ON THE EXTENDED DOUBLE POINT RELATION IN ALGEBRAIC COBORDISM

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Abstract. An obvious observation enables us to obtain a simpler proof of the extended double point relation stated by M. Levine and R. Pandharipande. The proof in [LP] is based on two consecutive blow ups, the first of which yields a non-smooth variety. We show that the relation can be proved by performing a single smooth blow up.

1. The double point relation. The work of Levine and Pandharipande [LP] begins with the following observations. Let Y be a smooth variety of pure dimension over a field k and $\pi : Y \to \mathbb{P}^1$ a projective morphism such that

(a) $p^{-1}(0) = A \cup B$, the union of two smooth Cartier divisors on Y intersecting transversally; let $D = A \cap B$;

(b) $p^{-1}(\xi) = C$, a smooth Cartier divisor on Y, where ξ is a regular value of π.

Then, first, the normal bundles to D in A and B are dual:

$$N_{A/D} \otimes N_{B/D} \cong O_D \,. \tag{1}$$

It follows that the varieties $\mathbb{P}(N_{A/D} \oplus O_D)$ and $\mathbb{P}(N_{B/D} \oplus O_D)$ are isomorphic over D; denote by $\mathbb{P}_D \to D$ either of them ($\mathbb{P}_\pi$ in [LP, Sect. 0.3]). Second,

$$[C \to Y] = [A \to Y] + [B \to Y] - [\mathbb{P}_D \to Y] \text{ in } \Omega(Y)\,, \tag{2}$$

which is called the *double point relation* in algebraic cobordism (DPR for short). Here Ω is the theory developed in [LM].

We claim that (1) and (2) remain true in a slightly larger situation. As above, let A, B, and C be smooth Cartier divisors on a smooth Y, with A and B intersecting transversally at D. We do not assume that Y projects to $\mathbb{P}^1$ replacing this by the following requirements:

(c) $A + B \sim C$ (linearly equivalent divisors)

(d) $C \cap D = \emptyset$.

Clearly (a) and (b) imply (c) and (d), but (d) is weaker than

(d') $C \cap (A \cup B) = \emptyset$

which also follows from (a) and (b).

Proposition. *Assertions (1) and (2) hold under the assumptions (c) and (d).*

This can be proved by exactly the same arguments as in [LP]; we provide some comments for convenience.

As $N_{Y/A} \cong O_Y(A)|_A$ and $N_{B/D} \cong N_{Y/A}|_D$, we have $N_{B/D} \cong O_Y(A)|_D$, and the same way $N_{A/D} \cong O_Y(B)|_D$. Thus

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$$N_{A/D} \otimes N_{B/D} \cong (O_Y(B) \otimes O_Y(A))|_D \cong O_Y(B+A)|_D \cong O_Y(C)|_D \cong O_D$$

since $C \cap D = \emptyset$.

As for the DPR (2), observe that the main point in the proof given in [LP, Sect. 5.4] is that the non-additivity term

$$[(A+B) \to Y] - [A \to Y] - [B \to Y]$$

depends solely on the normal bundles $N_{A/D}$ and $N_{B/D}$ (cf. [LP, Lemma 8]). Next, if $N_{A/D} \otimes N_{B/D} \cong O_D$, this term can be calculated as $-[\mathbb{P}_D \to Y]$ by the deformation space trick, see [LP, Lemma 9]. In none of these arguments we use the assumption $C \cap (A \cup B) = \emptyset$; $C \cap D = \emptyset$ suffices.

Thus one can write the DPR axiom in the definition of the new cobordism theory ω, see [LP, Sect. 0.6], under the assumptions (c) and (d), with no loss of the natural map $\omega \to \Omega$ (which remains an isomorphism of course - by the results of [LP]). The advantage of doing so is that the extended double point relation now admits a simpler proof, without non-smooth blow ups and explanations of related geometry.

2. The extended double point relation. Now C is allowed to intersect D under the assumptions

(c) $A + B \sim C$ in Y, as above, and

(e) $A + B + C$ is a reduced strict normal crossing divisor, as in [LP, Sect. 7.2].

Let

$$D = A \cap B\,,\ F = A \cap C\,,\ E = A \cap B \cap C\,,$$

and consider the blow up $p : \hat{Y} = \mathrm{Bl}_F Y \to Y$. Denote $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ the proper transforms of A, B, C, and D, then clearly

$$\hat{A} \cong A\,,\ \hat{B} \cong \mathrm{Bl}_E B\,,\ \hat{C} \cong C\,,\ \hat{A} \cap \hat{B} = \hat{D} \cong D\,.$$

Let $\hat{F} = p^{-1}(F)$ and $\hat{E} = p^{-1}(E)$ denote the exeptional divisors in $\hat{Y}$ and $\hat{B}$, then pulling (c) back to $\hat{Y}$ we get $(\hat{A} + \hat{F}) + \hat{B} \sim \hat{C} + \hat{F}$, whence

($\hat{c}$) $\hat{A} + \hat{B} \sim \hat{C}$ in $\hat{Y}$,

and clearly we have

($\hat{d}$) $\hat{C} \cap \hat{D} = \emptyset$

since already $\hat{C} \cap \hat{A} = \emptyset$.

By the proposition, we get

$$[\hat{C} \to \hat{Y}] = [\hat{A} \to \hat{Y}] + [\hat{B} \to \hat{Y}] - [\mathbb{P}_{\hat{D}} \to \hat{Y}]\,. \qquad (\hat{2})$$

At this point we have two options for $\mathbb{P}_{\hat{D}}$. First choose $\mathbb{P}_{\hat{D}} = \mathbb{P}(N_{\hat{A}/\hat{D}} \oplus O_{\hat{D}})$, observe that $N_{\hat{A}/\hat{D}} \cong p^* N_{A/D}$ and push forward to Y; this yields

$$[C \to Y] = [A \to Y] + [\hat{B} \to Y] - [\mathbb{P}(N_{A/D} \oplus O_D) \to Y]\,. \qquad (3)$$

Replacing $[\hat{B} \to Y]$ by the blow up formula, see [Ne] or [LP, Lemma 15], we get

$$[C \to Y] = [A \to Y] + [B \to Y] - [\mathbb{P}(N_{B/E} \oplus O_E) \to Y] + [\mathbb{P}(N_{\hat{B}/\hat{E}} \oplus O_{\hat{E}}) \to Y] - [\mathbb{P}(N_{A/D} \oplus O_D) \to Y] \qquad (4)$$

which is the *extended double point relation* of [LP, Lemma 16], with the roles of A and B interchanged.

If we choose $\mathbb{P}_{\hat{D}} = \mathbb{P}(N_{\hat{B}/\hat{D}} \oplus O_{\hat{D}})$, then we get a slightly different (but, of course, equivalent) form of (3):

$$[C \to Y] = [A \to Y] + [\hat{B} \to Y] - [\mathbb{P}(N_{\hat{B}/\hat{D}} \oplus O_{\hat{D}}) \to Y]\,. \tag{3'}$$

Substituting the blow up formula for $[\hat{B} \to Y]$, we get another form of (4).

One can show easily that now

$$N_{A/D} \otimes N_{B/D} \cong O_D(E) \text{ and } N_{\hat{B}/\hat{D}} \cong p^*(N_{B/D} \otimes O_D(-E))\,,$$

whence

$$\mathbb{P}(N_{\hat{B}/\hat{D}} \oplus O_{\hat{D}}) \cong \mathbb{P}(N_{B/D} \oplus O_D(E)) \cong \mathbb{P}(N_{A/D} \oplus O_D)\,,$$

which is one more way to see that the last terms in (3) and (3') coincide.

References

[LM] M. Levine and F. Morel, *Algebraic Cobordism, I*, preprint at www.math.neu.edu/ levine/mathindex.html.

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