

Gersten's conjecture for commutative discrete valuation rings

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Abstract

The purpose of this article is to prove that Gersten's conjecture for a commutative discrete valuation ring is true. Combining with the result of [GL87], we learn that Gersten's conjecture is true if the ring is a commutative regular local, smooth over a commutative discrete valuation ring.

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0 Introduction

In [Ger73] p.28, Gersten proposed the following conjecture

Conjecture 0.1 (Gersten’s conjecture).

For any commutative regular local ring A and natural numbers n, p , the canonical inclusion $\mathcal{M}^{p+1}(A) \hookrightarrow \mathcal{M}^p(A)$ induces the zero map on K -groups

$$K_n(\mathcal{M}^{p+1}(A)) \rightarrow K_n(\mathcal{M}^p(A)) ,$$

where $\mathcal{M}^i(A)$ is the category of finitely generated A -modules M with $\text{Codim}_{\text{Spec } A} \text{Supp } M \geq i$.

Historical Note 0.2.

The conjecture has been proved in the following cases:

(1) **The case of general dimension**

(a) If A is of equi-characteristic, then Gersten’s conjecture for A is true.

We refer to [Qui73] for special cases, and the general cases [Pan03] can be deduced from limit argument and Popescu’s general Néron desingularization [Pop86]. (For the case of commutative discrete valuation rings, it was first proved by Sherman [She78])

(b) If A is smooth over some commutative discrete valuation ring S and satisfies some condition, then Gersten’s conjecture for A is true. [Blo86]

(c) If A is smooth over some commutative discrete valuation ring S and if we accept Gersten’s conjecture for S , then Gersten’s conjecture for A is true. [GL87]

(2) **The case of that A is a commutative discrete valuation ring**

(d) For the cases of $n = 0, 1$, that conjecture is classically known and for $n = 2$, it was proved by Dennis and Stein, announced in [DS72] and proved in [DS75].

(e) If its residue field is algebraic over a finite field, then Gersten’s conjecture for A is true.

We refer to [Ger73] for the cases of that its residue field is a finite field, the proof of general cases [She82] is improved from that of special cases by using Swan’s result [Swa63], the universal property of algebraic K -theory [Hil81] and limit argument.

(3) **Other works**

(f) For torsion coefficient K -theory, Gersten’s conjecture for a commutative discrete valuation ring is true. [Gil86], [GL00]

(g) Some conjectures imply that Gersten’s conjecture for a commutative discrete valuation ring is true. [She89]

The purpose of this article is to prove the following theorem.

Theorem 0.3.

For any commutative discrete valuation rings, Gersten’s conjecture is true.

Combining **Historical Note 0.2** (c) with **Theorem 0.3**, we have the following result.

Corollary 0.4.

For any commutative regular local ring which is smooth over a commutative discrete valuation ring, Gersten’s conjecture is true.

Example 0.5 (Due to Kazuya Kato).

In **Theorem 0.3**, the assumption of commutativity is essential. (See the proof of **Lemma 5.2**.) Let D be a skew field finite over \mathbf{Q}_p , A its integer ring and a its prime element. As the inner automorphism of a induces non-trivial automorphism on its residue field, we have $x \in A^\times$ with $y = axa^{-1}x^{-1}$ is non vanishing in its residue field, a fortiori in $K_1(A) = (A^\times)^{ab}$. On the other hand y is a commutator in D^\times . Hence we learn $K_1(A) \rightarrow K_1(D) = (D^\times)^{ab}$ is not injective.

Notations 0.6.

In this paper, we will use the following notations.

- (1) R denote a commutative discrete valuation ring, and π its uniformizer element, $k = R/\pi$.
- (2) A topological space means a compactly generated topological space and their fiber product is considered in the category of compactly generated topological spaces. A pointed space means a pointed topological space with non-degenerate base point.
- (3) When we quote the paper [Qui73], we follow the page numbering by the numbering at the bottom. (In [Qui73], there are numbers both at the top and the bottom of each page, and the numberings do not agree.)

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1 The strategy of proving the main theorem

The proof of Gersten's conjecture in this paper requires a long, sustained argument. In this section, we will explain how to prove the main theorem and philosophy behind the proof. The author's proof of Gersten's conjecture is inspired from the works related about Serre's conjecture for intersection multiplicities [Ser65] by H. Gillet, M. Levine, and C. Soulé [GS87], [GS99] and [Lev85]. Their studying comparison between coniveau and γ -filtrations led him to the idea that "weight argument of the Adams operations implies Gersten's conjecture." Now, we explain more precisely. First we will review the following **Proposition 1.2**.

Notations 1.1.

As in the **Conjecture 0.1**, for noetherian finite Krull dimensional commutative ring with unit A , let $\mathcal{M}^p(A)$ be the category of finitely generated A -modules M with $\text{Codim}_{\text{Spec } A} \text{Supp } M \geq p$.

Proposition 1.2.

(c.f. [Lev85] P.452, Proposition 1.1 and [GS87] p.263 Proposition 4.12) *Let A be a regular domain.*

(1) *The following statements are equivalent.*

(i) *The maps $K_0(\mathcal{M}^p(A)) \rightarrow K_0(\mathcal{M}^{p-1}(A))$ are zero for $p = 1, \dots, \dim A$.*

(ii) *$K_0(\mathcal{M}^p(A))$ is generated by cyclic modules $A/(f_1, \dots, f_p)$ where f_1, \dots, f_p forms a regular sequence for $p = 1, \dots, \dim A$.*

(2) Moreover we consider the following statements

(i) $_{\mathbb{Q}}$ The maps $K_0(\mathcal{M}^p(A)) \otimes \mathbb{Q} \rightarrow K_0(\mathcal{M}^{p-1}(A)) \otimes \mathbb{Q}$ are zero for $p = 1, \dots, \dim A$.

(iii) For non-negative integer k , the Adams operation ψ_k acts on $K_0(\mathcal{M}^p(A))$ by multiplication of k^p for $p = 1, \dots, \dim A$.

Then (ii) implies (iii) and (iii) implies (i) $_{\mathbb{Q}}$.

Sketch of proof. (1) The implication (i) \Rightarrow (ii) is proved in [Lev85] Ibid by using the following surjections

$$K_1(\mathcal{M}^{p-1}(A)/\mathcal{M}^p(A)) = \bigoplus_{\substack{x \in \text{Spec } A \\ \text{ht } x = p-1}} k(x)^{\times} \rightarrow K_0(\mathcal{M}^p(A))$$

for $p = 1, \dots, \dim A$ and induction argument. The implication (ii) \Rightarrow (i) can be proved by the following exact sequence and additivity of K -theory.

$$0 \rightarrow A/(f_1, \dots, f_{p-1}) \xrightarrow{f_p} A/(f_1, \dots, f_{p-1}) \rightarrow A/(f_1, \dots, f_p) \rightarrow 0 \quad (1)$$

where f_1, \dots, f_p are regular sequence of A . (Compare the proof of **Lemma 5.2**.)

(2) The implication (iii) \Rightarrow (i) $_{\mathbb{Q}}$ can be proved by the weight argument of the Adams operations. The implication (ii) \Rightarrow (iii) will be proved as follows. First, for any closed set $Y \subset \text{Spec } A$, we put $\text{Perf}^Y(\text{Spec } A)$ the category of strictly perfect complexes which are acyclic on $\text{Spec } A - Y$, that is, whose any object is a bounded complex X of finitely generated projective A -modules such that $X|_{\text{Spec } A - Y}$ is acyclic. We also put $\text{Perf}^p(\text{Spec } A) := \bigcup_{\substack{Y \subset X \\ \text{Codim } Y \geq p}} \text{Perf}^Y(\text{Spec } A)$.

Claim

For any non-negative integer n , we have the following identity.

$$K_n(\mathcal{M}^p(A)) \xrightarrow{\sim} K_n(\text{Perf}^p(\text{Spec } A); \text{qis})$$

where qis means quasi-isomorphisms, that is, $K_n(\text{Perf}^p(\text{Spec } A); \text{qis})$ is a Waldhausen quasi-isomorphism ‘Klassen’ group.

Proof of Claim. We have the following identities

$$K_n(\mathcal{M}^p(A)) \xrightarrow{\text{I}} \underset{\text{Codim } Y \geq p}{\text{colim}}_{Y \subset \text{Spec } A} K'_n(Y) \xrightarrow{\text{II}} \underset{\text{Codim } Y \geq p}{\text{colim}}_{Y \subset \text{Spec } A} K_n(\text{Spec } A \text{ on } Y) \xrightarrow{\text{I}} K_n(\text{Perf}^p(\text{Spec } A); \text{qis})$$

where isomorphisms I are proved by continuity [Qui73], [TT90] and isomorphism II is proved by the Poincaré duality and comparing the following fibration sequences [Qui73] and [TT90]

$$K'(Y) \rightarrow K'(\text{Spec } A) \rightarrow K'(\text{Spec } A - Y),$$

$$K(\text{Spec } A \text{ on } Y) \rightarrow K(\text{Spec } A) \rightarrow K(\text{Spec } A - Y)$$

for any closed set $Y \subset \text{Spec } A$. □

For $n = 0$, through the isomorphism in **Claim**, a class of A -module $[A/(f_1, \dots, f_p)]$ correspond to a class of the Koszul complex $[\text{Kos}(f_1, \dots, f_p)]$ where f_1, \dots, f_p forms a regular sequence. So we learn (ii) and (iii) are equivalent to the following statement (ii)' and (iii)' respectively.

(ii)' $K_0(\text{Perf}^p(A); \text{qis})$ is generated by Koszul complexes $\text{Kos}(f_1, \dots, f_p)$ where f_1, \dots, f_p forms a regular sequence for $p = 1, \dots, \dim A$.

(iii)' For non-negative integer k , the Adams operation ψ_k acts on $K_0(\text{Perf}^p(\text{Spec } A); \text{qis})$ by

multiplication of k^p for $p = 1, \dots, \dim A$.

Similar argument in [GS87] shows that we can define the Adams operations ψ_k on $K_0(\text{Perf}^p(\text{Spec } A); \text{qis})$ by using the Dold-Puppe correspondence and the Serre's result about Grothendieck groups of reductive group schemes [Ser68], and we have the identity

$$\psi_k([\mathbf{Kos}(f_1, \dots, f_p)]) = k^p[\mathbf{Kos}(f_1, \dots, f_p)]$$

by [GS87] Ibid. Hence the assertion (ii)' implies the assertion (iii)'. \square

To prove Gersten's conjecture, we shall extend the argument above to a higher analogue of that. The author's starting point to prove Gersten's conjecture for commutative discrete valuation rings is the following computational result **Theorem 1.5**.

Notations 1.3.

Let $\mathbf{Kos}^1(R)$ be the full additive subcategory of complexes X in $\mathcal{M}(R)$ such that $X_i = 0$ for $i \neq 0, 1$ and X_i are free R modules for $i = 0, 1$ and its boundary morphism $d^X : X_1 \rightarrow X_0$ is injective and homology groups are in $\mathcal{M}^1(R)$.

Remark 1.4.

We say that a sequence $X \rightarrow Y \rightarrow Z$ in $\mathbf{Kos}^1(R)$ is an admissible short exact sequence if it is a short exact sequence in $C_b(\mathcal{M}(R))$. Then $\mathbf{Kos}^1(R)$ is an exact category and

$$H_0 : \mathbf{Kos}^1(R) \rightarrow \mathcal{M}^1(R)$$

is an exact functor.

Theorem 1.5 ([Moc07]).

In the notation above, for any non-negative integer n , H_0 induces an isomorphism

$$K_n(H_0) : K_n(\mathbf{Kos}^1(R); \text{qis}) \xrightarrow{\sim} K_n(\mathcal{M}^1(R))$$

where qis means quasi-isomorphisms, that is, $K_n(\mathbf{Kos}^1(R); \text{qis})$ is a quasi-isomorphisms Waldhausen 'Klassen' group.

This theorem means that higher K -groups of $\mathcal{M}^1(R)$ are also generated by Koszul complexes. We shall ask the following question.

Question 1.6.

For any regular local ring A , for any non-negative integer n and p , is $K_n(\mathcal{M}^p(A))$ generated by weight p pure objects? Here the phrase 'weight p pure objects' means Koszul complexes $\mathbf{Kos}(f_1, \dots, f_p)$ (resp. cyclic modules $A/(f_1, \dots, f_p)$) where f_1, \dots, f_p forms a regular sequence.

Observations 1.7.

(1) Now remember that there are several methods of constructing the Adams operations on higher algebraic K -theory for schemes.

(A) Categorical description of the Adams operations [Gra92], [Sch87], [Wal00]. (See also [Tho77] p.111.)

(B) First using group representations theory, we define them for affine case [Kra80] [Hil81], and by patching argument or Jouanolou's device, we extend to general (divisorial) schemes case. [Sou85] [Lec98] [Lev97].

(C) Stable homotopy theoretic approach. [Rio06].

- (2) Now consider what we need to do weight argument integrally.
- (a) We don't need to precisely compute the action of ψ_k on $K_n(\mathcal{M}^p(A))$, we only verify that the weight is changing through the map $K_n(\mathcal{M}^{p+1}(A)) \rightarrow K_n(\mathcal{M}^p(A))$.
- (b) This means that we shall only do the argument in the proof of **Proposition 1.2** (ii) implies (i). (Compare the Quillen's proof of Gersten's conjecture [Qui73] p.133-134.) So we shall only use the higher analogue of short exact sequence (1). (Compare the short exact sequence (3) in the proof of **Lemma 5.2**.)

In this paper we will choose the method (B) and do argument (b) in **Observation 1.7**. Next we will explain the difficulty of doing argument (b).

Observations 1.8.

It is well-known that the category of small categories, simplicial sets and (compactly generated) topological spaces are related the nerve functor [Gro59] and the geometric realization functor [Mil57]

$$\mathbf{Cat} \xrightarrow{N} \Delta^{\text{op}}\mathbf{Set} \xrightarrow{!} \mathbf{Top}$$

and they induce categories equivalence between their homotopy categories respectively

$$\mathcal{S}^{-1} \mathbf{Cat} \xrightarrow{\sim} \mathcal{S}^{-1} \Delta^{\text{op}}\mathbf{Set} \xrightarrow{\sim} \mathcal{S}^{-1} \mathbf{Top} \tag{2}$$

where the morphism sets \mathcal{S} are certain classes of weak equivalences respectively. (See the former equivalence for [Ilu72], [Tho77], and Appendix, and the latter one [Qui67].) In theoretically, we can approximate to topological spaces by combinatorial data (that is, simplicial sets) and by special type of simplicial sets (for example, categories) up to homotopies. This thought is a fundamental dogma in algebraic K -theory. But in practice, there are a few morphisms between categories as in the following **Example 1.9**. So many authors attempt to replace \mathbf{Cat} with a category of more higher categorical objects. (c.f. [Tho77], [Tho79], [HS85], [Sch87] and [TV04].) The author will also introduce a class of suitable 2-categories for our argument in §2.

Example 1.9.

Let A be a regular noetherian ring, $\mathcal{M}(A)$ the category of finitely generated A -modules and $C_b(\mathcal{P}(A))$ the category of bounded complexes of finitely generated projective A -modules. Then we have the following assignment

$$\text{Ob } \mathcal{M}(A) \ni M \mapsto (\text{a projective resolution of } M) \in \text{Ob } C_b(\mathcal{P}(A))$$

which induces a canonical isomorphism between two Grothendieck groups

$$K_0(\mathcal{M}(A)) \xrightarrow{\sim} K_0(C_b(\mathcal{P}(A))); \text{qis}$$

where qis means a quasi isomorphisms ‘‘Klassen’’ group. We also have isomorphisms between their higher K -groups, but to prove that result is more complicated. The main reason is that the assignment above is not a functor. (Only it is a *lax functor*. See **Definition 2.3** (1) and **Example 2.4**.) So in the usual theory, we can not construct maps between higher K -groups from this assignment directly. In the usual theory, we used to prove this result by inserting the following inclusion functors

$$\mathcal{M}(A) \hookrightarrow \mathcal{P}(A) \hookrightarrow C_b(\mathcal{P}(A))$$

and verifying that they induce isomorphisms on their higher algebraic K -groups

$$K_n(\mathcal{M}(A)) \xleftarrow{\sim} K_n(\mathcal{P}(A)) \xrightarrow{\sim} K_n(C_b(\mathcal{P}(A))); \text{qis}$$

for any non-negative integer n . (The former isomorphism can be proved by resolution theorem [Qui73] and the latter one can be done by Gillet-Waldhausen theorem [Gil81], [Wal85], [TT90].) This is just another typical example of lacking techniques on inducing morphisms between higher algebraic K -theory from exotic functors. (But see [Nee00].) If we discuss using sequence (1), we shall consider the following assignment

$$\mathcal{M}^p(A) \rightarrow C_b(\mathcal{M}^{p-1}(A))$$

$$R/(f_1, \dots, f_p) \mapsto \begin{bmatrix} R/(f_1, \dots, f_{p-1}) \\ \downarrow f_p \\ R/(f_1, \dots, f_{p-1}) \end{bmatrix}$$

In general, this assignment is not functorial in usual sense. (Compare the proof of **Lemma 5.2**, especially the short exact sequence (3). When A is a complete discrete valuation ring, making assignment above functorially shall be related with Teichmüller lifting. Notice the assertion that the Generalized Teichmüller lifting conjecture implies Gersten's conjecture for discrete valuation ring in [She89].)

Solution 1.10 (Rectification techniques).

Here is one of the solution of the problem above, called *rectification* techniques. Now we will explain it.

(1) Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a functor between small categories. we can consider \mathcal{X} as a family over \mathcal{Y} . So we can reconstruct f by gathering each fiber. That is, we have the following formula

$$f(x) = \operatorname{colim} f|x (= \operatorname{colim}_{y \rightarrow x \in \operatorname{Ob} \mathcal{X}/x} f(y)).$$

(2) Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a *lax functor* in some sense and assume \mathcal{Y} is enough closed under taking *homotopy colimits* in some sense. Then f can be rectified by the functor \tilde{f} defined by the following formula

$$\tilde{f}(x) = \operatorname{hocolim} f|x (= \operatorname{hocolim}_{y \rightarrow x \in \operatorname{Ob} \mathcal{X}/x} f(y)).$$

See for example [GR71] Exposé VI §8, [Tho77] p.37 II §4, [Tho82] p.1646-1647.

(3) So it is important to examine the behaviour of homotopy colimits through the equivalences in (2). (See through the intentions behind the proof of Theorem B in [Qui73] p.97, see also [Tho82] p.1625-1626 and appendix.)

In this paper, the author will use updated version of rectification technique by using the result of Bullesjos, Cegarra, Moerdijk and Svensson [BC03] [MS93] in §3. (See also **Remark 2.8** (3) and compare the argument in **Solution 1.10** (3).) Next we analyse the method (B) in **Observation 1.7**.

Observations 1.11.

Quillen had constructed the universal map using study of characteristic classes of representations. (See [Qui76], [Hil81], see also [Gro68].) In [She82], Sherman read more formal argument into Quillen's construction. (Compare **Example 4.4**.) Namely the construction of the universal map is only depend on the behaviour of internal hom objects through the equivalences in (2). It is an important fact that to state the universal property of algebraic K -theory, we shall only consider skeletons of connected groupoids (that is, groups) for the domain variable of internal hom functors. This is the reason why the universal property of algebraic K -theory can be related with group representations.

To combine the rectification techniques with the universal property of algebraic K -theory, we will introduce a notion of *lax group representations*. In §4-1, we will study the behaviour (lax)

internal hom objects and use this results, in §4-2, define the lax version of the universal map, in §4-3, compare this universal map and the original one in the certain cases.

Caution 1.12.

To use the universal property of algebraic K -theory associated with semi-simple exact categories, we replace $\mathcal{M}^1(R)$ with $\mathcal{M}(k)$. By dévissage theorem, we have a group isomorphism

$$K_n(\mathcal{M}(k)) \xrightarrow{\sim} K_n(\mathcal{M}^1(R))$$

for any non negative integer n . This is only group isomorphism and is not considered as special λ -ring morphism. So we shall not use the usual Adams operations on $K_n(k)$ for our purpose.

2 Categories with equivalence relations

In this section, we will introduce categories with equivalence relations - a special class of 2-categories - and lax functors. Using these notions, we will formulate lax group representations. We will also explain how to associate categories with equivalence relations with topological spaces.

Definition 2.1 (Category with equivalence relations).

(1) A *category with equivalence relations* is a category \mathcal{X} which has compatible equivalence relations in each hom set. That is, for each objects X, Y in \mathcal{X} , $\text{Hom}_{\mathcal{X}}(X, Y)$ has equivalence relation \sim and satisfies the following axiom.

Let $a : X \rightarrow Y, b : Y \rightarrow Z$ and $x, y : X \rightarrow Y$ be morphisms in \mathcal{X} and assume $x \sim y$. Then $x \circ a \sim y \circ a$ and $b \circ x \sim b \circ y$.

(2) A morphism between categories equivalence relations \mathcal{X}, \mathcal{Y} is a functor $f : \mathcal{X} \rightarrow \mathcal{Y}$ which preserves equivalence relations. That is, if $a, b : X \rightarrow Y$ are morphisms in \mathcal{X} and assume $a \sim b$, then we have $f(a) \sim f(b)$.

(3) We will denote the category of small categories with equivalence relations by \mathbf{Cat}_{er} .

Example 2.2.

(1) Any category \mathcal{X} can be considered as a category with equivalence relations where equivalence relations between morphisms $f, g : X \rightarrow Y$ in \mathcal{X} is defined by $f \sim g$ if and only if $f = g$. we will call it *trivial equivalence relations* in \mathcal{X}

(2) Let \mathcal{X} be category with equivalence relations. Then its subcategory \mathcal{Y} is considered as a category with equivalence relations in the natural way.

(3) Let \mathcal{X} be category with equivalence relations and $f : \mathcal{Y} \rightarrow \mathcal{X}$ a category equivalence. Then \mathcal{Y} can be considered as a category with equivalence relations such that $f : \mathcal{X} \rightarrow \mathcal{Y}$ is those morphisms in the obvious way.

(4) Let \mathcal{X} be an additive category and $C_b(\mathcal{X})$ be the category of bounded complexes in \mathcal{X} . Then $C_b(\mathcal{X})$ can be considered as a category with equivalence relations. Here $f \sim g$ if and only if f and g are chain homotopic.

(5) We define the category \mathcal{A} that is full subcategory of $\mathcal{M}(R)$. This plays crucial role in the proof of the main theorem. We put

$$\text{Ob } \mathcal{A} := \{M \in \text{Ob } \mathcal{M}(R); \pi T(M) = 0\}$$

where $T(M) := \{x \in M; rx = 0 \text{ for some } r \in R\}$. So every object in \mathcal{A} is isomorphic to the following type module $R^{\oplus n} \oplus (R/\pi)^{\oplus m}$. (If readers worry about set theoretic problems, you shall better replace the definition of \mathcal{A} with its skeletons. c.f. (3).) For any morphisms

$$a = \begin{pmatrix} a_{f \rightarrow f} & 0 \\ a_{f \rightarrow t} & a_{t \rightarrow t} \end{pmatrix}, b = \begin{pmatrix} b_{f \rightarrow f} & 0 \\ b_{f \rightarrow t} & b_{t \rightarrow t} \end{pmatrix} : R^{\oplus n} \oplus (R/\pi)^{\oplus m} \rightarrow R^{\oplus u} \oplus (R/\pi)^{\oplus v}$$

where the indexes f and t means ‘free’ and ‘torsion’ respectively, $a \sim b$ if and only if $a_{f \rightarrow f} = b_{f \rightarrow f} \pmod{\pi}$, $a_{f \rightarrow t} = b_{f \rightarrow t}$ and $a_{t \rightarrow t} = b_{t \rightarrow t}$. This definition does not depend on the choice of isomorphisms and \mathcal{A} is a category with equivalence relations.

Proof. In the notation above, we assume $a \sim b$. For any $c : R^{\oplus n'} \oplus (R/\pi)^{\oplus m'} \rightarrow R^{\oplus n} \oplus (R/\pi)^{\oplus m}$ and $d : R^{\oplus u} \oplus (R/\pi)^{\oplus v} \rightarrow R^{\oplus u'} \oplus (R/\pi)^{\oplus v'}$, we want to prove $ac \sim bc$ and $da \sim db$. (This fact implies independence of the choice of isomorphisms.) We will use similar index notations above for c and d . Obviously $a_{f \rightarrow f}c_{f \rightarrow f} = b_{f \rightarrow f}c_{f \rightarrow f} \pmod{\pi}$, $d_{f \rightarrow f}a_{f \rightarrow f} = d_{f \rightarrow f}b_{f \rightarrow f} \pmod{\pi}$, $a_{t \rightarrow t}c_{t \rightarrow t} = b_{t \rightarrow t}c_{t \rightarrow t}$, $d_{t \rightarrow t}a_{t \rightarrow t} = d_{t \rightarrow t}b_{t \rightarrow t}$ and $a_{f \rightarrow t}c_{f \rightarrow f} + a_{t \rightarrow t}c_{t \rightarrow t} = b_{f \rightarrow t}c_{f \rightarrow f} + b_{t \rightarrow t}c_{t \rightarrow t}$. Since $a_{f \rightarrow f} - b_{f \rightarrow f}$ can be divided by π , we have $d_{f \rightarrow t}(a_{f \rightarrow f} - b_{f \rightarrow f}) = 0$. So we learn $d_{f \rightarrow t}a_{f \rightarrow f} + d_{t \rightarrow t}a_{f \rightarrow t} = d_{f \rightarrow t}b_{f \rightarrow f} + d_{t \rightarrow t}b_{f \rightarrow t}$. \square

Definition 2.3 (Lax functors, Deformations).

Let I be a small category and \mathcal{X} a small category with equivalence relations.

(1) A *lax functor* $x : I \rightarrow \mathcal{X}$ consists of two maps which assign to each object $i \in I$ an object $x_i \in \mathcal{X}$, to each morphism $a : i \rightarrow j$ in I an morphism $x_a : x_i \rightarrow x_j$ in \mathcal{X} , respectively. These data are required to satisfy the following conditions:

LF1 for any object $i \in I$, $x_{\text{id}_i} = \text{id}_{x_i}$,

LF2 for any pair of composable morphisms $i \xrightarrow{a} j \xrightarrow{b} k$ in I , $x_{ba} \sim x_b x_a$.

(2) A (*lax*) *deformation* $f : x \Rightarrow y$, between lax functors $x, y : I \rightarrow \mathcal{X}$ consists of a map which assign to each object $i \in I$ a morphism $f_i : x_i \rightarrow y_i$ in \mathcal{X} . These data satisfy the condition that for each morphism $a : i \rightarrow j$ in I , $f_j x_a \sim y_a f_i$.

(3) In the notation above, f is called *strict* if each morphism $a : i \rightarrow j$ in I , $f_j x_a = y_a f_i$.

(4) Let Θ be a subset of $\text{Ob } I$ and $x, y : I \rightarrow \mathcal{X}$ a lax functors which coincide over Θ . A deformation $f : x \Rightarrow y$ is qualified as *relative* to Θ whenever $f_i = \text{id}_{x_i}$ for all $i \in \Theta$.

(5) For any deformations $x \xRightarrow{f} y \xRightarrow{g} z$, between lax functors $x, y, z : I \rightarrow \mathcal{X}$, we can define their composition $x \xRightarrow{gf} z$ by $(gf)_i = g_i f_i$ for any i in I . Let $\mathcal{LAX}(I, \mathcal{X})$ (resp. $\mathcal{LAX}(I, \mathcal{X})_s$) be the category whose objects are lax functors from I to \mathcal{X} and morphisms are deformations (resp. strict deformations). $\mathcal{HOM}(I, \mathcal{X})$ (resp. $\mathcal{HOM}(I, \mathcal{X})_s$) is a full subcategory of $\mathcal{LAX}(I, \mathcal{X})$ (resp. $\mathcal{LAX}(I, \mathcal{X})_s$) whose objects are just functors. $\mathcal{LAX}(I, \mathcal{X})_{\text{rel } \Theta}$ (resp. $\mathcal{HOM}(I, \mathcal{X})_{\text{rel } \Theta}$) is a subcategory of $\mathcal{LAX}(I, \mathcal{X})$ (resp. $\mathcal{HOM}(I, \mathcal{X})$) whose morphisms are deformations relative to Θ . And we will put $\text{Lax}(I, \mathcal{X}) := \text{Ob } \mathcal{LAX}(I, \mathcal{X})$.

Example 2.4.

In the notation **Example 1.9**, we can consider $C_b(\mathcal{P}(A))$ as a category with equivalence relations by **Example 2.2** (4). Then the assignment

$$\mathcal{M}(A) \rightarrow C_b(\mathcal{P}(A))$$

$$M \mapsto (\text{a projective resolution of } M)$$

$$f \mapsto (\text{a lifting of } f)$$

defines a lax functor.

Remark 2.5.

(1) Let I and J be small categories. We consider J as a category with trivial equivalence relations. Then a lax functor $I \rightarrow J$ is just a usual functor. So we have identities

$$\mathcal{LAX}(I, J) = \mathcal{LAX}(I, J)_s = \mathcal{HOM}(I, J) = \mathcal{HOM}(I, J)_s.$$

(2) Let I be a small category and \mathcal{X} a small category with equivalence relations. Then $\mathcal{LAX}(I, \mathcal{X})$ can be considered as a category with equivalence relations where equivalence relations between deformations defined by $f \sim g$ if and only if $f_i \sim g_i$ for any objects in $i \in I$.

(3) Let $f : I \rightarrow J$ be a functor between small categories, $g : \mathcal{X} \rightarrow \mathcal{Y}$ a morphism between small categories with equivalence relations, $x, y : J \rightarrow \mathcal{X}$ lax functors and $a : x \Rightarrow y$ a deformation. Then we can define their compositions. That is, a lax functor $gxf : I \rightarrow \mathcal{Y}$ and a deformation $gaf : gxf \Rightarrow gyf$ in the obvious ways.

(4) So we have functors

$$\mathcal{LAX}(?, ?)_{\square}, \mathcal{HOM}(?, ?)_{\square} : \mathbf{Cat}^{\text{op}} \times \mathbf{Cat}_{\text{er}} \rightarrow \mathbf{Cat}_{\text{er}}$$

where $\square = \emptyset, \text{s}, \text{or relOb}(?)$. The last one means the functors

$$(I, \mathcal{X}) \mapsto \mathcal{LAX}(I, \mathcal{X})_{\text{relOb } I} \text{ (resp. } \mathcal{HOM}(I, \mathcal{X})_{\text{relOb } I}\text{)}.$$

Example 2.6 (Lax group representations).

Let G be a group and \mathcal{X} be a small category with equivalence relations. As usual we will sometimes consider G as the category whose object set is only one element. We will denote the “category” G by \underline{G} . An object in $\mathcal{LAX}(\underline{G}, \mathcal{X})$ is a pair (X, ρ_X) consisting of an object X in \mathcal{X} and a map $\rho_X : G \rightarrow \text{Aut } X$ which satisfies $\rho_X(gh) \sim \rho_X(g)\rho_X(h)$ for any $g, h \in G$. It is called a *lax group representation* in \mathcal{X} . This concept plays crucial role in the proof of the main theorem.

Definition 2.7 (Various nerves).

Let \mathcal{X} be a small category with equivalence relations.

(1) ([Dus02], [Str87]) A *geometric nerve* $\Delta \mathcal{X}$ is a simplicial set defined by

$$\Delta \mathcal{X} := \text{Lax}(?, \mathcal{X}) : \Delta^{\text{op}} \rightarrow \mathbf{Set}.$$

(2) A *simplicial geometric nerve* $\underline{\Delta} \mathcal{X}$ is a simplicial category defined by

$$\underline{\Delta} \mathcal{X} := \mathcal{LAX}(?, \mathcal{X})_{\text{relOb}(?)} : \Delta^{\text{op}} \rightarrow \mathbf{Cat}.$$

(3) A *2-nerve* is a simplicial category defined by

$$\mathbf{N}^{\text{er}} \mathcal{X} := \mathcal{HOM}(?, \mathcal{X})_{\text{relOb}(?)} : \Delta^{\text{op}} \rightarrow \mathbf{Cat}.$$

(4) Every assignments are functorial in \mathcal{X} (cf. **Remark 2.5** (4)), so we get the functors

$$\Delta : \mathbf{Cat}_{\text{er}} \rightarrow \Delta^{\text{op}} \mathbf{Set} \text{ and } \underline{\Delta}, \mathbf{N}^{\text{er}} : \mathbf{Cat}_{\text{er}} \rightarrow \Delta^{\text{op}} \mathbf{Cat}.$$

Remark 2.8 (Fundamental properties of various nerves).

Let \mathcal{X} be a small category with equivalence relations.

(1) If the equivalence relations in \mathcal{X} are trivial, then $\Delta \mathcal{X}$ is just a usual nerve $\mathbf{N} \mathcal{X}$ by **Remark 2.5** (1).

(2) We will define a canonical natural transformation

$$\text{Lax}(?, ?) \Rightarrow \text{Hom}_{\Delta^{\text{op}} \mathbf{Set}}(\mathbf{N}(?), \Delta(?))$$

between functors $\mathbf{Cat}^{\text{op}} \times \mathbf{Cat}_{\text{er}} \rightarrow \mathbf{Set}$, as follows.

For each lax functor $x : I \rightarrow \mathcal{X}$ from a small category I to a small category with equivalence relations \mathcal{X} , we will assign a simplicial map $\hat{x} : \mathbf{N} I \rightarrow \Delta \mathcal{X}$ in the following way. For each non-negative integer n ,

$$\mathbf{N}_n I = \text{Hom}_{\mathbf{Cat}}([n], I) \ni \phi \mapsto x \circ \phi \in \text{Lax}([n], \mathcal{X}) = \Delta_n \mathcal{X}.$$

This define the assignment $x \mapsto \hat{x}$ and it is just the desired natural transformation.

(3) The following canonical inclusions of simplicial categories induce homotopy equivalence for their geometric realizations, respectively

$$\mathbf{N}^{\text{er}} \mathcal{X} \hookrightarrow \underline{\Delta} \mathcal{X} \hookrightarrow \Delta \mathcal{X}$$

where $\Delta \mathcal{X}$ is considered as a simplicial discrete category. (See [MS93], [BC03].)

3 Lax algebraic K -theory

In this section, we will define ad hoc lax algebraic K -theory. As explained in §1, to introduce this concept, especially **Theorem 3.13**, is one of the variant of rectification techniques on algebraic K -theory.

Definition 3.1 (Categories with fibrations and equivalence relations).

(1) A *category with fibrations* is the dual of a usual category with cofibrations. (c.f.[Wal85], [TT90], [TV04]). So a category with fibrations will be a pair $(\mathcal{X}, \text{fib}(\mathcal{X}))$ consisting of a category \mathcal{X} with subcategory $\text{fib}(\mathcal{X}) \hookrightarrow \mathcal{X}$ whose morphisms will be called *fibrations* satisfying the following axioms:

Fib 0 \mathcal{X} has a specific zero object $*$.

Fib 1 The subcategory $\text{fib}(\mathcal{X})$ contains all isomorphisms in \mathcal{X} .

Fib 2 For any object $x \in \mathcal{X}$, the morphism $x \rightarrow *$ is in $\text{fib}(\mathcal{X})$.

Fib 3 If $x \rightarrow y$ is fibration, then, for any morphism $z \rightarrow y$ in \mathcal{X} , the fiber product $x \times_y z$ exists in \mathcal{X} and the canonical morphism $x \times_y z \rightarrow z$ is again a fibration.

We always omit $\text{fib}(\mathcal{X})$ in the notation. The twoheadarrow \dashrightarrow will be used to denote the morphisms in $\text{fib}(\mathcal{X})$.

(2) A functor $f : \mathcal{X} \rightarrow \mathcal{Y}$ between categories with fibrations is called *exact* if it preserves all relevant structures, that is, if $f(\text{fib}(\mathcal{X})) \subset \text{fib}(\mathcal{Y})$ and if f preserves pull back along fibration. The last condition means that the canonical map $f(X \times_Z Y) \rightarrow f(X) \times_{f(Z)} f(Y)$ is an isomorphism whenever $X \dashrightarrow Z$ is in $\text{fib}(\mathcal{X})$. We will denote the category of small category with fibrations and exact functors by **Fib**.

(3) A *category with fibrations and equivalence relations* \mathcal{X} is a category with fibrations whose underlying category \mathcal{X} is a category with equivalence relations.

(4) A morphism of categories with fibrations and equivalence relations $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a functor which is both an exact functor and a morphism of categories with equivalence relations. We will denote the category of small categories with fibrations and equivalence relations by **Fib_{er}**.

Example 3.2.

(1) An exact category in the sense of Quillen [Qui73] (see also **Notation 4.20** (1)) can be considered as a category with fibrations by choosing a zero object, and defining the fibrations to be the admissible epimorphisms.

(2) Let \mathcal{X} be a category with fibrations and $f : \mathcal{Y} \rightarrow \mathcal{X}$ be a category equivalence. Then \mathcal{Y} can be considered as a category with fibrations such that $f : \mathcal{Y} \rightarrow \mathcal{X}$ is an exact functor in the obvious way.

(3) \mathcal{A} in **Example 2.2** (5) can be considered as a category with fibrations by choosing a zero object, and defining the fibrations to be the surjections.

Proof. Obviously \mathcal{A} is closed under the direct sum operations and taking submodules. Since a fiber product is a submodule of a certain direct sum module, these assertions above implies that \mathcal{A} is also closed under taking fiber products. To check other axioms is trivial. \square

We will use the dual S_{\bullet} -construction in [Wal85], denoted by S_{\bullet}^{op} .

Definition 3.3 (S_{\bullet}^{op} -construction).

(c.f [Wal85], [TV04]) Let \mathcal{X} be a small category with fibrations.

(1) A sequence in \mathcal{X}

$$X \xrightarrow{i} Y \xrightarrow{p} Z$$

is called a *fibration sequence*, if p is a fibration, and if the following diagram is a cartesian

square.

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow & & \downarrow p \\ * & \longrightarrow & Z \end{array}$$

(2) Let n be a non-negative integer. Consider the *arrow category* $\text{Ar}[n] := \mathcal{HOM}([1], [n])$ of $[n]$, that is, it is a partially ordered set of pairs (i, j) , $0 \leq i \leq j \leq n$, where $(i, j) \leq (i', j')$ if and only if $i \leq i'$ and $j \leq j'$. We will denote (i, j) as j/i .

(3) A functor $X : (\text{Ar}[n])^{\text{op}} \rightarrow \mathcal{X}$ is called the n -th *exact functor* if it satisfies **Ex 1** and **Ex 2** below

Ex 1 $X(i/i) = *$ for any $0 \leq i \leq n$.

Ex 2 For any $0 \leq i \leq j \leq k \leq n$, $X(k/j) \rightarrow X(k/i) \rightarrow X(j/i)$ is a fibration sequence.

(4) A morphism between n -th exact functors is a natural transformation. n -th exact functors and their morphisms form a category $S_n^{\text{op}} \mathcal{X}$. $iS_n^{\text{op}} \mathcal{X}$ denotes the subcategory of $S_n^{\text{op}} \mathcal{X}$. By definition here a morphism $f : X \rightarrow Y$ of $S_n^{\text{op}} \mathcal{X}$ is in $iS_n^{\text{op}} \mathcal{X}$ if and only if the $f(i/j) : X(i/j) \rightarrow Y(i/j)$ is an isomorphism for every $0 \leq j \leq i \leq n$. Since $S_n^{\text{op}} \mathcal{X}$ is subcategory of $\mathcal{HOM}((\text{Ar}[n])^{\text{op}}, \mathcal{X})_s$, $S_{\bullet}^{\text{op}} \mathcal{X}$ and $iS_{\bullet}^{\text{op}} \mathcal{X}$ can be considered as sub simplicial categories of $\mathcal{HOM}((\text{Ar}[-])^{\text{op}}, \mathcal{X})_s$. Since this construction is functorial in \mathcal{X} by **Remark 2.5** (4), we have functors

$$S_{\bullet}^{\text{op}}, iS_{\bullet}^{\text{op}} : \mathbf{Fib} \rightarrow \Delta^{\text{op}} \mathbf{Cat}.$$

(5) We will define a simplicial set $s_{\bullet}^{\text{op}} \mathcal{X}$ as $[n] \mapsto \text{Ob } S_n^{\text{op}} \mathcal{X}$. Since this construction is functorial in \mathcal{X} , we have a functor

$$s_{\bullet}^{\text{op}} : \mathbf{Fib} \rightarrow \Delta^{\text{op}} \mathbf{Set}.$$

Remark 3.4.

Let \mathcal{X} be a category with fibrations and equivalence relations. Then since for each non-negative integer n , $S_n^{\text{op}} \mathcal{X}$ is a subcategory of $\mathcal{HOM}((\text{Ar}[n])^{\text{op}}, \mathcal{X})_s$, it can be considered as a category with equivalence relations by **Example 2.2** (2) and **Remark 2.5** (2). So we get the functors

$$iS_{\bullet}^{\text{op}}, S_{\bullet}^{\text{op}} : \mathbf{Fib}_{\text{er}} \rightarrow \Delta^{\text{op}} \mathbf{Cat}_{\text{er}}.$$

Remark 3.5 (Fundamental properties for s_{\bullet}^{op} -construction).

Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be an exact functor between small categories with fibrations. Then we have the following properties.

(1) The simplicial set $s_{\bullet} \mathcal{X}$ is considered as a discrete simplicial category. Then we have the natural inclusion $s_n^{\text{op}} \mathcal{X} \ni X \mapsto X \in iS_n^{\text{op}} \mathcal{X}$ for each n . Hence we get a simplicial map $s_{\bullet}^{\text{op}} \mathcal{X} \hookrightarrow iS_{\bullet}^{\text{op}} \mathcal{X}$ and it is a homotopy equivalence. (See the dual version, [Wal85] p.335 Corollary (2) and p.376, [Sch87] p.272 (1.2) Proposition, see also [HS85] p.416 (3.6).)

(2) If f is a category equivalence, that is, it is fully faithful and essentially surjective, then $s_{\bullet}^{\text{op}} f$ is a homotopy equivalence. (See the dual version, [Wal85] p.335 Corollary (1).)

Definition 3.6 (Cogluing axiom).

Let \mathcal{X} be a category with fibrations and equivalence relations. We say \mathcal{X} satisfies the *cogluing axiom* if in the following commutative diagrams for $i = 1, 2$

$$\begin{array}{ccccc} X & \twoheadrightarrow & Y & \longleftarrow & Z \\ \downarrow a_i & & \downarrow b_i & & \downarrow c_i \\ X' & \twoheadrightarrow & Y' & \longleftarrow & Z' \end{array},$$

assertions $a_1 \sim a_2$, $b_1 \sim b_2$ and $c_1 \sim c_2$ imply $a_1 \times_{b_1} c_1 \sim a_2 \times_{b_2} c_2$.

Example 3.7.

- (1) Let \mathcal{X} be a category with fibrations. We consider it as a category with trivial equivalence relations. Then \mathcal{X} satisfies the cogluing axiom.
(2) \mathcal{A} in **Example 2.2** (5) satisfies the cogluing axiom.

Proof. Consider the diagram in **Definition 3.6** for the case of \mathcal{A} , if Y and Y' are 0-modules, the result is easy. For the free part of $X \oplus Z$ (resp. $X' \oplus Z'$) is the direct sum of each of the free parts and the torsion part is also. For a general case also follows this special case. For fiber products are submodules of certain direct sum modules and equivalence relations compatible with an inclusion map. □

Proposition 3.8.

- (1) Let G be a group and \mathcal{X} be a small category with fibrations and equivalence relations satisfying the cogluing axiom. Then $\mathcal{LAX}(\underline{G}, \mathcal{X})_s$ is a category with fibrations where a morphism $f : X \rightarrow Y$ in $\mathcal{LAX}(\underline{G}, \mathcal{X})_s$ is a fibration if and only if forgetting the action of G , it is in $\text{fib}(\mathcal{X})$.
(2) In the notation above, moreover let $\phi : G \rightarrow H$ be a group homomorphism and $\psi : \mathcal{X} \rightarrow \mathcal{Y}$ a morphism between categories with fibrations and equivalence relations satisfying the cogluing axiom. Then the induced functors

$$\begin{aligned} \mathcal{LAX}(\phi, \mathcal{X})_s &: \mathcal{LAX}(\underline{H}, \mathcal{X})_s \rightarrow \mathcal{LAX}(\underline{G}, \mathcal{X})_s \\ \mathcal{LAX}(\underline{G}, \psi)_s &: \mathcal{LAX}(\underline{G}, \mathcal{X})_s \rightarrow \mathcal{LAX}(\underline{G}, \mathcal{Y})_s \end{aligned}$$

are exact.

Proof. In a diagram of lax group representations

$$(X, \rho_X) \rightarrow (Z, \rho_Z) \leftarrow (Y, \rho_Y),$$

we define a lax action $\rho_{X \times_Z Y} : G \rightarrow \text{Aut}(X \times_Z Y)$ by $g \mapsto \rho_X(g) \times_{\rho_Z(g)} \rho_Y(g)$. By the cogluing axiom, we learn that $(X \times_Z Y, \rho_{X \times_Z Y})$ is a lax group representation and a pull back of diagram above. To check other axioms and assertions is trivial. □

Definition 3.9 (Lax algebraic K-theory).

- (1) Let \mathcal{X} be a small category with fibrations and equivalence relations. We put a pointed space called as a *lax algebraic K-theory space* associated with \mathcal{X} in the following way.

$$\mathbb{K}^{\text{lax}}(\mathcal{X}) := \Omega |\Delta i S_{\bullet}^{\text{op}} \mathcal{X}|$$

where Ω is the *loop space functor*, namely $\Omega = \mathbf{Hom}_{\mathbf{Top}_*}(S^1, ?)$. (See **Example 4.2** (2)_{*}).

- (2) In the notation above, assume \mathcal{X} satisfies the cogluing axiom. For any group G , we put a pointed space called as a *lax representations space* associated with G and \mathcal{X} in the following way.

$$\mathbb{R}^{\text{lax}}(G, \mathcal{X}) := \Omega B i S_{\bullet}^{\text{op}} \mathcal{LAX}(\underline{G}, \mathcal{X})_s$$

where B is the *classifying space functor*, that is, composition of $\mathbf{Cat} \xrightarrow{N} \Delta^{\text{op}} \mathbf{Set} \xrightarrow{|\cdot|} \mathbf{Top}$. We put its n -th homotopy group for any non-negative integer n as

$$R_n^{\text{lax}}(G, \mathcal{X}) := \pi_n(\mathbb{R}^{\text{lax}}(G, \mathcal{X}))$$

and call it as the n -th lax representations group associated with G and \mathcal{X} .

(3) In the notation above, by **Proposition 3.8** (2), for any non-negative integer n , the canonical homomorphism $G \rightarrow \text{pt}$ induces the group homomorphism

$$R_n(G, \mathcal{X}) \rightarrow R_n(\text{pt}, \mathcal{X})$$

where pt means the unit group. We put

$$\tilde{R}_n^{\text{lax}}(G, \mathcal{X}) := \text{Ker}(R_n(G, \mathcal{X}) \rightarrow R_n(\text{pt}, \mathcal{X}))$$

and call it as the n -th reduced lax group representation group associated with G and \mathcal{X} .

Excuse 3.10.

We can define a concept of Waldhausen categories with equivalence relations and associate them with lax algebraic K -theory in the obvious way. But we will not need them to prove the main theorem and the author abandon developing these theories and full functoriality of lax algebraic K -theory.

Example 3.11.

In the notation above, we assume \mathcal{X} is a category with trivial equivalence relations. Then

- (1) $\mathbb{K}^{\text{lax}}(\mathcal{X})$ is just a usual algebraic K -space associated with \mathcal{X} . We will denote it by $\mathbb{K}(\mathcal{X})$.
- (2) For any group G (and non-negative integer n), $\mathbb{R}^{\text{lax}}(G, \mathcal{X})$ (resp. $R_n^{\text{lax}}(G, \mathcal{X})$, $\tilde{R}_n^{\text{lax}}(G, \mathcal{X})$) is just a usual representation space (resp. n -th representation group, n -th reduced representation group) associated with G and \mathcal{X} . We will denote it by $\mathbb{R}(G, \mathcal{X})$ (resp. $R_n(G, \mathcal{X})$, $\tilde{R}_n(G, \mathcal{X})$).

Remark 3.12 (Edgewise subdivision).

We review the *edgewise subdivision* functor. (c.f. [Seg73] p.219, [Wal85] p.375.)

- (1) First we define the *doubling functor* $d : \Delta \rightarrow \Delta$ as follows. For each $[n]$, we put $d([n]) := [2n + 1]$ and for each ordered map $a : [n] \rightarrow [m]$, we put

$$d(a)(k) := \begin{cases} m - a(n - k) & \text{if } 0 \leq k \leq n \\ a(k - n + 1) + m + 1 & \text{if } n + 1 \leq k \leq 2n + 1. \end{cases}$$

- (2) For any simplicial object $X : \Delta^{\text{op}} \rightarrow \mathcal{X}$ in a category \mathcal{X} , associate another $X^e : \Delta^{\text{op}} \rightarrow \mathcal{X}$, namely the composite $X^e := X d^{\text{op}}$. This assignment define the *edgewise subdivision* functor

$$?^e : \Delta^{\text{op}} \mathcal{X} \rightarrow \Delta^{\text{op}} \mathcal{X}.$$

- (3) If X is a simplicial space then the geometric realizations $|X|$ and $|X^e|$ are naturally homeomorphic. (c.f. [Seg73] Ibid Proposition (A.1).)
- (4) For any small category with fibrations and equivalence relations \mathcal{X} , applying the fact in (3) to the simplicial space $[n] \mapsto |\Delta_i S_n^{\text{op}} \mathcal{X}|$ we obtain that $\mathbb{K}^{\text{lax}}(\mathcal{X})$ and $\mathbb{K}^{\text{lax,e}}(\mathcal{X}) := \Omega |\Delta_i S_{\bullet}^{\text{op,e}} \mathcal{X}|$ are homotopy equivalent. (See [Wal85] Ibid.) If \mathcal{X} satisfies the cogluing axiom and let G be a group, then we will also use $\mathbb{R}^{\text{lax,e}}(G, \mathcal{X}) := \Omega BiS_{\bullet}^{\text{op,e}} \mathcal{LAX}(\underline{G}, \mathcal{X})_s$ which is homotopy equivalent to $\mathbb{R}^{\text{lax}}(G, \mathcal{X})$.

To prove the following theorem, we will frequently use the results in Appendix.

Theorem 3.13.

Let \mathcal{X} be a category with fibrations and equivalence relations satisfying the cogluing axiom. Then the canonical map induced from the identity functor

$$\mathbb{K}(\mathcal{X}) \rightarrow \mathbb{K}^{\text{lax}}(\mathcal{X})$$

has a retraction up to homotopy.

Proof. First notice the following commutative diagram of trisimplicial sets

$$\begin{array}{ccccc} s_{\bullet}^{\text{op}} \mathcal{X} & \longrightarrow & \text{N } iS_{\bullet}^{\text{op}} \mathcal{X} & \longrightarrow & \text{NN}^{\text{er}} iS_{\bullet}^{\text{op}} \mathcal{X} \\ & & \downarrow & & \downarrow \\ & & \Delta iS_{\bullet}^{\text{op}} \mathcal{X} & \longrightarrow & \text{N } \underline{\Delta} iS_{\bullet}^{\text{op}} \mathcal{X} \end{array}$$

where $s_{\bullet}^{\text{op}} \mathcal{X}$ is considered as a constant bisimplicial simplicial set (namely trisimplicial set) and so on. By **Remark 2.8** (3), **Remark 3.5** (1), we shall only prove that the inclusion map $s_{\bullet}^{\text{op}} \mathcal{X} \hookrightarrow \text{NN}^{\text{er}} iS_{\bullet}^{\text{op}} \mathcal{X}$ has a retraction up to homotopy. Now we consider a simplicial category $M_{\bullet} \mathcal{X}$ which is a sub simplicial category of $[n] \mapsto \mathcal{L}\mathcal{A}\mathcal{X}([n] \times [n], \mathcal{X})$ as follows.

Object: A lax functor $X : [n] \times [n] \rightarrow \mathcal{X}$ satisfying the following conditions.

(i) For any $0 \leq i \leq n$ and $0 \leq s \leq t \leq n$, $X(i, s) = X(i, t)$ and $X(i, s \leq t) = \text{id} : X(i, s) \rightarrow X(i, t)$ and,

(ii) for any $0 \leq j \leq n$ and $0 \leq s \leq n-1$, $X_{s,j} := X(s \leq s+1, j) : X(s, j) \rightarrow X(s+1, j)$ is an isomorphism.

Morphism: A deformation $f : X \rightarrow Y$ satisfying the following conditions.

(iii) For any $0 \leq i \leq n$ and $0 \leq s \leq t \leq n$, $f(i, s) = f(i, t)$ and,

(iv) for any $0 \leq j \leq n$ and $0 \leq s \leq n-1$, the following diagram is commutative:

$$\begin{array}{ccc} X(s, j) & \xrightarrow[\sim]{X_{s,j}} & X(s+1, j) \\ f(s,j) \downarrow & & \downarrow f(s+1,j) \\ Y(s, j) & \xrightarrow[\sim]{Y_{s,j}} & Y(s+1, j). \end{array}$$

Claim

(1) For each n , $M_n \mathcal{X}$ can be considered as a category with fibrations where a morphism is a fibration if and only if it is a term-wised fibrations in \mathcal{X} .

(2) For each ordered map $\phi : [n] \rightarrow [m]$, the induced map $M_m \mathcal{X} \rightarrow M_n \mathcal{X}$ is exact.

(3) $\text{diag } \text{N } \text{N}^{\text{er}} iS_{\bullet}^{\text{op}} \mathcal{X} = \text{diag } s_{\bullet}^{\text{op}} M_{\bullet} \mathcal{X}$. Here diag means the *diagonalization functor*.

Proof of Claim. To prove **Fib 3**, we need the assumption that \mathcal{X} satisfies the cogluing axiom. Fiber product of a pull back diagram in $M_n \mathcal{X}$ is a term-wised pull back. To prove other axioms and assertions is trivial. \square

So we shall prove the natural map $s_{\bullet}^{\text{op}} \mathcal{X} \rightarrow s_{\bullet}^{\text{op}} M_{\bullet} \mathcal{X}$ induced from inclusion $\iota : \mathcal{X} \hookrightarrow M_{\bullet} \mathcal{X}$ possesses a retraction up to homotopy. Here \mathcal{X} is considered as the constant simplicial category. Now we consider the following diagram

$$\begin{array}{ccc} s_{\bullet}^{\text{op}} \mathcal{X} & \longrightarrow & iS_{\bullet}^{\text{op}} \mathcal{X} \\ s_{\bullet}^{\text{op}} \iota \downarrow & & \downarrow iS_{\bullet}^{\text{op}} \iota \\ s_{\bullet}^{\text{op}} M_{\bullet} \mathcal{X} & \longrightarrow & iS_{\bullet}^{\text{op}} M_{\bullet} \mathcal{X} \end{array}$$

where horizontal morphisms are canonical inclusions in **Remark 3.5** (1), and hence they are homotopy equivalences by *Ibid* and the **Realizaiton lemma 3.14** below. Therefore we shall prove that the $iS_{\bullet}^{\text{op}} \iota : iS_{\bullet}^{\text{op}} \mathcal{X} \rightarrow iS_{\bullet}^{\text{op}} M_{\bullet} \mathcal{X}$ possesses a retraction up to homotopy. Then, by **Proposition A.6**, we shall prove that $\iota : \mathcal{X} \rightarrow M_{\bullet} \mathcal{X}$ possesses a retraction as a strict lomt. (See the definition of strict lomt for **Definition A.2** (3), (4)).

Claim 2

$\mathcal{X} \rightarrow M_\bullet \mathcal{X}$ has a strict lomt retraction.

Proof of Claim 2. We define the lomt $\theta : M_\bullet \mathcal{X} \rightarrow \mathcal{X}$ as follows. First, for any non-negative integer n , we define

$$M_n \mathcal{X} \rightarrow \mathcal{X}$$

$$(X \xrightarrow{f} Y) \mapsto (X(0,0) \xrightarrow{f(0,0)} Y(0,0)).$$

This functor is obviously an exact functor. Next for any ordered map $a : [m] \rightarrow [n]$, we define the natural equivalence $\theta_a : \theta_n \xrightarrow{\sim} \theta_m M_a \mathcal{X}$ as follows. For each object $X \in M_n \mathcal{X}$, we put

$$\theta(X) := X_{a(0)-1, a(0)-1} X_{a(0)-2, a(0)-2} \cdots X_{1,1} X_{0,0} : X(0,0) \xrightarrow{\sim} X(a(0), a(0)).$$

Then we can easily verify that θ is a strict lomt and $\theta \iota = \text{id}$. □

Hence the canonical map $\mathbb{K}(\mathcal{X}) \rightarrow \mathbb{K}^{\text{lax}}(\mathcal{X})$ has a retraction up to homotopy. □

Lemma 3.14 (Realization lemma).

(c.f. [Seg74] Appendix A, [Wal78] p. 164 Lemma 5.1).

Let $X_{\bullet\bullet} \rightarrow Y_{\bullet\bullet}$ be a map of bisimplicial sets. Suppose that for every n , the map $X_{\bullet n} \rightarrow Y_{\bullet n}$ is a homotopy equivalence. Then $X_{\bullet\bullet} \rightarrow Y_{\bullet\bullet}$ is a homotopy equivalence.

4 The lax Sherman map

In this section, we will construct the *lax Sherman map* and by using them, state the universal property of algebraic K -theory associated with semi-simple exact categories.

4.1 The exponential law

As explained in §1, in this section, we will study behaviour of internal hom objects through the equivalences in (2). To do so, we will start to define *cartesian closed categories*.

Definition 4.1 (Cartesian closed categories).

Let \mathcal{V} be a category with a terminal object $*$, and finite products. We pick a functor

$$\times : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$$

sending V_1, V_2 to $V_1 \times V_2$, the product of V_1 and V_2 . Suppose that \mathcal{V} provided the internal hom functor, i.e., a functor $\mathbf{Hom}_{\mathcal{V}} : \mathcal{V}^{\text{op}} \times \mathcal{V} \rightarrow \mathcal{V}$ and we have an isomorphism, natural in U, V and $W \in \mathcal{V}$ of morphism sets:

$$\exp : \text{Hom}(U \times V, W) \xrightarrow{\sim} \text{Hom}(U, \mathbf{Hom}_{\mathcal{V}}(V, W)).$$

We call such a \mathcal{V} together with the structure $\times, \mathbf{Hom}_{\mathcal{V}}(,)$, \exp , a *cartesian closed category*.

Example 4.2.

- (1) $\mathcal{V} = \mathbf{Set}$, the category of sets, where $\mathbf{Hom}_{\mathcal{V}}(U, V)$ is the set of functions $U \rightarrow V$.
- (2) $\mathcal{V} = \mathbf{Top}$, the category of compactly generated spaces, where $\mathbf{Hom}_{\mathcal{V}}(U, V)$ is the space of continuous functions $U \rightarrow V$, with the compact-open topology strengthened to a compactly generated topology.
- (2)* $\mathcal{V} = \mathbf{Top}_*$, the category of pointed compactly generated spaces with non-degenerated base points, where $\mathbf{Hom}_{\mathcal{V}}(U, V)$ is the subspace of $\mathbf{Hom}_{\mathbf{Top}}(U, V)$ whose maps are base point preserving.
- (3) $\mathcal{V} = \Delta^{\text{op}}\mathbf{Set}$, the category of simplicial sets, where $\mathbf{Hom}_{\mathcal{V}}(U, V)$ is the usual simplicial function space $[n] \mapsto \mathbf{Hom}_{\Delta^{\text{op}}\mathbf{Set}}(N[n] \times U, V)$.
- (4) $\mathcal{V} = \mathbf{Cat}$, the category of small categories, where $\mathbf{Hom}_{\mathcal{V}}(U, V)$ is the category with objects functors $U \rightarrow V$, and morphisms the natural transforms, that is, $\mathcal{HOM}(U, V)_s$

Lemma 4.3.

Let $f : \mathcal{V} \rightarrow \mathcal{W}$ be a functor between cartesian closed categories. Assume f preserves products. Then we have a canonical natural transformation

$$f(\mathbf{Hom}_{\mathcal{V}}(?, ?)) \Rightarrow \mathbf{Hom}_{\mathcal{W}}(f(?), f(?))$$

between the functors $\mathcal{V}^{\text{op}} \times \mathcal{V} \rightarrow \mathcal{W}$.

Proof. For each objects V_1, V_2 in \mathcal{V} , we have the *evaluation morphism*

$$\text{ev}_{V_1, V_2} : \mathbf{Hom}_{\mathcal{V}}(V_1, V_2) \times V_1 \rightarrow V_2$$

which is functorial in V_1 and V_2 and corresponded to $\text{id}_{\mathbf{Hom}(V_1, V_2)}$ through the following isomorphism

$$\text{exp} : \mathbf{Hom}_{\mathcal{V}}(\mathbf{Hom}_{\mathcal{V}}(V_1, V_2) \times V_1, V_2) \xrightarrow{\sim} \mathbf{Hom}_{\mathcal{V}}(\mathbf{Hom}_{\mathcal{V}}(V_1, V_2), \mathbf{Hom}_{\mathcal{V}}(V_1, V_2)).$$

Since f preserves products, we have the following isomorphisms

$$\begin{aligned} \mathbf{Hom}_{\mathcal{W}}(f(\mathbf{Hom}_{\mathcal{V}}(V_1, V_2) \times V_1), f(V_2)) &\xrightarrow{\sim} \mathbf{Hom}_{\mathcal{W}}(f(\mathbf{Hom}_{\mathcal{V}}(V_1, V_2)) \times f(V_1), f(V_2)) \\ &\xrightarrow{\sim} \mathbf{Hom}_{\mathcal{W}}(f(\mathbf{Hom}_{\mathcal{V}}(V_1, V_2)), \mathbf{Hom}_{\mathcal{W}}(f(V_1), f(V_2))). \end{aligned}$$

Hence we get the morphism

$$f(\mathbf{Hom}_{\mathcal{V}}(V_1, V_2)) \rightarrow \mathbf{Hom}_{\mathcal{W}}(f(V_1), f(V_2))$$

which is corresponded to $f(\text{ev}_{V_1, V_2})$ through the isomorphisms above, functorial in V_1 and V_2 and the desired natural transformation. \square

Example 4.4.

- (1) It is well-known that the nerve functor $N : \mathbf{Cat} \rightarrow \Delta^{\text{op}}\mathbf{Set}$ preserves products. So by **Lemma 4.3**, we have the canonical natural transformation

$$N\mathcal{HOM}(?, ?)_s \Rightarrow \mathbf{Hom}_{\Delta^{\text{op}}\mathbf{Set}}(N(?), N(?)).$$

(Compare the argument in [She82] p.212.) We will construct this ‘lax version’ in **Example 4.10** (1).

- (2) It is well-know that the geometric realization functor $|?| : \Delta^{\text{op}}\mathbf{Set} \rightarrow \mathbf{Top}$ preserves products. (See [Mil57].) So by **Lemma 4.3**, we have the canonical natural transformation

$$|\mathbf{Hom}_{\Delta^{\text{op}}\mathbf{Set}}(?, ?)| \Rightarrow \mathbf{Hom}_{\mathbf{Top}}(|?|, |?|).$$

(Compare the argument in Ibid.) We will construct this ‘bisimplicial version’ in **Example 4.6**.

The following **Lemma 4.5** can be easily verified.

Lemma 4.5.

Let I be a category, \mathcal{V} a cartesian closed category and $f : I \rightarrow \mathcal{V}$ a functor. For each object V in I , we consider it as a constant functor $V : I \rightarrow \mathcal{V}$. Then the following functor

$$\mathbf{Hom}(V, f) : I \ni i \mapsto \mathbf{Hom}_{\mathcal{V}}(V, f(i)) \in \mathcal{V}$$

is an internal hom object in $\mathcal{HOM}(I, \mathcal{V})_{\mathbf{s}}$.

Example 4.6.

Let X be a simplicial set and Y a bisimplicial set. Then we have a simplicial simplicial set $[n] \mapsto \mathbf{Hom}_{\Delta^{\text{op}}\mathbf{Set}}(X, Y_{n\bullet})$. We define a continuous map

$$|\text{diag } \mathbf{Hom}_{\Delta^{\text{op}}\mathbf{Set}}(X, Y)| \rightarrow \mathbf{Hom}_{\mathbf{Top}}(|X|, |Y|)$$

natural in each variable, as follows.

First notice that we have two simplicial topological spaces $[n] \mapsto |\mathbf{Hom}_{\Delta^{\text{op}}\mathbf{Set}}(X, Y_{n\bullet})|$ and $[n] \mapsto \mathbf{Hom}_{\mathbf{Top}}(|X|, |Y_{n\bullet}|)$. By **Example 4.4** (2), we have the natural transformation between them. Therefore we also have that between their geometric realizations

$$|[n] \mapsto |\mathbf{Hom}_{\Delta^{\text{op}}\mathbf{Set}}(X, Y_{n\bullet})| \rightarrow |[n] \mapsto \mathbf{Hom}_{\mathbf{Top}}(|X|, |Y_{n\bullet}|)|.$$

Notice that by **Lemma 4.5**, $[n] \mapsto \mathbf{Hom}_{\mathbf{Top}}(|X|, |Y_{n\bullet}|)$ is an internal hom object in $\Delta^{\text{op}}\mathbf{Top}$ and it is well-known that $|\cdot| : \Delta^{\text{op}}\mathbf{Top} \rightarrow \mathbf{Top}$ also preserves products. Hence we have the following map

$$|[n] \mapsto \mathbf{Hom}_{\mathbf{Top}}(|X|, |Y_{n\bullet}|)| \rightarrow \mathbf{Hom}_{\mathbf{Top}}(|X|, |[n] \mapsto |Y_{n\bullet}|)$$

by **Lemma 4.3**. Composing the two maps above, we get the desired map. Notice that if Y is a constant simplicial simplicial set, then the construction above is coincided with that in **Example 4.4** (2).

Theorem 4.7 (Lax exponential law).

Let I, J be small categories and \mathcal{X} a category with equivalence relations. Then we have an ‘almost’ isomorphism as categories with equivalence relations

$$\mathcal{LAX}(I, \mathcal{LAX}(J, \mathcal{X})) \xrightarrow{\sim} \mathcal{LAX}(I \times J, \mathcal{X})$$

which is functorial in each variable.

Proof. The phrase ‘almost’ in the statement means the following sense. There are two morphisms of categories of equivalence relations

$$A : \mathcal{LAX}(I, \mathcal{LAX}(J, \mathcal{X})) \rightarrow \mathcal{LAX}(I \times J, \mathcal{X}),$$

$$B : \mathcal{LAX}(I \times J, \mathcal{X}) \rightarrow \mathcal{LAX}(I, \mathcal{LAX}(J, \mathcal{X}))$$

such that $BA = \text{id}$ and there is a deformation $\Theta : AB \Rightarrow \text{id}$ relative to $\text{Ob } \mathcal{LAX}(I \times J, \mathcal{X})$.

The definition of A :

For any lax functors $f, g : I \rightarrow \mathcal{LAX}(J, \mathcal{X})$, any deformation $x : f \Rightarrow g$ and any morphism $(a, b) : (i_1, j_1) \rightarrow (i_2, j_2)$ in $I \times J$, we put

$$\begin{cases} A(f)(i_1, j_1) := f(i_1)(j_1) \\ A(f)(a, b) := f(a)(j_2)f(i_1)(b) \\ A(x)(i_1, j_1) := x(i_1)(j_1). \end{cases}$$

(In the definition above, we may replace $A(f)(a, b) = f(a)(j_2)f(i_1)(b)$ with $A(f)(a, b) := f(i_2)(b)f(a)(j_1)$.)

The definition of B :

$f, g : I \times J \rightarrow \mathcal{X}$, any deformation $x : f \Rightarrow g$ and any morphism $a : i_1 \rightarrow i_2$ in I and $j_1 \rightarrow j_2$ in J , we put

$$\begin{cases} B(f)(i_1)(j_1) := f(i_1, j_1) \\ B(f)(a)(j_1) := f(a, j_1) \\ B(f)(i_1)(b) := f(\text{id}_i, b) \\ B(x)(i_1, j_1) := x(i_1, j_1). \end{cases}$$

The definition Θ is obvious. Then we can easily verify the assertions above. \square

Corollary 4.8.

As in the notation **Theorem 4.7**, ‘almost’ isomorphisms

$$\mathcal{LAX}(I, \mathcal{LAX}(J, \mathcal{X})) \xrightarrow{\sim} \mathcal{LAX}(I \times J, \mathcal{X}) \xrightarrow{\sim} \mathcal{LAX}(J, \mathcal{LAX}(I, \mathcal{X}))$$

induce an actual isomorphism

$$\mathcal{HOM}(I, \mathcal{LAX}(J, \mathcal{X})_s)_s \xrightarrow{\sim} \mathcal{LAX}(J, \mathcal{HOM}(I, \mathcal{X})_s)_s$$

natural in each variable.

Proof. The proof is straightforward. Notice that in the proof of **Theorem 4.7**, we have two choice of defining desired isomorphisms. But these isomorphisms induce same isomorphism on $\mathcal{HOM}(I, \mathcal{LAX}(J, \mathcal{X})_s)_s$. \square

Corollary 4.9.

Let G be a group and \mathcal{X} a category with fibrations and equivalence relations satisfying the cogluing axiom. Then by **Proposition 3.8**, $\mathcal{LAX}(\underline{G}, \mathcal{X})_s$ is a category with fibrations and equivalence relations. We have a canonical isomorphism of simplicial categories

$$iS_{\bullet}^{\text{op}} \mathcal{LAX}(\underline{G}, \mathcal{X})_s \xrightarrow{\sim} \mathcal{LAX}(\underline{G}, iS_{\bullet}^{\text{op}} \mathcal{X})_s$$

natural in each variable.

Proof. For each non-negative integer n , the isomorphism

$$\mathcal{HOM}((\text{Ar}[n])^{\text{op}}, \mathcal{LAX}(\underline{G}, \mathcal{X})_s)_s \xrightarrow{\sim} \mathcal{LAX}(\underline{G}, \mathcal{HOM}((\text{Ar}[n])^{\text{op}}, \mathcal{X})_s)_s$$

induces an isomorphism

$$iS_n^{\text{op}} \mathcal{LAX}(\underline{G}, \mathcal{X})_s \xrightarrow{\sim} \mathcal{LAX}(\underline{G}, iS_n^{\text{op}} \mathcal{X})_s$$

which is functorial in n and each variable and this is the desired isomorphism. \square

Example 4.10.

(1) Let I be a small category and \mathcal{X} a small category with equivalence relations. We have a simplicial map

$$\mathbb{N} \mathcal{LAX}(I, \mathcal{X})_s \rightarrow \mathbf{Hom}_{\Delta^{\text{op}} \mathbf{Set}}(\mathbb{N} I, \Delta \mathcal{X})$$

natural in each variable, as follows.

For each non-negative integer n , by **Theorem 4.7**, we have the canonical morphism

$$\begin{aligned} \mathbb{N}_n \mathcal{LAX}(I, \mathcal{X})_s &= \text{Ob } \mathcal{HOM}([n], \mathcal{LAX}(I, \mathcal{X})_s) = \text{Hom}_{\mathbf{Cat}}([n], \mathcal{LAX}(I, \mathcal{X})_s) \\ &\rightarrow \text{Ob } \mathcal{LAX}([n] \times I, \mathcal{X}) = \text{Lax}([n] \times I, \mathcal{X}) \end{aligned}$$

and by **Remark 2.8** (2), we have the canonical morphism

$$\text{Lax}([n] \times I, \mathcal{X}) \rightarrow \text{Hom}_{\Delta^{\text{op}}\mathbf{Set}}(\Delta[n] \times NI, \Delta \mathcal{X}).$$

Composing these morphisms, we get the desired morphism. Notice that if \mathcal{X} is a trivial category with equivalence relations, then the construction above is coincided with that in **Example 4.4** (1).

(2) Let G be a group and \mathcal{X} a small category with fibrations and equivalence relations satisfying the cogluing axiom. Then composing the maps in (1) and **Corollary 4.9**, we get the following morphism

$$\mathbb{N}iS_{\bullet}^{\text{op}} \mathcal{LAX}(\underline{G}, \mathcal{X})_s \xrightarrow{\sim} \mathbb{N} \mathcal{LAX}(\underline{G}, iS_{\bullet}^{\text{op}} \mathcal{X})_s \rightarrow \mathbf{Hom}_{\Delta^{\text{op}}\mathbf{Set}}(\mathbb{N} \underline{G}, \Delta iS_{\bullet}^{\text{op}} \mathcal{X})$$

natural in each variable.

Conclusion 4.11.

Let G be a group and \mathcal{X} be a small category with fibrations and equivalence relations satisfying the cogluing axiom. Then composing morphisms in **Example 4.6** and **Example 4.10**

$$\begin{aligned} \mathbb{R}^{\text{lax},e}(G, \mathcal{X}) &= \Omega BiS_{\bullet}^{\text{op},e} \mathcal{LAX}(\underline{G}, \mathcal{X})_s \rightarrow \Omega | \mathbf{Hom}_{\Delta^{\text{op}}\mathbf{Set}}(\mathbb{N} \underline{G}, \Delta iS_{\bullet}^{\text{op},e} \mathcal{X}) | \\ &\rightarrow \Omega \mathbf{Hom}_{\mathbf{Top}}(BG, |\Delta iS_{\bullet}^{\text{op},e} \mathcal{X}|), \end{aligned}$$

we get the morphism

$$\mathbb{R}^{\text{lax},e}(G, \mathcal{X}) \rightarrow \Omega \mathbf{Hom}_{\mathbf{Top}}(BG, |\Delta iS_{\bullet}^{\text{op},e} \mathcal{X}|)$$

natural in each variable.

Before constructing the *lax Sherman map*, we shall review fundamental facts about topological spaces. So we will do in the next subsection.

4.2 The construction

In this section, we will define the *lax Sherman map*. To do so, we will start to review fundamental facts about topologies.

Notations 4.12.

(1) The set of free homotopy classes of free maps from X to Y will be denoted $[X, Y]$ ($= \pi_0(\mathbf{Hom}_{\mathbf{Top}}(X, Y))$), while $[X, Y]_*$ ($= \pi_0(\mathbf{Hom}_{\mathbf{Top}_*}(X, Y))$) will denote the set of pointed homotopy classes of pointed maps.

(2) For a pointed space X , X_0 will denote the path-component containing the basepoint.

Review 4.13 (Well-known result for function spaces).

(1) If Y is a connected simple space, then the canonical map $[X, Y]_* \rightarrow [X, Y]$ is a bijection. In particular, in this case, if two pointed maps $f_1, f_2 : X \rightarrow Y$ are freely homotopic, then they are homotopic by a pointed homotopy.

(2) If Y is a connected pointed space, then we have a canonical bijection $\mathbf{Hom}_{\mathbf{Top}_*}(Y, X_0) \rightarrow \mathbf{Hom}_{\mathbf{Top}_*}(Y, X)$.

Definition 4.14 (H -spaces).

(1) An H -spaces will be a pointed space X equipped with a pointed addition map $\mu : X \times X \rightarrow X$

which, up to homotopy, is associative and has the base point as unit element.

(2) An H -homomorphism of H -spaces will be pointed map preserving the addition up to (pointed) homotopy.

(3) An H -isomorphism will be an H -homomorphism which is a homotopy equivalence.

(4) A homotopy inverse for an H -space X will be a pointed map $X \rightarrow X$ which is an inverse for the addition, up to pointed homotopy.

Review 4.15 (Well-known results for H -spaces).

Let X be an H -space.

(1) If X has a homotopy inverse, then $\pi_0(X)$ is a group. Conversely, if X is a CW-complex and $\pi_0(X)$ is a group, then X has a homotopy inverse.

(2) The addition restricts to an addition on X_0 . If X has a homotopy inverse, then it restricts to a homotopy inverse for X_0 .

Moreover let Y be an arbitrary pointed space.

(3) Pointwise addition of functions defines H -space structures on both $\mathbf{Hom}_{\mathbf{Top}}(Y, X)$ and $\mathbf{Hom}_{\mathbf{Top}_*}(Y, X)$. Moreover, if X has a homotopy inverse, then so does each of these function spaces. In this case, $[Y, X]$ and $[Y, X]_*$ are groups.

Review 4.16 (2-coskeleton map).

Let X be a connected pointed CW-complex and Y be an H -spaces.

(1) There is a canonical pointed homotopy class of maps $X \rightarrow B\pi_1(X)$ (the 2-coskeleton); more precisely, this is the class corresponding to the identity map of $\pi_1(X)$ under the isomorphism

$$[X, B\pi_1(X)]_* \xrightarrow{\sim} \text{Hom}(\pi_1(X), \pi_1(X)).$$

(2) Composition with this define a homomorphism

$$[B\pi_1(X), Y] \rightarrow [X, Y].$$

Definition 4.17 (Lax Sherman map).

(1) In the notation **Conclusion 4.11**, taking homotopy group we get a homomorphism, for any non negative integer n :

$$\begin{aligned} R_n^{\text{lax}}(G, \mathcal{X}) &= \pi_n(\mathbb{R}^{\text{lax}, e}(G, \mathcal{X})) \\ &\rightarrow \pi_n(\Omega \mathbf{Hom}_{\mathbf{Top}}(BG, |\Delta i S_{\bullet}^{\text{op}, e} \mathcal{X}|)) \\ &\xrightarrow{\sim} \pi_0(\Omega^{n+1} \mathbf{Hom}_{\mathbf{Top}}(BG, |\Delta i S_{\bullet}^{\text{op}, e} \mathcal{X}|)) \\ &\xrightarrow{\sim} \pi_0(\mathbf{Hom}_{\mathbf{Top}}(BG, \Omega^n \mathbb{K}^{\text{lax}, e}(\mathcal{X}))) \\ &= [BG, \Omega^n \mathbb{K}^{\text{lax}, e}(\mathcal{X})]. \end{aligned}$$

(Here we are using the fact that the canonical homomorphism

$$\Omega^{n+1} \mathbf{Hom}_{\mathbf{Top}}(BG, |\Delta i S_{\bullet}^{\text{op}, e} \mathcal{X}|) \xrightarrow{\sim} \mathbf{Hom}_{\mathbf{Top}}(BG, \Omega^{n+1} |\Delta i S_{\bullet}^{\text{op}, e} \mathcal{X}|)$$

is an H -homomorphism, hence induces an isomorphism on π_0 .)

This map is clearly natural in each variable.

(3) For any pointed connected CW-complex X , putting this together with the 2-coskeleton map above, we get a homomorphism

$$\text{Sh}^{\text{lax}}(X, \mathcal{X})_n : R_n^{\text{lax}}(\pi_1(X), \mathcal{X}) \rightarrow [X, \Omega^n \mathbb{K}^{\text{lax}, e}(\mathcal{X})],$$

representing a natural transformation of bifunctors

$$\mathcal{H}(\mathbf{CW}_*) \times \mathbf{Fib}_{\text{er}} \rightarrow \mathbf{Grp}$$

where $\mathcal{H}(\mathbf{CW}_*)$ is the pointed homotopy category of pointed CW-complexes and \mathbf{Grp} is the category of groups.

Variant 4.18 (Reduced cases).

Let pt denote the trivial space.

(1) Since $B\pi_1(X)$ is connected, the kernel of the homomorphism

$$[B\pi_1(X), \Omega^n \mathbb{K}^{\text{lax},e} \mathcal{X}] \rightarrow [\text{pt}, \Omega^n \mathbb{K}^{\text{lax},e} \mathcal{X}] \xrightarrow{\sim} \pi_0(\Omega^n \mathbb{K}^{\text{lax},e} \mathcal{X})$$

is clearly $[B\pi_1(X), (\Omega^n \mathbb{K}^{\text{lax},e} \mathcal{X})_0]$.

(3) Furthermore, since $(\Omega^n \mathbb{K}^{\text{lax},e} \mathcal{X})_0$ is an H -space (hence simple) and connected, the canonical map

$$[B\pi_1(X), (\Omega^n \mathbb{K}^{\text{lax},e} \mathcal{X})_0]_* \rightarrow [B\pi_1(X), (\Omega^n \mathbb{K}^{\text{lax},e} \mathcal{X})_0]$$

is an isomorphism.

(4) It follows that there is an induced map

$$\tilde{R}_n^{\text{lax}}(\pi_1(X), \mathcal{X}) \rightarrow [B\pi_1(X), (\Omega^n \mathbb{K}^{\text{lax},e} \mathcal{X})_0]_*,$$

and thus also a map

$$\tilde{\text{Sh}}^{\text{lax}}(X, \mathcal{X})_n : \tilde{R}_n^{\text{lax}}(\pi_1(X), \mathcal{X}) \rightarrow [X, (\Omega^n \mathbb{K}^{\text{lax},e} \mathcal{X})_0]_*.$$

Remark 4.19.

In the notation above, if equivalence relations in \mathcal{X} are trivial, we get maps

$$\text{Sh}(X, \mathcal{X})_n : R_n(G, \mathcal{X}) \rightarrow [BG, \Omega^n \mathbb{K}^e(\mathcal{X})],$$

$$\tilde{\text{Sh}}(X, \mathcal{X})_n : \tilde{R}_n(G, \mathcal{X}) \rightarrow [BG, (\Omega^n \mathbb{K}^e(\mathcal{X}))_0]_*.$$

4.3 Compatibility with the original one and the universal property

In this subsection, we will prove that the lax Sherman map is compatible with the original one defined in [She82] in the suitable case. First we will review the dual notion of Q -construction in [Qui73].

Notations 4.20 (Q^{op} -construction).

(1) First we shall quickly review the definition of *exact categories*. An exact category is a triple of $(\mathcal{E}, \text{fib}(\mathcal{E}), \text{cof}(\mathcal{E}))$ consisting of an additive category \mathcal{E} with subcategories $\text{fib}(\mathcal{E}), \text{cof}(\mathcal{E}) \hookrightarrow \mathcal{E}$ whose morphisms will be called *admissible epimorphisms* and *admissible monomorphisms* respectively satisfying the following axioms:

(a) Pairs $(\mathcal{E}, \text{fib}(\mathcal{E}))$ and $(\mathcal{E}^{\text{op}}, (\text{cof}(\mathcal{E}))^{\text{op}})$ are categories with fibrations.

(b) A sequence in \mathcal{E}

$$X \xrightarrow{i} Y \xrightarrow{p} Z$$

is a fibration sequence in $(\mathcal{E}, \text{fib}(\mathcal{E}))$ if and only if it is in $(\mathcal{E}^{\text{op}}, (\text{cof}(\mathcal{E}))^{\text{op}})$.

(Compare [Kel90] p.405 A.1, this definition is equivalent to Quillen's one [Qui73]. See also [TT90] p.253 1.2.2). We will always omit $\text{fib}(\mathcal{E})$ and $\text{cof}(\mathcal{E})$ in the notation.

(2) For each small exact category \mathcal{E} , we form a new category $Q^{\text{op}} \mathcal{E}$, the dual notion in [Qui73]. That is, $Q^{\text{op}} \mathcal{E}$ has the same objects as \mathcal{E} and morphisms are defined in the following way. A morphism from X to Y in the category $Q^{\text{op}} \mathcal{E}$ is by definition an isomorphism class of the following diagrams

$$X \xrightarrow{i} W \xleftarrow{p} Y$$

where i is an admissible monomorphism and p is an admissible epimorphism. Here we consider isomorphisms of these diagrams which induces the identity on X and Y . Given a morphism from Y to Z represented by the diagram

$$Y \xrightarrow{j} V \xleftarrow{q} Z$$

the composition of this morphism with the morphism from X to Y above is the morphism represented by the pair $\iota_1 i, \iota_2 q$ in the diagram:

$$\begin{array}{ccccc}
 W & \coprod_Y & V & \xleftarrow{\iota_2} & V & \xleftarrow{q} & Z \\
 & & \uparrow \iota_1 & & \uparrow j & & \\
 & & W & \xleftarrow{p} & Y & & \\
 & & \uparrow i & & & & \\
 & & X & & & &
 \end{array}$$

It is clear that composition is well-defined and associative.

(3) In the notation above, we also form a new simplicial category $iQ_{\bullet}^{\text{op}} \mathcal{E}$, the dual notion in [Wal85] p.375. $iQ_{\bullet}^{\text{op}} \mathcal{E}$ is a sub simplicial category of $[n] \mapsto \mathcal{HOM}([n], Q^{\text{op}} \mathcal{E})_s$ whose morphisms are all isomorphisms.

We will review the original definition of the Sherman map for exact categories.

Example 4.21 (The original Sherman map).

We will review the original construction of the Sherman maps. Let fix an small exact category \mathcal{E} .

(1) For any group G , as a variant of **Corollary 4.9**, we have the following canonical identities

$$\begin{aligned}
 iQ_{\bullet}^{\text{op}} \mathcal{HOM}(\underline{G}, \mathcal{E})_s &\xrightarrow{\sim} \mathcal{HOM}(\underline{G}, iQ_{\bullet}^{\text{op}} \mathcal{E})_s, \\
 Q^{\text{op}} \mathcal{HOM}(\underline{G}, \mathcal{E})_s &\xrightarrow{\sim} \mathcal{HOM}(\underline{G}, Q^{\text{op}} \mathcal{E})_s
 \end{aligned}$$

natural in each variable. (Here we use the fact that the automorphisms in $Q^{\text{op}} \mathcal{E}$ may be identified with the automorphisms in \mathcal{E} .)

(2) Taking the classifying space functor and composing the natural transformation

$$B\mathcal{HOM}(?, ?) \Rightarrow \mathbf{Hom}_{\mathbf{Top}}(B(?), B(?)),$$

we get the maps

$$\begin{aligned}
 BiQ_{\bullet}^{\text{op}} \mathcal{HOM}(\underline{G}, \mathcal{E})_s &\rightarrow \mathbf{Hom}_{\mathbf{Top}}(BG, BiQ_{\bullet}^{\text{op}} \mathcal{E}) \\
 BQ^{\text{op}} \mathcal{HOM}(\underline{G}, \mathcal{E})_s &\rightarrow \mathbf{Hom}_{\mathbf{Top}}(BG, BQ^{\text{op}} \mathcal{E})
 \end{aligned}$$

natural in each variable.

(3) Take the n -th homotopy group and the same argument in **Definition 4.17** and **Variante 4.18**, for any connected pointed CW-complexes X , we get the following maps which shall be called the n -th original Sherman map (resp. n -th reduced original Sherman map)

$$\begin{aligned}
 \text{Sh}^{\text{ori}}(X, \mathcal{E})_n &: R_n(\pi_1(X), \mathcal{E}) \rightarrow [B\pi_1(X), \Omega^{n+1} BQ^{\text{op}} \mathcal{E}] \\
 \tilde{\text{Sh}}^{\text{ori}}(X, \mathcal{E})_n &: \tilde{R}_n(\pi_1(X), \mathcal{E}) \rightarrow [B\pi_1(X), (\Omega^{n+1} BQ^{\text{op}} \mathcal{E})_0]_*
 \end{aligned}$$

natural in each variables. Here to prove that there is a functorial isomorphism

$$R_n(\pi_1(X), \mathcal{E}) \xrightarrow{\sim} \pi_n(\Omega BQ^{\text{op}} \mathcal{HOM}(\underline{G}, \mathcal{E})_s),$$

we are using the $iS_{\bullet}^{\text{op}} = Q^{\text{op}}$ theorem below. (c.f. **Review 4.22**)

To compare the two Sherman maps, we will review the proof of $iS_{\bullet}^{\text{op}} = Q^{\text{op}}$ in [Wal85]. (See also [Gil81].)

Notations 4.24.

- (1) We will let \mathcal{F} denote the category of finite pointed connected CW-complexes.
(2) Consider the functors F and G from \mathcal{F} to the category of commutative monoids, a natural transformation $t : F \rightarrow G$ is said to be *universal* if for any connected H -space Z and any natural transformation $F \rightarrow [-, Z]_*$, the following diagram can be uniquely completed:

$$\begin{array}{ccc} F & \xrightarrow{t} & G \\ & \searrow & \downarrow \\ & & [-, Z]_* \end{array}$$

Theorem 4.25 ([She92] p.173 Corollary 5.2.).

If \mathcal{E} is a semi-simple exact category, that is, all short exact sequences in \mathcal{E} split, then the natural transformation

$$\tilde{\text{Sh}}^{\text{ori}}(? , \mathcal{E})_0 : \tilde{R}_0(\pi_1(?), \mathcal{E}) \rightarrow [?, (\Omega BQ^{\text{op}} \mathcal{E})_0]_*$$

is universal. By **Proposition 4.23**, we learn that $\tilde{\text{Sh}}(? , \mathcal{E})_0$ is also universal.

Historical Note 4.26.

The original version of the statement above is first announced in [Ger73], and used to prove Gersten's conjecture for the case of discrete valuation rings whose residue fields are finite, and proved by Quillen in [Hil81]. The variant statement is appeared in [Lev97] and used to define the Adams operations on algebraic K -theory for (divisorial) schemes.

5 Proof of the main theorem

In this section, we will prove the main theorem by using the results in previous sections. The argument is divided the following a pair of lemmas.

Lemma 5.1.

To prove main theorem, we shall only check the following assertion:

For any group G , the canonical map induced from the inclusion map $\mathcal{M}(k) \hookrightarrow \mathcal{A}$,

$$R_0(G, \mathcal{M}(k)) \rightarrow R_0^{\text{lux}}(G, \mathcal{A})$$

is zero. Here \mathcal{A} is defined in **Example 2.2** (5).

Proof. First by the dévissage theorem [Qui73] p.112 Theorem 4, the inclusion functor $\mathcal{M}(k) \hookrightarrow \mathcal{M}^1(R)$ induces isomorphisms on higher algebraic K -groups. So we may prove the inclusion functor $\mathcal{M}(k) \hookrightarrow \mathcal{M}(R)$ induces the zero maps on higher algebraic K -groups. We have the following commutative diagram for each $X \in \mathcal{F}$:

$$\begin{array}{ccccc} \tilde{R}_0(\pi_1(X), \mathcal{M}(k)) & \longrightarrow & \tilde{R}_0(\pi_1(X), \mathcal{M}(R)) & \longleftarrow & \tilde{R}_0(\pi_1(X), \mathcal{P}(R)) \\ \downarrow \tilde{\text{sh}} & & \downarrow \tilde{\text{sh}} & & \downarrow \tilde{\text{sh}} \\ [X, (\mathbb{K}(\mathcal{M}(k)))_0]_* & \longrightarrow & [X, (\mathbb{K}(\mathcal{M}(R)))_0]_* & \xleftarrow{\text{I}} & [X, (\mathbb{K}(\mathcal{P}(R)))_0]_* \end{array}$$

where the morphism I is an isomorphism by the resolution theorem [Qui73] p.108 Theorem 3. It is well-known that $\mathbb{K}(\mathcal{P}(R))$ is a H -space (See for example [Gra76]), $\mathcal{M}(k)$ is semi-simple and by the universal property **Theorem 4.25**, we learn that we shall only prove the composition

$$\tilde{R}_0(\pi_1(X), \mathcal{M}(k)) \rightarrow \tilde{R}_0(\pi_1(X), \mathcal{M}(R)) \xrightarrow{\tilde{\text{sh}}} [X, (\mathbb{K}(\mathcal{M}(R)))_0]_*$$

is the zero map for any $X \in \mathcal{F}$. (The argument above is due to [She82] p.240 and originally [Ger73].) Now remember that \mathcal{A} is a category with fibrations and equivalence relations satisfying the cogluing axiom by **Example 2.2** (5), **Example 3.2** (3), and **Example 3.7** (2). Notice the inclusion functor factor through $\mathcal{M}(k) \hookrightarrow \mathcal{A} \hookrightarrow \mathcal{M}(R)$ and we have the following commutative diagram for each $X \in \mathcal{F}$:

$$\begin{array}{ccccc}
& & \tilde{R}_0(\pi_1(X), \mathcal{M}(R)) & \xrightarrow{\tilde{\text{Sh}}} & [X, (\mathbb{K}^e(\mathcal{M}(R)))_0]_* \\
& & \uparrow & & \uparrow \\
\tilde{R}_0(\pi_1(X), \mathcal{M}(k)) & \longrightarrow & \tilde{R}_0(\pi_1(X), \mathcal{A}) & \xrightarrow{\tilde{\text{Sh}}} & [X, (\mathbb{K}^e(\mathcal{A}))_0]_* \\
& & \downarrow & & \downarrow \Pi \\
& & \tilde{R}_0^{\text{Lax}}(\pi_1(X), \mathcal{A}) & \xrightarrow{\tilde{\text{Sh}}^{\text{Lax}}} & [X, (\mathbb{K}^{\text{Lax}, e}(\mathcal{A}))_0]_*
\end{array}$$

where the morphism Π is an injection by **Theorem 3.13**. Hence we get the result. \square

Lemma 5.2.

The assertion in **Lemma 5.1** is true.

Proof. Let G be a group and (X, ρ_X) be a representation in $\mathcal{HOM}(\underline{G}, \mathcal{M}(k))_s$. Assume rank of X as a k -vector space is m . Then there is a short exact sequence

$$0 \rightarrow R^{\oplus m} \xrightarrow{\pi} R^{\oplus m} \xrightarrow{p} X \rightarrow 0.$$

For each $g \in G$, we have a lifting of $\rho_X(g)$, that is, a R -module homomorphism $\tilde{\rho}(g) : R^{\oplus m} \rightarrow R^{\oplus m}$ such that $\tilde{\rho}(g) \bmod \pi = \rho_X(g)$. By Nakayama's lemma, we learn $\tilde{\rho}(g)$ is an isomorphism of R -modules. Obviously assignment $\tilde{\rho} : G \rightarrow \text{Aut}(R^{\oplus m})$ defines a lax representation $(R^{\oplus m}, \tilde{\rho})$ in $\mathcal{LAX}(\underline{G}, \mathcal{A})_s$ and we have a short exact sequence

$$(R^{\oplus m}, \tilde{\rho}) \xrightarrow{\pi} (R^{\oplus m}, \tilde{\rho}) \xrightarrow{p} (X, \rho_X) \tag{3}$$

in $\mathcal{LAX}(\underline{G}, \mathcal{A})_s$. Notice that proving π is a strict deformation, we need the assumption that R is commutative!! (Compare **Example 0.5**.) So we have an identity

$$[(X, \rho_X)] = [(R^{\oplus m}, \tilde{\rho})] - [(R^{\oplus m}, \tilde{\rho})] = 0$$

in $R_0^{\text{Lax}}(G, \mathcal{A})$. Hence we get the result. \square

A The extended algebraic K -spaces functor

For simplicial category with fibrations, we have an obvious association of its algebraic K -spaces. Namely the composition

$$\Delta^{\text{op}} \mathbf{Fib} \xrightarrow{iS^{\text{op}}} \Delta^{\text{op}} \Delta^{\text{op}} \mathbf{Cat} \xrightarrow{\text{diag}} \Delta^{\text{op}} \mathbf{Cat} \xrightarrow{B} \Delta^{\text{op}} \mathbf{Top} \xrightarrow{[?]} \mathbf{Top}.$$

The purpose of this appendix is extending to this functoriality. That is, we will define $\widetilde{\Delta^{\text{op}} \mathbf{Fib}}$ the (2-)category of simplicial categories with fibrations whose 1-morphisms are strict lons (See **Definition A.2** (3), (4)) and the *extended algebraic K -spaces functor*

$$\mathbb{K}^{\text{ext}} : \widetilde{\Delta^{\text{op}} \mathbf{Fib}} \rightarrow \mathbf{Top}.$$

We will also prove the compatibility of \mathbb{K} and \mathbb{K}^{ext} . (See **Proposition A.6**.)

A.1 The languages in 2-category theory

Notice that there are many choice of notations in higher category theory. The purpose of this subsection is to keep the notations in 2-category theory and to define the extended algebraic K -spaces functor. So we will first start the following definition.

Caution A.1.

We will neglect the set theoretic problem.

Definition A.2 (2-category, 2-functor, lonts, modifications).

(1) A *2-category* \mathcal{X} is a category enriched in **Cat**. That is, \mathcal{X} is a category in which the hom-set between any two objects $x, y \in \mathcal{X}$ is the set of objects of a category $\mathcal{X}(x, y)$. The objects of $\mathcal{X}(x, y)$ are called *1-morphism*, and are denoted $u : x \rightarrow y$, and its morphisms are called *2-morphism* and are denoted $f : u \Rightarrow v$. Moreover, the composition is a bifunctor $\mathcal{X}(x, y) \times \mathcal{X}(y, z) \rightarrow \mathcal{X}(x, z)$ which is associative and has identities $\text{id}_x \in \mathcal{X}(x, x)$.

(2) A *2-functor* $f : \mathcal{X} \rightarrow \mathcal{Y}$ between 2-categories is an enriched functor and so it takes objects, 1-morphisms and 2-morphisms in \mathcal{X} to objects, 1-morphisms and 2-morphisms in \mathcal{Y} , respectively. In such a way that all the 2-category structure preserved. We can define the composition of 2-functors in the obvious way. So let **2-Cat** be the category of small 2-categories and 2-functors.

(3) ([Tho77] p.25 II 1.1.2) Let I be a small category and \mathcal{X} be a 2-category, then a *lont* between functor $x, y : I \rightarrow \mathcal{X}$ consist of functions assigning:

- (a) to each object i of I , a functor $f_i : x_i \rightarrow y_i$,
- (b) to each morphism $a : i \rightarrow j$ in I , a 2-morphism

$$f_a : y_a f_j \Rightarrow f_i x_a \quad ,$$

these must satisfy the conditions that

Lont 1 for $i \xrightarrow{a} j \xrightarrow{b} k$,

$$\begin{array}{ccc} \begin{array}{ccc} x_k & \xrightarrow{f_k} & y_k \\ x_b \downarrow & \Downarrow f_b & \downarrow y_b \\ x_j & \xrightarrow{f_j} & y_j \\ x_a \downarrow & \Downarrow f_a & \downarrow y_a \\ x_i & \xrightarrow{f_i} & y_i \end{array} & = & \begin{array}{ccc} x_k & \xrightarrow{f_k} & y_k \\ x_{ba} \downarrow & \Downarrow f_{ba} & \downarrow y_{ba} \\ x_i & \xrightarrow{f_i} & y_i \end{array} \end{array}$$

Lont 2 and that for i in I ,

$$f_{\text{id}_i} = \text{id}_{f_i} .$$

(4) In the notation above, a lont $f : x \rightarrow y$ is *strict* if for each $a : i \rightarrow j$, f_a is an isomorphism.

(5) ([Tho77] p.26 II.1.3) Let x, y be functors $I \rightarrow \mathcal{X}$ and $f, g : x \rightarrow y$ lonts. A *modification* $\theta : f \rightarrow g$ consists of a function assigned each object i in I , a 2-morphism

$$\begin{array}{ccc} & \xrightarrow{f_i} & \\ x_i & \Downarrow \theta_i & y_i \\ & \xleftarrow{g_i} & \end{array}$$

We require that for each morphism $u : i \rightarrow j$ in I satisfies the following diagram

$$\begin{array}{ccc} \begin{array}{ccc} x_j & \xrightarrow{f_j} & y_j \\ x_u \downarrow & \Downarrow \theta_j & \downarrow y_u \\ x_i & \xrightarrow{f_i} & y_i \end{array} & = & \begin{array}{ccc} x_j & \xrightarrow{f_j} & y_j \\ x_u \downarrow & \Downarrow f_u & \downarrow y_u \\ x_i & \xrightarrow{f_i} & y_i \\ & \Downarrow \theta_i & \\ & \xleftarrow{g_i} & \end{array} \end{array}$$

(6) Let x, y and z be functors $I \rightarrow \mathcal{X}$, $f : x \rightarrow y$, $g : y \rightarrow z$ lonts. We define a lont $gf : x \rightarrow z$ as follows.

For each i , $(gf)_i : x_i \rightarrow z_i$ is composition of 1-morphisms.

For each $a : i \rightarrow j$, 2-morphism $(gf)_a : z(a)g_i f_i \Rightarrow g_j f_j x(a)$ is defined as $(gf)_a := g_j(f_a) \circ g_a(f_i)$.

(7) Let I be a small category and \mathcal{X} a 2-category, then we can define the 2-category $\mathcal{OP}\mathcal{L}\mathcal{A}\mathcal{X}(I, \mathcal{X})$ as follows.

Object: functors $I \rightarrow \mathcal{X}$

1-morphism: lonts

2-morphism: modifications.

The composition of modifications are term-wised compositions. $\mathcal{OP}\mathcal{L}\mathcal{A}\mathcal{X}(?, ?)$ is functorial in each variable, that is, there is the functor

$$\mathcal{OP}\mathcal{L}\mathcal{A}\mathcal{X} : \mathbf{Cat}^{\text{op}} \times \mathbf{2-Cat} \rightarrow \mathbf{2-Cat}.$$

Example A.3 (Various examples).

(1) \mathbf{Cat} , \mathbf{Fib} are considered as 2-categories. Here 2-morphisms in \mathbf{Cat} are natural transformations and in \mathbf{Fib} are natural equivalences.

(2) We put $\widetilde{\Delta^{\text{op}}\mathbf{Cat}} := \mathcal{OP}\mathcal{L}\mathcal{A}\mathcal{X}(\Delta^{\text{op}}, \mathbf{Cat})$ and $\widetilde{\Delta^{\text{op}}\mathbf{Fib}} := \mathcal{OP}\mathcal{L}\mathcal{A}\mathcal{X}(\Delta^{\text{op}}, \mathbf{Fib})$. $\Delta^{\text{op}}\mathbf{Cat}$ are considered as sub 2-category of $\widetilde{\Delta^{\text{op}}\mathbf{Cat}}$.

(3) $iS_{\bullet}^{\text{op}} : \mathbf{Fib} \rightarrow \Delta^{\text{op}}\mathbf{Cat}$ is a 2-functor. Let $x : f \Rightarrow g$ be a natural equivalence between exact functors $f, g : \mathcal{X} \rightarrow \mathcal{Y}$. Then $iS_{\bullet}^{\text{op}}x : iS_{\bullet}^{\text{op}}f \Rightarrow iS_{\bullet}^{\text{op}}g$ will be defined as follows. For any non-negative integer n , and the n -th exact functor $X \in iS_n^{\text{op}}\mathcal{X}$ and $(i/j) \in \text{Ar}[n]$, we put

$$iS_n^{\text{op}}x(X)(i/j) := x(X(i/j)) : f(X(i/j)) \xrightarrow{\sim} g(X(i/j)).$$

We can easily check this assignment is 2-functorial.

Review A.4 (Grothendieck construction).

(c.f [Tho77] p.31 II 2.1.1., [Tho79] p.92 Definition 1.1.)

(1) Let $\mathcal{X} : I \rightarrow \mathbf{Cat}$ a functor. The *Grothendieck construction* on \mathcal{X} , $\int_I \mathcal{X}$, is the category with objects the pair (i, x) with i an object of I and x an object of $\mathcal{X}(i)$, and with morphisms $(u, a) : (i_1, x_1) \rightarrow (i_2, x_2)$ given by a morphism $u : i_1 \rightarrow i_2$ in I and an $a : \mathcal{X}(u)(x_1) \rightarrow x_2$ in $\mathcal{X}(i_2)$. Composition is defined by

$$(u, a) \circ (v, b) := (u \circ v, a \circ \mathcal{X}(u)(b)).$$

(2) Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a lont between functors $I \rightarrow \mathbf{Cat}$. Then we define the functor

$$\int_I f : \int_I \mathcal{X} \rightarrow \int_I \mathcal{Y},$$

$$((i, x) \xrightarrow{(u, a)} (j, y)) \mapsto (i, f(i)(x)) \xrightarrow{(u, f(j)(a))} (j, f(j)(y)).$$

(3) Let $x : f \rightarrow g$ be a modification between lonts $f, g : \mathcal{X} \rightarrow \mathcal{Y}$, where $\mathcal{X}, \mathcal{Y} : I \rightarrow \mathbf{Cat}$ are functors. Then we can define a natural transformation $\int_I x : \int_I f \rightarrow \int_I g$ putting for every (i, X) in $\int_I \mathcal{X}$,

$$\int_I x(i, X) := (\text{id}, x(i)) : (i, f(i)(x)) \rightarrow (i, g(i)(x)).$$

(4) The associations above makes 2-functors

$$\int_I : \mathcal{OP}\mathcal{L}\mathcal{A}\mathcal{X}(I, \mathbf{Cat}) \rightarrow \mathbf{Cat}.$$

Definition A.5 (The extended algebraic K -spaces functor \mathbb{K}^{ext}).

We define the functor \mathbb{K}^{ext} as the following composition

$$\begin{array}{ccc} \widetilde{\Delta^{\text{op}}\mathbf{Fib}} = \mathcal{OPLAX}(\Delta^{\text{op}}, \mathbf{Fib}) & \xrightarrow{\mathcal{OPLAX}(\Delta^{\text{op}}, iS^{\text{op}})} & \mathcal{OPLAX}(\Delta^{\text{op}}, \Delta^{\text{op}}\mathbf{Cat}) \\ & \xrightarrow{\mathcal{OPLAX}(\Delta^{\text{op}}, f_{\Delta^{\text{op}}})} & \mathcal{OPLAX}(\Delta^{\text{op}}, \mathbf{Cat}) \\ & \xrightarrow{f_{\Delta^{\text{op}}}} & \mathbf{Cat} \\ & \xrightarrow{B} & \mathbf{Top}. \end{array}$$

The rest of appendix, we will devote to prove the compatibility of \mathbb{K} and \mathbb{K}^{ext} , that is, to prove the following **Proposition A.6**. It will be used in the proof of **Theorem 3.13**.

Proposition A.6 (Compatibility).

The following diagram is commutative up to homotopy

$$\begin{array}{ccc} \Delta^{\text{op}}\mathbf{Fib} & & \\ \downarrow \iota & \searrow \mathbb{K} & \\ \widetilde{\Delta^{\text{op}}\mathbf{Fib}} & \xrightarrow{\mathbb{K}^{\text{ext}}} & \mathbf{Top}. \end{array}$$

where the vertical arrow is the canonical inclusion, that is, between $\mathbb{K}^{\text{ext}} \circ \iota$ and \mathbb{K} , there are zigzag natural transformations which are homotopy equivalent for every sections.

To do so, in the next subsection, we will review the Thomason's thesis [Tho77], especially VI §4.

A.2 Review of [Tho77]

The purpose of this subsection is proving **Proposition A.6**. First we will review the concept of Bousfield and Kan's homotopy colimits. (But our notation is slightly different from the original one.)

Definition A.7 (Homotopy colimit).

(1) Let $X : I \rightarrow \Delta^{\text{op}}\mathbf{Set}$ be a functor. The *homotopy colimit* of X , $\text{hocolim}_I X$, is the simplicial set defined as follows.

First we consider X as a functor $\Delta^{\text{op}} \rightarrow \mathcal{HOM}(I, \mathbf{Set})$ and define the composition

$$\int_I X : \Delta^{\text{op}} \xrightarrow{X} \mathcal{HOM}(I, \mathbf{Set}) \xrightarrow{f_I} \mathbf{Cat}.$$

We put $\text{hocolim}_I X := \text{diag N } \int_I X$. Therefore n -simplex of $\text{hocolim}_I X$ is a pair of functors $(i : [n] \rightarrow I, x : [n] \rightarrow X(i(0)))$.

(2) For each morphism $u : X \rightarrow Y$ between the functor $I \rightarrow \Delta^{\text{op}}\mathbf{Set}$, we define a natural transformation $\int_I u : \int_I X \rightarrow \int_I Y$ between simplicial categories by the formula $(\int_I u)_n := \int_I u_n$ for every non-negative integer n . Since this association is functorial, we get the functor

$$\text{hocolim} : \mathcal{HOM}(I, \Delta^{\text{op}}\mathbf{Set}) \rightarrow \Delta^{\text{op}}\mathbf{Set}.$$

Next we explain the main theorem in [Tho79] which mentions relation between the Grothendieck constructions and homotopy colimits.

Example A.8 (Thomason map).

(1) Let $\mathcal{X} : I \rightarrow \mathbf{Cat}$ be a functor. Then we will define the *Thomason map* (c.f. [Tho77] p.33-35 II 2.2.2., [Tho79] p.92 Lemma 1.2.1.)

$$\mathrm{Th}(\mathcal{X}) : \mathrm{hocolim}_I \mathbf{N} \mathcal{X} \rightarrow \mathbf{N} \int_I \mathcal{X}$$

as follows. First notice that for each non-negative integer n , n -simplex of $\mathrm{hocolim}_I \mathbf{N} \mathcal{X}$ is a pair of functors $(i : [n] \rightarrow I, x : [n] \rightarrow \mathcal{X}(i(0)))$. We define a functor

$$\int_i x : [n] \rightarrow \int_I \mathbf{N}_n \mathcal{X}$$

$$(s \leq t) \mapsto ((i(s), \mathcal{X}(i(0 \leq s))) \circ x) \xrightarrow{(i(s \leq t), \mathcal{X}(i(0 \leq t))) \circ x(s \leq t)} (i(t), \mathcal{X}(i(0 \leq t))) \circ x).$$

We put $\mathrm{Th}(\mathcal{X})_n(i, x) := \int_i x$. Next notice for each ordered map $[m] \xrightarrow{\phi} [n]$, we have the following formula

$$\left(\int_i x \right) \circ \phi = \int_{i \circ \phi} (\mathcal{X}(i(0 \leq \phi(0))) \circ x \circ \phi).$$

This means $\mathrm{Th}(\mathcal{X})$ is a simplicial map. It is well known that this map is weak homotopy equivalence. (c.f. [Tho77] p.65-66 IV 4.1., [Tho79] p.92 Theorem 1.2.).

(2) $\mathrm{Th} : \mathrm{hocolim}_I \mathbf{N} \rightarrow \int_I \mathbf{N}$ is a natural transformation, that is, for every natural transformation $u : \mathcal{X} \rightarrow \mathcal{Y}$ between functors $I \rightarrow \mathbf{Cat}$, the following diagram is commutative.

$$\begin{array}{ccc} \mathrm{hocolim}_I \mathbf{N} \mathcal{X} & \xrightarrow{\mathrm{hocolim}_I \mathbf{N} f} & \mathrm{hocolim}_I \mathbf{N} \mathcal{Y} \\ \mathrm{Th}(\mathcal{X}) \downarrow & & \downarrow \mathrm{Th}(\mathcal{Y}) \\ \mathbf{N} \int_I \mathcal{X} & \xrightarrow{\mathbf{N} \int_I f} & \mathbf{N} \int_I \mathcal{Y}. \end{array}$$

Next we will define the homotopy geometric realization functor and compare it with the usual geometric realization functor. This attempt is essentially done in [Tho77] (Especially, VI §4.)

Definition A.9 (Homotopy geometric realization).

The *homotopy geometric realization* functor HGR is a composition of the following morphisms.

$$\mathrm{HGR} : \widetilde{\Delta^{\mathrm{op}} \mathbf{Cat}} \xrightarrow{f_{\Delta^{\mathrm{op}}}} \mathbf{Cat} \xrightarrow{B} \mathbf{Top}$$

where B is the usual classifying space functor.

From now on, we will explain the reason why the functor above is called homotopy geometric realization functor.

Example A.10 (Last vertex map).

(1) Let X be a simplicial set. We define the *last vertex map*

$$\mathrm{LV}(X) : \mathbf{N} \int_{\Delta^{\mathrm{op}}} X \rightarrow X$$

as follows. (c.f. [BK72] p.293 Ch. XI §2 Example 2.6 (ii), [Wal85] p.359) First notice that for every non-negative integer n , n -simplex of $N \int_{\Delta^{\text{op}}} X$ is a pair of a functor $[n] \xrightarrow{u} \Delta$ and $x \in X_{u(n)}$. We define an ordered map

$$\text{LV}(u) : [n] \ni s \mapsto u(s \leq n)(u(s)) \in [u(n)]$$

and put $\text{LV}(X)_n(u, x) := X(\text{LV}(u))(x)$. Next notice that for each ordered map $[m] \xrightarrow{\phi} [n]$,

$$u(\phi(n) \leq n) \circ \text{LV}(u \circ \phi) = \text{LV}(u) \circ \phi.$$

This implies $\text{LV}(X)$ is a simplicial map. This map is a homotopy equivalence. (c.f. [BK72] p.293 Ch. XI §2 Example 2.6 (iii), [Wal85] p.359)

(2) $\text{LV} : N \int_{\Delta^{\text{op}}} \rightarrow \text{id}_{\Delta^{\text{op}}\mathbf{Set}}$ is a natural transformation. That is, for every simplicial map $u : X \rightarrow Y$, the following diagram is commutative.

$$\begin{array}{ccc} N \int_{\Delta^{\text{op}}} X & \xrightarrow{\text{LV}(X)} & X \\ N \int_{\Delta^{\text{op}}} u \downarrow & & \downarrow u \\ N \int_{\Delta^{\text{op}}} Y & \xrightarrow{\text{LV}(Y)} & Y. \end{array}$$

(3) Every bisimplicial set can be consider as a simplicial simplicial set $\Delta^{\text{op}} \rightarrow \Delta^{\text{op}}\mathbf{Set}$ and therefore we have a natural transformation

$$\text{diag}(\text{LV}) : \text{hocolim}_{\Delta^{\text{op}}} \rightarrow \text{diag}$$

between functors from $(\Delta^{\text{op}})^2\mathbf{Set}$ to $\Delta^{\text{op}}\mathbf{Set}$. We will abbreviate $\text{diag}(\text{LV})$ as LV .

Corollary A.11.

(c.f. [BK72] p.331 Ch. XII §3 3.4) *For each bisimplicial set X ,*

$$\text{LV}(X) : \text{hocolim}_{\Delta^{\text{op}}} X \rightarrow \text{diag } X$$

is a homotopy equivalence.

Proof. This is an easy consequence of **Example A.10** (1) and the **Realization lemma 3.14**. \square

The following corollary is explained the reason why the naming of HGR.

Corollary A.12.

There are natural transformations

$$\text{HGR}(?) \xleftarrow{B(\text{Th})} |(\text{hocolim}_{\Delta^{\text{op}}} N(?))| \xrightarrow{|\text{LV}(N(?))|} B(?).$$

For every simplicial category \mathcal{X} , $B(\text{Th}(\mathcal{X}))$ and $|\text{LV}(N \mathcal{X})|$ are homotopy equivalences.

Proof of Proposition A.6. We have the following (non-commutative) diagrams:

$$\begin{array}{ccccccc} \Delta^{\text{op}} \mathbf{Fib} & \xrightarrow{\mathcal{HOM}(\Delta^{\text{op}}, iS^{\text{op}})} & \Delta^{\text{op}} \Delta^{\text{op}} \mathbf{Cat} & \xrightarrow{\text{diag}} & \Delta^{\text{op}} \mathbf{Cat} & \xrightarrow{B} & \mathbf{Top} \\ \downarrow \iota & & \downarrow & & \downarrow & & \parallel \\ \widetilde{\Delta^{\text{op}} \mathbf{Fib}} & \xrightarrow{\mathcal{OPLA}\mathcal{X}(\Delta^{\text{op}}, iS^{\text{op}})} & \mathcal{OPLA}\mathcal{X}(\Delta^{\text{op}}, \Delta^{\text{op}} \mathbf{Cat}) & \xrightarrow{\mathcal{OPLA}\mathcal{X}(\Delta^{\text{op}}, f_{\Delta^{\text{op}}})} & \widetilde{\Delta^{\text{op}} \mathbf{Cat}} & \xrightarrow{\text{HGR}} & \mathbf{Top} \end{array}$$

where the vertical morphisms are the natural inclusions and the compositions of the top line is the functor \mathbb{K} and the bottom one is the functor \mathbb{K}^{ext} and the diagram I is commutative and diagram III is commutative up to homotopy, in the sense that between HGR and B , there are zig-zag natural transformations whose every sections are homotopy equivalences by **Corollary A.12**. For diagram II, notice that there is the following commutative diagram

$$\begin{array}{ccc}
 \Delta^{\text{op}} \Delta^{\text{op}} \mathbf{Cat} & \xrightarrow{\mathcal{HOM}(\Delta^{\text{op}}, f_{\Delta^{\text{op}}})} & \Delta^{\text{op}} \mathbf{Cat} \\
 \downarrow & \text{II'} & \downarrow \\
 \mathcal{OPLEX}(\Delta^{\text{op}}, \Delta^{\text{op}} \mathbf{Cat}) & \xrightarrow{\mathcal{OPLEX}(\Delta^{\text{op}}, f_{\Delta^{\text{op}}})} & \widetilde{\Delta^{\text{op}} \mathbf{Cat}} .
 \end{array}$$

So we shall only prove the existence of zig-zag natural homotopy equivalences between B diag and $B\mathcal{HOM}(\Delta^{\text{op}}, f_{\Delta^{\text{op}}}) = \mathcal{HOM}(\Delta^{\text{op}}, \text{HGR}(?))$. It is well-know that B diag is canonically equivalent to $\text{Hom}(\Delta^{\text{op}}, B)$. Now we have the following zig-zag natural transformations

$$\mathcal{HOM}(\Delta^{\text{op}}, \text{HGR}(?)) \xrightarrow{\mathcal{HOM}(\Delta^{\text{op}}, B(\text{Th}(?)))} \mathcal{HOM}(\Delta^{\text{op}}, |(\text{hocolim}_{\Delta^{\text{op}}} N(?))|) \xrightarrow{\mathcal{HOM}(\Delta^{\text{op}}, |LV(N(?))|)} \mathcal{HOM}(\Delta^{\text{op}}, B(?)).$$

They are homotopy equivalent for each sections by **Corollary A.12** and the **Realization lemma 3.14**. \square

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