CHOW-WITT GROUPS AND GROTHENDIECK-WITT GROUPS OF REGULAR SCHEMES

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Abstract. We show that the Chow-Witt groups can be defined using only the notion of Grothendieck-Witt groups of triangulated categories. As an application, we show that there is always a homomorphism $\widetilde{CH}^n(X) \to GW^n(X)$ if X is a regular scheme of dimension n. In dimension 3, we use this fact to prove that the vanishing of its Euler class is sufficient for a projective module of rank 3 with trivial determinant to have a free factor of rank one.

1. Introduction

Let A be a commutative noetherian ring of Krull dimension d and P a projective A-module of rank r. Serre proved a long time ago that one can find a projective module Q such that $P \simeq Q \oplus A$ provided $r > d$. When $r = d$, this result is wrong in general. Let X be a regular scheme over $\mathbb{Z}[\frac{1}{2}]$ of dimension d. To deal with the above question, Barge and Morel introduced the Chow-Witt groups $\widetilde{CH}^j(X)$ of X (see [BM]), called *oriented Chow groups* at that time, and associated to each vector bundle E of rank d an Euler class $\tilde{c}_d(E)$ in $\widetilde{CH}^d(X)$. Morel proved recently that if $X = \mathrm{Spec}(A)$, where A is a smooth k-algebra, we have $\tilde{c}_d(P) = 0$ if and only if $P \simeq Q \oplus A$ given $d = 2$ or $d \geq 4$ (see [Mo1], or [Fa1] for the case $d = 2$). In this paper, we prove the following theorem (Theorem 5.7 in the text):

Theorem. *Let A be a regular $\mathbb{Z}[\frac{1}{2}]$-algebra of dimension 3. Let P be a projective A-module of rank 3. Then $\tilde{c}_3(P) = 0$ if and only if $P \simeq Q \oplus A$.*

The proof is very simple and requires only basic facts on Witt and Grothendieck-Witt groups. Note also that the algebra A is not necessarily defined over a field. Moreover, our method provides a very simple proof of the above theorem in dimension 2 whereas the proof of [Fa1] was a little bit long. It is not obvious that the same method will give results in higher dimension. The ideas of the paper are the following:

In a first time, we observe that the Chow-Witt groups can be defined using only Grothendieck-Witt groups of triangulated categories (see [Wa] for more information). This description could be seen as an analogue of the fact that usual Chow groups can be defined using only K_0, at least when the scheme is nice. Indeed, Let X be a regular scheme and let $D(X)$ be the triangulated category of bounded complexes of locally free $\mathcal{O}_X$-modules over X. Then this category has a filtration by subcategories D^i, where the latter denotes the subcategory of objects whose support is of codimension $\geq i$. It is straightforward to see that $CH^i(X)$ has a simple description in terms of K_0 of some categories D^j or some quotients of them. This

The first author is supported by Swiss National Science Foundation, grant PP002-112579.

fact is recalled in section 2. Now Grothendieck-Witt groups can be seen as the K_0 of some category and Chow-Witt groups are philosophically just Chow groups with additional quadratic information. This observation led us to the following theorem (Theorem 3.3):

Theorem. *Let X be a regular scheme over $\mathbb{Z}[\frac{1}{2}]$. For any i, consider the following sequence of Grothendieck-Witt groups*

$$GW^i(D^i/D^{i+2}) \xrightarrow{\alpha} GW^i(D^i/D^{i+1}) \xrightarrow{\beta} GW^i(D^{i-1}/D^{i+1})$$

obtained from the usual duality preserving functors $D^i/D^{i+2} \to D^i/D^{i+1}$ and $D^i/D^{i+1} \to D^{i-1}/D^{i+1}$. Then $\ker(\beta) \subset \mathrm{Im}(\alpha)$ and $\widetilde{CH}^i(X) \simeq \mathrm{Im}(\alpha)/\ker(\beta)$.

The proof of this theorem uses higher Grothendieck-Witt (also called Hermitian K-theory by some authors) groups and a localization sequence associated to any open subscheme U of X ([Sch2, Section 5.1]). The identification of the groups appearing in that long exact sequence then naturally gives the result. Observe that the same arguments shows that Chow-Witt groups can be defined as the homology of some Gersten complex in higher Grothendieck-Witt groups. This also reinforces the links between Chow-Witt groups and Grothendieck-Witt groups, as already discovered in [Fa2, Theorem 7.6] and remarked by Hornbostel ([Ho]). As far as we understand, it seems that Chow-Witt groups and higher Grothendieck-Witt groups satisfy more or less the same relations as Chow groups and K-theory do.

A straightforward consequence of the theorem is the identification $\widetilde{CH}^n(X) \simeq GW^n(D^{n-1})$ if X is of dimension n. Thus there is always a homomorphism $\widetilde{CH}^n(X) \to GW^n(X)$ and this fact is of crucial importance for the proof in dimension 3. Remark that the same homomorphism is obtained directly from the spectral sequence in higher Grothendieck-Witt groups.

This article is organized as follows: In section 2, we briefly recall the fact that Chow groups can be defined using only K_0. Section 3 contains the definition of Chow-Witt groups and some very basic results. We also state the fact that Chow-Witt groups can be defined using only Grothendieck-Witt groups. The next section contains the proof of this statement. It must be signaled that we use results in higher Grothendieck-Witt groups that are pretty recent. Our main source is a paper of Schlichting ([Sch2]) that is still in preparation. Due to this fact, we didn't use the Gersten complex and the spectral sequence in higher Grothendieck-Witt groups as much as we should have. A complete treatment will probably take place in Schlichting's paper. Section 5 is devoted to the study of Chow-Witt group on regular schemes of dimension at most 3. We prove there the main theorem of this paper, namely the proof that, over a regular algebra where 2 is invertible, the vanishing of its Euler class is sufficient for a projective module of rank 3 with trivial determinant to have a free factor of rank one. The fact that our technique does not immediately generalize to higher dimensions is a priori not due to some degeneracy in low dimensions of the spectral sequence whose first page is the Gersten complex in higher Grothendieck-Witt groups.

In this paper, we make frequent use of Witt groups and Grothendieck-Witt groups of triangulated categories. We don't want to recall the definition of these groups. The reader is referred to [Bal] or to [Wa] for more information. The source for higher Grothendieck-Witt groups is [Sch2] and the source for Chow-Witt groups is [Fa1]. By convention, the word *scheme* will be used for a separated noetherian scheme.

The first author wishes to thank Marco Schlichting for many useful conversations on Grothendieck-Witt theory. He also thanks Paul Balmer and Ivo Dell'Ambrogio for their comments on earlier versions of this work. Many thanks are also due to the Tata Insitute of Fundamental Research, where part of this work was done.

2. Chow groups and K-theory

Let X be a regular scheme of dimension n and let $D(X)$ be the derived category of bounded complexes of locally free $\mathcal{O}_X$-modules (of finite rank). Denote by $D^i(X)$ (or sometimes simply by D^i when there is no possible confusion) the full subcategory of objects whose support is of codimension $\geq i$. We then have a filtration

$$0 = D^{n+1}(X) \subset D^n(X) \subset \ldots \subset D^1(X) \subset D^0(X) = D(X).$$

From this filtration and localization sequences in K-theory, we obtain the Brown-Gersten-Quillen spectral sequence ([Qu] or [Sr]), whose E_1 page contains the following row (the i-th Gersten complex of X)

$$\bigoplus_{x_0 \in X^{(0)}} K_i(k(x_0)) \to \bigoplus_{x_1 \in X^{(1)}} K_{i-1}(k(x_1)) \to \ldots \to \bigoplus_{x_i \in X^{(i)}} K_0(k(x_i)) \to 0.$$

The last differential on the right was shown by Quillen to coincide with the divisor map ([Qu, Lemma 5.16, Remark 5.17]), so that the homology of this complex at the last term is $CH^i(X)$.

The construction of the spectral sequence also leads to a presentation of $CH^i(X)$ using only K_0. Indeed, it is not difficult to see that

$$K_0(D^i/D^{i+1}) \simeq \bigoplus_{x_i \in X^{(i)}} K_0(k(x_i))$$

which is the group of codimension i cycles. The use of a localization sequence and the fact that $K_0(D^{i-1}/D^i)$ is a free abelian group give

$$K_0(D^{i-1}/D^{i+1}) \simeq CH^i(X) \oplus K_0(D^{i-1}/D^i)$$

and the same arguments show the homomorphism

$$\beta : K_0(D^i/D^{i+1}) \to K_0(D^{i-1}/D^{i+1})$$

induced by the inclusion $D^i/D^{i+1} \to D^{i-1}/D^{i+1}$ sends a cycle to its class in the Chow group $CH^i(X)$. In other terms, we get the following presentation of $CH^i(X)$:

$$CH^i(X) = \ker(K_0(D^{i-1}/D^{i+1}) \to K_0(D^{i-1}/D^i)).$$

The natural functor $D^i/D^{i+2} \to D^i/D^{i+1}$ induces a surjective homomorphism

$$\alpha : K_0(D^i/D^{i+2}) \to K_0(D^i/D^{i+1})$$

and we get the following composition:

$$K_0(D^i/D^{i+2}) \xrightarrow{\alpha} K_0(D^i/D^{i+1}) \xrightarrow{\beta} K_0(D^{i-1}/D^{i+1}).$$

The above discussion shows that $\ker(\beta) \subset \mathrm{Im}(\alpha)$ and that

$$CH^i(X) \simeq \mathrm{Im}(\alpha)/\ker(\beta).$$

Of course, this presentation of $CH^i(X)$ is a little bit artificial, but leads to a version for Chow-Witt groups as well. This is the object of the next section.

3. Chow-Witt groups and Grothendieck-Witt groups

Let X be a d dimensional regular scheme where 2 is invertible. As in the previous section, we denote by $D(X)$ the derived category of bounded complexes of locally free $\mathcal{O}_X$-modules and by $D^i(X)$ (or simply D^i) the full subcategory of objects whose support is of codimension $\geq i$. The filtration

$$0 = D^{d+1}(X) \subset D^d(X) \subset \ldots \subset D^1(X) \subset D^0(X) = D(X)$$

yields a complex

$$0 \longrightarrow W^0(D^0/D^1) \longrightarrow \cdots \longrightarrow W^{d-1}(D^{d-1}/D^d) \longrightarrow W^d(D^d) \longrightarrow 0$$

called the *Gersten-Witt complex* of X ([BW]). By dévissage, one can prove that ([BW, Theorem 6.1])

$$W^i(D^i/D^{i+1}) \simeq \bigoplus_{x_i \in X^{(i)}} W^{fl}(\mathcal{O}_{X,x_i})$$

where $W^{fl}(\mathcal{O}_{X,x_i})$ denotes the Witt group of finite length modules of $\mathcal{O}_{X,x_i}$, which is non canonically isomorphic to $W(k(x_i))$. Now there is a "rank mod 2" homomorphism

$$\overline{f} : W^i(D^i/D^{i+1}) \to K_0(D^i/D^{i+1}) \otimes \mathbb{Z}/2$$

defined by sending a symmetric complex to its class in $K_0(D^i/D^{i+1}) \otimes \mathbb{Z}/2$. Using [BW, Theorem 6.1] again, we see that this homomorphism is surjective.

Definition 3.1. For any i, denote by $I_i(D^i/D^{i+1})$ the kernel of $\overline{f}$.

Observe that $I_i(D^i/D^{i+1})$ is non canonically isomorphic to $\bigoplus_{x_i \in X^{(i)}} I(k(x_i))$ because $W^i(D^i/D^{i+1})$ is non canonically isomorphic to $\bigoplus_{x_i \in X^{(i)}} W(k(x_i))$. Now for any i, we have a complex

$$I_{i-1}(D^{i-1}/D^i) \xrightarrow{d_I} W^i(D^i/D^{i+1}) \longrightarrow \cdots \longrightarrow W^d(D^d) \longrightarrow 0.$$

There is a "discriminant homomorphism"

$$\overline{h} : I_{i-1}(D^{i-1}/D^i) \to \bigoplus_{x_{i-1} \in X^{(i-1)}} K_1(k(x_{i-1})) \otimes \mathbb{Z}/2$$

such that the following diagram commutes ([Fa1, Theorem 10.2.6]):

$$\begin{array}{ccc} I_{i-1}(D^{i-1}/D^i) & \longrightarrow & W^i(D^i/D^{i+1}) \\ \downarrow \bar{h} & & \downarrow \bar{f} \\ \bigoplus_{x_{i-1}\in X^{(i-1)}} K_1(k(x_{i-1}))\otimes \mathbb{Z}/2 & \xrightarrow{div} & \bigoplus_{x_i\in X^{(i)}} K_0(k(x_i))\otimes \mathbb{Z}/2. \end{array}$$

Hence we can consider the fiber product of the complexes

$$I_{i-1}(D^{i-1}/D^i) \xrightarrow{d_I} W^i(D^i/D^{i+1}) \longrightarrow \cdots \longrightarrow W^d(D^d) \longrightarrow 0$$

and

$$\bigoplus_{x_{i-1}\in X^{(i-1)}} K_1(k(x_{i-1})) \xrightarrow{div} \bigoplus_{x_i\in X^{(i)}} K_0(k(x_i)) \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0$$

over

$$\bigoplus_{x_{i-1}\in X^{(i-1)}} K_1(k(x_{i-1}))\otimes \mathbb{Z}/2 \xrightarrow{div} \bigoplus_{x_i\in X^{(i)}} K_0(k(x_i))\otimes \mathbb{Z}/2 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0$$

to get a complex

$$J \longrightarrow G \longrightarrow W^{i+1}(D^{i+1}/D^{i+2}) \longrightarrow \cdots \longrightarrow W^d(D^d) \longrightarrow 0.$$

Observe that G, being the fiber product of $W^i(D^i/D^{i+1})$ and $K_0(D^i/D^{i+1})$ over $K_0(D^i/D^{i+1})\otimes \mathbb{Z}/2$, is isomorphic to $GW^i(D^i/D^{i+1})$. Indeed, we have a commutative diagram with exact rows ([Wa, Proposition 2.2])

$$\begin{array}{ccccccccc} & & K_0(D^i/D^{i+1}) & \xrightarrow{H} & GW^i(D^i/D^{i+1}) & \longrightarrow & W^i(D^i/D^{i+1}) & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \downarrow \bar{f} & & \\ 0 & \longrightarrow & K_0(D^i/D^{i+1}) & \xrightarrow[\cdot 2]{} & K_0(D^i/D^{i+1}) & \longrightarrow & K_0(D^i/D^{i+1})\otimes \mathbb{Z}/2 & \longrightarrow & 0, \end{array}$$

where H is the hyperbolic map as defined in [Wa, Section 1]. The diagram shows that H is injective and that $GW^i(D^i/D^{i+1})$ is the fiber product.

Definition 3.2. The *Chow-Witt group of codimension i cycles* is defined as the homology of the complex

$$J \longrightarrow GW^i(D^i/D^{i+1}) \longrightarrow W^{i+1}(D^{i+1}/D^{i+2}) \longrightarrow \cdots \longrightarrow W^d(D^d) \longrightarrow 0$$

at $GW^i(D^i/D^{i+1})$. It is denoted by $\widetilde{CH}^i(X)$.

A natural question arises from this definition: Can we define the Chow-Witt group $\widetilde{CH}^i(X)$ by using only the notion of Grothendieck-Witt group? This question is clearly the analogue of the question considered in the previous section. We are going to prove the following theorem:

Theorem 3.3. *Let X be a regular scheme over $\mathbb{Z}[\frac{1}{2}]$. For any i, consider the following sequence of Grothendieck-Witt groups*

$$GW^i(D^i/D^{i+2}) \xrightarrow{\alpha} GW^i(D^i/D^{i+1}) \xrightarrow{\beta} GW^i(D^{i-1}/D^{i+1})$$

obtained from the usual duality preserving functors $D^i/D^{i+2} \to D^i/D^{i+1}$ and $D^i/D^{i+1} \to D^{i-1}/D^{i+1}$. Then $\ker(\beta) \subset \mathrm{Im}(\alpha)$ and $\widetilde{CH}^i(X) \simeq \mathrm{Im}(\alpha)/\ker(\beta)$.

The next section is devoted to the proof of this theorem.

4. The proof of Theorem 3.3

First we identify the group J coming in the definition of the Chow-Witt groups. From the very definition, we have that J is the fibre product of $I_{i-1}(D^{i-1}/D^i)$ and $\bigoplus_{x_{i-1}\in X^{(i-1)}} K_1(k(x_{i-1}))$ over $\bigoplus_{x_{i-1}\in X^{(i-1)}} K_1(k(x_{i-1}))\otimes \mathbb{Z}/2$. Using dévissage, we have a canonical isomorphism (use [Fa1, Lemma 2.6, Proposition E.2.1])

$$I_{i-1}(D^{i-1}/D^i) \simeq \bigoplus_{x_{i-1}\in X^{(i-1)}} I(k(x_{i-1}), \wedge^{i-1}(\mathfrak{m}_{x_{i-1}}/\mathfrak{m}^2_{x_{i-1}})^\vee)$$

where $\mathfrak{m}_{x_{i-1}}$ is the maximal ideal in the local ring $\mathcal{O}_{X,x_{i-1}}$. So the fibre product J is in fact a direct sum of fibre products

$$\begin{array}{ccc} J(k(x_{i-1}), \wedge^{i-1}(\mathfrak{m}_{x_{i-1}}/\mathfrak{m}^2_{x_{i-1}})^\vee) & \longrightarrow & I(k(x_{i-1}), \wedge^{i-1}(\mathfrak{m}_{x_{i-1}}/\mathfrak{m}^2_{x_{i-1}})^\vee) \\ \downarrow & & \downarrow \\ K_1^M(k(x_{i-1})) & \longrightarrow & K_1^M(k(x_{i-1}))\otimes\mathbb{Z}/2. \end{array}$$

Consider the triangulated category D^{i-1}/D^i with the i-th shifted duality, i.e the duality $T^i\mathrm{Hom}_{\mathcal{O}_X}(\_,\mathcal{O}_X)$ where T is the translation functor. Then one can see this category as the homotopy category of the $\mathbb{Z}[\frac{1}{2}]$-complicial exact category with duality $(D^{i-1}, \omega, \otimes, [\_,\_], \eta, can)$, where ω is the family of morphisms whose cone is in D^i, η is the natural isomorphism between an object to its bidual associated to the i-th shifted duality and the other symbols reflect an action of the bounded chain complexes of free $\mathbb{Z}[\frac{1}{2}]$-modules on D^{i-1} ([Sch2, Section 4, Definition 4.7]). Thus we can define the *Grothendieck-Witt theory* of this category, whose first group is denoted by $GW^i_1(D^{i-1}/D^i)$ ([Sch2]). By dévissage again, we have an isomorphism (use [BW, Proposition 7.1] for example)

$$GW^i_1(D^{i-1}/D^i) \simeq \bigoplus_{x_{i-1}\in X^{(i-1)}} GW^i_1(D^b_{fl}(\mathcal{P}(\mathcal{O}_{X,x_{i-1}})))$$

where $\mathcal{P}(\mathcal{O}_{X,x_{i-1}})$ is the category of free $\mathcal{O}_{X,x_{i-1}}$-modules and $D^b_{fl}(\mathcal{P}(\mathcal{O}_{X,x_{i-1}}))$ is the homotopy category of the obvious complicial category. The exact category $\mathcal{O}_{X,x_{i-1}} - fl$ of finite length $\mathcal{O}_{X,x_{i-1}}$-modules is endowed with the duality $\mathrm{Ext}^{i-1}_{\mathcal{O}_{X,x_{i-1}}}(\_,\mathcal{O}_{X,x_{i-1}})$. The category of bounded complexes of finite length modules, with quasi-isomorphisms as weak equivalences and first shifted duality of the duality $\mathrm{Ext}^{i-1}_{\mathcal{O}_{X,x_{i-1}}}(\_,\mathcal{O}_{X,x_{i-1}})$ is again a complicial exact category. Its homotopy category is the triangulated category $D^b(\mathcal{O}_{X,x_{i-1}}-fl)$. Now there is a canonical equivalence of triangulated categories with duality ([BW, Lemma 6.4])

$$D^b_{fl}(\mathcal{P}(\mathcal{O}_{X,x_{i-1}}))\simeq D^b(\mathcal{O}_{X,x_{i-1}}-fl)$$

and such an equivalence induces an isomorphism in Grothendieck-Witt theory. Consider the full subcategory $\mathcal{O}_{X,x_{i-1}}-ss\subset\mathcal{O}_{X,x_{i-1}}-fl$ of semi-simple objects. Again, we can endow the category of bounded complexes of semi-simple objects with a structure of complicial exact category. Then we have a induced functor on triangulated categories (seen as homotopy categories)

$$D^b(\mathcal{O}_{X,x_{i-1}}-ss)\to D^b(\mathcal{O}_{X,x_{i-1}}-fl).$$

This functor induces an equivalence in K-theory and an isomorphism of Witt groups ([BW, Theorem 1.4, Proposition 5.2, Proposition 5.1]). Therefore it induces an isomorphism in Grothendieck-Witt theory by Karoubi induction ([Sch2, Lemma 7.11]). Now we have an equivalence of triangulated categories with duality

$$D^b(\mathcal{O}_{X,x_{i-1}}-ss)\simeq D^b(k(x_{i-1})),$$

where the latter denotes the triangulated category of bounded complexes of finite dimensional vector spaces, with the first shifted duality of the duality

$$V^\sharp=\mathrm{Hom}_{k(x_{i-1})}(V,\wedge^{i-1}(\mathfrak{m}_{x_{i-1}}/\mathfrak{m}^2_{x_{i-1}})^\vee)$$

and usual canonical isomorphism ϖ. In the sequel, we will denote the group $GW^1_1(D^b(k(x_{i-1})),\wedge^{i-1}(\mathfrak{m}_{x_{i-1}}/\mathfrak{m}^2_{x_{i-1}})^\vee)$ simply by $GW^1_1(k(x_{i-1}),^\sharp)$. Therefore we get

$$GW^i_1(D^{i-1}/D^i)\simeq\bigoplus_{x_{i-1}\in X^{(i-1)}}GW^1_1(k(x_{i-1}),^\sharp).$$

Let F be a field. Then $GW^1_1(F)$ has a very explicit presentation ([Sch1, Section 1.9]). Consider elements (V,q_1,q_2) where V is a finite dimensional vector space and q_i are quadratic forms defined on V. Two symbols (V,q_1,q_2) and (W,r_1,r_2) are isomorphic if there exists an isomorphism $f:V\to W$ which is an isometry between q_i and r_i $(i=1,2)$. The group $GW^1_1(F)$ is then generated by isomorphism classes of symbols $[V,q_1,q_2]$ under the relations $[V,q_0,q_1]+[V,q_1,q_2]+[V,q_2,q_0]=0$. It is easily checked that there is a homomorphism

$$g:J(F)\to GW^1_1(F)$$

defined by $g(u,<u,-1>)=[F,u,1]$ for any $u\in F^\times$.

Proposition 4.1. *The homomorphism $g:J(F)\to GW^1_1(F)$ is an isomorphism.*

Proof. Let $K^{MW}_1(F)$ be the first Milnor-Witt K-group of F ([Mo2, Definition 5.1]). Then there is an isomorphism $f:K^{MW}_1(F)\to J(F)$ defined by $f([u])=(u,<u,-1>)$ for any $u\in F^\times$ ([Mo2, Theorem 5.3]). Moreover, there is an isomorphism $h:K^{MW}_1(F)\to GW^1_1(F)$ defined by $h([u])=[F,u,1]$ for any $u\in F^\times$ ([Sch1, Theorem 1.21] where $GW^1_1(F)$ is denoted by $K^h_1(F,1)$). By definition, $g=hf^{-1}$. □

Proposition 4.2. *The isomorphism* $g : J(k(x_{i-1}) \to GW^1_1(k(x_{i-1}))$ *induces an isomorphism*

$$g : J(k(x_{i-1}), \wedge^{i-1}(\mathfrak{m}_{x_{i-1}}/\mathfrak{m}^2_{x_{i-1}})^\vee) \simeq GW^1_1(k(x_{i-1}),^\sharp).$$

Proof. Choose a generator ξ of $\wedge^{i-1}(\mathfrak{m}_{x_{i-1}}/\mathfrak{m}^2_{x_{i-1}})^\vee$. Such a choice induces isomorphisms

$$J(k(x_{i-1}), \wedge^{i-1}(\mathfrak{m}_{x_{i-1}}/\mathfrak{m}^2_{x_{i-1}})^\vee) \simeq J(k(x_{i-1}))$$

and between $GW^1_1(k(x_{i-1}),^\sharp)$ and $GW^1_1(k(x_{i-1}))$ with untwisted duality. The dependences of the two terms under the choice of a generator ξ are clearly the same. □

Corollary 4.3. *There is a canonical isomorphism* $g : J \simeq GW^i_1(D^{i-1}/D^i)$.

Consider the following exact sequence of triangulated categories with duality

$$D^i/D^{i+1} \longrightarrow D^{i-1}/D^{i+1} \longrightarrow D^{i-1}/D^i.$$

It induces a long exact sequence in Grothendieck-Witt theory ([Sch2, Theorem 7.15, Lemma B.2]):

$$\cdots \longrightarrow GW^i_1(D^{i-1}/D^i) \xrightarrow{\partial} GW^i(D^i/D^{i+1}) \longrightarrow GW^i(D^{i-1}/D^{i+1}) \longrightarrow \cdots$$

Recall that we have a homomorphism $d : J \to GW^i(D^i/D^{i+1})$ that we used to define the Chow-Witt groups. By construction d is the fiber product of the differentials

$$d_I : I_{i-1}(D^{i-1}/D^i) \to W^i(D^i/D^{i+1})$$

and

$$div : \bigoplus_{x_{i-1}\in X^{(i-1)}} K_1(k(x_{i-1})) \to \bigoplus_{x_i\in X^{(i)}} K_0(k(x_i))$$

over their respective reduction modulo 2. Define $\tilde{d}$ to be the fibre product of $-d_I$ and *div* over their respective reduction modulo 2. Then:

Proposition 4.4. *The following diagram commutes*

$$\begin{array}{ccc} GW^i_1(D^{i-1}/D^i) & \xrightarrow{\partial} & GW^i(D^i/D^{i+1}) \\ {\scriptstyle g}\uparrow & & \| \\ J & \xrightarrow[\tilde{d}]{} & GW^i(D^i/D^{i+1}). \end{array}$$

Proof. We will use here the notations of [Sch2]. For any exact category ε with weak equivalences and duality $(\varepsilon, w, *, \eta)$, we denote by $(\mathcal{M}(\varepsilon), w, *, \eta)$ the exact category of morphisms in ε with natural weak equivalences and duality ([Sch2, Section 5.1]). We have a commutative diagram where the rows and columns are fibration sequences ([Sch2, Theorem 7.6, Theorem 7.15])

$$\begin{array}{ccccc}
(D^i/D^{i+1}, \sharp(i-1), \eta^{i-1}) & \longrightarrow & (D^{i-1}/D^{i+1}, \sharp(i-1), \eta^{i-1}) & \longrightarrow & (D^{i-1}/D^i, \sharp(i-1), \eta^{i-1}) \\
\downarrow & & \downarrow & & \downarrow \\
(\mathcal{M}(D^i/D^{i+1}), \sharp(i-1), \eta^{i-1}) & \to & (\mathcal{M}(D^{i-1}/D^{i+1}), \sharp(i-1), \eta^{i-1}) & \to & (\mathcal{M}(D^{i-1}/D^i), \sharp(i-1), \eta^{i-1}) \\
\downarrow & & \downarrow & & \downarrow \\
(D^i/D^{i+1}, \sharp i, \eta^i) & \longrightarrow & (D^{i-1}/D^{i+1}, \sharp i, \eta^i) & \longrightarrow & (D^{i-1}/D^i, \sharp i, \eta^i)
\end{array}$$

Each row and column gives a long exact sequence in Grothendieck-Witt theory. Considering the lower right corner of the diagram and taking the first Grothendieck-Witt group of that category, we get an anti-commutative square (because of Verdier's exercise)

$$\begin{array}{ccc}
GW^i_1(D^{i-1}/D^i) & \xrightarrow{\partial} & GW^i(D^i/D^{i+1}) \\
{\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho'} \\
GW^{i-1}(D^{i-1}/D^i) & \xrightarrow[d]{} & W^i(D^i/D^{i+1})
\end{array}$$

where ρ and ρ' are the connecting homomorphisms appearing in the long exact sequences in Hermitian K-theory. Now ρ fits in the exact sequence (as a part of the long exact sequence)

$$GW^i_1(D^{i-1}/D^i) \xrightarrow{\rho} GW^{i-1}(D^{i-1}/D^i) \longrightarrow K_0(D^{i-1}/D^i) \longrightarrow 0.$$

But for any field F, we have an exact sequence

$$0 \longrightarrow I(F) \longrightarrow GW(F) \longrightarrow K_0(F) \longrightarrow 0,$$

thus we see that the image of ρ is precisely $I_{i-1}(D^{i-1}/D^i)$. Now d coincides with d_I on $I_{i-1}(D^{i-1}/D^i)$ by construction ([Sch2, Remark B.4]), thus we get a commutative diagram

$$\begin{array}{ccc}
GW^i_1(D^{i-1}/D^i) & \xrightarrow{\partial} & GW^i(D^i/D^{i+1}) \\
{\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho'} \\
I_{i-1}(D^{i-1}/D^i) & \xrightarrow[-d_I]{} & W^i(D^i/D^{i+1})
\end{array}$$

Similar arguments show that we also have a commutative diagram

$$\begin{array}{ccc}
GW^i_1(D^{i-1}/D^i) & \xrightarrow{\partial} & GW^i(D^i/D^{i+1}) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
K_1(D^{i-1}/D^i) & \xrightarrow[div]{} & K_0(D^i/D^{i+1})
\end{array}$$

where f and f' are the forgetful functors. The result then clearly follows from Corollary 4.3. $\square$

Now we prove Theorem 3.3. Consider the two filtrations

$$D^{i+1}/D^{i+2} \longrightarrow D^i/D^{i+2} \longrightarrow D^i/D^{i+1}$$

and

$$D^i/D^{i+1} \longrightarrow D^{i-1}/D^{i+1} \longrightarrow D^{i-1}/D^i.$$

Using the long exact sequence in Grothendieck-Witt theory for both filtrations, we get a diagram where the row and the column are exact

$$\begin{array}{ccccc} & & GW_1^i(D^{i-1}/D^i) & & \\ & & \downarrow \beta' & & \\ GW^i(D^i/D^{i+2}) & \xrightarrow[\alpha]{} & GW^i(D^i/D^{i+1}) & \xrightarrow[\alpha']{} & W^{i+1}(D^{i+1}/D^{i+2}) \\ & & \downarrow \beta & & \\ & & GW^i(D^{i-1}/D^{i+1}) & & \end{array}$$

From Proposition 4.4 and the definition of Chow-Witt groups (Definition 3.2), we get $\widetilde{CH}^i(X) = \ker(\alpha')/\mathrm{Im}(\beta')$. Since the sequences are exact, this yields $\widetilde{CH}^i(X) = \mathrm{Im}(\alpha)/\ker(\beta)$.

Remark 4.5. Of course, one can twist the duality by a line bundle L over X. The same result holds for $\widetilde{CH}^i(X, L)$ (where the duality appearing in the Grothendieck-Witt groups are also twisted by L).

There is a remarkable difference between the result in K-theory and the result in Grothendieck-Witt theory. In K-theory, the homomorphism

$$K_0(D^i/D^{i+1}) \to K_0(D^{i-1}/D^{i+1})$$

is in general not surjective, whereas

$$\beta : GW^i(D^i/D^{i+1}) \to GW^i(D^{i-1}/D^{i+1})$$

is surjective. This is the next lemma:

Lemma 4.6. *The homomorphism $\beta : GW^i(D^i/D^{i+1}) \to GW^i(D^{i-1}/D^{i+1})$ induced by localization is surjective.*

Proof. Using the localization long exact sequence, we see that it is sufficient to prove that $GW^i(D^{i-1}/D^i)$ is zero. We know that the forgetful homomorphism $f : GW^{i-1}(D^{(i-1)}/D^i) \to K_0(D^{(i-1)}/D^i)$ is surjective (section 3). From [Wa, Theorem 2.6], there is an exact sequence

$$GW^{i-1}(D^{(i-1)}/D^i) \xrightarrow{f} K_0(D^{(i-1)}/D^i) \xrightarrow{H} GW^i(D^{(i-1)}/D^i) \longrightarrow W^i(D^{(i-1)}/D^i).$$

But $W^i(D^{(i-1)}/D^i) = 0$ by [BW, Theorem 6.1].

□

Thus we get:

Proposition 4.7. *Let*

$$\alpha : GW^i(D^i/D^{i+2}) \to GW^i(D^i/D^{i+1})$$

and

$$\beta : GW^i(D^i/D^{i+1}) \to GW^i(D^{i-1}/D^{i+1})$$

be the homomorphisms induced by the functors

$$D^i/D^{i+2} \to D^i/D^{i+1}$$

and

$$D^i/D^{i+1} \to D^{i-1}/D^{i+1}.$$

Then $\widetilde{CH}^i(X) \simeq \mathrm{Im}(\beta\alpha) \subset GW^i(D^{i-1}/D^{i+1})$. *In particular, if* $n = \dim X$ *then* $\widetilde{CH}^n(X) = GW^n(D^{n-1})$.

Proof. Both statements are obvious consequences of Theorem 3.3 and the above lemma. □

5. Some consequences in low dimensions

Suppose that X is a regular surface over $\mathbb{Z}[\frac{1}{2}]$. Then Proposition 4.7 shows that there is a homomorphism

$$\widetilde{CH}^2(X) \to GW^2(X).$$

The exact sequence of triangulated categories $D^1 \to D^0 \to D^0/D^1$ yields an exact sequence of (Grothendieck-)Witt groups:

$$GW^2(D^1) \longrightarrow GW^2(X) \longrightarrow GW^2(D^0/D^1) \longrightarrow W^3(D^1).$$

Using [BW, Theorem 6.1], we see that $W^3(D^1) = 0$. Moreover, using [Wa, Theorem 2.6], we find an isomorphism $GW^2(D^0/D^1) \simeq K_0(D^0/D^1)$ induced by H. Since $K_0(D^0/D^1) \simeq \mathbb{Z}$, we have

Lemma 5.1. *There is a surjective homomorphism* $\widetilde{CH}^2(X) \to GW^2(X)/H(\mathcal{O}_X)$.

Remark 5.2. This homomorphism has already been constructed in [Fa1] following the ideas of Barge and Morel. The construction was significantly longer than the one presented here.

Remark 5.3. Easy computations on the spectral sequence show that the homomorphism $\widetilde{CH}^2(X) \to GW^2(X)/H(\mathcal{O}_X)$ is in fact an isomorphism. If X is affine, there is a more direct way to see it. This is the next theorem.

To state the following theorem, we just need a definition:

Definition 5.4. Let A be a ring. We denote by $K_0^{sp}(A)$ the ring $GW^-(A)/H(A)$.

Theorem 5.5. *Let A be a regular $\mathbb{Z}[\frac{1}{2}]$-algebra of dimension* 2. *Then the Euler class induces an isomorphism* $\widetilde{CH}^2(A) \simeq K_0^{sp}(A)$.

Proof. By [Wa, Theorem 6.1], we have an isomorphism $GW^2(A) \simeq GW^-(A)$. From the above lemma and Proposition 4.7, we get a surjective homomorphism

$$\widetilde{CH}^2(A) \to K_0^{sp}(A).$$

Since we are in dimension 2, $K_0^{sp}(A)$ is generated by isomorphism classes of elements (P, χ) where P is a projective module of rank 2 and χ is an anti-symmetric form on P. If (Q, ψ) and (P, χ) are such elements, then $(P \oplus Q, \chi \perp \psi)$ can be decomposed uniquely as $(R, \theta) + H(A)$. Define a homomorphism $K_0^{sp}(A) \to \widetilde{CH}^2(A)$ by sending (P, χ) to $\tilde{c}_2(P)$ (see [Fa1, Theorem 14.3.1] for more information on the Euler class and [Fa1, Proposition 15.3.10] for a proof that this is a homomorphism). One checks that both homomorphisms are inverse to each other. □

Corollary 5.6. *Let A be a regular $\mathbb{Z}[\frac{1}{2}]$-algebra of dimension 2 and P be a projective module of rank 2 with trivial determinant. Then $\tilde{c}_2(P) = 0$ if and only if $P \simeq A^2$.*

Let A be a regular ring (containing $\frac{1}{2}$) of dimension 3. From Proposition 4.7, we see that there is a homomorphism $\widetilde{CH}^3(A) \to GW^3(A)$. In the sequel, we will prove the following theorem:

Theorem 5.7. *Let A be a regular ring of dimension 3 and let P be a projective module of rank 3. Then $\tilde{c}_3(P) = 0$ if and only if $P \simeq P' \oplus A$ for some P'.*

The proof will require some lemmas:

Lemma 5.8. *Let P be a projective module of rank 3 such that* $\det(P) \simeq A$. *Then $\tilde{c}_3(P) = H(A - P)$ in $GW^3(A)$.*

Proof. Let $s : P \to A$ be a section such that $ht(s(P)) = 3$. Then $\tilde{c}_3(P)$ is given by the class of the Koszul complex with its usual symmetric isomorphism in $\widetilde{CH}^3(A)$ ([Fa1, Theorem 14.3.1]):

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A & \longrightarrow & \wedge^2 P & \xrightarrow{t} & P & \xrightarrow{s} & A & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\ 0 & \longrightarrow & A^\vee & \xrightarrow[-s^\vee]{} & P^\vee & \xrightarrow[-t^\vee]{} & (\wedge^2 P)^\vee & \longrightarrow & A^\vee & \longrightarrow & 0. \end{array}$$

Denote by K this complex and by $\varphi : K \to T^3 K^\vee$ the symmetric isomorphism. Consider next the complexes L:

$$0 \longrightarrow 0 \longrightarrow P \xrightarrow{s} A \longrightarrow 0$$

and $T^2 L^\vee$:

$$0 \longrightarrow A^\vee \xrightarrow[s^\vee]{} P^\vee \longrightarrow 0 \longrightarrow 0.$$

The symmetric isomorphism φ gives a homomorphism $T^2 L^\vee \to L$ that we still denote by t. The cone of t is the complex K

$$0 \longrightarrow A \longrightarrow \wedge^2 P \xrightarrow{t} P \xrightarrow{s} A \longrightarrow 0$$

and the following diagram commutes:

$$\begin{array}{ccccccc}
T^2L^\vee & \xrightarrow{t} & L & \xrightarrow{v} & K & \xrightarrow{w} & T^3L^\vee \\
\| & & \downarrow{\scriptstyle -\varpi} & & \downarrow{\scriptstyle \varphi} & & \| \\
T^2L^\vee & \xrightarrow[-T^2t^\vee]{} & L^{\vee\vee} & \xrightarrow[T^3w^\vee]{} & K^\vee & \xrightarrow[T^3v^\vee]{} & T^3L^\vee.
\end{array}$$

This shows that L is a Lagrangian of (K, φ), hence $(K, \varphi) = H(L)$ in $GW^3(A)$. But $L = A - P \in K_0(A)$. □

This shows that if $\tilde{c}_3(P) = 0$, then $H(A - P) = 0$ in $GW^3(A)$. From the exact sequence

$$GW^2(A) \xrightarrow{f} K_0(A) \xrightarrow{H} GW^3(A) \xrightarrow{p} W^3(A) \longrightarrow 0$$

we see that there exists an element $\beta \in GW^2(A)$ such that $f(\beta) = P - A$. Recall that $GW^2(A)$ is isomorphic to $GW^-(A)$ ([Wa, Theorem 6.1]). We have then:

Lemma 5.9. *Any element $\beta \in GW^2(A)$ can be written $\beta = (R, \theta) + H(A^r) - H(A^s)$ for some projective module R of rank 2, some anti-symmetric form θ on R and some $r, s \in \mathbb{N}$.*

Proof. A general $\beta \in GW^-(A)$ is a sum $\sum(R_i, \theta_i) - \sum(S_j, \rho_j)$ for projective modules R_i, S_j and anti-symmetric forms θ_i, ρ_j. First observe that $(S_j, \rho_j) + (S_j, -\rho_j) = H(S_j)$. Let T_j be such that $S_j \oplus T_j = A^n$ for some n. Then

$$-(S_j, \rho_j) = (S_j, -\rho_j) + H(T_j) - H(A^n).$$

So we can suppose that $\beta = \sum(R_i, \theta_i) - H(A^s)$ for some s. Because θ_i is anti-symmetric, then R_i is of even rank. Suppose that R_i is of rank ≥ 4. Because of Serre's theorem, R_i has a unimodular element. Therefore there exists an injective homomorphism $A \to R_i$ and because θ_i is anti-symmetric we see that $A \subset A^\perp$. This shows that there exists a projective module R_i' of rank $\mathrm{rk}(R_i) - 2$ and an anti-symmetric form ρ_i' such that $(R_i, \rho_i) = (R_i', \rho_i') + H(A)$ in $GW^-(A)$ (see also the proof of the next lemma). Thus the lemma is proved. □

This shows that $P - A = R + A^r - A^s$ in $K_0(A)$ for some rank two projective module R carrying an anti-symmetric form if $\tilde{c}_3(P) = 0$. The conclusion follows from the next lemma:

Lemma 5.10. *Let A be a regular ring of odd dimension d. Let P be a rank d projective module such that P is stably isomorphic to $R \oplus A$ where R is a rank $d-1$ projective module carrying an anti-symmetric form. Then $P \simeq P' \oplus A$ for some projective module P'.*

Proof. First observe that if $P \oplus A^r \simeq R \oplus A^{r+1}$ then $P \oplus A \simeq R \oplus A^2$ by Bass' cancellation theorem. Since R has an anti-symmetric form, then $R \oplus A^2$ also has one. Therefore we see that $P \oplus A$ has an anti-symmetric form. Denote it by θ. Let $i : A \to P \oplus A$ be the natural injection. Then $i^\vee \theta i = 0$ and there exist homomorphisms $A \to P^\vee$ and $P \to A^\vee$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & P \oplus A & \xrightarrow{\pi} & P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \theta} & & \downarrow & & \\ 0 & \longrightarrow & P^{\vee} & \xrightarrow[\pi^{\vee}]{} & (P \oplus A)^{\vee} & \xrightarrow[i^{\vee}]{} & A^{\vee} & \longrightarrow & 0. \end{array}$$

Since θ is surjective, the homomorphism $P \to A^{\vee}$ is also surjective. Therefore P has a unimodular element. □

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