

**K -REGULARITY,
 cdh -FIBRANT HOCHSCHILD HOMOLOGY,
AND A CONJECTURE OF VORST**

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ABSTRACT. In this paper we prove that for an affine scheme essentially of finite type over a field F and of dimension d , K_{d+1} -regularity implies regularity, assuming that the characteristic of F is zero. This verifies a conjecture of Vorst.

INTRODUCTION

It is a well-known fact that algebraic K -theory is homotopy invariant as a functor on regular schemes; if X is a regular scheme then the natural map $K_n(X) \rightarrow K_n(X \times \mathbb{A}^1)$ is an isomorphism for all $n \in \mathbb{Z}$. This is false in general for nonregular schemes and rings.

To express this failure, Bass introduced the terminology that, for any contravariant functor \mathcal{P} defined on schemes, a scheme X is called \mathcal{P} -regular if the pullback maps $\mathcal{P}(X) \rightarrow \mathcal{P}(X \times \mathbb{A}^r)$ are isomorphisms for all $r \geq 0$. If $X = \text{Spec}(R)$, we also say that R is \mathcal{P} -regular. Thus regular schemes are K_n -regular for every n . In contrast, it was observed as long ago as [2] that a nonreduced affine scheme can never be K_1 -regular. In particular, if A is an Artinian ring (that is, a 0-dimensional Noetherian ring) then A is regular (that is, reduced) if and only if A is K_1 -regular. In [15], Vorst conjectured that for an affine scheme X , of finite type over a field F and of dimension d , regularity and K_{d+1} -regularity are equivalent; Vorst proved this conjecture for $d = 1$ (by proving that K_2 -regularity implies normality).

In this paper, we prove Vorst's conjecture in all dimensions provided the characteristic of the ground field F is zero. In fact we prove a stronger statement. We say that X is *regular in codimension $< n$* if $\text{Sing}(X)$ has codimension $\geq n$ in X .

Let \mathcal{F}_K denote the presheaf of spectra such that $\mathcal{F}_K(X)$ is the homotopy fiber of the natural map $K(X) \rightarrow KH(X)$, where $K(X)$ is the algebraic K -theory spectrum of X and $KH(X)$ is the homotopy K -theory of X defined in [16]. We write $\mathcal{F}_K(R)$ for $\mathcal{F}_K(\text{Spec}(R))$.

Theorem 0.1. *Let R be a commutative ring which is essentially of finite type over a field F of characteristic 0. Then:*

- (a) *If $\mathcal{F}_K(R)$ is n -connected, then R is regular in codimension $< n$.*
- (b) *If R is K_n -regular, then R is regular in codimension $< n$.*
- (c) *(Vorst's conjecture) If R is $K_{1+\dim(R)}$ -regular, then R is regular.*

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It was observed in [16] that if X is K_n -regular then $K_i(X) \rightarrow KH_i(X)$ is an isomorphism for $i \leq n$, and a surjection for $i = n+1$, so that $\mathcal{F}_K(X)$ is n -connected. Thus (a) implies (b) in this theorem, and (c) is a special case of (b).

The bounds in (a) and (b) are the best possible, because it follows from Vorst's results ([15, Thm. A], [14, Thm. 3.6]) that for an affine singular seminormal curve X , $\mathcal{F}_K(X)$ is 1-connected, but not 2-connected. The converse of (c) is trivial, but those of (a) and (b) are false. Indeed, affine normal surfaces are regular in codimension 1 but may not be K_{-1} -regular, much less K_2 -regular; see [20, 5.8.1].

Finally the analogue of (c) –and thus also of (a) and (b)– for K -theory of general nonaffine schemes is false. Indeed we give the following example of a nonreduced (and in particular nonregular) projective curve which is K_n -regular for all n .

Theorem 0.2. *Let (X, Q) be an elliptic curve over a field of characteristic 0, and P a rational point on X such that the line bundle $L = L(P - Q)$ does not have odd order in the Picard group $\text{Pic}(X)$. Write Y for the nonreduced scheme with the same underlying space as X but with structure sheaf $\mathcal{O}_Y = \mathcal{O} \oplus L$, where L is regarded as a square-zero ideal.*

Then Y is K_n -regular for all n , and $\mathcal{F}_K(Y)$ is contractible.

The proof of Theorem 0.1 employs results from our paper with M. Schlichting [4] that allow us to describe \mathcal{F}_K in terms of cyclic homology; the necessary statements will be recalled in Section 1. In Section 2, we study the cdh -fibrant version of Hochschild homology and its Hodge decomposition. Section 3 contains a smoothness criterion (Theorem 3.1) using cdh -fibrant Hochschild homology, which is of independent interest and generalized in Theorem 4.11. The proof of part (a) of Theorem 0.1 is contained in Section 4 (see Theorem 4.12). As explained above, parts (b) and (c) follow from this. Finally Section 5 is devoted to the counterexample stated in Theorem 0.2 (and restated as Theorem 5.2).

Notation

All rings considered in this paper are commutative and noetherian. We shall write Sch/F for the category of schemes essentially of finite type over a field F . Objects of Sch/F shall be called F -schemes.

The category of spectra we use in this paper will not be critical. In order to minimize technical issues, we will use the terminology that a *spectrum* \mathcal{E} is a sequence of simplicial sets \mathcal{E}_n together with bonding maps $b_n : \mathcal{E}_n \rightarrow \Omega\mathcal{E}_{n+1}$. We say that \mathcal{E} is an Ω -spectrum if all bonding maps are weak equivalences. A map of spectra is a strict map. We will use the model structure on the category of spectra defined in [3]. Note that in this model structure, every fibrant spectrum is an Ω -spectrum.

We say that a presheaf E of spectra on Sch/F satisfies the *Mayer-Vietoris-property* (or MV-property, for short) for a cartesian square of schemes

$$\begin{array}{ccc}
 Y' & \longrightarrow & X' \\
 \downarrow & & \downarrow \\
 Y & \longrightarrow & X
 \end{array}$$

(□)

if applying E to this square results in a homotopy cartesian square of spectra. We say that E satisfies the Mayer-Vietoris property for a class of squares provided it satisfies the MV-property for each square in the class.

We say that E satisfies *Nisnevich descent* for Sch/F if E satisfies the MV-property for all elementary Nisnevich squares in Sch/F ; an *elementary Nisnevich*

square is a cartesian square of schemes (\square) for which $Y \rightarrow X$ is an open embedding, $X' \rightarrow X$ is étale and $(X' - Y') \rightarrow (X - Y)$ is an isomorphism. By [12, 4.4], this is equivalent to the assertion that $E(X) \rightarrow \mathbb{H}_{\text{nis}}(X, E)$ is a weak equivalence for each scheme X , where $\mathbb{H}_{\text{nis}}(-, E)$ is a fibrant replacement for E in a suitable model structure.

We say that E satisfies *cdh-descent* for Sch/F if E satisfies the MV-property for all elementary Nisnevich squares (Nisnevich descent) and for all abstract blow-up squares in Sch/F ; an *abstract blow-up square* is a square (\square) such that $Y \rightarrow X$ is a closed embedding, $X' \rightarrow X$ is proper and the induced morphism $(X' - Y')_{\text{red}} \rightarrow (X - Y)_{\text{red}}$ is an isomorphism. With M. Schlichting, we showed in Theorem 3.7 of [4] that *cdh-descent* is equivalent to the assertion that $E(X) \rightarrow \mathbb{H}_{\text{cdh}}(X, E)$ is a weak equivalence for each scheme X , where $\mathbb{H}_{\text{cdh}}(-, E)$ is a fibrant replacement for the presheaf E in a suitable model structure. We abbreviate $\mathbb{H}_{\text{cdh}}(-, E)$ as $\mathbb{H}(-, E)$ if no confusion can arise, and write $\mathbb{H}_n(X, E) = \mathbb{H}^{-n}(X, E)$ for $\pi_n \mathbb{H}(X, E)$.

We use cohomological indexing for all chain complexes in this paper; for a complex A , $A[p]^q = A^{p+q}$. It is well-known that there is an Eilenberg-Mac Lane functor $A \mapsto |A|$ from chain complexes of abelian groups to spectra, and from presheaves of chain complexes of abelian groups to presheaves of spectra. This functor sends quasi-isomorphisms of complexes to weak homotopy equivalences of spectra, and satisfies $\pi_n(|A|) = H^{-n}(A)$. In this spirit, we will use descent terminology for presheaves of complexes.

For example, the Hochschild, cyclic, periodic and negative cyclic homology of schemes over a field k (such as F -schemes over a field $F \supseteq k$) can be defined using the Zariski hypercohomology of certain presheaves of complexes; see [18] and [4, 2.7] for precise definitions. We shall write these presheaves as $HH(/k)$, $HC(/k)$, $HP(/k)$ and $HN(/k)$, respectively and regard them as presheaves of either cochain complexes or spectra. When k is omitted, it is understood that $k = \mathbb{Q}$ is intended. Finally, we write $\Omega_{/k}^i$ for the presheaf $X \mapsto \Omega_{X/k}^i$, while $\Omega_{/k}^\bullet$ denotes the presheaf of algebraic de Rham complexes and $\Omega_{/k}^{\leq i}$ denotes its brutal truncation in degree i .

1. *cdh*-DESCENT

In this section we recall the main results from [4], and prove that the failure of K -theory to be homotopy invariant can be measured using cyclic homology. We work on the category Sch/F of F -schemes essentially of finite type over a field F of characteristic 0.

Here are two of the main results of [4]. Recall that infinitesimal K -theory $\mathcal{K}^{\text{inf}}(X)$ is the homotopy fiber of the Jones-Goodwillie Chern character $K(X) \rightarrow HN(X)$. The first one is Theorem 4.6 of [4]:

Theorem 1.1. *The presheaf of spectra \mathcal{K}^{inf} satisfies *cdh-descent*.*

The second one is a slight modification of Corollary 3.13 of [4].

Theorem 1.2. *For each subfield $k \subseteq F$, the presheaf $HP(/k)$ satisfies *cdh-descent* on Sch/F . In particular, HP satisfies *cdh-descent*.*

Proof. For $k = F$, this is proven in [4, Corollary 3.13]. As in *loc. cit.*, the result for $k \subset F$ follows from [4, Theorem 3.12], once we verify that the hypotheses hold. But this follows from three observations: (1) the Cuntz-Quillen excision theorem holds over k (see [5, 5.3], noting that the condition that the base field be algebraically

closed is not needed, see [5, p. 3]); (2) Goodwillie's theorem that periodic cyclic homology is invariant under infinitesimal extension does not require rings of finite type over k (see [6, II.5.1] or [10, 4.1.15]); and (3) the results of [4, Section 2] also hold for (Hochschild, cyclic, periodic, negative) homology over k . \square

Example 1.3. Consider the presheaf of spectra KH associated to Homotopy K -theory. The main theorem of [7] states that $\mathbb{H}_{cdh}(-, K) \simeq KH$, and that KH satisfies cdh descent. Thus the following definition of \mathcal{F}_K is compatible with the definition of \mathcal{F}_K given in the introduction.

Definition 1.4. For any presheaf of spectra E , we write \mathcal{F}_E for the homotopy fiber of $E \rightarrow \mathbb{H}_{cdh}(-, E)$. If $\mathcal{F}_E(X)$ is n -connected for some scheme X (for all X in some subcategory of Sch/F), we say that E satisfies n - cdh -descent on X (resp., on the subcategory). Note that if E satisfies n - cdh -descent for all n on a subcategory, then E satisfies cdh -descent on that subcategory.

Since the fibrant replacement functor \mathbb{H}_{cdh} preserves (objectwise) fibration sequences, it follows that \mathcal{F} does too. (See the first paragraph of [4, Section 5].) We record this as the following observation.

Lemma 1.5. *Let $E_1 \rightarrow E_2 \rightarrow E_3$ be a fibration sequence of presheaves of spectra. Then there is a natural induced fibration sequence*

$$\mathcal{F}_{E_1} \rightarrow \mathcal{F}_{E_2} \rightarrow \mathcal{F}_{E_3}.$$

Theorem 1.6. *For any scheme X , essentially of finite type over a field of characteristic 0, the Chern character $K \rightarrow HN = HN(/Q)$ induces natural weak equivalences*

$$\mathcal{F}_K(X) \xrightarrow{\simeq} \mathcal{F}_{HN}(X) \xleftarrow{\simeq} \Omega^{-1}\mathcal{F}_{HC}(X).$$

Proof. The first weak equivalence follows from Lemma 1.5 and Theorem 1.1. The second weak equivalence follows from Lemma 1.5, Theorem 1.2 and the SBI fibration sequence $\Omega HP \rightarrow \Omega^{-1}HC \rightarrow HN \rightarrow HP$. \square

Corollary 1.7. *Let $X \in Sch/F$. Then K satisfies $(n+1)$ - cdh -descent on X if and only if HH satisfies n - cdh -descent on X .*

Proof. Since $HH(X)$ and $\mathbb{H}(X, HH)$ are n -connected for $n < -\dim(X)$ by [18], this follows from Theorem 1.6 and the SBI sequence $\Omega^{-1}HC \rightarrow HH \rightarrow HC$. \square

2. cdh -FIBRANT HOCHSCHILD AND CYCLIC HOMOLOGY.

In this section we study the cdh -fibrant version of Hochschild homology and its Hodge decomposition, and establish some of their basic properties.

For legibility, we will write a for the natural morphism of sites from the cdh -site to the Zariski site on Sch/F . If A is a Zariski sheaf, its associated cdh sheaf will be written as A_{cdh} or a^*A . We will simplify notation and write $H_{cdh}^*(X, A)$ for the cohomology of A_{cdh} . In particular this applies to the sheaf $X \mapsto \Omega_{X/k}^i$ of Kähler i -differential forms ($i \geq 0$); $H_{cdh}^*(X, \Omega_{X/k}^i)$ is the cohomology of $a^*\Omega_{X/k}^i$. If A is a complex of presheaves of abelian groups on Sch/F , then we write $\mathbb{H}_{cdh}(A)$ for a cdh -fibrant replacement of A , and $\mathbb{H}_{cdh}(X, A)$ for its complex of sections over X ; the usual hypercohomology $\mathbb{H}_{cdh}^n(X, A)$ is just $H^n\mathbb{H}_{cdh}(X, A)$. For example, if A

is a presheaf, considered as a complex concentrated in degree zero, then $\mathbb{H}_{cdh}(A)$ is just an injective resolution of the cdh -sheafification A_{cdh} , and $\mathbb{H}_{cdh}^n(X, A)$ is the usual cohomology $H_{cdh}^n(X, A_{cdh})$ of A_{cdh} .

When A is an unbounded complex, such as a complex representing Hochschild homology, then $\mathbb{H}_{cdh}(A)$ may be constructed using product total complexes of flasque Cartan-Eilenberg resolutions. This works because the columns of the Cartan-Eilenberg double complex are locally cohomologically bounded by [13].

The cdh site is Noetherian (every covering has a finite subcovering), so cdh cohomology commutes with filtered direct limits of sheaves. A typical application of this fact is that if M is a sheaf of F -modules and V is a vector space then $H_{cdh}^n(X, V \otimes_F M) \cong V \otimes_F H_{cdh}^n(X, M)$. (See [1], Exp. VI, 2.11 and 5.2.)

The Hochschild and cyclic homology of schemes over a field k (such as F -schemes over a field $F \supseteq k$) can be defined using the Zariski hypercohomology of certain presheaves of mixed complexes; see [18] and [4, 2.7] for precise definitions. It was observed in [19, 3.0] that, because the mixed complexes already admit a Hodge decomposition, so do the complexes $HH(/k)$, $HC(/k)$, $HP(/k)$ and $HN(/k)$. Taking fibrant replacements for any Grothendieck topology respects such product decompositions; the following proposition records this for the cdh -topology.

Proposition 2.1. *Let X be a scheme over a field k of characteristic 0 (such as an F -scheme for $F \supseteq k$). Then the cdh -fibrant Hochschild, cyclic, negative cyclic and periodic cyclic homology of X over k admit natural Hodge decompositions. That is, if H denotes any of $HH(/k)$, $HC(/k)$, $HP(/k)$ or $HN(/k)$ then:*

$$\mathbb{H}_{cdh}(X, H) \cong \prod \mathbb{H}_{cdh}(X, H^{(i)}).$$

Moreover, using the computations of these decompositions provided in [19, Theorem 3.3] and the fact that all F -schemes are locally smooth in the cdh -topology, it is possible to compute the Hodge decomposition explicitly in terms of the cdh -hypercohomology of the de Rham complex.

Theorem 2.2. *Let $k \subseteq F$ be a subfield. There are natural isomorphisms for every F -scheme X :*

$$\begin{aligned} \pi_n \mathbb{H}_{cdh}(X, HH^{(i)}(/k)) &\cong H_{cdh}^{i-n}(X, \Omega_{/k}^i); \\ \pi_n \mathbb{H}_{cdh}(X, HC^{(i)}(/k)) &\cong \mathbb{H}_{cdh}^{2i-n}(X, \Omega_{/k}^{\leq i}); \\ \pi_n \mathbb{H}_{cdh}(X, HN^{(i)}(/k)) &\cong \mathbb{H}_{cdh}^{2i-n}(X, \Omega_{/k}^{\geq i}); \\ \pi_n \mathbb{H}_{cdh}(X, HP^{(i)}(/k)) &\cong \mathbb{H}_{cdh}^{2i-n}(X, \Omega_{/k}^\bullet). \end{aligned}$$

Proof. Let $\mathbf{C}(X)$ denote the mixed complex computing the Hochschild and cyclic homology of X over k . The functor $\mathbf{C} : X \mapsto \mathbf{C}(X)$ is a presheaf of mixed complexes. By [17, 9.8.12], there is a Hodge decomposition $\mathbf{C} \cong \prod_i \mathbf{C}^{(i)}$ and a natural map of mixed complexes $e : \mathbf{C} \rightarrow (\Omega_{X/k}^*, 0, d)$ that sends the Hochschild chain complex $HH^{(i)}(X) = (\mathbf{C}^{(i)}(X), b)$ to $\Omega_{X/k}^i[i]$. As observed in [19], the induced map on Connes' double complexes sends $\mathcal{B}_{**}^{(i)}$ to $\Omega_{X/k}^{\leq i}[2i]$. It suffices to prove that these are locally quasi-isomorphisms for the cdh topology. This boils down to showing that e induces a quasi-isomorphism $HH^{(i)}(R/k) \rightarrow \Omega_{R/k}^i[i]$ for every regular local F -algebra R . For $k = F$, this is the Hochschild-Kostant-Rosenberg theorem ([17, 9.4.7]). The general case follows from this and the fact that R is the union of

smooth k -algebras. (It also follows from the Kassel-Sledsjøe spectral sequence of [9, 4.3a], which we recall in 4.1 below.) \square

Lemma 2.3. *Let R be an F -algebra essentially of finite type, $k \subseteq F$ a subfield. Then for each n , $HH(/k)$ satisfies n -cdh descent on $X = \text{Spec}(R)$ if and only if the following three conditions hold simultaneously:*

$$(2.3a) \quad HH_m^{(q)}(R/k) = 0 \quad \text{if } 0 \leq q < m \leq n;$$

$$(2.3b) \quad \Omega_{R/k}^q \rightarrow H_{cdh}^0(X, \Omega_{/k}^q) \quad \text{is bijective if } q \leq n \text{ and onto if } q = n + 1;$$

$$(2.3c) \quad H_{cdh}^p(X, \Omega_{/k}^q) = 0 \quad \text{if } p > 0 \text{ and } 0 \leq q \leq p + n + 1.$$

Note that (2.3a) is vacuous if $n \leq 0$, and (2.3b) is vacuous if $n \leq -2$. In particular, $HH(/k)$ satisfies (-2) -cdh-descent just in case $H_{cdh}^p(X, \Omega_{/k}^q) = 0$ for all $p > q \geq 0$.

Proof. This follows easily from the Hodge decomposition and the isomorphisms

$$HH_q^{(q)}(R/k) \cong \Omega_{R/k}^q, \quad \text{and} \quad HH_m^{(q)}(R/k) = 0 \text{ for } q > m.$$

In more detail, we see from 2.1 and 2.2 that the maps $HH_m^{(q)}(R/k) \rightarrow H_{cdh}^{q-m}(X, \Omega_{/k}^q)$ must be isomorphisms for $m \leq n$ and onto for $m = n + 1$. \square

On smooth schemes, all our functors are well-behaved. Recall from [13] that the $scdh$ topology on Sm/F is the restriction of the cdh topology on Sch/F . Since every scheme is locally smooth, it follows that $\mathbb{H}_{scdh}(X, A)$ is just $\mathbb{H}_{cdh}(X, A)$ for every presheaf A . (See the argument of the first part of the proof of [4, 3.12].) If A satisfies $scdh$ -descent then $A(X) \cong \mathbb{H}_{cdh}(X, A)$ for all smooth X .

Recall that k is a subfield of F .

Theorem 2.4. *Let H denote any of: $HH(/k)$, $HC(/k)$, $HN(/k)$ or $HP(/k)$, and let $H^{(i)}$ denote the i^{th} component in the Hodge decomposition of H .*

Then H and $H^{(i)}$ satisfy $scdh$ -descent on Sm/F . In particular, if X is smooth over F , then $H^{(i)}(X) \cong \mathbb{H}_{cdh}(X, H^{(i)})$.

Proof. Since every smooth scheme over F is locally a filtered limit of smooth affine schemes over k , and H commutes with limits of affine schemes, we may assume that $k = F$. By [4, 3.9, 2.9, and 2.10], Hochschild, cyclic, negative and periodic cyclic homology (relative to F) all satisfy $scdh$ -descent on Sm/F .

By Proposition 2.1, the quasi-isomorphisms $H(X) \cong \mathbb{H}_{cdh}(X, H) = \mathbb{H}_{scdh}(X, H)$ induce quasi-isomorphisms $H^{(i)}(X) \cong \mathbb{H}_{cdh}^{(i)}(X, H) = \mathbb{H}_{scdh}^{(i)}(X, H)$ for all i . \square

The special case $H_{Zar}^*(X, \mathcal{O}) \cong H_{cdh}^*(X, \mathcal{O})$ (for smooth X) of the following corollary was proven in [4, 6.3].

Corollary 2.5. *If X is smooth over F , then $H_{Zar}^p(X, \Omega_{/k}^i) \cong H_{cdh}^p(X, \Omega_{/k}^i)$ for all p and i . In particular, $\Omega_{X/k}^i \cong H_{cdh}^0(X, \Omega_{/k}^i)$.*

Proof. Consider the map $e^{(i)} : HH^{(i)} = (\mathbf{C}^{(i)}, b) \rightarrow \Omega_{/F}^i[i]$ of complexes of Zariski sheaves. By [19, 3.3], it is a quasi-isomorphism over every smooth scheme X over F , inducing $HH_{i-p}^{(i)}(X) \cong H_{Zar}^{p-i}(X, \mathbf{C}^{(i)}) \cong H_{Zar}^p(X, \Omega_{/k}^i)$. The map $e^{(i)}$ remains a quasi-isomorphism after sheafifying for the cdh topology, so that $\mathbb{H}_{cdh}^{p-i}(X, HH^{(i)}) \cong H_{cdh}^{p-i}(X, \mathbf{C}^{(i)}) \cong H_{cdh}^p(X, \Omega_{/k}^i)$. By Theorem 2.4, $HH_n^{(i)}(X) \cong \mathbb{H}_{cdh}^{-n}(X, HH^{(i)})$, whence the result. \square

The next result is proven by copying the proof of [4, 6.1], replacing \mathcal{O} with $\Omega_{/k}^i$.

Proposition 2.6. *If X is a d -dimensional scheme, essentially of finite type over F , and $k \subseteq F$ is a subfield, then*

$$H_{Zar}^d(X, \Omega_{/k}^i) \rightarrow H_{cdh}^d(X, \Omega_{/k}^i)$$

is surjective. In particular, if X is affine and $d > 0$ then $H_{cdh}^d(X, \Omega_{/k}^i) = 0$.

The following useful theorem is proven in [13, 12.1].

Theorem 2.7. *For every abstract blow-up square (\square) , and for every complex of sheaves of abelian groups A , there is a long exact Mayer-Vietoris sequence:*

$$\cdots H_{cdh}^n(X, A) \rightarrow H_{cdh}^n(X', A) \oplus H_{cdh}^n(Y, A) \rightarrow H_{cdh}^n(Y', A) \rightarrow H_{cdh}^{n+1}(X, A) \cdots$$

Consider the change-of-topology morphism $a : (\text{Sch}/F)_{cdh} \rightarrow (\text{Sch}/F)_{Zar}$.

Lemma 2.8. *If a Zariski sheaf M on Sch/F is a quasi-coherent sheaf (resp., coherent sheaf) on each X_{Zar} , and M satisfies $scdh$ -descent on Sm/F , then the cohomology sheaves $R^q a_*(a^* M)$ are also quasi-coherent (resp., coherent) on each X_{Zar} .*

If $X = \text{Spec}(R)$ is affine, then $R^q a_(a^* M)$ is the quasi-coherent sheaf associated to the R -module $H_{cdh}^q(X, M)$, and the natural map $M(X) \rightarrow H_{cdh}^0(X, M)$ is R -linear.*

Proof. We proceed by induction on $\dim X$, the case $\dim(X) = 0$ being clear. Pick a smooth proper birational cdh cover X' of X , as in [13, 5.9] or [11, 12.23], and form the abstract blow-up square (\square) . By Theorem 2.7, we get a triangle on X_{Zar} : $Ra_*(a^* M)|_X \rightarrow Ra_*(a^* M)|_{X'} \amalg_Y \rightarrow Ra_*(a^* M)|_{Y'}$. As the latter two terms have quasi-coherent (resp., coherent) cohomology sheaves, by induction and $scdh$ -descent on X' , so does the first.

If X is affine, then $H_{Zar}^p(X, R^q a_* M) = 0$ for $p > 0$. Hence the Leray spectral sequence collapses to yield $H_{cdh}^q(X, M) = H_{Zar}^0(X, R^q a_*(a^* M))$. \square

Corollary 2.9. *Suppose that $X = \text{Spec}(R)$ is affine. Then $U \mapsto \pi_n \mathcal{F}_{HH(/k)}(U)$ and $U \mapsto \pi_n \mathbb{H}_{cdh}(U, HH(/k))$ are quasi-coherent Zariski sheaves on X for all n .*

3. A CRITERION FOR SMOOTHNESS.

In this section we present a local criterion for smoothness of schemes over a field F , in terms of the Hochschild homology and cdh -fibrant Hochschild homology of their local rings over F (see 3.1). As an application we prove Vorst's conjecture for algebras of finite type over \mathbb{Q} and their localizations at maximal ideals (see 3.3).

A stronger global version of the following result shall be proved in Section 4 below (see Theorem 4.11).

Recall that F is a field of characteristic 0.

Theorem 3.1. *Let R be the local ring of a d -dimensional F -algebra of finite type at a maximal ideal. If $HH(/F)$ satisfies d - cdh -descent on R , then R is smooth over F .*

Proof. Recall that $\Omega_{/F}^\bullet$ denotes the de Rham complex, whose terms are the Zariski sheaves $\Omega_{/F}^i$, while $\Omega_{/F}^{\leq i}$ denotes its brutal truncation in degrees at most i . By 2.2 and 1.2, we have isomorphisms

$$HP_n^{(j)}(X/F) \xrightarrow{\cong} \mathbb{H}_{cdh}^{2j-n}(X, \Omega_{/F}^\bullet)$$

for any $X \in \text{Sch}/F$, and all n and j . Moreover, by the proof of 2.2, this isomorphism factors through a natural map $e : HP_n^{(j)}(X/F) \rightarrow \mathbb{H}_{\text{Zar}}^{2j-n}(X, \Omega_{/F}^\bullet)$. Now specialize to the case $X = \text{Spec } R$, where R is as in the theorem. Since every cdh cover of X has a d -dimensional refinement, we have

$$\mathbb{H}_{cdh}^*(X, \Omega_{/F}^\bullet) = \mathbb{H}_{cdh}^*(X, \Omega_{/F}^{\leq d}).$$

Moreover, Lemma 2.3 implies that the hypercohomology spectral sequence for \mathbb{H}_{cdh}^* degenerates to yield an isomorphism

$$H^*(\Omega_{R/F}^{\leq d}, d) \rightarrow \mathbb{H}_{cdh}^*(X, \Omega_{/F}^{\leq d}).$$

The canonical map $S : HP_n^{(j)}(R/F) \rightarrow HC_{n-2}^{(j-1)}(R/F)$ fits into the commutative diagram

$$\begin{array}{ccccc} HP_{d+2}^{(d+1)}(R/F) & \xrightarrow{e} & H_{dR}^d(R/F) & \longrightarrow & \mathbb{H}_{cdh}^d(X, \Omega_{/F}^\bullet) \\ \downarrow S & & \downarrow & & \downarrow \wr \\ HC_d^{(d)}(R/F) & \xrightarrow{e} & \Omega_{R/F}^d/d\Omega_{R/F}^{d-1} & \xrightarrow{\cong} & \mathbb{H}_{cdh}^d(X, \Omega_{/F}^{\leq d}). \end{array}$$

We have seen that the top composite, the right vertical and both bottom arrows are isomorphisms. It follows that the middle vertical inclusion is the identity map, i.e., that $d\Omega_{R/F}^d = 0$. On the other hand, $d\Omega_{R/F}^d$ generates $\Omega_{R/F}^{d+1}$ as an R -module; therefore we can infer that $\Omega_{R/F}^{d+1} = 0$. By Lemma 3.2 below, R is regular, and hence smooth over F . \square

Lemma 3.2. *Let F be any perfect field. Suppose R is the local ring of a d -dimensional F -algebra of finite type at a maximal ideal. If $\Omega_{R/F}^{d+1} = 0$, then R is regular.*

Proof. Let \mathfrak{m} be the maximal ideal of R . Since $L := R/\mathfrak{m}$ is smooth over F , the Second Fundamental Theorem [17, 9.3.5] shows that there is an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \rightarrow L \otimes_R \Omega_{R/F}^1$ sending x to dx . Consequently, there is a surjection from $\Omega_{R/F}^{d+1}$ onto $\Lambda_L^{d+1}(\mathfrak{m}/\mathfrak{m}^2)$, which is a nonzero vector space unless R is regular. \square

As an application, we can now verify Vorst's Conjecture for algebras of finite type over \mathbb{Q} and their localizations at maximal ideals.

Theorem 3.3. *Let R be a d -dimensional \mathbb{Q} -algebra which is either of finite type over \mathbb{Q} , or a localization of a \mathbb{Q} -algebra of finite type at a maximal ideal.*

If R is K_{d+1} -regular, then R is regular.

Proof. First assume R is of finite type over \mathbb{Q} , and K_{d+1} -regular. To prove R is regular, we may replace R by its localization at a maximal ideal; these local rings are also K_{d+1} -regular, by Vorst's localization theorem [14, 1.90]. Thus we are reduced to proving the theorem in the local case.

As remarked in the introduction, if R is K_{d+1} -regular, then $\mathcal{F}_K(R)$ is $(d+1)$ -connected (see [16]). By Corollary 1.7, $\mathcal{F}_{HH(/F)}(R)$ is d -connected. Now Theorem 3.1 applies to prove that R is smooth over \mathbb{Q} and hence regular. \square

4. VORST'S CONJECTURE.

In this section we will prove Theorem 0.1. Throughout, F will be a fixed field of characteristic zero, $k \subseteq F$ a subfield, R an F -algebra essentially of finite type, and $X = \text{Spec}(R)$. Note that we write HH for $HH(/\mathbb{Q})$.

Lemma 4.1. (*Kassel-Sledsjae*, [9, 4.3a]) *Let $k \subseteq F$ and $p \geq 1$ be fixed. Then there is a bounded second quadrant homological spectral sequence ($0 \leq i < p$, $j \geq 0$):*

$${}^p E_{-i, i+j}^1 = \Omega_{F/k}^i \otimes_F HH_{p-i+j}^{(p-i)}(R/F) \Rightarrow HH_{p+j}^{(p)}(R/k)$$

Lemma 4.2. *Let $k \subseteq F$ and $p \geq 1$ be fixed. Then there is a spectral sequence:*

$${}^p E_1^{i,j} = \Omega_{F/k}^i \otimes_F H_{cdh}^{i+j}(X, \Omega_{F/k}^{p-i}) \Rightarrow H_{cdh}^{i+j}(X, \Omega_{F/k}^p).$$

Proof. Consider the sheaf of ideals $I := \ker(\Omega_{F/k}^* \rightarrow \Omega_{F/k}^p)$. The I -adic filtration of $\Omega_{F/k}^*$ induces a filtration $\mathcal{G} = \mathcal{G}(p)$ on $\Omega_{F/k}^p$. If R is any F -algebra essentially of finite type, we have a natural surjection

$$(4.3) \quad \Omega_{F/k}^i \otimes_F \Omega_{R/F}^{p-i} \rightarrow \mathcal{G}^i(R)/\mathcal{G}^{i+1}(R),$$

which is an isomorphism if R is smooth. Thus the *cdh*-sheafification of (4.3) is an isomorphism. Since $\Omega_{F/k}^i$ is a vector space, the spectral sequence of the lemma is the one associated to the corresponding filtration of the *cdh* sheaf $a^*\Omega_{F/k}^p$. \square

Lemma 4.4. *Let $X = \text{Spec}(R)$ be affine, and fix $n \geq 0$. Assume that*

$$(4.4a) \quad \Omega_{R/k}^q \rightarrow H_{cdh}^0(X, \Omega_{R/k}^q) \text{ is bijective if } q \leq n \text{ and onto if } q = n+1,$$

$$(4.4b) \quad H_{cdh}^1(X, \Omega_{R/k}^q) = 0 \text{ if } q \leq n+1.$$

Then $\Omega_{R/F}^q \rightarrow H_{cdh}^0(X, \Omega_{R/F}^q)$ is bijective if $q \leq n$, and onto if $q = n+1$.

Proof. By induction on q . If $q = 0$, there is nothing to prove. Fix $q > 0$, and consider the filtration $\mathcal{G}^i = \mathcal{G}^i(q)$, $0 \leq i \leq q$, considered in the proof of Lemma 4.2. We have a commutative diagram

$$(4.5) \quad \begin{array}{ccc} \Omega_{F/k}^i \otimes_F \Omega_{R/F}^{q-i} & \longrightarrow & \mathcal{G}^i(R)/\mathcal{G}^{i+1}(R) \\ \downarrow & & \downarrow \\ \Omega_{F/k}^i \otimes_F H_{cdh}^0(X, \Omega_{R/F}^{q-i}) & \longrightarrow & H_{cdh}^0(X, \mathcal{G}^i/\mathcal{G}^{i+1}). \end{array}$$

The top arrow is surjective for all i , and an isomorphism for $i = 0$. The bottom arrow is an isomorphism by the proof of 4.2. By the inductive hypothesis, the left vertical arrow is an isomorphism for $0 < i$. It follows that the top arrow is an isomorphism for all $0 \leq i \leq q$, and that the arrow on the right is an isomorphism for $i > 0$. By (4.4b) we have an exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{cdh}^0(X, \mathcal{G}^{i+1}) & \longrightarrow & H_{cdh}^0(X, \mathcal{G}^i) & \longrightarrow & \Omega_{F/k}^i \otimes_F H_{cdh}^0(X, \Omega_{R/F}^{q-i}) \\ & & & & & & \downarrow \\ & & & & 0 & \longleftarrow & H_{cdh}^1(X, \mathcal{G}^i) & \longleftarrow & H_{cdh}^1(X, \mathcal{G}^{i+1}) \end{array}$$

Since $\mathcal{G}^{q+1} = 0$, we deduce, by descending induction on i , that for all $i > 0$,

$$(4.6) \quad H_{cdh}^1(X, \mathcal{G}^i) = 0$$

Consider the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{G}^{i+1}(R) & \longrightarrow & \mathcal{G}^i(R) & \longrightarrow & \Omega_{F/k}^i \otimes \Omega_{R/F}^{q-i} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_{cdh}^0(X, \mathcal{G}^{i+1}) & \longrightarrow & H_{cdh}^0(X, \mathcal{G}^i) & \longrightarrow & \Omega_{F/k}^i \otimes H_{cdh}^0(X, \Omega_{R/F}^{q-i}) \longrightarrow 0
\end{array}$$

Using descending induction on i again, we obtain from this diagram that

$$(4.7) \quad \mathcal{G}^i(R) \cong H_{cdh}^0(X, \mathcal{G}^i) \quad (i > 0).$$

We have a map of exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{G}^1(R) & \longrightarrow & \Omega_{R/k}^q & \longrightarrow & \Omega_{R/F}^q \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_{cdh}^0(X, \mathcal{G}^1) & \longrightarrow & H_{cdh}^0(X, \Omega_{R/k}^q) & \longrightarrow & H_{cdh}^0(X, \Omega_{R/F}^q) \longrightarrow 0
\end{array}$$

The third map in the bottom row is onto by (4.6). The first vertical map is an isomorphism by (4.7). By (4.4a), the second is an isomorphism if $q \leq n$ and onto if $q = n + 1$. It follows that the same is true of the third vertical map. \square

Proposition 4.8. *Assume $n \geq 0$. If $HH(/k)$ satisfies n -cdh-descent on R , then so does $HH(/F)$.*

Proof. By Lemma 2.3, the hypothesis is equivalent to saying that the following conditions hold simultaneously.

$$(4.8a) \quad HH_m^{(q)}(R/k) = 0 \quad \text{if } 0 \leq q < m \leq n$$

$$(4.8b) \quad \Omega_{R/k}^q \rightarrow H_{cdh}^0(X, \Omega_{R/k}^q) \quad \text{is bijective if } q \leq n \text{ and onto if } q = n + 1$$

$$(4.8c) \quad H_{cdh}^p(X, \Omega_{R/k}^q) = 0 \quad \text{if } p > 0 \text{ and } q \leq p + n + 1.$$

We have to prove that the following conditions hold

$$(4.9a) \quad HH_m^{(q)}(R/F) = 0 \quad \text{if } 0 \leq q < m \leq n$$

$$(4.9b) \quad \Omega_{R/F}^q \rightarrow H_{cdh}^0(X, \Omega_{R/F}^q) \quad \text{is bijective if } q \leq n \text{ and onto if } q = n + 1$$

$$(4.9c) \quad H_{cdh}^p(X, \Omega_{R/F}^q) = 0 \quad \text{if } p > 0 \text{ and } q \leq p + n + 1.$$

Using (4.8c), the spectral sequence of 4.2, and induction, we obtain (4.9c). Hence (4.8b) implies (4.9b), by Lemma 4.4. To prove (4.9a) we proceed by induction on q . The case $q = 0$ is just the fact that $HH_m^{(0)}(A/k) = 0$ for any $m > 0$, any field k and any k -algebra A . Assume $n \geq q \geq 1$, and that we have $HH_m^{(q')}(R/F) = 0$ for all $m \leq n$ and all $q' < \min\{m, q\}$. By (4.9b) and (4.5), the spectral sequence of Lemma 4.1 collapses for $j = 0$ to yield

$${}_q E_{i,-i}^\infty = {}_q E_{i,-i}^1.$$

Given this, (4.9a) follows from (4.8a) by induction. \square

Lemma 4.10. *Let F be a field, and R a local F -algebra essentially of finite type. Then there exists a field $F \subset E \subset R$ such that R is isomorphic to the localization of a finite type algebra over E at a maximal ideal.*

Proof. The hypothesis on R means that there exist an F -algebra A of finite type and a prime ideal $P \subset A$ such that $R = A_P$. Suppose that $\dim(A) = k + h$ and $ht(P) = h$. By Noether normalization, there is a polynomial subring $S = F[t_1, \dots, t_k]$ of A meeting P in 0, and the field $R/P = A_P/PA_P$ is a finite extension of $E = F(t_1, \dots, t_k)$. There is an evident inclusion of E in R , and R is the localization of the finite type E -algebra $A \otimes_S E$ at a maximal ideal. \square

The results of this section, together with Theorem 3.1, allow us to prove the following global regularity criterion.

Theorem 4.11. *Let $k \subseteq F$ be fields of characteristic 0, and let R be an F -algebra essentially of finite type.*

*If $HH(/k)$ satisfies h -*cdh*-descent on R , then R is smooth in codimension h . That is, for every prime ideal P of height h , the local ring R_P is regular.*

Proof. Let P be a prime ideal of R of height h . Since $HH(/F)$ satisfies h -*cdh* descent on R by Proposition 4.8, it also satisfies h -*cdh* descent on the localization R_P , by Corollary 2.9. By Lemma 4.10, there is a field $F \subset E \subset R_P$ such that R_P is the localization at a maximal ideal of an algebra of finite type over E . By 4.8, $HH(/E)$ satisfies h -*cdh* descent on R_P . Because $\dim(R_P) = h$, Theorem 3.1 implies that R_P is smooth over E , and hence is regular. \square

Once again, let F be a field of characteristic 0.

Theorem 4.12. *Suppose R is an F -algebra essentially of finite type. If R is K_{h+1} -regular for some $h \geq 0$, then R is regular in codimension h . In particular, if R is $K_{\dim(R)+1}$ -regular, then R is regular, and hence smooth over F .*

Proof. If R is K_{h+1} -regular, then $HH(/\mathbb{Q})$ satisfies h -descent on R , by Corollary 1.7. The assertion now follows from Theorem 4.11. \square

5. A NONREDUCED SCHEME WHICH IS K -REGULAR

This section is devoted to the counterexample stated in Theorem 0.2, which reappears here as Theorem 5.2.

Lemma 5.1. *Let X be a smooth projective elliptic curve over a field F with base-point Q , and let L be a degree zero line bundle L on X . Assume that L is not an element of odd order in the Picard group. Then $H^*(X, L^{\otimes 2n+1}) = 0$ for all $n \in \mathbb{Z}$.*

Proof. Because \mathcal{O}_X is a dualizing sheaf, we are reduced by Serre duality to proving that $H^0(X, L^{2n+1}) = 0$. Because X is elliptic, there exists a rational point $P \in X$ such that $L := L(P - Q)$. Now if $H^0(X, L^{\otimes 2n+1})$ were nonzero, there would exist an element f in the function field of X with $\operatorname{div}(f) = (2n+1)(P - Q)$. But because L is not an odd torsion element, there is no such f . \square

Theorem 5.2. *Let X be a smooth projective elliptic curve over a field F of characteristic 0, and let L be as in Lemma 5.1. Write Y for the nonreduced scheme with the same underlying space as X but with structure sheaf $\mathcal{O}_Y = \mathcal{O}_X \oplus L$, where L is regarded as a square-zero ideal.*

Then for all n , $K_n(Y) = K_n(X)$ and Y is K_n -regular.

Proof. As X is regular, and hence K_n -regular, it suffices to show that $K(Y \times \mathbb{A}^m) \rightarrow K(X \times \mathbb{A}^m) \cong K(X)$ is an equivalence for all $m \geq 0$. We shall prove the equivalent assertion that the relative homotopy groups $K(Y \times \mathbb{A}^m, X \times \mathbb{A}^m)$ are zero. By Goodwillie's theorem [6] and Zariski descent, these relative K -groups are isomorphic to the corresponding relative cyclic homology groups over \mathbb{Q} . By base-change (see [8]) it suffices to show that the relative groups $HH_n^{rel} = HH_n(Y, X)$ vanish for all n . By Zariski descent, it suffices to show that $H^0(X, HH_n^{rel})$ and $H^1(X, HH_n^{rel})$ vanish for all n . From Lemma 5.3 below and the fact that $\Omega_{X/F}^1 \cong \mathcal{O}_X$, we see that the Zariski sheaves HH_n^{rel} are sums of odd tensor powers of L when F is a number field, and odd tensor powers of L tensored over F with vector spaces $\Omega_{F/\mathbb{Q}}^i$ in general. But the cohomology of such powers vanishes by Lemma 5.1. \square

The following lemma is well-known, at least in the case when L is free. We include a proof for the sake of completeness. For simplicity, we write $HH_*(R)$ for $HH_*(R/k)$.

Lemma 5.3. *Let k be a field with $\text{char}(k) \neq 2$, R a commutative k -algebra, and L a projective R -module of rank 1. Put $A = R \oplus L$. Let M_* denote the graded R -module*

$$M_p = \begin{cases} L^{\otimes_R(p+1)} & p \geq 0 \text{ even,} \\ L^{\otimes_R p} & p > 0 \text{ odd.} \end{cases}$$

Then for relative Hochschild homology over k ,

$$HH_n(A, L) = \bigoplus_{p+q=n} M_p \otimes_R HH_q(R).$$

Proof. Let $C_*(A, L)$ be the relative Hochschild complex; the subspace $L^{\otimes_k 2m+1}$ of $C_{2m}(A, L)$ consists of cycles, and induces a map $M_{2m} = L^{\otimes_R 2m+1} \rightarrow HH_{2m}(A, L)$, because for $x_i \in L$ and $r \in R$ we have

$$(-1)^i b(x_0 \otimes \cdots \otimes x_i \otimes r \otimes x_{i+1} \cdots) = (x_0 \otimes \cdots \otimes x_i r \otimes x_{i+1} \cdots) - (x_0 \otimes \cdots \otimes x_i \otimes r x_{i+1} \cdots).$$

Because $Bb + bB = 0$, where $B : C_*(A, L) \rightarrow C_{*+1}(A, L)$ is the Connes operator, the subspace $B(L^{\otimes_k 2m+1})$ of $C_{2m+1}(A, L)$ also consists of cycles and induces a map $M_{2m+1} = L^{\otimes_R 2m+1} \rightarrow HH_{2m+1}(A, L)$. Thus we have a graded map $M_* \rightarrow HH_*(A, L)$. Because $HH_*(A, L)$ is a graded module over $HH_*(R)$, we get a canonical R -module map from $M_* \otimes_R HH_*(R)$ to $HH_*(A, L)$. To see that it is an isomorphism, we may assume R is local, whence $A = R[x]/\langle x^2 \rangle$. By the Künneth formula, we are reduced to the case $R = k$, which is straightforward. \square

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