A BLOW-UP RELATION IN ALGEBRAIC COBORDISM

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Let $i : Y \hookrightarrow X$ be a closed embedding of smooth quasi-projective varieties, and let $N = N_{X/Y}$ denote the normal bundle to Y in X and $\pi : \mathbb{P}_Y(N \oplus 1) \to Y$ the projection. Define

$$\phi(X,Y) = [X] - i_*\pi_*[\mathbb{P}_Y(N \oplus 1)] \in \Omega(X)\,.$$

We think of $\mathbb{P}_Y(N \oplus 1)$ as a (compactified) turbular neighbourhood of Y, thus roughly speaking, $\phi(X,Y)$ can be thought of as "the class of X with a turbular neighbourhood of Y removed".

Let $p : \widehat{X}_Y \to X$ denote the blow-up of X at Y. Since "X with a turbular neighbourhood of Y removed" can be likely identified with "$\widehat{X}_Y$ with a turbular neighbourhood of $p^{-1}(Y) = \mathbb{P}(N)$ removed", one can expect that $\phi(X,Y)$ is pretty much the same as $\phi(\widehat{X}_Y, \mathbb{P}(N))$. This proves to be the case.

Theorem. $p_*(\phi(\widehat{X}_Y, \mathbb{P}(N))) = \phi(X,Y)$.

Lemma. *Blowing up and taking the "turbular neighbourhood" are interchangeable in the following sense: the blow-up of* $\mathbb{P}_Y(N\oplus 1)$ *at* Y *(embedded as the zero section of* N*) is canonically isomorphic to the "turbular neighbourhood" of* $\mathbb{P}(N)$ *in the blow-up* $\widehat{X}_Y$*,*

$$\mathbb{P}_Y\widehat{(N \oplus 1)}_Y \cong \mathbb{P}_{\mathbb{P}(N)}(\widehat{N} \oplus 1)\,, \tag{aa}$$

where $\widehat{N} = N_{\widehat{X}_Y/\mathbb{P}(N)}$. *(Clearly* $\widehat{N} \cong \mathcal{O}_{\mathbb{P}(N)}(-1)$*, which we do not use.)*

Proof of the lemma. (i) If $Y = pt$, then (aa) becomes $\widehat{\mathbb{P}^n}_{pt} \cong \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}(-1) \oplus 1)$, which is elementary. (ii) As the isomorphism in (i) is canonical, it gives rise to (aa) over a base Y. Details are left to the reader.

Let $\widehat{\pi} : \mathbb{P}_{\mathbb{P}(N)}(\widehat{N} \oplus 1) \to \mathbb{P}(N)$ denote the projection. By abuse of notation, we will also write $\widehat{\pi}$ for the map $\mathbb{P}_Y\widehat{(N \oplus 1)}_Y \to \mathbb{P}(N)$ obtained from the former $\widehat{\pi}$ by composing with (aa). We introduce more notation for arrows by the diagram

$$\begin{array}{ccccc} \widehat{X}_Y & \xleftarrow{\widehat{i}} & \mathbb{P}(N) & \xleftarrow{\widehat{\pi}} & \mathbb{P}_Y\widehat{(N \oplus 1)}_Y \\ {\scriptstyle p}\downarrow & & {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle p_N} \\ X & \xleftarrow{i} & Y & \xleftarrow{\pi} & \mathbb{P}_Y(N \oplus 1) \end{array} \tag{bb}$$

Proof of the theorem. We want to prove that

$$p_*([\widehat{X}_Y] - \widehat{i}_*\widehat{\pi}_*[\mathbb{P}_{\mathbb{P}(N)}(\widehat{N} \oplus 1)]) = [X] - i_*\pi_*[\mathbb{P}_Y(N \oplus 1)] \text{ in } \Omega(X)\,,$$

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or equivalently,

$$p_*[\widehat{X}_Y] - [X] = p_*\widehat{i}_*\widehat{\pi}_*[\mathbb{P}_{\mathbb{P}(N)}(\widehat{N} \oplus 1)] - i_*\pi_*[\mathbb{P}_Y(N \oplus 1)] \, .$$

Applying $p \circ \widehat{i} \circ \widehat{\pi} = i \circ \pi \circ p_N$, see (bb), and the lemma, we rewrite the last equation as

$$\begin{aligned}[\widehat{X}_Y \to X] - [X] &= i_*\pi_*(p_N)_*[\mathbb{P}_Y\widehat{(N \oplus 1)}_Y] - i_*\pi_*[\mathbb{P}_Y(N \oplus 1)] \\ &= i_*\pi_*([\mathbb{P}_Y\widehat{(N \oplus 1)}_Y \to \mathbb{P}_Y(N \oplus 1)] - [\mathbb{P}_Y(N \oplus 1)]) \, .\end{aligned} \tag{cc}$$

By [LM, Prop. 3.2], the left-hand side of (cc) can be written as

$$[\widehat{X}_Y \to X] - [X] = i_*\pi_*(e(N)) \, ,$$

where $e(N)$ is an element of $\Omega(\mathbb{P}_Y(N \oplus 1))$. In [LM] an explicit formula is given, expressing $e(N)$ as a power series of the first Chern class of the sheaf $\mathcal{O}(-1)$ on $\mathbb{P}_Y(N \oplus 1)$, which we do not need for our purposes. We only need to know that $e(N)$ depends solely on N as a vector bundle over Y. Now observe that the normal bundle to Y in $\mathbb{P}_Y(N \oplus 1)$ is canonically the same N, so applying the same formula to the blow-up p_N, we get

$$[\mathbb{P}_Y\widehat{(N \oplus 1)}_Y \to \mathbb{P}_Y(N \oplus 1)] - [\mathbb{P}_Y(N \oplus 1)] = z_*\pi_*(e(N)) \ \text{ in } \ \Omega(\mathbb{P}_Y(N \oplus 1)) \, ,$$

where $z : Y \to \mathbb{P}_Y(N \oplus 1)$ is the zero section of N. Thus the right-hand side of (cc) becomes $i_*\pi_*z_*\pi_*(e(N)) = i_*\pi_*(e(N))$ as $\pi \circ z = \mathrm{id}_Y$. The theorem is proved.

As Proposition 3.2 of [LM] holds for any *oriented Borel-Moore functor of geometric type*, our theorem is also valid in such a more general context.

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References

[LM] M. Levine and F. Morel, *Algebraic Cobordism, I*, preprint at www.math.neu.edu/ levine/mathindex.html.