k -INVARIANTS FOR K-THEORY OF CURVES OVER GLOBAL FIELDS

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1. Introduction.

Let X be a connected simple CW-complex and let $\alpha_n : X \rightarrow X[n]$ denote the n -th *Postnikov section* of X for any positive integer n : $X[n]$ is the CW-complex obtained from X by killing the homotopy groups of X in dimensions $> n$, more precisely by adjoining cells of dimensions $\geq n + 2$ such that $\pi_j(X[n]) = 0$ for $j > n$ and $(\alpha_n)_* : \pi_j(X) \rightarrow \pi_j(X[n])$ is an isomorphism for $j \leq n$. The Postnikov k -invariants of X are cohomology classes $k^{n+1}(X) \in H^{n+1}(X[n-1]; \pi_n(X))$, for $n \geq 2$, which provide the necessary information for the reconstruction of X , up to a weak homotopy equivalence, from its homotopy groups (see for instance [W], Section IX.2 for a definition).

In this paper we extend our results of [AB] concerning k -invariants to the case of algebraic curves over global fields. In section 2 we state general results about k -invariants and we discuss the cases when they are of finite order. Then in section 3 we consider the case of k -invariants of Quillen and Étale K-theory spaces and we explain why they are of finite order. In section 4 we prove our main result:

Theorem 4.9. *Let X be a smooth, proper and geometrically irreducible curve of genus g , defined over a global field F and $\mathcal{X}$ be a regular and proper model of X over O_{F,S_l} . Let $\mathcal{X}_v$ be the special fibre of $\mathcal{X}$ at a prime v . Then for $n \geq 1$ and $l \gg 0$ the order of $k_{et}^{2n+1}(\mathcal{X})$ divides the order of $k_{et}^{2n+1}(\mathcal{X}_v)$ for infinitely many $v \notin S_l$.*

To prove Theorem 4.9 we need a technical result about Galois cohomology of a compatible family of l -adic representations (see Lemma 4.1 and Remark 4.2). From this we conclude that for $l \gg 0$ the $\mathbb{Z}_l$ module $K_{2n}^{et}(\mathcal{X})$ has no l -torsion (see Lemma 4.3). Note that Lemma 4.3 gives some evidence that the conjecture of Quillen-Lichtenbaum for $K_{2n}^{et}(\mathcal{X})$ (see Remark 4.5) and the conjecture about finite generation of $K_{2n}^{et}(\mathcal{X})$ might be true. Other important technical ingredients in the proof of Th. 4.9 are: the compatibility of k -invariants (cf. eg. diagrams 2.4 and 4.11) and good control of the reduction map for étale K-theory [BGK3, Th. 3.1]. Theorem 4.9 has the following corollary in which we numerically estimate the orders of k -invariants.

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Corollary 4.19. *Let X be a smooth, proper and geometrically irreducible curve of genus g , defined over a global field F and $\mathcal{X}$ be a regular and proper model of X over O_{F,S_l} . For every $n \geq 1$ and $l \gg 0$ the order of $k_{et}^{2n+1}(\mathcal{X})$ divides the number $|P_v(q_v^{-n})|_l^{-1}$ for infinitely many $v \notin S_l$, where $P_v(t) = \det(t \text{Id}_{T_l(J_X)} - Fr_v)$ is the characteristic polynomial of the Frobenius Fr_v acting on the Tate module of the Jacobian of X . If the Quillen-Lichtenbaum conjecture holds for $\mathcal{X}$ and $\mathcal{X}_v$ for all $v \notin S_l$, then for every $n \geq 1$ and $l \gg 0$ the l -part of the order of $k^{2n+1}(\mathcal{X})$ divides the number $|P_v(q_v^{-n})|_l^{-1}$ for infinitely many $v \notin S_l$.*

It is explained in Remark 4.2 that the assumption $l \gg 0$ can be made numerically explicit. We finish the paper with discussion of k -invariants associated with odd K -groups and we get similar results to Th. 4.9 and Corollary 4.19 (cf. Th. 4.25 and Corollary 4.26).

2. Basic facts about k -invariants.

The main property of k -invariants is that $X[n]$ is the homotopy fiber of the map $X[n-1] \rightarrow K(\pi_n(X), n+1)$ corresponding to the cohomology class $k^{n+1}(X) \in H^{n+1}(X[n-1]; \pi_n(X))$, for $n \geq 2$. In other words, there is a commutative diagram of fibrations

$$\begin{array}{ccc}
 K(\pi_n(X), n) & \xrightarrow{\cong} & K(\pi_n(X), n) \\
 \downarrow & & \downarrow \\
 X[n] & \longrightarrow & PK(\pi_n(X), n+1) \\
 \downarrow & & \downarrow p \\
 X[n-1] & \xrightarrow{k^{n+1}(X)} & K(\pi_n(X), n+1),
 \end{array}$$

in which the right column is the path fibration over the Eilenberg-MacLane space $K(\pi_n(X), n+1)$ and the bottom square is a homotopy pull-back. Consequently, the knowledge of $X[n-1]$, $\pi_n(X)$ and $k^{n+1}(X)$ enables us to construct the next homotopical approximation $X[n]$ of X .

It is then obvious that the vanishing of $k^{n+1}(X)$ implies that $X[n] \cong X[n-1] \times K(\pi_n(X), n)$ and that the Hurewicz homomorphism $h_n : \pi_n(X) \rightarrow H_n(X; \mathbb{Z})$ is split injective (see [A], Lemma 7.3). It is of particular interest to get information on the order of the k -invariant $k^{n+1}(X)$ in the group $H^{n+1}(X[n-1]; \pi_n(X))$, because the previous splitting result can be generalized as follows for nonvanishing k -invariants which are of finite order (see [A], Proposition 7.10).

Theorem 2.1. *Let X be a connected simple CW-complex, n an integer ≥ 2 and ρ a positive integer. The following assertions are equivalent:*

- (1) $\rho k^{n+1}(X) = 0$ in $H^{n+1}(X[n-1]; \pi_n(X))$.
- (2) There is a map

$$f_n : X \rightarrow K(\pi_n(X), n)$$

such that the induced homomorphism $(f_n)_* : \pi_n(X) \rightarrow \pi_n(X)$ is multiplication by ρ .

- (3) There is a homomorphism $\theta_n : H_n(X; \mathbb{Z}) \rightarrow \pi_n(X)$ such that the composition

$$\pi_n(X) \xrightarrow{h_n} H_n(X; \mathbb{Z}) \xrightarrow{\theta_n} \pi_n(X),$$

where h_n denotes the Hurewicz homomorphism, is multiplication by ρ .

Lemma 2.2. *If X is an infinite loop space then $k^{n+1}(X)$ is of finite order for $n \geq 2$.*

Proof. It follows by Theorem 7.15 of [A]. $\square$

Theorem 2.3. *Let*

$$F \rightarrow E \rightarrow B$$

be a fibration. Assume that both $\pi_{n-1}(F)$ and $\pi_n(F)$ are torsion groups and that $k^{n+1}(B)$ has finite order. Then $k^{n+1}(E)$ has finite order.

Proof. Consider the exact sequence

$$\pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{\pi_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F)$$

Define the groups $I_1 := \text{Im}(i_*)$, $I_2 := \text{Im}(\pi_*)$, $I_3 := \text{Im}(\partial)$. Observe that I_1 and I_3 are torsion groups. In addition we get two short exact sequences

$$0 \rightarrow I_1 \rightarrow \pi_n(E) \rightarrow I_2 \rightarrow 0,$$

$$0 \rightarrow I_2 \rightarrow \pi_n(B) \rightarrow I_3 \rightarrow 0.$$

These exact sequences yield exact sequences in cohomology

$$H^{n+1}(E[n-1], I_1) \rightarrow H^{n+1}(E[n-1], \pi_n(E)) \rightarrow H^{n+1}(E[n-1], I_2),$$

$$H^n(E[n-1], I_3) \rightarrow H^{n+1}(E[n-1], I_2) \rightarrow H^{n+1}(E[n-1], \pi_n(B)).$$

Because I_1 and I_3 are torsion groups, it is clear from the above sequences that for $k^{n+1}(E)$ to have finite order it is enough to check that its image via the composition of the right maps in the exact sequences has finite order in $H^{n+1}(E[n-1], \pi_n(B))$. Now the Theorem follows by applying the following diagram:

$$(2.4) \quad \begin{array}{c} H^{n+1}(B[n-1], \pi_n(B)) \\ \downarrow \alpha \\ H^{n+1}(E[n-1], \pi_n(B)) \\ \uparrow \beta \\ H^{n+1}(E[n-1], \pi_n(E)) \end{array}$$

and the assumption that $k^{n+1}(B)$ has finite order because $\alpha(k^{n+1}(B)) = \beta(k^{n+1}(E))$ by [W] p. 424. $\square$

Example 2.5. *We observe that if all k -invariants of F and all k -invariants of B are of finite order it does not imply that all k -invariants of E are of finite order. Namely we can take F and B to be Eilenberg-McLane spaces then the only k -invariant of E can be of infinite order.*

3. Finite order of k -invariants for K-theory of schemes.

Let $\mathcal{E}$ be an exact category. Quillen [Q] constructed the space $BQ\mathcal{E}$ and defined K-theory of $\mathcal{E}$ as follows: $K_n(\mathcal{E}) := \pi_n(BQ\mathcal{E})$. We will denote the k -invariant for the space $BQ\mathcal{E}$ as follows: $k^{n+1}(\mathcal{E}) := k^{n+1}(BQ\mathcal{E})$.

Lemma 3.1. *If $\mathcal{E}$ is an exact category then $BQ\mathcal{E}$ is an infinite loop space.*

Proof. It follows by application of Waldhausen's construction (see e.g. [We], chapter IV, Remark 4.5.1). $\square$

In this way by Lemma 2.2 the k -invariants $k^n(\mathcal{E})$ are of finite order for any $n \geq 3$ and any exact category $\mathcal{E}$. Let X be a scheme and let $\mathbb{M}(X)$ be the category of finitely generated coherent sheaves on X and let $\mathbb{P}(X)$ be the category of finitely generated, locally free $\mathcal{O}_X$ modules on X . Both $\mathbb{P}(X)$ and $\mathbb{M}(X)$ are exact categories and Quillen [Q] defined K' -theory and K -theory of a scheme X as follows: $K'_n(X) := \pi_n(BQ\mathbb{M}(X))$ and $K_n(X) := \pi_n(BQ\mathbb{P}(X))$. When X is a regular scheme Quillen [Q] proved that $K'_n(X) = K_n(X)$. We will simplify the notation for k -invariants of K-theory spaces of schemes in the following way: $k'^n(X) := k^n(BQ\mathbb{M}(X))$ and $k^n(X) := k^n(BQ\mathbb{P}(X))$. By lemma 2.2 the k -invariants $k^n(X)$ and $k'^n(X)$ have finite order for any $n \geq 3$. If $X = \text{spec } R$, where R is a unital, commutative ring, then we will write $k^n(R) := k^n(X)$. For a prime l let $\mathcal{K}^{et}(X)$ denote the Dwyer and Friedlander étale K-theory space of a scheme $X/\mathbb{Z}[1/l]$. Let $K_n^{et}(X)$ denote the étale K-theory group of a scheme X over $\text{spec } \mathbb{Z}[1/l]$ cf. [DF]. Let $k_{et}^n(X) := k^n(\mathcal{K}^{et}(X))$. By Proposition 4.5 [DF] $\mathcal{K}^{et}(X)$ is homotopy equivalent to an r -th loop space for any r so for any $n \geq 3$ the k -invariant $k_{et}^n(X)$ is of finite order for any scheme $X/\mathbb{Z}[1/l]$. Using a technique from [AB] we can prove the following basic properties of k -invariants of schemes.

Proposition 3.2. *Let X/S be a scheme over a scheme S . Assume that X has a point over S . Then for $n \geq 2$ the order of $k^{n+1}(X)$ ($k_{et}^{n+1}(X)$ resp.) is divisible by the order of $k^{n+1}(S)$ ($k_{et}^{n+1}(S)$ resp.).*

Proof. We do the proof for algebraic K-theory. The étale K-theory case is treated analogously. Since the composition of maps $S \rightarrow X \rightarrow S$ is the identity by assumption, it follows that the composition

$$BQ\mathbb{P}(S)[n-1] \rightarrow BQ\mathbb{P}(X)[n-1] \rightarrow BQ\mathbb{P}(S)[n-1]$$

is also the identity. Hence the map α in the diagram below is split injective

$$\begin{array}{c} H^{n+1}(BQ\mathbb{P}(S)[n-1], K_n(S)) \\ \downarrow \alpha \\ H^{n+1}(BQ\mathbb{P}(X)[n-1], K_n(S)) \\ \uparrow \beta \\ H^{n+1}(BQ\mathbb{P}(X)[n-1], K_n(X)) \end{array}$$

□

Corollary 3.3. *Let $X/\mathbb{F}$ be a scheme over a finite field $\mathbb{F}$. Assume that X has a point over $\mathbb{F}$. Then for $n \geq 2$ the order of $k^{2n}(X)$ is divisible by the order of $k^{2n}(\mathbb{F})$ which is equal to $\gcd((n-1)!, |\mathbb{F}| - 1)$. Also the order of $k_{et}^{2n}(X)$ is divisible by the order of $k_{et}^{2n}(\mathbb{F})$*

Proof. The proof follows from Lemma 2.2, Lemma 3.2 and Theorem C of [H]. □

4. k -invariants for K-theory of curves over global fields.

Let us start with some remarks on Galois cohomology and l -adic representations.

Let F be a global field and consider for each l a free, finitely generated $\mathbb{Z}_l$ module T_l and an l adic representation $\rho_l : G_F \rightarrow GL(T_l)$. For each l let S_l be a finite set of valuations of F containing all primes over l in the number field case. Let $V_l := T_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ and $A_l := V_l/T_l$. The representation ρ_l induces the following l -adic representation: $\rho_l^0 : G_F \rightarrow GL(V_l)$. Representation ρ_l^0 is called *rational* (cf. [Se]) if ρ_l^0 is unramified outside S_l and for each finite valuation $v \notin S_l$ the characteristic polynomial

$$P_{v, \rho_l}(t) = \det(t \text{Id}_{V_l} - Fr_v) = \det(t \text{Id}_{T_l} - Fr_v)$$

of the Frobenius at v has coefficients in $\mathbb{Q}$. Since ρ_l^0 is unramified at a given prime v iff ρ_l is unramified at this prime, we will also say that ρ_l is rational whenever ρ_l^0 is rational. The family of l -adic representations $(\rho_l^0)_l$ is called *strictly compatible* [Se] I, 2.3 if there exists a finite set of places S of F such that each representation ρ_l^0 is rational with $S_l = S \cup \{\text{primes over } l\}$ in number field case and $S_l = S$ in function field case and for any primes l and l' and $v \notin S_l \cup S_{l'}$ there is an equality $P_{v, \rho_l}(t) = P_{v, \rho_{l'}}(t)$. For strictly compatible family of representations the polynomial $P_{v, \rho_l}(t)$ is independant of prime number l hence for each $v \notin S$ we can put:

$$P_v(t) := P_{v, \rho_l}(t)$$

for any l such that $v \notin S_l$. Again we easily observe that it is justified to say that the family $(\rho_l)_l$ is strictly compatible instead of saying that $(\rho_l^0)_l$ is strictly compatible.

Lemma 4.1. *Assume that the family of representations $(\rho_l)_l$ is strictly compatible and all roots of $P_v(t)$ in $\bar{\mathbb{Q}}$ are different from 1. Then $H^1(G_{F,S_l}; T_l)_{l\text{-tor}} = 0$ for almost all l .*

Proof. Let $\bar{\mathbb{Z}}_l$ be the integers in $\bar{\mathbb{Q}}_l$. We can write

$$P_v(t) = (t - \alpha_1)(t - \alpha_2) \dots (t - \alpha_r) \in \bar{\mathbb{Q}}[t].$$

Put $L := \mathbb{Q}(\alpha_1, \alpha_2, \dots, \alpha_r)$. L is the splitting field of $P_v(t)$. Choose $l \gg 0$ such that α_i are integers over l and $\alpha_i - 1$ are units over l for all $1 \leq i \leq r$. Let $\bar{F}r_v : A_l[l] \rightarrow A_l[l]$ be the automorphism of $\mathbb{F}_l$ vector space $A_l[l]$ induced by the automorphism $Fr_v : T_l \rightarrow T_l$ of free $\mathbb{Z}_l$ module T_l . Let $\wp$ be a prime in O_L over l . Put $k_\wp := O_L/\wp$. Consider the automorphism

$$\bar{F}r_v \otimes 1 : A_l[l] \otimes_{\mathbb{F}_l} k_\wp \rightarrow A_l[l] \otimes_{\mathbb{F}_l} k_\wp$$

of $k_\wp$ vector space $A_l[l] \otimes_{\mathbb{F}_l} k_\wp$. Observe that by definition of the splitting field L we have:

$$\bar{P}_v(t) = \det(t \text{Id}_{A_l[l] \otimes_{\mathbb{F}_l} k_\wp} - \bar{F}r_v \otimes 1) = (t - \bar{\alpha}_1)(t - \bar{\alpha}_2) \dots (t - \bar{\alpha}_r) \in k_\wp[t]$$

where $\bar{P}_v(t)$ is the polynomial obtained from $P_v(t)$ by reducing its coefficients modulo l and $\bar{\alpha}_i$ are the images of α_i in $k_\wp$. We observe that the automorphism $\bar{F}r_v$ cannot have a fix point because it would imply that the automorphism $\bar{F}r_v \otimes 1$ has a fix point which is equivalent to the statement that the polynomial $\bar{P}_v(t)$ has $1 \in k_\wp$ as a root. This means that $\bar{\alpha}_i = 1$ for some $1 \leq i \leq r$. This would imply that $\alpha_i - 1$ is not a unit over l , a contradiction. Hence $\bar{F}r_v$ cannot have a fix point which imply that $A_l[l]^{G_{F,S_l}} = 0$ for $l \gg 0$ since for $v \notin S_l$ we can consider Fr_v as an element of G_{F,S_l} . So applying cohomology to the exact sequence

$$0 \rightarrow T_l \rightarrow V_l \rightarrow A_l \rightarrow 0$$

we get the exact sequence:

$$0 \rightarrow A_l^{G_{F,S_l}} \rightarrow H^1(G_{F,S_l}; T_l) \rightarrow H^1(G_{F,S_l}; V_l)$$

cf. [BGK1], which shows that $H^1(G_{F,S_l}; T_l)_{l\text{-tor}} = A_l^{G_{F,S_l}} = 0$. $\square$

Remark 4.2. *The statement that $l \gg 0$ in Lemma 4.1 can be made much more explicit. It is clear from the proof of Lemma 4.1. Simply we can pick any place $v_0 \notin S$. In the number field case we take l_0 to be the prime in $\mathbb{Z}$ below v_0 . Then we do the proof with the eigenvalues of Fr_{v_0} which are the roots $\alpha_1^0, \dots, \alpha_r^0$ of the polynomial $P_{v_0}(t)$.*

In the number field case the assumption $l \gg 0$ means that l is not below any prime in S_{l_0} , and $\alpha_i^0 - 1 \in \bar{\mathbb{Z}}_l^\times$, $\alpha_i^0 \in \bar{\mathbb{Z}}_l$ for each $1 \leq i \leq r$.

In the function field case the assumption $l \gg 0$ means that $l \neq \text{char } F$ and $\alpha_i^0 - 1 \in \overline{\mathbb{Z}_l}^\times$, $\alpha_i^0 \in \overline{\mathbb{Z}_l}$ for each $1 \leq i \leq r$.

In all results below the assumption $l \gg 0$ can be made explicit in the same way as we did it above for the case of Lemma 4.1.

Now consider X a smooth, proper and geometrically irreducible algebraic variety defined over a global field F . Let l be an odd rational prime different from the characteristic of the field F . Let S be the set of places of F which consists of places of bad reduction of X and places at infinity. Denote by $\mathcal{X}$ a regular and proper model of X over the ring $O_{F,S}$ of S -integers of F . Let $S_l = S$ in function field case and $S_l = S \cup \{\text{places lying over } l\}$ in the number field case. For a prime v of O_{F,S_l} let F_v denote the completion of F at v $k_v := O_{F,S_l}/v$. Let $\bar{F}$ (resp. $\bar{k}_v$) denote algebraic closure of F (resp. k_v). Let $\mathcal{X}_v$ be the fiber of $\mathcal{X}$ over a prime v of O_{F,S_l} . By [Mi], VI, Cor. 4.2 there are natural isomorphisms of $\mathbb{Z}_l[G(\bar{F}_v/F_v)]$ -modules:

$$H^i(\bar{X}; \mathbb{Z}_l(j)) \cong H^i(\bar{\mathcal{X}}_v; \mathbb{Z}_l(j))$$

for all $i \geq 0$ and all $j \in \mathbb{Z}$, where $\bar{X} = X \otimes_F \bar{F}$ and $\bar{\mathcal{X}}_v = \mathcal{X}_v \otimes_{k_v} \bar{k}_v$. Since the inertia subgroup $I_v \subset G(\bar{F}_v/F_v)$ acts trivially on the group on the right side of the above isomorphisms, it shows that the action of G_F on the group on the left factors through G_{F,S_l} . Taking $T_l := H^i(\bar{X}; \mathbb{Z}_l(j))/l - \text{tor}$ we get a family of l -adic representations. This family is strictly compatible because of the proof of the Weil conjecture by Deligne [D] and the discussion above. It was conjectured by Jannsen [J], Conjecture 1, p. 317 that $H^2(G_{F,S_l}; H^i(\bar{X}; \mathbb{Z}_l(j)))$ is finite for $i+1 < j$ or $i+1 > 2j$. Jannsen verified this conjecture in function field case [J], Theorem 1, p. 335. Now consider the special case when X is a smooth, proper and geometrically irreducible curve of genus g , defined over a global field F . Let J_X denote the Jacobian variety of X and let $T_l(J_X)$ denote the Tate module of the Jacobian. Then one checks that there is a natural isomorphism of $\mathbb{Z}_l[G_{F,S_l}]$ -modules:

$$H^1(\bar{X}; \mathbb{Z}_l(1)) \cong T_l(J_X).$$

Hence according to Jannsen [J], Conjecture 1, p. 317, we expect the group $H^2(G_{F,S_l}; T_l(J_X)(n+1))$ to be finite for all $n > 0$. Hence if F is a function field then by [J], Theorem 1, p. 335 the group $H^2(G_{F,S_l}; T_l(J_X)(n+1))$ is finite for all $n > 0$.

Lemma 4.3. *Let X be a smooth, proper and geometrically irreducible curve of genus g , defined over a global field F and let $\mathcal{X}$ be a regular and proper model of X over O_{F,S_l} . For $n > 0$ and for $l \gg 0$ the $\mathbb{Z}_l$ -modules $K_{2n}^{\text{et}}(\mathcal{X})$ are finitely generated, torsion free.*

Proof. Put $T_l := T_l(J_X)(n)$. By the diagrams (3.1) and (3.2) of the proof of Theorem 3.1 in [BGK3] there is a natural, surjective map

$$(4.4) \quad K_{2n}^{\text{et}}(\mathcal{X}) \rightarrow H^1(G_{F,S_l}; T_l).$$

Observe that in the number field case by Th. 4 [Sol] and Th. 8.7 [DF] the $H^2(G_{F,S_l}; \mathbb{Z}_l(n+1))$ is zero for $l \gg 0$. In the function field case $H^2(G_{F,S_l}; \mathbb{Z}_l(n+1))$ is zero for $l \gg 0$ because of Th. 8.9 [DF], Prop. 3.2 [Q1], Cor. 1, p.330 [So2]. Hence by diagrams (3.1) and (3.2), of the proof of Theorem 3.1 in [BGK3], the map (4.4) is an isomorphism for $l \gg 0$. By theorem of Deligne [D], the characteristic polynomial of Fr_v acting on V_l has rational coefficients and its eigenvalues in this case are algebraic numbers with absolute values $N(v)^{1/2-n}$. Hence the eigenvalues are different from 1. In addition by [ST] (or the discussion before this lemma) and by [D] the family of l -adic representations for $T_l := T_l(J_X)(n)$ is strictly compatible. So by lemma 4.1 we get $H^1(G_{F,S_l}; T_l)_{l-tor} = 0$ for $l \gg 0$, hence the isomorphism (4.4) for $l \gg 0$ shows that $K_{2n}^{et}(\mathcal{X})_{l-tor} = 0$ for $l \gg 0$. Finite generation of the groups $K_{2n}^{et}(\mathcal{X})$ follows by [DF] Prop. 5.1, p. 260. $\square$

Remark 4.5. *Let us return for a while to the general case when X is a smooth, proper and geometrically irreducible algebraic variety defined over a global field F . Note that according to the Beilinson conjectures the Quillen K -group $K_m(\mathcal{X})$ is expected to be finitely generated. The Quillen-Lichtenbaum conjecture (cf. [FW] p. 57) predicts that the Dwyer-Friedlander map [DF]:*

$$(4.6) \quad K_m(X; \mathbb{Z}/l^k) \rightarrow K_m^{et}(X; \mathbb{Z}/l^k)$$

is an isomorphism for $m \geq 3$. Hence by the localization sequences in K -theory the conjectures imply that for $m \geq 3$ there is an isomorphism:

$$(4.7) \quad K_m(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Z}_l \rightarrow K_m^{et}(\mathcal{X}).$$

Observe that if (4.7) is an isomorphism for $l \gg 0$ and $K_m(\mathcal{X})$ is finitely generated then Lemma 4.3 follows. Our proof of Lemma 4.3 does not appeal to these conjectures. Let us mention that by [DFST] the map (4.6) is surjective for $m \gg 0$.

From now on X is a smooth, proper and geometrically irreducible curve of genus g , defined over a global field F and $\mathcal{X}$ is a regular and proper model of X over O_{F,S_l} .

For places $v \notin S_l$, we consider the reduction map:

$$(4.8) \quad r_v : K_{2n}^{et}(\mathcal{X}) \rightarrow K_{2n}^{et}(\mathcal{X}_v)$$

induced on étale K -theory by the injection $\mathcal{X}_v \hookrightarrow \mathcal{X}$ of the special fiber at v .

Theorem 4.9. *For $n \geq 1$ and $l \gg 0$ the order of $k_{et}^{2n+1}(\mathcal{X})$ divides the order of $k_{et}^{2n+1}(\mathcal{X}_v)$ for infinitely many v .*

Proof. Consider the reduction map r_v (4.8). Since $K_{2n}^{et}(\mathcal{X}) \cong \bigoplus_{i=1}^r \mathbb{Z}_l$ for some r and $l \gg 0$, we get decomposition:

$$(4.10) \quad H^{2n+1}(\mathcal{K}^{et}(\mathcal{X})[2n-1], K_{2n}^{et}(\mathcal{X})) \cong \bigoplus_{i=1}^r H^{2n+1}(\mathcal{K}^{et}(\mathcal{X})[2n-1], \mathbb{Z}_l).$$

If $r = 0$, so $K_{2n}^{et}(\mathcal{X}) = 0$, then the theorem obviously holds. Hence we can consider the case $r > 0$. We observe that at least one of the r projections of $k_{et}^{2n+1}(\mathcal{X}) \in H^{2n+1}(\mathcal{K}^{et}(\mathcal{X})[2n-1], K_{2n}^{et}(\mathcal{X}))$ onto the summands of the right hand side of (4.10) must have the same order as $k_{et}^{2n+1}(\mathcal{X})$. Take the nontorsion element $P_i \in K_{2n}^{et}(\mathcal{X})$ corresponding to this projection. By [BGK3] Th. 3.1 there are infinitely many v such that $r_v(P_i)$ has order equal to l^k which is bigger then the order of $H^{2n+1}(\mathcal{K}^{et}(\mathcal{X})[2n-1], K_{2n}^{et}(\mathcal{X}))_{tor}$. Observe that the groups $H^{2n+1}(\mathcal{K}^{et}(\mathcal{X})[2n-1], K_{2n}^{et}(\mathcal{X}))$ are finitely generated $\mathbb{Z}_l$ modules since $K_m^{et}(\mathcal{X})$ is a finitely generated $\mathbb{Z}_l$ module for $m > 0$ and for $m = 0$ we see that $H^1(\mathcal{K}^{et}(\mathcal{X})[-1], K_0^{et}(\mathcal{X})) = 0$. Consider the following diagram

$$\begin{array}{c}
 H^{2n+1}(\mathcal{K}^{et}(\mathcal{X})[2n-1], K_{2n}^{et}(\mathcal{X})) \\
 \downarrow \alpha \\
 (4.11) \quad H^{2n+1}(\mathcal{K}^{et}(\mathcal{X})[2n-1], K_{2n}^{et}(\mathcal{X}_v)) \\
 \uparrow \beta \\
 H^{2n+1}(\mathcal{K}^{et}(\mathcal{X}_v)[2n-1], K_{2n}^{et}(\mathcal{X}_v))
 \end{array}$$

Let us extract from the top vertical arrow α the arrow induced by the reduction of the point P_i :

$$(4.12) \quad H^{2n+1}(\mathcal{K}^{et}(\mathcal{X})[2n-1], \mathbb{Z}_l) \rightarrow H^{2n+1}(\mathcal{K}^{et}(\mathcal{X})[2n-1], \mathbb{Z}/l^k).$$

We notice that the map (4.12) is given by reduction mod l^k on coefficients. Hence the top vertical arrow of the diagram (4.11) sends $k_{et}^{2n+1}(\mathcal{X})$ to the element of the same order. $\square$

Theorem 4.13. *If Quillen-Lichtenbaum conjecture holds for $\mathcal{X}$ and $\mathcal{X}_v$ for all $v \notin S_l$ then for $n \geq 1$ and $l \gg 0$ the l -part of order of $k^{2n+1}(\mathcal{X})$ divides the l -part of the order of $k^{2n+1}(\mathcal{X}_v)$ for infinitely many v .*

Proof. It follows by Th. 4.9 since the the reduction map

$$(4.14) \quad K_{2n}(\mathcal{X}) \rightarrow K_{2n}(\mathcal{X}_v)$$

is naturally compatible with the corresponding étale K-theory reduction map. $\square$

Before proceeding any further let us notice the following general lemma left as an exercise for the reader.

Lemma 4.15. *Let R be an infinite PID. Let M be a free finitely generated R module and let $\phi \in \text{End}_R(M)$. Put $P_\phi(t) = \det(t \text{Id}_M - \phi)$. If $M/(\phi - 1)M$ is finite then the following numbers are equal:*

$$|M/(\phi - 1)M| = |R/P_\phi(1)|,$$

where $|A|$ denotes the number of elements of a finite set A .

Corollary 4.16. *For an l -adic representation $\rho_l : G_F \rightarrow GL(T_l)$ such that $T_l^{Fr_v} = 0$ for $v \notin S_l$ we have equalities:*

$$(4.17) \quad |H^1(g_v, T_l)| = |P_v(1)|_l^{-1}$$

$$(4.18) \quad |H^1(g_v, T_l(n))| = |P_v(q_v^{-n})|_l^{-1},$$

where $g_v = G(\bar{k}_v/k_v)$.

Proof. Use Lemma 4.15 with $R = \mathbb{Z}_l$. To get the equality (4.17) put $M = T_l$ and to get the equality (4.18) put $M = T_l(n)$ where in both cases $\phi = Fr_v$. The assumption $T_l^{Fr_v} = 0$ guaranties that the groups on the left sides of equations (4.17) and (4.18) are finite. $\square$

Corollary 4.19. *For every $n \geq 1$ and $l \gg 0$ the order of $k_{et}^{2n+1}(\mathcal{X})$ divides the number $|P_v(q_v^{-n})|_l^{-1}$ for infinitely many $v \notin S_l$. If the Quillen-Lichtenbaum conjecture holds for $\mathcal{X}$ and $\mathcal{X}_v$ for all $v \notin S_l$ then for every $n \geq 1$ and $l \gg 0$ the l -part of the order of $k^{2n+1}(\mathcal{X})$ divides the number $|P_v(q_v^{-n})|_l^{-1}$ for infinitely many $v \notin S_l$.*

Proof. Consider the l -adic representation $\rho_l : G_F \rightarrow GL(T_l(J_X))$. By [BGK3] Prop. 2.5 (b) and Prop. 2.6 (b) there is a natural isomorphism

$$K_{2n}^{et}(\mathcal{X}_v) \cong H^1(g_v; T_l(J_X)(n)).$$

Hence by Corollary 4.16 we get equality $|K_{2n}^{et}(\mathcal{X}_v)| = |P_v(q_v^{-n})|_l^{-1}$. Now the Corollary follows by Theorems 4.9 and 4.13. $\square$

Let us now consider the case of odd K-theory. For places $v \notin S_l$, we consider the reduction map:

$$(4.20) \quad r_v : K_{2n+1}^{et}(\mathcal{X}) \rightarrow K_{2n+1}^{et}(\mathcal{X}_v).$$

Lemma 4.21. *Let $n > 0$ and let l be an odd prime number. Assume that the group $H^2(G_{F, S_l}; T_l(J_X)(n+1))$ is finite. Let $k \in \mathbb{N}$ be any fixed number. Then for any nontorsion element $P \in K_{2n+1}^{et}(\mathcal{X})$ there are infinitely many v such that $r_v(P)$ has order divisible by l^k .*

Proof. The proof has been suggested in [BGK3] Remark 3.5. Namely we take into account Propositions 2.4, 2.5, 2.6 and exact sequences (3.7) and (3.8) of [BGK3] to observe that the claim of Lemma 4.21 about the reduction map (4.20) reduces to the corresponding claim about the reduction map:

$$(4.22) \quad r_v : H^1(G_{F, S_l}; \mathbb{Z}_l(n+1)) \rightarrow H^1(g_v; \mathbb{Z}_l(n+1)).$$

But this claim about the map (4.22) has already been resolved for example in [AB]. $\square$

Lemma 4.23. *Let $H^2(G_{F,S_l}; T_l(J_X)(n+1)) = 0$ for $n > 0$. Then for $n > 0$ and for $l \gg 0$ the $\mathbb{Z}_l$ modules $K_{2n+1}^{et}(\mathcal{X})$ are finitely generated, torsion free.*

Proof. By Propositions 2.4, 2.6 and exact sequences (3.7) and (3.8) of [BGK3] and the assumption $H^2(G_{F,S_l}; T_l(J_X)(n+1)) = 0$, we have the following exact sequence:

$$(4.24) \quad 0 \rightarrow H^1(G_{F,S_l}; \mathbb{Z}_l(n+1)) \rightarrow K_{2n+1}^{et}(\mathcal{X}) \rightarrow H^1(G_{F,S_l}; \mathbb{Z}_l(n+1)) \rightarrow 0.$$

If we take $T_l := \mathbb{Z}_l(n+1)$, the resulting representation is the $n+1$ twist of cyclotomic character and obviously satisfies assumptions of Lemma 4.1. Hence $H^1(G_{F,S_l}; \mathbb{Z}_l(n+1))_{l-tor} = 0$ for $l \gg 0$. Hence for $l \gg 0$ the $\mathbb{Z}_l$ -module $K_{2n+1}^{et}(\mathcal{X})$ does not have torsion by the exact sequence (4.24). The $\mathbb{Z}_l$ -module $K_{2n+1}^{et}(\mathcal{X})$ is finitely generated by [DF] Prop. 5.1, p. 260. $\square$

Theorem 4.25. *Let $H^2(G_{F,S_l}; T_l(J_X)(n+1)) = 0$ for $n > 0$. Then for $n \geq 1$ and $l \gg 0$ the order of $k_{et}^{2n+2}(\mathcal{X})$ divides the order of $k_{et}^{2n+2}(\mathcal{X}_v)$ for infinitely many v . If the Quillen-Lichtenbaum conjecture holds for $\mathcal{X}$ and $\mathcal{X}_v$ for all $v \notin S_l$, then for all $n \geq 1$ and $l \gg 0$ the l -part of the order of $k^{2n+2}(\mathcal{X})$ divides the l -part of the order of $k^{2n+2}(\mathcal{X}_v)$ for infinitely many v .*

Proof. The proof is the same as the proof of Theorems 4.9 and 4.13. $\square$

Corollary 4.26. *Let $H^2(G_{F,S_l}; T_l(J_X)(n+1)) = 0$ for $n \geq 1$. Then for all $n \geq 1$ and $l \gg 0$ the order of $k_{et}^{2n+2}(\mathcal{X})$ divides the number $(|1 - q_v^{n+1}|_l^{-1})^2$ for infinitely many v . If the Quillen-Lichtenbaum conjecture holds for $\mathcal{X}$ and $\mathcal{X}_v$ for all $v \notin S_l$, then for all $n \geq 1$ and $l \gg 0$ the l -part of the order of $k^{2n+2}(\mathcal{X})$ divides the number $(|1 - q_v^{n+1}|_l^{-1})^2$ for infinitely many v .*

Proof. By [BGK3] Prop. 2.5 (a), (c) and Prop. 2.6 (b) there is the following exact sequence:

$$0 \rightarrow H^1(g_v; \mathbb{Z}_l(n+1)) \rightarrow K_{2n+1}^{et}(\mathcal{X}_v) \rightarrow H^1(g_v; \mathbb{Z}_l(n+1)) \rightarrow 0.$$

But $|H^1(g_v; \mathbb{Z}_l(n+1))| = |1 - q_v^{n+1}|_l^{-1}$ by Corollary 4.16, hence $|K_{2n+1}^{et}(\mathcal{X}_v)| = (|1 - q_v^{n+1}|_l^{-1})^2$. Now the Corollary follows by Theorem 4.25. $\square$

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