Note that $d+b+1 > d = \dim X$.) Thus the homomorphism

$$s^* : H^{2d+2,d+1}(M_{l-2}, Z/l) \to H^{2(d+b+1)+1,d+b+1}(\mathcal{X}, Z/l)$$

is injective. (In fact it is an isomorphism.) Finally we conclude that

$$\operatorname{im}(S^{l-1}x)^* = \ker\left(H^{1,1}(\mathcal{X}, \boldsymbol{Z}/l) \xrightarrow{s^*r^*} H^{2(d+b+1)+1,d+b+1}(\mathcal{X}, Z/l)\right)$$

The homomorphism s^*r^* coincides clearly with multiplication by the element $r \circ s$ of $H^{2(d+b+1),d+b}(\mathcal{X}, \boldsymbol{Z}/l)$. Let $\bar{\mu}$ denote the reduction of μ modulo l. As in [MC/l] we view $\bar{\mu}$ as an element of $H^{2b+1,b}(\mathcal{X}, \boldsymbol{Z}/l)$. According to [*ibid.* discussion preceding lemma 3.2 and theorem 3.8] the cohomology class $r \circ s$ coincides with $cQ_0P^b(\bar{\mu})$ for some $c \in (\boldsymbol{Z}/l)^*$.

To finish the computation we utilize the following relation between the Steenrod and the Milnor operations established in [MC/l, lemma 5.13]:

$$Q_0P^b = P^bQ_0 + P^{b-1}Q_1 + P^{b-l-1}Q_2 + \cdots + P^0Q_n.$$

Since $\bar{\mu} = Q_0Q_1 \ldots Q_{n-1}(\delta)$ is annihilated by $Q_0, Q_1, \ldots, Q_{n-1}$ we conclude that

$$Q_0P^b(\bar{\mu}) = Q_0Q_1 \ldots Q_n(\delta)$$

and hence that $\operatorname{im}(S^{l-1}x)^*$ coincides with the kernel of the homomorphism

$$k^*/(k^*)^l = H^{1,1}(\mathcal{X}, \boldsymbol{Z}/l) \xrightarrow{Q_0 \ldots Q_n(\delta)} H^{2(d+b+1)+1,d+b+1}(\mathcal{X}, Z/l).$$

To compute the kernel of this map we note that all operations Q_i are $K^M_*(k)$-linear maps and in particular that

$$\{a\} \cup Q_0 \ldots Q_n(\delta) = Q_0 \ldots Q_n(\delta \cup \{a\}).$$

However the standard application of Margolis acyclicity implies that the homomorphism

$$Q_0 \ldots Q_n : H^{n+1,n}(\mathcal{X}, \boldsymbol{Z}/l) \longrightarrow H^{2(d+b+1)+1,d+b+1}(\mathcal{X}, Z/l)$$

is injective and hence that the kernel of multiplication by $Q_0 \ldots Q_n(\delta)$ coincides with the kernel of multiplication by δ.

This final reduction completes the proof.

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Andrei Suslin:
Department of Mathematics
Northwestern University
2033 Sheridan Road
Evanston, IL 60208-2730
E-mail address: suslin@math.northwestern.edu

Seva Joukhovitski:
Department of Mathematics
University of California, Los Angeles
520 Portola Plaza
Los Angeles, CA 90095-1555
E-mail address: seva@member.ams.org