

Motivic interpretation of Milnor K -groups attached to Jacobian varieties

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Abstract

In the paper [Som90] p.105, Somekawa conjectures that his Milnor K -group $K(k, G_1, \dots, G_r)$ attached to semi-abelian varieties G_1, \dots, G_r over a field k is isomorphic to $\text{Ext}_{\mathcal{M}_k}^r(\mathbb{Z}, G_1[-1] \otimes \dots \otimes G_r[-1])$ where \mathcal{M}_k is a certain category of motives over k . The purpose of this note is to give remarks on this conjecture, when we take \mathcal{M}_k as Voevodsky's category of motives $\text{DM}^{\text{eff}}(k)$.

Key words: motivic cohomology, 1-motives, Milnor K -groups, Weil reciprocity law

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0 Introduction

To unify the Moore exact sequence and the Bloch exact sequence, K. Kato defined the generalized Milnor K -groups attached to finite family of semi-abelian varieties over a base field k in [Som90]. (See also [Akh00], [Kah92].) That is, for semi-abelian varieties G_1, \dots, G_r over k , he associated the group $K(k, G_1, \dots, G_r)$. (For precise definition, see 1.4) This group is a generalization of the Milnor K -group as the following example shows.

Example 0.1. In the notation above, if $G_1 = G_2 = \dots = G_r = \mathbb{G}_m$, the following equality holds.

$$K(k, \mathbb{G}_m, \dots, \mathbb{G}_m) = K_r^M(k)$$

On the other hand this group is also a generalization of the Bloch group V .

Example 0.2. Let C be a projective smooth curve over k such that $C(k) \neq \emptyset$. We have the following equality

$$K(k, \text{Jac } C, \mathbb{G}_m) = V(C).$$

where $V(C)$ is defined by S. Bloch (c.f. [Blo81]) as the following way

$$V(C) = \frac{\text{Ker}(\bigoplus_{x \in C^1} k(x)^\times \xrightarrow{\sum N_{k(x)/k}} k^\times)}{\text{Im}(K_2(K(C)) \xrightarrow{\partial_x} \bigoplus_{x \in C^1} k(x)^\times)}.$$

As is explained in [Som90], there is the generalized Bloch-Moore exact sequence.

0.3. Let k be a number field and A a semi-abelian variety over k . We write $G = \text{Gal}(\bar{k}/k)$ and $G_v = \text{Gal}(\bar{k}_v/k_v)$ for a place v . Let $T(A)$ be the Tate module of A . S is a finite set of places including all Archimedean and places where A has bad reduction. Then $T(A)_G$ is a finite group by owing to [KL81]. Let m be a nonzero integer divisible by the order of $T(A)_G$. Somekawa proves the following generalized Moore-Bloch exact sequence (c.f. [Som90] Theorem 4.1):

$$K(k, A, \mathbb{G}_m) \rightarrow \bigoplus_{v \notin S} T(A)_{G_v} \oplus \bigoplus_{v \in S} K(k_v, A_v, \mathbb{G}_m)/m \rightarrow T(A)_G \rightarrow 0$$

In the case of $A = \mathbb{G}_m$, the above exact sequence is proved by Moore (c.f. [Moo69])

$$K_2(k) \rightarrow \bigoplus_{v:\text{not complex}} \mu(k_v) \rightarrow \mu(k) \rightarrow 0.$$

In the case of $A = \text{Jac } C$ in the notation in 0.2, the above exact sequence is proved by S. Bloch, K. Kato and S. Saito (c.f. [Blo81], [KS83])

$$V(C) \rightarrow \bigoplus_{v \notin S} T(\text{Jac } C)_{G_v} \oplus \bigoplus_{v \in S} V(C \times_k k_v)/m \rightarrow T(\text{Jac } C)_G \rightarrow 0.$$

In [Som90], Somekawa conjectures that the Somekawa K -groups should be motivic cohomology groups attached to semi-abelian varieties. More precisely

Conjecture 0.4. (*Somekawa conjecture*)

Let G_1, \dots, G_r be semi-abelian varieties over k , then $K(k, G_1, \dots, G_r)$ is isomorphic to $\text{Ext}_{\mathcal{M}_k}^r(\mathbb{Z}, G_1[-1] \otimes \dots \otimes G_r[-1])$, where \mathcal{M}_k is a certain category of motives over k and $G_i[-1]$ means 1-motif (c.f [Del74]).

In this paper we will examine this conjecture, if we take \mathcal{M}_k as Voevodsky's category of motives DM_k^{eff} .

Main Theorem 0.5. (*Somekawa conjecture for Jacobian varieties*)

Let $(C_1, a_1), \dots, (C_n, a_n)$ be pointed projective smooth curves over perfect field k which admits resolution of singularities. Then

$$K(k, \text{Jac } C_1, \dots, \text{Jac } C_n) \xrightarrow{\sim} \text{Hom}_{\text{DM}_k^{\text{eff}}}(M_{\text{gm}}(\text{Spec } k), \mathbb{Z}(\bigwedge_{i=1}^n (C_i, a_i))[n]).$$

0.6. In this paper, let k be a perfect field which admits resolution of singularity.

1 Milnor K -groups attached to semi-abelian varieties

1.1 Extension of valuations and tame symbols

1.1. Suppose k is a field and G is a semi-abelian variety defined over k , that is, there is an exact sequence of group schemes (viewed as sheaves in the flat topology) over k :

$$0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0$$

where T is a torus and A is an abelian variety.

1.2. In the notation above, let K/k be an algebraic function field and v a place of K/k . Let L/K_v be a finite unramified Galois extension such that $T \times_k F \xrightarrow{\sim} \mathbb{G}_m^n$ for the residue field F of L and some n ; let w be the unique extension of v of L . We obtain the following commutative diagram of exact sequences defining a

map $r_w = (r_w^1, \dots, r_w^n)$;

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T(\mathcal{O}_w) & \longrightarrow & G(\mathcal{O}_w) & \longrightarrow & A(\mathcal{O}_w) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \wr \\
0 & \longrightarrow & T(L) & \longrightarrow & G(L) & \longrightarrow & A(L) \longrightarrow 0 \\
& & \downarrow \text{ord}_w & & \downarrow r_w = (r_w^1, \dots, r_w^n) & & \\
& & \mathbb{Z}^n & \xrightarrow{\text{id}} & \mathbb{Z}^n & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

1.3. In the notation above, we are going to construct a map

$$\partial_v : G(K_v) \otimes K_v^\times \rightarrow G(k(v)).$$

Fix $g \in G(K_v)$ and $h \in K_v^\times$. For each $i = 1, \dots, n$, we define $h_i \in T(L)$ to be the n -th tuple having h in the i -th coordinate and 1 elsewhere. Then set

$$\varepsilon(g, h) = ((-1)^{\text{ord}_w(h)r_w^1(g)}, \dots, (-1)^{\text{ord}_w(h)r_w^n(g)}) \in T(\mathcal{O}_w) \subset G(\mathcal{O}_w)$$

and

$$\tilde{\partial}_v(g, h) = \varepsilon(g, h) g^{\text{ord}_w(h)} \prod_{i=1}^n h_i^{-r_w^i(g)} \in G(\mathcal{O}_w).$$

We define the “extended tame symbol” $\partial_v(g, h)$ to be the image of $\tilde{\partial}_v(g, h)$ under the canonical map $G(\mathcal{O}_w) \rightarrow G(F)$; Then $\partial_v(g, h)$ is invariant under the action of $\text{Gal}(F/k(v))$, so that it belongs to $G(k(v))$. This definition of ∂_v is independent of the choice of L and of the isomorphism from the torus to $\mathbb{G}_m^{\oplus n}$.

1.2 Definition of the Milnor K -groups attached to semiabelian varieties

1.4. Let k be a field and G_1, \dots, G_r a finite (possibly empty) family of semiabelian varieties defined over k . We define Milnor K -groups attached to semiabelian varieties $K(k, G_1, \dots, G_r)$ as follows. If $r = 0$, we write $K(k, \phi)$ for our groups and set $K(k, \phi) = \mathbb{Z}$.

For $r \geq 1$, we define

$$K(k, G_1, \dots, G_r) = F/R$$

where

$$F = \bigoplus_{E/k:\text{finite}} G_1(E) \otimes \dots \otimes G_r(E)$$

and $R \subset F$ is the subgroup generated by the relation **R1-R2** below.

R1 For any finite extensions $k \hookrightarrow E_1 \xrightarrow{\phi} E_2$, let $g_{i_0} \in G_{i_0}(E_2)$ and $g_i \in G_i(E_1)$ for $i \neq i_0$, the relation

$$(\phi^*(g_1) \otimes \dots \otimes g_{i_0} \otimes \dots \otimes \phi^*(g_r))_{E_2} - (g_1 \otimes \dots \otimes N_{E_2/E_1}(g_{i_0}) \otimes \dots \otimes g_r)_{E_1}$$

(Here N_{E_2/E_1} denotes the norm map on the group scheme G_{i_0})

R2 For every algebraic function field K/k and all choices $g_i \in G_i(K)$, $h \in K^\times$ such that for each place v of K/k , there exists $i(v)$ such that $g_i \in G_i(\mathcal{O}_v)$ for all $i \neq i(v)$, the relation

$$\sum_{v: \text{place of } K/k} (g_1(v) \otimes \dots \otimes \partial_v(g_{i(v)}, h) \otimes \dots \otimes g_r(v))_{k(v)/k}$$

Here $g_i(v) \in G_i(k(v))$ ($i \neq i(v)$) denotes the reduction of $g_i \in \mathcal{O}_v$ modulo m_v and $\partial_v(g_{i(v)}, h)$ is the extended tame symbol as defined in 1.3

The class in F/R of an element $a_1 \otimes \dots \otimes a_r \in G_1(E) \otimes \dots \otimes G_r(E)$ will be denoted $\{a_1, \dots, a_r\}_{E/k}$.

Remark 1.5. By the relation **R1**, if ϕ is a k -isomorphism $E_1 \xrightarrow{\sim} E_2$, then $\{g_1, \dots, g_r\}_{E_1/k} = \{\phi^*(g_1), \dots, \phi^*(g_r)\}_{E_2/k}$. This shows that symbols form a set.

2 Triangulated categories of motives

In this section, we will briefly review the definition of the triangulated categories of motives. (c.f. [TriCa]).

2.1 Triangulated category of effective geometric motives

First we will review the construction of the category of geometric motives.

2.1. 1. Let Sm/k be the category of schemes which are separated smooth, and of finite type over k .

2. Recall the definition of the category $\text{SmCor}(k)$: its objects are those of Sm/k . The set of morphism from Y to X is given by the group $\text{Cor}(Y, X)$ of finite correspondences from Y to X , defined as the free abelian group on the symbol (Z) , where Z runs through the integral closed subschemes of $Y \times_k X$ which are finite over Y and surjective over a connected component of Y . We will denote the object of $\text{SmCor}(k)$ which corresponds to a smooth scheme X by $[X]$.

2.2. The category $\text{SmCor}(k)$ is an additive category. Consider the homotopy category $\mathcal{H}^b(\text{SmCor}(k))$ of bounded complexes over $\text{SmCor}(k)$. Let \mathcal{T} be the class of complexes of the following two forms:

1. For any smooth scheme X over k the complex

$$[X \times \mathbb{A}_k^1] \xrightarrow{[pr_1]} [X]$$

belongs to \mathcal{T} .

2. For any smooth scheme X over k and an open covering $X = U \cup V$ of X the complex

$$[U \cap V] \xrightarrow{[j_U] \oplus [j_V]} [U] \oplus [V] \xrightarrow{[i_U] \oplus (-[i_V])} [X]$$

belongs to \mathcal{T} . (here j_U, j_V, i_U, i_V are the obvious open embeddings.)

Denote by $\tilde{\mathcal{T}}$ the minimal thick subcategory of $\mathcal{H}^b(\text{SmCor}(k))$ which contains \mathcal{T} .

The triangulated category $\text{DM}_{\text{gm}}^{\text{eff}}(k)$ of effective geometric motives over k is the pseudo-Abelian envelope of the localization of $\mathcal{H}^b(\text{SmCor}(k))$ with respect to the thick subcategory $\tilde{\mathcal{T}}$. We denote the obvious functor $\text{Sm}/k \rightarrow \text{DM}_{\text{gm}}^{\text{eff}}(k)$ by M_{gm} .

2.3. For a pair of smooth schemes X, Y over k , we set

$$[X] \otimes [Y] := [X \times Y].$$

For any smooth schemes X_1, Y_1, X_2, Y_2 the external product of cycles defines a homomorphism:

$$\text{Cor}(X_1, Y_1) \otimes \text{Cor}(X_2, Y_2) \rightarrow \text{Cor}(X_1 \times X_2, Y_1 \times Y_2)$$

which gives us a definition of tensor product of morphisms in $\text{SmCor}(k)$. This structure defines in the usual way a tensor category structure on $\mathcal{H}^b(\text{SmCor}(k))$ which can be descended to the category $\text{DM}_{\text{gm}}^{\text{eff}}(k)$ by the universal property of localization.

Note that the unit object our tensor structure is $M_{\text{gm}}(\text{Spec } k)$. We will denote it by \mathbb{Z} .

Example 2.4. Let $x, y : \text{Spec } k \rightarrow \mathbb{P}_k^1$ be two k -rational points. Then $M_{\text{gm}}(x) = M_{\text{gm}}(y) : M_{\text{gm}}(\text{Spec } k) \rightarrow M_{\text{gm}}(\mathbb{P}_k^1)$.

Proof. We take an affine open set $\mathbb{A}_k^1 \xrightarrow{j} \mathbb{P}_k^1$ which contains x and y . That is, there are $\tilde{x}, \tilde{y} : \text{Spec } k \rightarrow \mathbb{A}_k^1$ such that $x = j \circ \tilde{x}$ and $y = j \circ \tilde{y}$. Then we have $M_{\text{gm}}(x) = M_{\text{gm}}(j) \circ M_{\text{gm}}(\tilde{x}) = M_{\text{gm}}(j) \circ M_{\text{gm}}(p)^{-1} = M_{\text{gm}}(j) \circ M_{\text{gm}}(\tilde{y}) = M_{\text{gm}}(y)$, where $p : \mathbb{A}_k^1 \rightarrow \text{Spec } k$ is the structure morphism. \square

2.2 Triangulated category of effective motivic complexes

To study the fundamental property of $\text{DM}_{\text{gm}}^{\text{eff}}(k)$, Voevodsky uses sheaf theoretic method in [TriCa]. More precisely, he constructs another category $\text{DM}^{\text{eff}}(k)$ using a sheaf category and he proves $\text{DM}_{\text{gm}}^{\text{eff}}(k)$ admits a natural full embedding as a tensor category and a triangulated category to the category $\text{DM}^{\text{eff}}(k)$. We will review the construction of $\text{DM}^{\text{eff}}(k)$.

2.5. 1. A presheaf with transfers on Sm/k is an additive contravariant functor from the category $\text{SmCor}(k)$ to the category of abelian groups. We denote by $\text{PST}(k)$ the category of presheaf with transfers on Sm/k .

2. A presheaf with transfers on Sm/k is called a Nisnevich sheaf with transfers if the corresponding presheaf of abelian groups on Sm/k is a sheaf in the Nisnevich topology. We denote by $\text{Shv}_{\text{Nis}}(\text{SmCor}(k))$ the category of Nisnevich sheaves with transfers.

Example 2.6. For any smooth scheme X over k , a presheaf $\mathbb{Z}_{\text{tr}}(X) := \text{Cor}(?, X)$ is a Nisnevich sheaf with transfers on Sm/k . (c.f. [TriCa] Lemma 3.1.2). For a k -rational point $x : \text{Spec } k \rightarrow X$, we put

$$\mathbb{Z}_{\text{tr}}(X, x) := \text{Coker}(\mathbb{Z}_{\text{tr}}(\text{Spec } k) \xrightarrow{\mathbb{Z}_{\text{tr}}(x)} \mathbb{Z}_{\text{tr}}(X)).$$

2.7. 1. A presheaf with transfers F is called homotopy invariant if for any smooth scheme X over k the projection $X \times \mathbb{A}_k^1 \rightarrow X$ induces the isomorphism $F(X) \rightarrow F(X \times \mathbb{A}_k^1)$.

2. A Nisnevich sheaf with transfers is called homotopy invariant if it is homotopy invariant as a presheaf with transfers.

2.8. $\text{Shv}_{\text{Nis}}(\text{SmCor}(k))$ is an abelian category. (c.f. [TriCa] Theorem 3.1.4) Inside the derived category $D^-(\text{Shv}_{\text{Nis}}(\text{SmCor}(k)))$ of complexes bounded from above, one defines the full subcategory $\text{DM}_{\text{eff}}^{\text{eff}}(k)$ of effective motivic complexes over k as the one consisting of objects whose cohomology sheaves are homotopy invariant. This subcategory is triangulated. (Need the assumption of perfectness of k .) (c.f. [TriCa] Proposition 3.1.13).

2.9. 1. Let F be a presheaf with transfers. There is a canonical surjection of presheaves

$$\bigoplus_{(X, x \in F(X))} \mathbb{Z}_{\text{tr}}(X) \rightarrow F.$$

Iterating this construction we get a canonical left resolution $\mathcal{L}(F)$ of F which consists of direct sums of presheaves of the form $\mathbb{Z}_{\text{tr}}(X)$ for smooth schemes X over k .

2. We set for two smooth schemes X, Y :

$$\mathbb{Z}_{\text{tr}}(X) \otimes \mathbb{Z}_{\text{tr}}(Y) := \mathbb{Z}_{\text{tr}}(X \times Y)$$

and for two presheaves with transfers F, G :

$$F \otimes G := \underline{\mathbf{H}}_0(\mathcal{L}(F) \otimes \mathcal{L}(G)).$$

3. This construction provides us with a tensor structure on the derived category $D^-(\text{Shv}_{\text{Nis}}(\text{SmCor}(k)))$.

To define the tensor structure on $\text{DM}_{\text{eff}}^{\text{eff}}(k)$, we will need an alternative description of $\text{DM}_{\text{eff}}^{\text{eff}}(k)$.

2.10. Let Δ^\bullet be the standard cosimplicial object in Sm/k . For any presheaf with transfers F on Sm/k let $C_*(F)$ be the complex of presheaves on Sm/k of the form $C_n(F) = \underline{\mathbf{H}}\text{om}(\Delta^n, F)$ with differentials given by alternated sums of morphisms which correspond to the boundary morphisms of Δ^\bullet . This complex

is called the singular simplicial complex of F .

The following properties are fundamental.

1. If F is a presheaf with transfers (resp. a Nisnevich sheaf with transfers) then $C_*(F)$ is a complex of presheaves with transfers (resp. Nisnevich sheaves with transfers).
2. For any presheaf with transfers F over k , the cohomology presheaf $h_i(F)$ of the complex $C_*(F)$ and its Nisnevich sheafification $h_i^{Nis}(F)$ are homotopy invariant. (Need the assumption of perfectness of k .) (c.f. [TriCa] Lemma 3.2.1).
3. In view of 1. and 2., $C_*(?)$ is a functor from the category of Nisnevich sheaves with transfers on Sm/k to $DM_-^{\text{eff}}(k)$.

Proposition 2.11. (c.f. [TriCa] Proposition 3.2.3)

The functor $C_*(?)$ can be extended to a functor

$$\mathcal{RC} : D^-(\text{Shv}_{\text{Nis}}(\text{SmCor}(k))) \rightarrow DM_-^{\text{eff}}(k)$$

which is left adjoint to the natural embedding. The functor \mathcal{RC} identifies $DM_-^{\text{eff}}(k)$ with localization of $D^-(\text{Shv}_{\text{Nis}}(\text{SmCor}(k)))$ with respect to the localizing subcategory \mathcal{A} generated by complexes of the form

$$\mathbb{Z}_{\text{tr}}(X \times \mathbb{A}_k^1) \xrightarrow{\mathbb{Z}_{\text{tr}}(\text{pr}_1)} \mathbb{Z}_{\text{tr}}(X)$$

for smooth schemes X over k .

2.12. 1. In the notation above, \mathcal{A} is a \otimes -ideal, that is, for any object T of $D^-(\text{Shv}_{\text{Nis}}(\text{SmCor}(k)))$ and an object S of \mathcal{A} the object $T \otimes S$ belongs to \mathcal{A} . (c.f. [TriCa] Lemma 3.2.4).

2. We define tensor structure on $DM_-^{\text{eff}}(k)$ as the descent of the tensor structure on $D^-(\text{Shv}_{\text{Nis}}(\text{SmCor}(k)))$ with respect to the projector \mathcal{RC} . Note that such a descent exists by the universal property of localization and 1.

Theorem 2.13. (c.f. [TriCa] Theorem 3.2.6)

There is a commutative diagram of functors of the form such that the following conditions hold:

$$\begin{array}{ccc} \mathcal{H}^b(\text{SmCor}(k)) & \xrightarrow{L} & D^-(\text{Shv}_{\text{Nis}}(\text{SmCor}(k))) \\ \downarrow & & \downarrow \mathcal{RC} \\ DM_{\text{gm}}^{\text{eff}}(k) & \xrightarrow{i} & DM_-^{\text{eff}}(k) \end{array}$$

1. The functor i is a full embedding with a dense image.
2. For any smooth scheme X over k the object $\mathcal{RC}(L(X))$ is canonically isomorphic to the $C_*(\mathbb{Z}_{\text{tr}}(X))$.
3. All functors preserve tensor and triangulated structures.

Example 2.14. Let $x : \text{Spec } k \rightarrow X$ be a k -rational point of a smooth scheme X . Then we have an identification

$$C_*(\mathbb{Z}_{\text{tr}}(X, x)) \xrightarrow{\sim} \text{Cone}(M_{\text{gm}}(\text{Spec } k) \xrightarrow{M_{\text{gm}}(x)} M_{\text{gm}}(X))$$

defined by

$$\begin{array}{ccc} \left[\begin{array}{c} \mathbb{Z}_{\text{tr}}(\text{Spec } k) \\ \mathbb{Z}_{\text{tr}}(x) \downarrow \\ \mathbb{Z}_{\text{tr}}(X) \end{array} \right] & \rightarrow & \left[\begin{array}{c} 0 \\ \downarrow \\ \mathbb{Z}_{\text{tr}}(X, x) \end{array} \right]. \end{array}$$

2.3 Motives with compact support

In this subsection, we will briefly review the notation and fundamental result of [FV00], [RelCy] and [TriCa].

2.15. For any scheme of finite type X over k and any $r \geq 0$ we denote by $\mathcal{Z}_{\text{equi}}(X, r)$ the presheaf on the category of smooth schemes over k which takes a smooth scheme Y to free abelian groups generated by closed integral subschemes Z of $X \times Y$ which are equidimensional of relative dimension r over Y .

This presheaf has the following property.

1. $\mathcal{Z}_{\text{equi}}(X, r)$ is a sheaf in the Nisnevich topology.
2. It has a canonical structure of a presheaf with transfers.
3. The presheaf with transfers $\mathcal{Z}_{\text{equi}}(X, r)$ is covariantly functorial with respect to proper morphisms of X by means of the usual proper push-forward of cycles.
4. It is contravariantly functorial with an appropriate dimension shift with respect to flat equidimensional morphisms.
5. There is a pairing

$$\times : \mathcal{Z}_{\text{equi}}(X, r) \otimes \mathcal{Z}_{\text{equi}}(X', r') \rightarrow \mathcal{Z}_{\text{equi}}(X \times X', r + r')$$

of presheaves.:

Let U be a smooth scheme over k . For any pair of integral closed subschemes $Z \subset X \times U, Z' \subset X' \times U$ equidimensional over U . Sending Z, Z' to the cycle associated to the subscheme $Z \times_U Z' \subset X \times X' \times U$ determines a pairing.

6. (Need the assumption of 0.6.) The flat pull-back morphism induces a quasi-isomorphism

$$C_*(\mathcal{Z}_{\text{equi}}(X, 0)) \rightarrow C_*(\mathcal{Z}_{\text{equi}}(X \times \mathbb{A}^n, n)).$$

2.16. For any scheme of finite type over k the object $C_*(\mathcal{Z}_{\text{equi}}(X, 0))$ belongs to $\text{DM}_{-}^{\text{eff}}(k)$. Moreover it belongs to $\text{DM}_{\text{gm}}^{\text{eff}}(k)$. (Using the assumption 0.6) (c.f. [TriCa], Corollary 4.1.6). We will denote $M_{\text{gm}}^c(X) := C_*(\mathcal{Z}_{\text{equi}}(X, 0))$ and call it a motivic complex of X with compact support.

Since $\mathbb{Z}_{\text{tr}}(X)$ is a subsheaf of $\mathcal{Z}_{\text{equi}}(X, 0)$, the inclusion morphism induces the natural morphism $M_{\text{gm}}(X) \rightarrow M_{\text{gm}}^c(X)$.

The following properties are fundamental. (c.f. [TriCa], Proposition 4.1.5, Proposition 4.1.7)

1. If X is proper then the canonical morphism $M_{\text{gm}}(X) \rightarrow M_{\text{gm}}^c(X)$ is the isomorphism.
2. (Need the assumption 0.6.) Let Z be a closed subscheme of X . Then there is a canonical distinguished triangle of the form

$$M_{\text{gm}}^c(Z) \rightarrow M_{\text{gm}}^c(X) \rightarrow M_{\text{gm}}^c(X - Z) \rightarrow M_{\text{gm}}^c(Z)[1].$$

3. For X, Y of finite type over k , the pairing $\mathcal{Z}_{\text{equi}}(X, 0) \otimes \mathcal{Z}_{\text{equi}}(Y, 0) \rightarrow \mathcal{Z}_{\text{equi}}(X \times_k Y, 0)$ induces an isomorphism

$$M_{\text{gm}}^c(X) \otimes M_{\text{gm}}^c(Y) \xrightarrow{\sim} M_{\text{gm}}^c(X \times_k Y).$$

2.4 Tate object

2.17. For any smooth scheme X over k , the morphism $X \rightarrow \text{Spec } k$ gives us a morphism in $\text{DM}_{\text{gm}}^{\text{eff}}(k)$ of the form $M_{\text{gm}}(X) \rightarrow \mathbb{Z}$. There is a canonical split distinguished triangle

$$\widetilde{M}_{\text{gm}}(X) \rightarrow M_{\text{gm}}(X) \rightarrow \mathbb{Z} \rightarrow \widetilde{M}_{\text{gm}}(X)[1]$$

where $\widetilde{M}_{\text{gm}}(X)$ is the reduced motif of X represented in $\mathcal{H}^b(\text{SmCor}(k))$ by the complex $[X] \rightarrow [\text{Spec } k]$.

Example 2.18. In the notation above, for any k -rational point $x : \text{Spec } k \rightarrow X$, we have the canonical identification $M_{\text{gm}}(X, x) \xrightarrow{\sim} \widetilde{M}_{\text{gm}}(X)$ as the following way.

$$\begin{array}{ccc} \left[\begin{array}{c} [\text{Spec } k] \\ \downarrow x \\ [X] \\ \downarrow \\ 0 \end{array} \right] & \xrightarrow{\text{id} - M_{\text{gm}}(x \circ p)} & \left[\begin{array}{c} 0 \\ \downarrow \\ [X] \\ \downarrow p \\ [\text{Spec } k] \end{array} \right] \end{array}$$

where $p : X \rightarrow \text{Spec } k$ is the structure morphism. $x : \text{Spec } k \rightarrow X$ defines splitting $M_{\text{gm}}(X) \xrightarrow{\sim} M_{\text{gm}}(X, x) \oplus \mathbb{Z}$.

2.19. We define the Tate object $\mathbb{Z}(1)$ of $\text{DM}_{\text{gm}}^{\text{eff}}(k)$ as $\widetilde{M}_{\text{gm}}(\mathbb{P}^1)[-2]$. We further define $\mathbb{Z}(n)$ to be the n -th tensor power of $\mathbb{Z}(1)$. For any object A of $\text{DM}_{\text{gm}}^{\text{eff}}(k)$ we put

$$A(n) = A \otimes \mathbb{Z}(n)$$

$$A\{n\} = A \otimes \mathbb{Z}(n)[n]$$

$$A((n)) = A \otimes \mathbb{Z}(n)[2n].$$

By Example 2.18, for any $x : \text{Spec } k \rightarrow \mathbb{P}_k^1$, we have the canonical isomorphism $M_{\text{gm}}(\mathbb{P}_k^1, x) \xrightarrow{\sim} \mathbb{Z}((1))$.

2.20. Let $x : \text{Spec } k \rightarrow \mathbb{P}_k^1$ be a k -rational point. Comparing the following split distinguished triangles

$$\begin{array}{ccccccc}
M_{\text{gm}}^c(\text{Spec } k) & \longrightarrow & M_{\text{gm}}^c(\mathbb{P}^1) & \longrightarrow & M_{\text{gm}}^c(\mathbb{A}^1) & \longrightarrow & M_{\text{gm}}^c(\text{Spec } k)[1] \\
\text{id} \downarrow & & \text{id} \downarrow & & \downarrow & & \text{id} \downarrow \\
M_{\text{gm}}(\text{Spec } k) & \xrightarrow{M_{\text{gm}}(x)} & M_{\text{gm}}(\mathbb{P}_k^1) & \longrightarrow & \mathbb{Z}((1)) & \longrightarrow & M_{\text{gm}}(\text{Spec } k)[1],
\end{array}$$

where $M_{\text{gm}}(\mathbb{P}_k^1) \rightarrow \mathbb{Z}((1))$ is defined by

$$M_{\text{gm}}(\mathbb{P}_k^1) \xrightarrow{\text{can}} M_{\text{gm}}(\mathbb{P}^1, x) \xrightarrow{\sim} \mathbb{Z}((1)),$$

we know that there is a natural isomorphism $\mathbb{Z}((1)) \xrightarrow{\sim} M_{\text{gm}}^c(\mathbb{A}^1)$. It does not depend on the choice of a k -rational point by Example 2.4. Similarly using Mayer-Vietoris sequence for canonical covering of \mathbb{P}_k^1 , we know also that there is a natural isomorphism $\mathbb{Z}\{1\} \xrightarrow{\sim} \widehat{M_{\text{gm}}}(\mathbb{A}^1 - \{0\})$.

2.5 The triangulated category of geometric motives

In this subsection, we will define the triangulated category $\text{DM}_{\text{gm}}(k)$ of geometric motives over k .

2.21. 1. We define the category $\text{DM}_{\text{gm}}(k)$: its objects are pairs of the form (A, n) where A is an object of $\text{DM}_{\text{gm}}^{\text{eff}}(k)$ and $n \in \mathbb{Z}$ and morphisms are defined by the following formula

$$\text{Hom}_{\text{DM}_{\text{gm}}(k)}((A, n), (B, m)) := \lim_{k \geq -n, -m} \text{Hom}_{\text{DM}_{\text{gm}}^{\text{eff}}(k)}(A(k+n), B(k+m)).$$

2. The category $\text{DM}_{\text{gm}}(k)$ with the obvious shift functor and class of distinguished triangles is a triangulated category.

3. The permutation involution on $\mathbb{Z}(1) \otimes \mathbb{Z}(1)$ is identity in $\text{DM}_{\text{gm}}^{\text{eff}}(k)$. (c.f. [TriCa] Corollary 2.1.5)

4. Using the fact of 3. and general theory, $\text{DM}_{\text{gm}}(k)$ has a natural tensor structure.

Theorem 2.22. (c.f. [Voe02] *The cancellation theorem*)

(Need the assumption of perfectness of k .) For objects A, B in $\text{DM}_{\text{gm}}^{\text{eff}}(k)$ the natural map

$$? \otimes \text{id}_{\mathbb{Z}(1)} : \text{Hom}_{\text{DM}_{\text{gm}}^{\text{eff}}(k)}(A, B) \rightarrow \text{Hom}_{\text{DM}_{\text{gm}}^{\text{eff}}(k)}(A(1), B(1))$$

is an isomorphism. Thus the canonical functor

$$\text{DM}_{\text{gm}}^{\text{eff}}(k) \rightarrow \text{DM}_{\text{gm}}(k)$$

is a full embedding.

3 Various morphisms between motives

In this section, we will briefly review [TriCa] and [MotGe]

3.1 Transpose for finite equidimensional morphisms

3.1. Let X, Y be smooth schemes over k and $f : X \rightarrow Y$ a finite equidimensional morphism. Then we have the transpose of f , ${}^t f : Y \rightarrow X$ in $\text{SmCor}(k)$. That is $s(\Gamma_f) \in \text{Hom}_{\text{SmCor}(k)}(Y, X)$, where $s : X \times Y \rightarrow Y \times X$ is the switch morphism.

Example 3.2. Let $L/K/k$ be finite field extensions. Then we have a canonical morphism $i : \text{Spec } L \rightarrow \text{Spec } K$. Hence we get $M_{\text{gm}}({}^t i) : M_{\text{gm}}(\text{Spec } K) \rightarrow M_{\text{gm}}(\text{Spec } L)$ in $\text{DM}_{\text{gm}}(k)$. We shall write this map as $N_{L/K}$ because there is the following commutative diagram

$$\begin{array}{ccc} \text{Hom}(M_{\text{gm}}(\text{Spec } L), \mathbb{Z}\{n\}^{\sim}) & \longrightarrow & K_n^M(L) \\ \text{Hom}(N_{L/K}, \mathbb{Z}\{n\}) \downarrow & & \downarrow N_{L/K} \\ \text{Hom}(M_{\text{gm}}(\text{Spec } K), \mathbb{Z}\{n\}^{\sim}) & \longrightarrow & K_n^M(K). \end{array}$$

(c.f. [BKcon] Lemma 3.4.4. See also Theorem 5.16)

3.2 Pull back for flat equidimensional morphisms

3.3. Let X, Y be smooth schemes and $f : X \rightarrow Y$ a flat equidimensional morphism of relative dimension n . Then one can define a morphism $f^* : M_{\text{gm}}^c(Y)((n)) \rightarrow M_{\text{gm}}^c(X)$ as follows. (This is slightly different from the definition in [TriCa] Corollary 4.2.4).

$$\begin{array}{ccc} M_{\text{gm}}^c(Y)((n)) & = & C_*(\mathcal{Z}_{\text{equi}}(Y \times \mathbb{A}^n, 0)) \\ & \xrightarrow{C_*(f \times \text{id}_{\mathbb{A}^n})^*} & C_*(\mathcal{Z}_{\text{equi}}(X \times \mathbb{A}^n, n)) \\ & \xleftarrow{\text{qis}} & C_*(\mathcal{Z}_{\text{equi}}(X, 0)) = M_{\text{gm}}^c(X) \end{array}$$

where every quasi-isomorphisms are induced from flat pull backs.

3.4. The following properties are easily proved.

1 In the notation above, if f is an open immersion, this morphism coincides with the canonical morphism $M_{\text{gm}}^c(Y) \rightarrow M_{\text{gm}}^c(X)$.

2 In the notation above, if X and Y are proper over $\text{Spec } k$ and f is flat finite equidimensional, then $M_{\text{gm}}({}^t f) = f^*$. (c.f. [MotGe] Lemma 1.1.2)

3 Let X, Y and Z be smooth schemes and $f : X \rightarrow Y$, $g : Y \rightarrow Z$ flat equidimensional morphisms of relative dimension n and m respectively. Then we have $f^* \circ g^*((m)) = (g \circ f)^*$.

Lemma 3.5.

Let $p : \mathbb{A}_k^1 \rightarrow \text{Spec } k$ be the structure morphism. Then $p^* : M_{\text{gm}}^c(\text{Spec } k)((1)) = M_{\text{gm}}^c(\mathbb{A}_k^1) \rightarrow M_{\text{gm}}^c(\mathbb{A}_k^1)$ coincides with $\text{id}_{M_{\text{gm}}^c(\mathbb{A}_k^1)}$.

Proof. By definition (See 3.3) ,

$$p^* = C_*(\mathcal{Z}_{\text{equi}}(\mathbb{A}_k^1, 0)) \xrightarrow{C_*(pr_1^*)} C_*(\mathcal{Z}_{\text{equi}}(\mathbb{A}_k^2, 1)) \xrightarrow{(C_*(pr_2^*))^{-1}} C_*(\mathcal{Z}_{\text{equi}}(\mathbb{A}_k^1, 0))$$

where pr_1, pr_2 are two projections $pr_1, pr_2 : \mathbb{A}_k^2 \rightarrow \mathbb{A}_k^1$. We assert that two projections pr_1, pr_2 induce the same morphism $pr_1^* = pr_2^* : C_*(\mathcal{Z}_{\text{equi}}(\mathbb{A}_k^1, 0)) \rightarrow C_*(\mathcal{Z}_{\text{equi}}(\mathbb{A}_k^2, 1))$ in $\text{DM}_-^{\text{eff}}(k)$. Since

$$pr_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ pr_2,$$

it suffices to prove that the action of $\text{GL}_2(k)$ on \mathbb{A}_k^2 induces trivial action on $C_*(\mathcal{Z}_{\text{equi}}(\mathbb{A}_k^2, 1))$ in $\text{DM}_-^{\text{eff}}(k)$ by flat pull back. On the other hand $\text{GL}_2(k)$ is generated by the elements of conjugate of $\begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix}$. For $A = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \in \text{GL}_2(k)$, considering the following diagram

$$\begin{array}{ccc} C_*(\mathcal{Z}_{\text{equi}}(\mathbb{A}_k^2, 1)) & \xrightarrow{A^*} & C_*(\mathcal{Z}_{\text{equi}}(\mathbb{A}_k^2, 1)) \\ \uparrow pr_1^* \wr & & \wr \uparrow pr_1^* \\ C_*(\mathcal{Z}_{\text{equi}}(\mathbb{A}_k^1, 0)) & \xrightarrow{\text{id}} & C_*(\mathcal{Z}_{\text{equi}}(\mathbb{A}_k^1, 0)), \end{array}$$

we get the result. \square

3.3 Motives with closed support

3.6. We call (X, Z) a closed pair if X is a smooth scheme over k and Z is a closed subscheme. If Z is smooth over k , we call (X, Z) a smooth pair. We call a pair of morphisms of scheme $(f, g) : (Y, T) \rightarrow (X, Z)$ a morphism of closed pair if a commutative square $\begin{array}{ccc} T & \longrightarrow & X \\ g \downarrow & & \downarrow f \\ Z & \longrightarrow & Y \end{array}$ is a Cartesian square as underlying topolog-

$$\begin{array}{ccc} T & \longrightarrow & X \\ g \downarrow & & \downarrow f \\ Z & \longrightarrow & Y \end{array}$$

ical spaces. Such a morphisms called Cartesian (resp. excisive) if the diagram above is a Cartesian square (resp. f is étale and g_{red} is an isomorphism.)

3.7. Let X be a smooth scheme and U its open subset. Then we define

$$M_{\text{gm}}(X/U) := C^*(\text{Coker}(\mathbb{Z}_{\text{tr}}(U) \rightarrow \mathbb{Z}_{\text{tr}}(X))).$$

For any closed pair (X, Z) , we define relative motif associated (X, Z) by $M_Z(X) := M_{\text{gm}}(X/X - Z)$. By definition there is a canonical distinguished triangle of the form

$$M_{\text{gm}}(X - Z) \xrightarrow{M_{\text{gm}}(j)} M_{\text{gm}}(X) \xrightarrow{i^\sharp} M_Z(X) \rightarrow M_{\text{gm}}(X - Z)[1].$$

where $i : Z \hookrightarrow X$ is a closed immersion and $j : X - Z \hookrightarrow X$ is an open immersion.

3.8. For any morphisms of closed pair $(f, g) : (Y, T) \rightarrow (X, Z)$, we associate a morphism $(f, g)_* : M_T(Y) \rightarrow M_Z(X)$ which makes the following diagram commute

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_{\text{gm}}(Y - T) & \longrightarrow & M_{\text{gm}}(Y) & \longrightarrow & M_T(Y) \longrightarrow 0 \\ & & \downarrow M_{\text{gm}}(h) & & \downarrow M_{\text{gm}}(f) & & \downarrow (f, g)_* \\ 0 & \longrightarrow & M_{\text{gm}}(X - Z) & \longrightarrow & M_{\text{gm}}(X) & \longrightarrow & M_Z(X) \longrightarrow 0 \end{array}$$

where $h : Y - T \rightarrow X - Z$ is a induced morphism from f .

3.9. In the notation above, if f is finite equidimensional, we associate a morphism $(f, g)^* : M_T(Y) \rightarrow M_Z(X)$ which makes the following diagram commute

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_{\text{gm}}(Y - T) & \longrightarrow & M_{\text{gm}}(Y) & \longrightarrow & M_T(Y) \longrightarrow 0 \\ & & \uparrow M_{\text{gm}}(h) & & \uparrow M_{\text{gm}}(f) & & \uparrow (f, g)^* \\ 0 & \longrightarrow & M_{\text{gm}}(X - Z) & \longrightarrow & M_{\text{gm}}(X) & \longrightarrow & M_Z(X) \longrightarrow 0. \end{array}$$

Proposition 3.10. (c.f. [IntMo] Proposition 2.3)

(Red) Reduction: If (X, Z) is a closed pair, the canonical morphism $(X, Z_{\text{red}}) \rightarrow (X, Z)$ induces identity map $M_{Z_{\text{red}}}(X) \rightarrow M_Z(X)$.

(Add) Additivity: Let X be a smooth scheme, and Z, W disjoint closed subschemes of X . Then induced morphism $M_{Z \sqcup W}(X) \rightarrow M_Z(X) \oplus M_W(X)$ is an isomorphism.

(Exc) Excision: Any excisive morphism $(Y, T) \rightarrow (X, Z)$ induces an isomorphism $M_T(Y) \rightarrow M_Z(X)$.

(MV) Mayer-Vietoris: Let X be a smooth scheme over k , U and V two open subsets of X such that $X = U \cup V$, and Z a closed subscheme of X . Then we have a distinguished triangle of the form

$$M_{Z \cap U \cap V}(U \cap V) \rightarrow M_{Z \cap U}(U) \oplus M_{Z \cap V}(V) \rightarrow M_Z(X) \xrightarrow{\pm 1}.$$

(Htp) Homotopy invariance: A Cartesian morphism $\pi : (\mathbb{A}_X^1, \mathbb{A}_Z^1) \rightarrow (X, Z)$ induced from the canonical projection induces an isomorphism

$$M_{\text{gm}}(\pi) : M_{\mathbb{A}_Z^1}(\mathbb{A}_X^1) \rightarrow M_Z(X).$$

3.4 Thom isomorphism

3.11. Let X be a scheme and E/X a vector bundle. We consider X as a closed subscheme of E by zero section. We define the Thom motif of E/X by $M_{\text{gm}}(ThE) := M_X(E)$.

3.12. In the notation above, if rank of E is n , there is the Thom isomorphism $\theta(E) : M_{\text{gm}}(ThE) \xrightarrow{\sim} M_{\text{gm}}(X)((n))$. We will briefly review the construction of this isomorphism.

3.13. Let X be a smooth scheme and $\Delta_X : X \rightarrow X \times_k X$ the diagonal immersion. Let \mathcal{M}, \mathcal{N} be objects of $\mathrm{DM}_{-}^{\mathrm{eff}}(k)$ and

$$\alpha : \mathrm{M}_{\mathrm{gm}}(X) \rightarrow \mathcal{M}$$

$$\beta : \mathrm{M}_{\mathrm{gm}}(X) \rightarrow \mathcal{N}$$

are morphisms in $\mathrm{DM}_{-}^{\mathrm{eff}}(k)$.

We define external cup product of α and β over X is composition of the following morphisms

$$\mathrm{M}_{\mathrm{gm}}(X) \xrightarrow{\mathrm{M}_{\mathrm{gm}}(\Delta_X)} \mathrm{M}_{\mathrm{gm}}(X) \otimes \mathrm{M}_{\mathrm{gm}}(X) \xrightarrow{\alpha \otimes \beta} \mathcal{M} \otimes \mathcal{N}.$$

We denote this morphism by $\alpha \boxtimes_X \beta$ or $\alpha \boxtimes \beta$

3.14. In the notation above, if $\mathcal{M} = \mathbb{Z}((m))$ and $\mathcal{N} = \mathbb{Z}((n))$, then we have the canonical isomorphism $\varepsilon : \mathbb{Z}((m)) \otimes \mathbb{Z}((n)) \xrightarrow{\sim} \mathbb{Z}((m+n))$.

We put $\alpha \cup \beta := \varepsilon \circ \alpha \boxtimes \beta$ and call it internal cup product of α and β .

3.15. There is a natural isomorphism as a presheaf with transfers (c.f. [MotGe] Corollary 2.2.7)

$$c_1 : \mathrm{Pic}(?) \rightarrow \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(\mathrm{M}_{\mathrm{gm}}(?), \mathbb{Z}((1))).$$

For any smooth scheme X and $\mathcal{L} \in \mathrm{Pic}(X)$, we will call $c_1(\mathcal{L})$ a motivic Chern class of \mathcal{L} .

3.16. Let X be a smooth scheme, E a vector bundle over X of rank n , λ_E the canonical invertible sheaf on $\mathbb{P}(E)$ and $p : \mathbb{P}(E) \rightarrow X$ the canonical projection. We define the r -th motivic Lefschetz projector of E by

$$l_r(E) := c_1(\lambda_E)^{\cup r} \boxtimes \mathrm{M}_{\mathrm{gm}}(p) : \mathrm{M}_{\mathrm{gm}}(\mathbb{P}(E)) \rightarrow \mathrm{M}_{\mathrm{gm}}(X)((r))$$

We put the motivic Lefschetz operator as

$$l(E) := \sum_{r=0}^{n-1} l_r(E).$$

Proposition 3.17. (c.f. [TriCa] Proposition 3.5.1)

In the notation above, the morphism

$$l(E) : \mathrm{M}_{\mathrm{gm}}(\mathbb{P}(E)) \rightarrow \bigoplus_{r=0}^{n-1} \mathrm{M}_{\mathrm{gm}}(X)((r))$$

is an isomorphism.

3.18. Let X be a smooth scheme and E/X a vector bundle of rank n . Put $\hat{E} := E \times_X \mathbb{A}_X^1$. Then we have the canonical isomorphisms

$$\mathrm{M}_{\mathrm{gm}}(ThE) \xrightarrow{1} \mathrm{M}_X(\mathbb{P}(\hat{E})) \xrightarrow{2} \mathrm{M}_{\mathrm{gm}}(\mathbb{P}(\hat{E})/\mathbb{P}(E)).$$

The first morphism is induced from an open immersion $E \rightarrow \mathbb{P}(\hat{E})$ which is an isomorphism by **(Exc)**. The second morphism is induced from the projection $\mathbb{P}(\hat{E}) - X \rightarrow \mathbb{P}(E)$ which is an isomorphism by **(MV)** and **(Htp)**. From this isomorphisms, we get the following distinguished triangle

$$\mathrm{M}_{\mathrm{gm}}(\mathbb{P}(E)) \rightarrow \mathrm{M}_{\mathrm{gm}}(\mathbb{P}(\hat{E})) \xrightarrow{\pi_E} \mathrm{M}_{\mathrm{gm}}(ThE) \xrightarrow{\pm 1}.$$

Using this distinguished triangle and Proposition 3.17, we get the following isomorphism

$$\mathrm{M}_{\mathrm{gm}}(X)((n)) \rightarrow \bigoplus_{r=0}^n \mathrm{M}_{\mathrm{gm}}(X)((r)) \xrightarrow{l(\hat{E})^{-1}} \mathrm{M}_{\mathrm{gm}}(\mathbb{P}(\hat{E})) \xrightarrow{\pi_E} \mathrm{M}_{\mathrm{gm}}(ThE).$$

We call the inverse of this isomorphism the Thom isomorphism and denote it by $\theta(E)$.

3.5 Normal cone deformation

3.19. Let (X, Z) be a smooth pair of pure codimension c over k such that dimension of X is n . We will write $B_Z X$ by the blow up of X in Z . Put $D_Z X := B_{0 \times Z}(\mathbb{A}^1 \times X) - B_Z X$. There are canonical isomorphisms

$$\mathrm{M}_Z(X) \rightarrow \mathrm{M}_{\mathbb{A}^1_Z}(D_Z X) \leftarrow \mathrm{M}_{\mathrm{gm}}(ThN_Z X).$$

Hence we get the isomorphism $\mathrm{M}_Z(X) \rightarrow \mathrm{M}_{\mathrm{gm}}(ThN_Z X)$.

3.6 Gysin triangles

3.20. Let (X, Z) be a smooth pair of pure codimension c over k and $i : Z \rightarrow X$ a closed immersion. In [IntMo], Dèglise constructs the following functorial Gysin triangle in $\mathrm{DM}_{\mathrm{gm}}(k)$

$$\mathrm{M}_{\mathrm{gm}}(X - Z) \rightarrow \mathrm{M}_{\mathrm{gm}}(X) \xrightarrow{i^*} \mathrm{M}_{\mathrm{gm}}(Z)((c)) \xrightarrow{\partial_{X,Z}[1]} \mathrm{M}_{\mathrm{gm}}(X - Z)[1].$$

This is constructed from the following distinguished triangle

$$\mathrm{M}_{\mathrm{gm}}(X - Z) \rightarrow \mathrm{M}_{\mathrm{gm}}(X) \rightarrow \mathrm{M}_Z(X) \rightarrow \mathrm{M}_{\mathrm{gm}}(X - Z)[1]$$

and the following isomorphisms

$$\mathrm{M}_Z(X) \xrightarrow{\sim} \mathrm{M}_{\mathrm{gm}}(Th(N_Z(X))) \xrightarrow{\theta(N_Z(X))} \mathrm{M}_{\mathrm{gm}}(Z)((c))$$

where the first isomorphism is induced from the normal cone deformation.

3.21. Let $(f, g) : (Y, T) \rightarrow (X, Y)$ be a morphism of closed pairs. We assume Z (resp. T) is connected and smooth over k of codimension n in X (resp. m in Y).

Then we define Gysin morphism associated to (f, g) , denote $(f, g)!$ by the following commutative diagram:

$$\begin{array}{ccccc} M_T(Y) & \xrightarrow{\sim (1)} & M_{\text{gm}}(Th(N_T Y)) & \xrightarrow{\sim \theta(N_T Y)} & M_{\text{gm}}(T)((m)) \\ (f, g)_* \downarrow & & \downarrow & & \downarrow (f, g)! \\ M_Z(X) & \xrightarrow{\sim (1)} & M_{\text{gm}}(Th(N_Z X)) & \xrightarrow{\sim \theta(N_Z X)} & M_{\text{gm}}(Z)((n)) \end{array}$$

where morphisms (1) are isomorphisms induced from the morphisms of normal cone deformations.

3.22. In the notation above, consider $i : Z \rightarrow X$, $j : X - Z \rightarrow X$, $k : T \rightarrow Y$ and $l : Y - T \rightarrow T$ the canonical immersions. The following diagram is commutative:

$$\begin{array}{ccccccc} M_{\text{gm}}(Y - T) & \xrightarrow{M_{\text{gm}}(l)} & M_{\text{gm}}(Y) & \xrightarrow{k^*} & M_{\text{gm}}(T)((m)) & \xrightarrow{\partial_{Y, T}[1]} & M_{\text{gm}}(Y - T)[1] \\ M_{\text{gm}}(h) \downarrow & & \downarrow M_{\text{gm}}(f) & & (f, g)! \downarrow & & \downarrow M_{\text{gm}}(h)[1] \\ M_{\text{gm}}(X - Z) & \xrightarrow{M_{\text{gm}}(j)} & M_{\text{gm}}(X) & \xrightarrow{i^*} & M_{\text{gm}}(Z)((n)) & \xrightarrow{\partial_{X, Z}[1]} & M_{\text{gm}}(X - Z)[1] \end{array}$$

Lemma 3.23. *Let $x : \text{Spec } k \rightarrow \mathbb{A}_k^1 \xrightarrow{j} \mathbb{P}^1$ be a k -rational point. Then the following diagram is commutative.*

$$\begin{array}{ccc} M_{\text{gm}}(\mathbb{P}^1) & \xrightarrow{x^*} & M_{\text{gm}}(\text{Spec } k)((1)) \\ \text{id} \downarrow & & \downarrow \wr \\ M_{\text{gm}}^c(\mathbb{P}^1) & \xrightarrow{j^*} & M_{\text{gm}}^c(\mathbb{A}_k^1) \end{array}$$

where the vertical isomorphism $M_{\text{gm}}(\text{Spec } k)((1)) \xrightarrow{\sim} M_{\text{gm}}^c(\mathbb{A}_k^1)$ is defined in 2.20.

Proof. It is just a matter of considering two split distinguished triangles below

$$\begin{array}{ccccccc} M_{\text{gm}}(\mathbb{A}_k^1) & \longrightarrow & M_{\text{gm}}(\mathbb{P}_k^1) & \xrightarrow{x^*} & M_{\text{gm}}(\text{Spec } k)((1)) & \longrightarrow & M_{\text{gm}}(\mathbb{A}_k^1)[1] \\ M_{\text{gm}}(p) \downarrow & & \downarrow \text{id} & & \downarrow & & \downarrow M_{\text{gm}}(p)[1] \\ M_{\text{gm}}(\text{Spec } k) & \xrightarrow{M_{\text{gm}}(x)} & M_{\text{gm}}(\mathbb{P}_k^1) & \xrightarrow{j^*} & M_{\text{gm}}^c(\mathbb{A}_k^1) & \longrightarrow & M_{\text{gm}}(\text{Spec } k)[1] \end{array}$$

where the commutativity of 1 follows from Example 2.4. \square

3.24. In the notation above, if f is finite equidimensional, then we define $(f, g)!$ by the following commutative diagram:

$$\begin{array}{ccccc} M_T(Y) & \xrightarrow{\sim (1)} & M_{\text{gm}}(Th(N_T Y)) & \xrightarrow{\sim \theta(N_T Y)} & M_{\text{gm}}(T)((m)) \\ (f, g)_* \downarrow & & \downarrow & & \downarrow (f, g)! \\ M_Z(X) & \xrightarrow{\sim (1)} & M_{\text{gm}}(Th(N_Z X)) & \xrightarrow{\sim \theta(N_Z X)} & M_{\text{gm}}(Z)((n)) \end{array}$$

where the morphisms (1) are the isomorphisms induced from the morphisms of normal cone deformations.

The following diagram is commutative:

$$\begin{array}{ccccccc}
\mathrm{M}_{\mathrm{gm}}(Y - T) & \xrightarrow{\mathrm{M}_{\mathrm{gm}}(l)} & \mathrm{M}_{\mathrm{gm}}(Y) & \xrightarrow{k^*} & \mathrm{M}_{\mathrm{gm}}(T)((m)) & \xrightarrow{\partial_{Y,T}[1]} & \mathrm{M}_{\mathrm{gm}}(Y - T)[1] \\
\mathrm{M}_{\mathrm{gm}}({}^t h) \downarrow & & \downarrow \mathrm{M}_{\mathrm{gm}}({}^t f) & & (f,g)^\dagger \downarrow & & \downarrow \mathrm{M}_{\mathrm{gm}}(h)[1] \\
\mathrm{M}_{\mathrm{gm}}(X - Z) & \xrightarrow{\mathrm{M}_{\mathrm{gm}}(j)} & \mathrm{M}_{\mathrm{gm}}(X) & \xrightarrow{i^*} & \mathrm{M}_{\mathrm{gm}}(Z)((n)) & \xrightarrow{\partial_{X,Z}[1]} & \mathrm{M}_{\mathrm{gm}}(X - Z)[1]
\end{array}$$

Proposition 3.25. (*[MotGe] Proposition 2.5.2*)
In the notation above, if (f, g) is Cartesian, then

$$(f, g)^\dagger = \mathrm{M}_{\mathrm{gm}}({}^t g)((m)).$$

Next we cite the some proposition in [MotGe]. This is needed to prove that the motivic reciprocity law implies the Weil reciprocity law for Milnor K -groups.

3.26. By 2.20, we have the following distinguished triangle

$$\mathbb{Z} \rightarrow \mathrm{M}_{\mathrm{gm}}(\mathbb{G}_m) \xrightarrow{\rho_1} \mathbb{Z}\{1\} \xrightarrow{\pm 1}$$

where $\mathbb{Z} \rightarrow \mathrm{M}_{\mathrm{gm}}(\mathbb{G}_m)$ is induced from the unit morphism $\mathrm{Spec} k \xrightarrow{1} \mathbb{G}_m$.

Proposition 3.27. (*c.f. [MotGe] Proposition 2.6.6*)

Let (X, Z) be a smooth closed pair of codimension 1. We denote $i : Z \rightarrow X$ a closed immersion and $j : X - Z \rightarrow X$ a canonical open immersion.

Suppose there is a regular function $\pi : X \rightarrow \mathbb{A}_k^1$ which parameterizes Z . Hence we have a morphism $\pi = \pi|_{X-Z} : X - Z \rightarrow \mathbb{G}_m$. Then the following diagram is commutative

$$\begin{array}{ccc}
\mathrm{M}_{\mathrm{gm}}(Z)\{1\} & \xrightarrow{\partial_{X,Z}} & \mathrm{M}_{\mathrm{gm}}(X - Z) \\
\mathrm{M}_{\mathrm{gm}}(i)\{1\} \downarrow & & \downarrow \mathrm{M}_{\mathrm{gm}}(\pi) \circ \rho_1 \boxtimes \mathrm{id}_{\mathrm{M}_{\mathrm{gm}}(X - Z)} \\
\mathrm{M}_{\mathrm{gm}}(X)\{1\} & \xleftarrow{\mathrm{M}_{\mathrm{gm}}(j)\{1\}} & \mathrm{M}_{\mathrm{gm}}(X - Z)\{1\}.
\end{array}$$

4 Motivic cohomology groups attached to pointed smooth curves

4.1 Definition

4.1. For pointed smooth curves $(C_1, x_1), \dots, (C_r, x_r)$ over field k , we define a motivic complex $\mathbb{Z}((C_1, x_1) \wedge \dots \wedge (C_r, x_r))$, or $\mathbb{Z}(C_1 \wedge \dots \wedge C_r)$ in $\mathrm{DM}_k^{\mathrm{eff}}(k)$ as follows

$$\mathbb{Z}(C_1 \wedge \dots \wedge C_r) = C^*(\mathbb{Z}_{\mathrm{tr}}(C_1, x_1) \otimes \dots \otimes \mathbb{Z}_{\mathrm{tr}}(C_r, x_r))[-r]$$

4.2. The restriction $\mathbb{Z}(\bigwedge_{i=1}^r (C_i, x_i))|_X$ of $\mathbb{Z}(\bigwedge_{i=1}^r (C_i, x_i))$ to the Zariski site of $X \in Sm/k$ is a complex of sheaves in Zariski topology and the motivic cohomology groups $H_{\mathcal{M}}^n(X, \bigwedge_{i=1}^r (C_i, x_i))$, or $H_{\mathcal{M}}^n(X, \bigwedge_{i=1}^r C_r)$ are defined to be the hyper cohomology of the motivic complexes $\mathbb{Z}(\bigwedge_{i=1}^r (C_i, x_i))$ with respect to Zariski topology:

$$H_{\mathcal{M}}^n(X, \bigwedge_{i=1}^r (C_i, x_i)) = \mathbb{H}_{Zar}^n(X, \mathbb{Z}(\bigwedge_{i=1}^r (C_i, x_i))|_X).$$

4.3. As in [TriCa] we have

$$\mathbb{H}_{Nis}^n(X, \mathbb{Z}(\bigwedge_{i=1}^r (C_i, x_i))|_X) = \text{Hom}_{\text{DM}^{\text{eff}}(k)}(\text{M}_{\text{gm}}(X), \mathbb{Z}(\bigwedge_{i=1}^r (C_i, x_i))[n])$$

Moreover if k is a perfect field as in [CohTh] Proposition 3.1.11, we have

$$\mathbb{H}_{Zar}^n(X, \mathbb{Z}(\bigwedge_{i=1}^r (C_i, x_i))|_X) = \mathbb{H}_{Nis}^n(X, \mathbb{Z}(\bigwedge_{i=1}^r (C_i, x_i))|_X).$$

4.2 Fundamental properties

4.4. (Product structure) Let $(C_1, a_1), \dots, (C_s, a_s), (D_1, b_1), \dots, (D_t, b_t)$ be pointed smooth curves over field k . As usual motivic complexes, we have canonical morphism

$$\mathbb{Z}(\bigwedge_{i=1}^s (C_i, a_i)) \otimes \mathbb{Z}(\bigwedge_{j=1}^t (D_j, b_j)) \rightarrow \mathbb{Z}(\bigwedge_{i=1}^s (C_i, a_i) \wedge \bigwedge_{j=1}^t (D_j, b_j)).$$

Hence we get for any $X \in Sm/k$ the pairing

$$H_{\mathcal{M}}^p(X, \bigwedge_{i=1}^s (C_i, a_i)) \otimes H_{\mathcal{M}}^q(X, \bigwedge_{j=1}^t (D_j, b_j)) \rightarrow H_{\mathcal{M}}^{p+q}(X, \bigwedge_{i=1}^s (C_i, a_i) \wedge \bigwedge_{j=1}^t (D_j, b_j)).$$

4.5. Let $(C_1, a_1), \dots, (C_r, a_r)$ be pointed smooth curves over field k . For any field extension L/k , we abbreviate $H_{\mathcal{M}}^p(\text{Spec } L, \bigwedge_{i=1}^r (C_i \times_k L, a_i \times_k \text{id}_L))$ as $H_{\mathcal{M}}^p(L, \bigwedge_{i=1}^r C_i)$. By definition, we have

$$H_{\mathcal{M}}^p(L, \bigwedge_{i=1}^r C_i) = H_{i-p}(C_* (\bigotimes_{i=1}^r \mathbb{Z}_{\text{tr}}(C_i \times_k L, a_i \times_k \text{id}_L))(\text{Spec } L)).$$

4.6. (Norm map) In 4.5, if we assume L/k is a finite field extension, then using the description of 4.5 and the proper push-forward of cycles induces a map

$$N_{L/k} : H_{\mathcal{M}}^p(L, \bigwedge_{i=1}^r C_i) \rightarrow H_{\mathcal{M}}^p(k, \bigwedge_{i=1}^r C_i).$$

From the corresponding properties of proper push-forward, the following properties are immediately verified.

For finite field extension $k \subset L \subset M$ and $x \in H_{\mathcal{M}}^p(M, \bigwedge_{i=1}^r C_i)$ and $y \in H_{\mathcal{M}}^p(L, \bigwedge_{i=1}^r C_i)$

then we have

- (1) $N_{M/L}(y_M \cdot x) = y \cdot N_{M/L}(x)$ and $N_{M/L}(x \cdot y_M) = N_{M/L}(x) \cdot y$
- (2) $N_{M/k}(x) = N_{M/L}(N_{L/k}(x))$
- (3) If M/k is a normal extension, we have

$$N_{L/k}(x)_M = [L : k]_{insep} \sum_{j: M \hookrightarrow L} j^*(x).$$

Example 4.7. Let (C, x) be a pointed projective smooth curve over k . Then we have

$$H_{\mathcal{M}}^1(k, (C, x)) = \text{Ker}(\text{CH}_0(C) \xrightarrow{\text{deg}} \mathbb{Z}).$$

Example 4.8. Let X be a smooth curve over k . A good compactification of X is a pair (\bar{X}, X_∞) such that there is an open embedding $X \xrightarrow{j} \bar{X}$, \bar{X} is proper non-singular curve over k and $X_\infty = \bar{X} - X$ has an affine open neighborhood in \bar{X} .

Let (C, x) be a pointed smooth affine curve with a good compactification (\bar{X}, X_∞) , then

$$H_{\mathcal{M}}^1(k, (C, x)) = \text{Ker}(\text{Pic}(\bar{X}, X_\infty) \xrightarrow{\text{deg}} \mathbb{Z})$$

where $\text{Pic}(\bar{X}, X_\infty)$ is the relative Picard group. The elements of $\text{Pic}(\bar{X}, X_\infty)$ are the isomorphism classes (\mathcal{L}, t) of line bundle \mathcal{L} on \bar{X} with a trivialization t on X_∞ .

4.9. In 4.7, for finite field extension L/k , using the property 4.6 (3), we know through isomorphisms in 4.7, norm maps in 4.6 and classical one are compatible.

5 Calculation of motivic cohomology groups attached to pointed smooth curves

5.1 Pro-motives

In this subsection, we will briefly review the result of [MotGe].

5.1. Let \mathcal{A} be a tensor triangulated category. We consider $\text{Pro-}\mathcal{A}$ the pro-category of \mathcal{A} . Then the following facts are fundamental.

1. $\text{Pro-}\mathcal{A}$ is additive.
2. The shift functor of \mathcal{A} induces an auto-functor of $\text{Pro-}\mathcal{A}$.
3. There is a unique tensor structure over $\text{Pro-}\mathcal{A}$ such that \otimes commutes projective limits.

5.2. In the notation above, we call any triangle in $\text{Pro-}\mathcal{A}$ isomorphic to formal projective limit of distinguished triangles of \mathcal{A} a pro-distinguished triangle. Let $H : \mathcal{A}^{op} \rightarrow \mathcal{A}b$ be a cohomological functor. Then the functor

$$\bar{H} : (\text{Pro-}\mathcal{A})^{op} \ni (X_i)_{i \in I} \mapsto \text{injlim}_{i \in I^{op}} H(X_i) \in \mathcal{A}b$$

sends pro-distinguished triangles to long exact sequences.

5.3. Let \mathcal{O} be a k -algebra. We say \mathcal{O} is local smooth over k iff there is an formally smooth of finite type k -algebra A , and a prime ideal x of A and an isomorphism $\mathcal{O} \xrightarrow{\sim} A_x$.

Since k is perfect, \mathcal{O} is local smooth iff it is regular and essentially of finite type.

5.4. Let \mathcal{O} be a local smooth k -algebra. A model of \mathcal{O}/k is a pair (X, x) consist of a smooth scheme X and a morphism $x : \text{Spec } \mathcal{O} \rightarrow X$ such that, if we write the image of closed point of $\text{Spec } \mathcal{O}$ to x , induced morphism $x^\# : \mathcal{O}_{X,x} \rightarrow \mathcal{O}$ is an isomorphism.

We put

$$\mathcal{M}^{\text{smo}}(\mathcal{O}/k) := \{A \subset \mathcal{O}; (\text{Spec } A, \text{Spec } \mathcal{O} \xrightarrow{\text{obvious map}} \text{Spec } A) \text{ is a model of } \mathcal{O}/k\}.$$

$\mathcal{M}^{\text{smo}}(\mathcal{O}/k)$ is not empty and filtrant for inclusion. (c.f. [MotGe] Lemma 3.1.5).

5.5. 1. Let \mathcal{O} be a local smooth k -algebra. We consider a pro-object of Sm/k

$$(\mathcal{O}) := \{\text{Spec } A\}_{A \in \mathcal{M}^{\text{smo}}(\mathcal{O}/k)}.$$

2. Let X be a smooth scheme and $x \in X$, we define localization of X in x as a pro-object of Sm/k .

$$X_x := \{U\}_{x \in U \subset X}$$

where U runs through the open neighborhood of x .

5.6. (c.f. [MotGe] Lemma 3.1.8) Let \mathcal{O} be a local smooth k -algebra, and (X, x) a model of \mathcal{O} . Then x induces a canonical isomorphism

$$(\mathcal{O}) \rightarrow X_x.$$

5.7. Let \mathcal{O} be a local smooth k -algebra and $n, m \in \mathbb{Z}$. We consider a pro-object of $\text{Pro-DM}_{\text{gm}}(k)$

$$\text{M}_{\text{gm}}(\text{Spec } \mathcal{O})(n)[m] := \{\text{M}_{\text{gm}}(\text{Spec } A)(n)[m]\}_{A \in \mathcal{M}^{\text{smo}}(\mathcal{O}/k)}.$$

Next we define a residue morphism associated to a discrete valuation.

5.8. Let E/k be a field extension of finite type, v a valuation of E/k , \mathcal{O}_v a valuation ring of v and (X, t) a k -model of \mathcal{O}_v . We say a special point of (X, t) for image of closed point of $\text{Spec } \mathcal{O}_v$ for t and denote by \bar{s} .

We say that (X, t) is a strict k -model of \mathcal{O}_v iff closure $\{\bar{s}\}$ in X is a smooth scheme.

Any discrete valuation ring \mathcal{O}_v essentially of finite type over k admits a strict k -model. (c.f. [MotGe] Lemma 4.5.3).

5.9. Let E/k be a field extension of finite type, v a valuation of E/k , \mathcal{O}_v a valuation ring of v and (X, t) a strict k -model of \mathcal{O}_v . Put $Z := \{\bar{s}\}$. Since (X, Z) is a smooth closed pair of codimension 1. We have a distinguished triangle of the form

$$\text{M}_{\text{gm}}(Z)\{1\} \xrightarrow{\partial_{X,Z}} \text{M}_{\text{gm}}(X - Z) \xrightarrow{j_*} \text{M}_{\text{gm}}(X) \xrightarrow{+1}.$$

Since this triangle is natural for inclusions of open sets in X . (c.f. 3.22) Considering a cofiltrant system of open neighborhoods of s in X , we get a pro-distinguished triangle

$$\mathrm{M}_{\mathrm{gm}}(Z_s)\{1\} \xrightarrow{\partial_{X_s, Z_s}} \mathrm{M}_{\mathrm{gm}}(X_s - Z_s) \xrightarrow{j_{s*}} \mathrm{M}_{\mathrm{gm}}(X_s) \xrightarrow{+1}.$$

Since (X, s) is a k -model of \mathcal{O}_v , a morphism $s : \mathrm{Spec} \mathcal{O}_v \rightarrow X$ induces an isomorphism of pro-object $(\mathcal{O}_v) \rightarrow X_s$; Hence we get a pro-distinguished triangle isomorphic to the form

$$\mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} k(v))\{1\} \xrightarrow{\partial_{X_s, Z_s}} \mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} E) \xrightarrow{i^\sharp} \mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} \mathcal{O}_v) \xrightarrow{+1}$$

where E (resp. $k(v)$) is a fraction field (resp. residue field) of v , $i : \mathcal{O}_v \rightarrow E$ is a canonical inclusion.

Lemma 5.10. (c.f. [MotGe] Lemma 4.5.5.)

Let E/k be a field extension of finite type, v a discrete valuation of E/k , \mathcal{O}_v a valuation ring of v and $k(v)$ is this residue field.

Then adopting the notation above, if (X, s) and (Y, t) are two strict k -models of \mathcal{O}_v , put $Z := \overline{\{s\}}$ closure in X and $T := \overline{\{t\}}$ closure in Y . Then we have

$$\partial_{X_s, Z_s} = \partial_{Y_t, T_t}.$$

5.11. Let E/k be a field extension of finite type, v a discrete valuation of E/k . We define a residue morphism associated to v , denoted by ∂_v defined by $\partial_v := \partial_{(X_s, Z_s)}$ where (X, s) is a strict k -model of valuation ring of v . By Lemma 5.10 this does not depend on a choice of a strict k -model of valuation ring of v . So we have the following pro-distinguished triangle of the form

$$\mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} k(v))\{1\} \xrightarrow{\partial_v} \mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} E) \xrightarrow{i^\sharp} \mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} \mathcal{O}_v) \xrightarrow{+1}.$$

Having defined residue morphisms, we explain the connection of Milnor K -groups and Hom sets in $\mathrm{Pro}\text{-DM}_{\mathrm{gm}}(k)$.

5.12. Using the distinguished triangle in 3.26 and the definition of tensor structure in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$, for any $n \in \mathbb{N}$, there is a distinguished triangle of the form

$$\bigoplus_{i=1}^n \mathrm{M}_{\mathrm{gm}}(\mathbb{G}_m^{n-1}) \rightarrow \mathrm{M}_{\mathrm{gm}}(\mathbb{G}_m^n) \xrightarrow{\rho_n} \mathbb{Z}\{n\} \xrightarrow{+1}.$$

where the first morphism is induced from sum of n closed immersions

$$\iota_i := \mathrm{id} \times \mathrm{id} \times \dots \times \overset{i}{1} \times \dots \times \mathrm{id} \times \mathrm{id}.$$

5.13. Let E/k be a field extension of finite type. $f : \mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} E) \rightarrow \mathcal{M}$ and $g : \mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} E) \rightarrow \mathcal{N}$ are morphisms in $\mathrm{Pro}\text{-DM}_{\mathrm{gm}}(k)$. Then we can extend the definition of external cup product $f \boxtimes g : \mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} E) \rightarrow \mathcal{M} \otimes \mathcal{N}$.

If $\mathcal{M} = \mathbb{Z}\{p\}$ and $\mathcal{N} = \mathbb{Z}\{q\}$, we have a canonical isomorphism $\mathbb{Z}\{p\} \otimes \mathbb{Z}\{q\} \xrightarrow{\sim} \mathbb{Z}\{p+q\}$. Then we have the following identity

$$\alpha \boxtimes \beta = -\beta \boxtimes \alpha.$$

(c.f. [MotGe] Remarque 4.4.2.)

5.14. Let E/k be a field extension of finite type. Then we have a morphism

$$\begin{array}{ccc} (E^\times)^n & \xrightarrow{\sim} & \text{Hom}(\text{Spec } E, \mathbb{G}_m^n) \\ & \xrightarrow{\text{M}_{\text{gm}}} & \text{Hom}_{\text{Pro-DM}_{\text{gm}}(k)}(\text{M}_{\text{gm}}(\text{Spec } E), \text{M}_{\text{gm}}(\mathbb{G}_m^n)) \\ \text{Hom}(\text{M}_{\text{gm}}(\text{Spec } E), \rho_n) & \xrightarrow{\sim} & \text{Hom}_{\text{Pro-DM}_{\text{gm}}(k)}(\text{M}_{\text{gm}}(\text{Spec } E), \mathbb{Z}\{n\}). \end{array}$$

This map induces a morphism

$$\alpha : K_n^M(E) \rightarrow \text{Hom}_{\text{Pro-DM}_{\text{gm}}(k)}(\text{M}_{\text{gm}}(\text{Spec } E), \mathbb{Z}\{n\}).$$

5.15. In the notation above, for any $x \in (E^\times)^{\otimes n}$ and $y \in (E^\times)^{\otimes m}$, we have

$$\alpha(x \otimes y) = \alpha(x) \boxtimes \alpha(y).$$

Theorem 5.16. (c.f. [MotGe] Theorem 4.4.4)
(Need the assumption of perfectness of k .) In the notation above,

$$\alpha : K_*^M(E) \rightarrow \text{Hom}_{\text{Pro-DM}_{\text{gm}}(k)}(\text{M}_{\text{gm}}(\text{Spec } E), \mathbb{Z}\{*\})$$

is an algebra isomorphism.

5.2 Motivic reciprocity law

The classical theorems “Weil reciprocity law” and “residue formula” are unified using Milnor K-groups. More precisely, the following statement is known. (c.f. [Sus82])

5.17. (Reciprocity law for Milnor K-groups)

Let K be an algebraic function field over a field k . Then the following composition are the zero maps for all non-negative integers n .

$$K_{n+1}^M(K) \xrightarrow{\oplus \partial_v} \bigoplus_v K_n^M(k(v)) \xrightarrow{\Sigma N_{k(v)/k}} K_n^M(k)$$

In this subsection, we will prove more fundamental style of the following reciprocity law.

Theorem 5.18. (Motivic reciprocity law)

The following composition

$$\text{M}_{\text{gm}}(\text{Spec } k)\{1\} \xrightarrow{\Sigma N_{k(v)/k}\{1\}} \prod_v \text{M}_{\text{gm}}(\text{Spec } k(v))\{1\} \xrightarrow{\prod \partial_v} \text{M}_{\text{gm}}(\text{Spec } K)$$

is the zero map in $\text{Pro-DM}_{\text{gm}}(k)$.

5.19. Let K/k be a field extension of transcendental degree one. Let C/k be a projective nonsingular curve such that $K(C) = K$. As in the previous subsection, we can construct the following pro-distinguished triangle in $\text{Pro-DM}_{\text{gm}}(k)$.

$$\begin{aligned} \text{M}_{\text{gm}}(\text{Spec } K) &\rightarrow \text{M}_{\text{gm}}(C) \\ &\rightarrow \prod_{x \in C: \text{closed points}}^{\sim} \text{M}_{\text{gm}}(\text{Spec } k(x))((1)) \xrightarrow{\tilde{\Pi}^{\partial_x[1]}} \text{M}_{\text{gm}}(\text{Spec } K)[1] \end{aligned}$$

This is constructed as follows: For any closed set $Z \subset C$, there is the Gysin triangle

$$\text{M}_{\text{gm}}(C - Z) \rightarrow \text{M}_{\text{gm}}(C) \rightarrow \bigoplus_{x \in Z} \text{M}_{\text{gm}}(\text{Spec } k(x))((1)) \xrightarrow{\partial_{C,Z}[1]} \text{M}_{\text{gm}}(C - Z)[1]$$

and we consider $\text{M}_{\text{gm}}(\text{Spec } K) = \{\text{M}_{\text{gm}}(C - Z)\}_{Z \subset C: \text{closed subsets}} \in \text{Pro-DM}_{\text{gm}}(k)$

Lemma 5.20.

In the notation above, for any closed point $x \in C$, the diagram of structure morphisms

$$\begin{array}{ccc} \text{Spec } k(x) & \xrightarrow{i} & C \\ & \searrow & \downarrow p \\ & & \text{Spec } k \end{array}$$

induces the following commutative diagram:

$$\begin{array}{ccc} \text{M}_{\text{gm}}(\text{Spec } k(x))((1)) & \xleftarrow{i^*} & \text{M}_{\text{gm}}(C) \\ & \searrow N_{k(x)/k}((1)) & \uparrow p^* \\ & & \text{M}_{\text{gm}}(\text{Spec } k)((1)). \end{array}$$

Proof. First choose a finite equidimensional morphism $C \xrightarrow{\pi} \mathbb{P}^1$ which is unramified at every points over $\pi(x)$. (This can be done by using Bertini theorem.) Next using 3.4 and Proposition 3.25, we may assume $C = \mathbb{P}^1$. Replacing \mathbb{P}^1 by $\mathbb{P}_{k(x)}^1$ and using Proposition 3.25 again, we may assume $k(x) = k$. In this case, $i^* \circ p^* = \text{id}$ by Lemma 3.5 and 3.23. \square

5.21. Hence we get the following diagram:

$$\begin{array}{ccccc} & & \text{M}_{\text{gm}}(\text{Spec } k)((1)) & & \\ & & \downarrow \Sigma N_{k(x)/k}((1)) & & \\ \text{M}_{\text{gm}}(C) & \xrightarrow{p^*} & \bigoplus_{x \in Z} \text{M}_{\text{gm}}(\text{Spec } k(x))((1)) & \longrightarrow & \text{M}_{\text{gm}}(C - Z)[1]. \end{array}$$

Taking a limit with respect to Z , we get the motivic reciprocity law.

Next we prove that the motivic reciprocity law implies the Weil reciprocity law for Milnor K -groups.

Lemma 5.22.

In the notation above, let v be a valuation of K/k , \mathcal{O}_v a valuation ring of v , π a uniformizer element of \mathcal{O}_v . Then the following diagram is commutative.

$$\begin{array}{ccc} \mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} k(v))\{1\} & \xrightarrow{\partial_v} & \mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} K) \\ \downarrow & & \downarrow \alpha(\pi) \boxtimes \mathrm{id}_{\mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} K)} \\ \mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} \mathcal{O}_v)\{1\} & \longleftarrow & \mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} K)\{1\} \end{array}$$

Proof. Take a strict k -model of $\mathrm{Spec} \mathcal{O}_v$. Denote it by (X, s) . If we take X sufficiently small, $\pi \in \mathcal{O}_v \xrightarrow{\sim} \mathcal{O}_{X,s}$ determines a regular function $X \rightarrow \mathbb{A}_k^1$ which parameterizes Z . Using Proposition 3.27 and considering a cofiltrant system of open neighborhoods of s in X , we get the following commutative diagram of pro-motives

$$\begin{array}{ccc} \mathrm{M}_{\mathrm{gm}}(Z_s)\{1\} & \xrightarrow{\partial_{X_s, Z_s}} & \mathrm{M}_{\mathrm{gm}}(X_s - Z_s) \\ \downarrow & & \downarrow \mathrm{M}_{\mathrm{gm}}(\pi) \circ \rho_1 \boxtimes \mathrm{id}_{\mathrm{M}_{\mathrm{gm}}(X_s - Z_s)} \\ \mathrm{M}_{\mathrm{gm}}(X_s)\{1\} & \longleftarrow & \mathrm{M}_{\mathrm{gm}}(X_s - Z_s)\{1\}. \end{array}$$

Hence we get the result. \square

Example 5.23. In the notation above, for any discrete valuation v of K/k , there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} K), \mathbb{Z}\{n+1\})_{\mathcal{L}} & \longleftarrow & K_{n+1}^M(K) \\ \mathrm{Hom}(\partial_v, \mathbb{Z}\{n+1\}) \downarrow & & \downarrow (-1)^n \partial_v \\ \mathrm{Hom}(\mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} k(v))\{1\}, \mathbb{Z}\{n+1\}) & \longleftarrow & K_n^M(k(v)). \end{array}$$

Proof. For $u_1, \dots, u_{n+1} \in \mathcal{O}_v^\times$ and a uniformizer π , it is enough to check the following two conditions.

1 $\mathrm{Hom}(\partial_v, \mathbb{Z}\{n+1\})(\alpha(\{u_1, \dots, u_{n+1}\})) = 0$

2 $\mathrm{Hom}(\partial_v, \mathbb{Z}\{n+1\})(\alpha(\{u_1, \dots, u_n, \pi\})) = (-1)^n \alpha(\{\overline{u_1}, \dots, \overline{u_n}\})$

To prove 1: Notice that there is a pro-distinguished triangle as follows (c.f. 5.11)

$$\mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} k(v))\{1\} \xrightarrow{\partial_v} \mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} K) \rightarrow \mathrm{M}_{\mathrm{gm}}(\mathrm{Spec} \mathcal{O}_v)^{\pm 1}.$$

To prove 2: Anti-commutativity of \boxtimes (c.f. 5.13) and Lemma 5.22. \square

Example 5.24. Let K/k be a field extension of transcendental degree 1, (C, x) be a pointed smooth curve and v a place of K/k . There is a tame symbol $\partial_v : \text{Jac } C(K_v) \otimes K_v^\times \rightarrow \text{Jac } C(k(v))$. Then the following diagram is commutative.

$$\begin{array}{ccc}
\text{Jac } C(K_v) \otimes K_v^\times & \longrightarrow & \text{Hom}_{\text{DM}^{\text{eff}}(k)}(\text{M}_{\text{gm}}(\text{Spec } K_v), \mathbb{Z}(C, x)[1]) \\
\downarrow \partial_v & & \downarrow \otimes \\
& & \text{Hom}_{\text{DM}^{\text{eff}}(k)}(\text{M}(\text{Spec } K_v), \mathbb{Z}(\mathbb{G}_m, 1)[1]) \\
& & \downarrow \boxtimes \\
& & \text{Hom}_{\text{DM}^{\text{eff}}(k)}(\text{M}(\text{Spec } K_v)[1], \mathbb{Z}(C, x)(1)[3]) \\
& & \downarrow -\text{Hom}(\partial_v, \mathbb{Z}(C, x)(1)[3]) \\
\text{Jac } C(k(v)) & \longrightarrow & \text{Hom}_{\text{DM}^{\text{eff}}(k)}(\text{M}(\text{Spec } k(v))(1)[2], \mathbb{Z}(C, x)(1)[3])
\end{array}$$

This is proved in the same way as Example 5.23.

Corollary 5.25.

The motivic reciprocity law implies the Weil reciprocity law for Milnor K -groups.

Proof. Take $\text{Hom}(?, \mathbb{Z}\{n+1\})$ and use Theorem 2.22, Theorem 5.16 and notice Example 3.2 and Example 5.23. \square

5.3 Main result

5.26. In this section, let k be a perfect field which admits resolution of singularities and $(C_1, a_1), \dots, (C_n, a_n)$ pointed projective smooth curves over k .

5.27. Let $p : Z \rightarrow \mathbb{A}_k^1$ be a finite surjective morphism and suppose that Z is integral. Let $f_i \in \text{Hom}(Z, C_i)$ and

$$p^{-1}(j) = \coprod n_i^j z_i^j \quad (j = 0, 1)$$

where n_i^j are the multiplicities of points $z_i^j = \text{Spec } L_i^j$. Define:

$$\phi_j = \Sigma n_i^j \{f_1, \dots, f_n\}_{L_i^j/k}$$

then we have

$$\phi_0 = \phi_1$$

in $K(k, \text{Jac } C_1, \dots, \text{Jac } C_n)$.

The proof is similar to [MVW02], p.45 Corollary 5.5.

5.28. As $\bigotimes_{i=1}^n \mathbb{Z}_{\text{tr}}(C_i, a_i)(\text{Spec } k)$ is a quotient of the free abelian groups generated by the closed points of $C_1 \times \dots \times C_n$ modulo the subgroup generated by all points of the form $(x_1, \dots, a_i, \dots, x_n)$ where the a_i 's can be any position. If x is a closed point of $C_1 \times \dots \times C_n$ with residue field L then x is defined by a

canonical sequence $(x_1, \dots, x_n) \in \text{Jac } C_1(L) \times \dots \times \text{Jac } C_n(L)$.
Since

$$\mathbb{H}_{\mathcal{M}}^n(k, \bigwedge_{i=1}^n C_i) = \text{Coker}\left(\bigotimes_{i=1}^n \mathbb{Z}_{\text{tr}}(C_i, a_i)(\mathbb{A}_k^1) \xrightarrow{\partial_0 - \partial_1} \bigotimes_{i=1}^n \mathbb{Z}_{\text{tr}}(C_i, a_i)(\text{Spec } k)\right)$$

using 5.27, we have a natural map $\mathbb{H}_{\mathcal{M}}^n(k, \bigwedge_{i=1}^n C_i) \rightarrow K(k, \text{Jac } C_1, \dots, \text{Jac } C_n)$.

5.29. Using 4.7, for every finite field extension L/k we have an isomorphism

$$\bigotimes_{i=1}^n \text{Jac } C_i(L) \xrightarrow{\sim} \bigotimes_{i=1}^n \mathbb{H}_{\mathcal{M}}^1(L, C_i).$$

Combining a natural pairing in 4.4

$$\bigotimes_{i=1}^n \mathbb{H}_{\mathcal{M}}^1(L, C_i) \rightarrow \mathbb{H}_{\mathcal{M}}^n(L, \bigwedge_{i=1}^n C_i)$$

and a norm map (c.f 4.6)

$$\mathbb{N}_{L/k} : \mathbb{H}_{\mathcal{M}}^n(L, \bigwedge_{i=1}^n C_i) \rightarrow \mathbb{H}_{\mathcal{M}}^n(k, \bigwedge_{i=1}^n C_i)$$

we get a canonical map

$$\bigoplus_{L/k: \text{finite extension}} \bigotimes_{i=1}^n \text{Jac } C_i(L) \rightarrow \mathbb{H}_{\mathcal{M}}^n(k, \bigwedge_{i=1}^n C_i).$$

If we use 4.6 (1), Theorem 4.9, Theorem 5.18 and Example 5.24, this map should factor through the map $K(k, \text{Jac } C_1, \dots, \text{Jac } C_n) \rightarrow \mathbb{H}_{\mathcal{M}}^n(k, \bigwedge_{i=1}^n C_i)$.

5.30. Obviously the morphisms above are inverse to each other. Hence we get the following result.

Theorem 5.31. *(Somekawa conjecture for Jacobian varieties)*

Let $(C_1, a_1), \dots, (C_n, a_n)$ be pointed projective smooth curves over perfect field k which admits resolution of singularities. Then

$$K(k, \text{Jac } C_1, \dots, \text{Jac } C_n) \xrightarrow{\sim} \text{Hom}_{\text{DM}_{\text{eff}}^{\text{eff}}(k)}(\mathbb{M}(\text{Spec } k), \mathbb{Z}(\bigwedge_{i=1}^n C_i)[n]).$$

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