

# PROJECTIVE PUSH-FORWARDS IN THE WITT THEORY OF ALGEBRAIC VARIETIES

ALEXANDER NENASHEV<sup>1</sup>

ABSTRACT. We define push-forwards along projective morphisms in the Witt theory of smooth quasi-projective varieties over a field. We prove that they have standard properties such as functoriality, compatibility with pull-backs and projection formulas.

## 1. INTRODUCTION

Let  $k$  be a field with  $\text{char} k \neq 2$  and  $Sm_k$  denote the category of smooth quasi-projective varieties over  $k$ . We consider the Witt theory of such varieties developed by P. Balmer [Ba1-3], which the reader is supposed to be familiar with.

**1.1. Twisted pull-backs of line bundles.** For  $X \in Sm_k$ , let  $\omega_X = \bigwedge^{\dim X} \Omega_X$  denote the canonical sheaf of  $X$ . For a morphism  $f : Y \rightarrow X$  in  $Sm_k$ , let  $\omega_f = \omega_{Y/X} = \omega_Y \otimes f^* \omega_X^\vee$  be the relative canonical sheaf; clearly we have  $\omega_{fg} \cong g^* \omega_f \otimes \omega_g$  (canonically) for any composable  $f$  and  $g$ . For a line bundle  $L$  on  $X$ , we introduce the *twisted pull-back*  $L^f = f^* L \otimes \omega_f$ ; one checks that  $(L^f)^g \cong L^{fg}$  canonically.

**1.2. The general objective.** Our objective is to construct push-forwards along projective morphisms in Witt theory: for every equi-codimensional projective  $f : Y \rightarrow X$  in  $Sm_k$  and a line bundle  $L$  on  $X$ , we define maps

$$f_* : W^q(Y; L^f) \rightarrow W^{q+c}(X; L),$$

where  $c = \dim X - \dim Y$  is the codimension of  $f$ . We also define push-forwards with support

$$f_* = f_*^{S,T} : W_S^q(Y; L^f) \rightarrow W_T^{q+c}(X; L),$$

where  $S$  and  $T$  are closed subschemes in  $Y$  and  $X$  respectively, not necessarily smooth, satisfying  $S \subset f^{-1}(T)$ . The following properties of push-forwards are established:

- (a) functoriality (Section 4.5);
  - (b) compatibility with pull-backs in transversal squares (change of base, Section 4.7);
  - (c) compatibility with the product structure (projection formulas, Section 4.11).
- Push-forwards will be also referred to as trace maps or trace operators.

---

*Key words and phrases.* Witt theory of algebraic varieties, projective morphism, push-forward, Gysin operator, Thom isomorphism, deformation to the normal bundle.

<sup>1</sup>Department of Mathematics, York University - Glendon College, Toronto, Canada; E-Mail: nenashev@glendon.yorku.ca. Supported by NSERC Grant 313294-05

**1.3. The closed embedding case.** In the case of a closed embedding  $i : Y \hookrightarrow X$  one has  $\omega_{Y/X} \cong \det N$  (canonically), where  $N = N_{X/Y}$  is the normal bundle of  $i$ , and the respective push-forward takes the form

$$i_* : W^q(Y; L_Y \otimes \det N) \rightarrow W^{q+c}(X; L),$$

where  $L_Y = i^*L$ . Such push-forwards, referred to as Gysin operators, were constructed in [Ne1] in terms of Thom (dévissage) isomorphisms and deformations to the normal cone. For the reader's convenience we survey on our construction of Gysin operators in Section 2.

**1.4. The case of projections.** In the present paper we first define push-forwards along projections of the form  $p : X \times \mathbb{P}^n \rightarrow X$ . Here, the situation essentially differs from the respective part of the work of I. Panin and A. Smirnov on push-forwards in *oriented theories* [PS][PS1][SP]. Recall that in an oriented theory, there are two approaches to the traces of projections: the one being based on residues [PS, Section 4.3.2], the other on the use of the cobordism ring  $MU$  and the respective formal group law [PS, Section 4.3.1]. In the Witt theory, however, there is no need to apply such advanced techniques. For the situation in  $W$  is simpler: for even  $n$ , the Gysin operator along a constant section  $X \hookrightarrow X \times \mathbb{P}^n$  proves to be an isomorphism, and we can define  $p_*$  as its inverse. In this part we use our computation of the Witt groups of projective bundles performed in [Ne2]; the same groups were calculated by a different method by C. Walter, see [W].

The case of an odd  $n$  can then be reduced to this one. The proofs of the expected properties of the  $p_*$ 's can be obtained on the basis of the properties of Gysin operators proved in [Ne1]. This is done in Section 3.

**1.5. The general case.** In Section 4 we proceed to the case of an arbitrary projective morphism  $f : Y \rightarrow X$ . Factoring such a morphism as  $Y \xrightarrow{i} X \times \mathbb{P}^n \xrightarrow{p} X$ , where  $i$  is a closed embedding, we define  $f_*$  as  $p_*i_*$ . We prove that the result does not depend on the choice of a factorization and establish the standard properties of projective push-forwards.

S. Gille introduced push-forwards (transfers) for coherent Witt groups of commutative rings with dualizing complexes [G1]. This yields transfers for the usual Witt groups if we consider finite morphisms of regular rings of finite Krull dimension. (More on coherent Witt groups can be found in [G2].) B. Calmès and J. Hornbostel constructed push-forwards along proper morphisms of smooth varieties by using dualities and adjunctions in derived categories and also working in the coherent Witt theory, see [CH]. This is quite different from our work in which we avoid any use of triangulated or derived categories. Nor do we use the coherent Witt theory. Our approach is based on the general cohomology theory properties of Witt groups and is as geometric as possible. We follow the guidelines of I. Panin and A. Smirnov given in [PS] and [SP] for *oriented theories*. However we have to considerably modify their machinery to the case of Witt theory which is not orientable in the sense of their work.

**Acknowledgements.** A part of this work was done at the University of Bielefeld and supported by the SFB-701, to which I express my gratitude. I am indebted to Stefan Gille and Ivan Panin for their interest in my work and various help.

## 2. REVIEW OF GYSIN OPERATORS

In this section we remind the reader how the push-forwards along closed embeddings are defined in [Ne1]. We provide a brief account of the properties of such push-forwards which are also referred to as *Gysin operators*.

**2.1. Definition.** Let  $i : Y \hookrightarrow X$  be a codimension  $c$  closed embedding of smooth varieties and let  $p : N_{X/Y} \rightarrow Y$  denote the normal vector bundle to  $Y$  in  $X$ . Let  $L$  be a line bundle on  $X$  and  $L_Y = i^*L$  its restriction to  $Y$ .

(i) We define the Gysin operator

$$i_* : W^q(Y; L_Y \otimes \det N_{X/Y}) \rightarrow W^{q+c}(X; L)$$

as the composition

$$W^q(Y; L_Y \otimes \det N) \xrightarrow{th(N)} W_Y^{q+c}(N; p^*L_Y) \xrightarrow{d(X,Y)} W_Y^{q+c}(X; L) \rightarrow W^{q+c}(X; L),$$

where  $N = N_{X/Y}$ ,  $th(N)$  and  $d(X, Y)$  are the Thom and the deformation to the normal cone isomorphisms, respectively (see [Ne1, Sections 2 and 3]), and the last arrow is an extension of support.

The Thom (dévissage) isomorphisms in Witt theory were also considered by S. Gille, see [G3]. A general reference on the deformations to the normal cone is Fulton's book [Fu, Chapter 4]; see also [PS][PS1][SP] in the context of oriented cohomology theories.

(ii) If  $T$  is a closed subscheme in  $X$  and  $S$  is a closed subscheme in  $Y$  such that  $S \subset T_Y = T \cap Y$ , then we define the *Gysin map with support*

$$i_* = i_*^{S,T} : W_S^q(Y; L_Y \otimes \det N_{X/Y}) \rightarrow W_T^{q+c}(X; L)$$

by

$$W_S^q(Y; L_Y \otimes \det N) \xrightarrow{th_S(N)} W_S^{q+c}(N; p^*L_Y) \xrightarrow{d_S(X,Y)} W_S^{q+c}(X; L) \rightarrow W_T^{q+c}(X; L).$$

Here  $S$  and  $T$  do not need to be smooth.

(iii) If  $S = T = Y$ , we will write  $i_*^Y$  for the map  $i_*^{Y,Y} : W^q(Y; L_Y \otimes \det N_{X/Y}) \rightarrow W_Y^{q+c}(X; L)$  which is an isomorphism (the first two steps in (i)).

(iv) Observe that if  $i : Y \rightarrow X$  is an isomorphism, then  $i_* = (i^{-1})^*$ , both on the Witt groups with and without support.

(v) In this section we use the more explicit [Ne1]-notation  $L_Y \otimes \det N_{X/Y}$  to denote the twist on  $Y$ . In Section 3 we'll switch to the notation introduced in 1.1, which is shorter and more functorial, and will write  $L^i$  for the same.

Gysin operators enjoy the following properties.

**2.2. Functoriality.** (See [Ne1, Proposition 5.1].)

(a)  $(id_X)_* = id_{W_T^q(X; L)}$

(b) Let  $Z \xrightarrow{i} Y \xrightarrow{j} X$  be equicodimensional closed embeddings of smooth quasiprojective varieties, and let  $r = \text{codim } i$ ,  $s = \text{codim } j$ ,  $t = \text{codim } (ji) = r + s$ .

Let  $L$  be a line bundle on  $X$  and  $L_Y, L_Z$  its restrictions to  $Y$  and  $Z$ . Then the diagram

$$\begin{array}{ccc} W^{q+r}(Y; L_Y \otimes \det N_{X/Y}) & \xrightarrow{j_*} & W^{q+t}(X; L) \\ \uparrow i_* & \nearrow (ji)_* & \\ W^q(Z; L_Z \otimes \det N_{X/Z}) & & \end{array}$$

commutes. Moreover, if  $R \subset S \subset T$  are compatible closed subvarieties in  $Z, Y, X$  respectively, then the diagram

$$\begin{array}{ccc} W_S^{q+r}(Y; L_Y \otimes \det N_{X/Y}) & \xrightarrow{j_*^{S,T}} & W_T^{q+t}(X; L) \\ \uparrow i_*^{R,S} & \nearrow (ji)_*^{R,T} & \\ W_R^q(Z; L_Z \otimes \det N_{X/Z}) & & \end{array}$$

commutes. (The natural isomorphism  $\det N_{X/Z} \cong \det N_{Y/Z} \otimes \det(N_{X/Y}|_Z)$  is involved implicitly in the definition of  $i_*$  in both diagrams.)

### 2.3. Compatibility with extensions of support. (See [Ne1, Section 4.2].)

Given closed embeddings

$$\begin{array}{ccccc} S' & \longrightarrow & S & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ T' & \longrightarrow & T & \longrightarrow & X \end{array}$$

the following diagram commutes:

$$\begin{array}{ccc} W_{S'}^q(Y; L_Y \otimes \det N) & \xrightarrow{i_*^{S',T'}} & W_{T'}^{q+c}(X; L) \\ \downarrow & & \downarrow \\ W_S^q(Y; L_Y \otimes \det N) & \xrightarrow{i_*^{S,T}} & W_T^{q+c}(X; L) \end{array}$$

Here the vertical maps are extensions of support.

### 2.4. Transversal base changes. (See [Ne1, Section 6.1].)

Consider a transversal square of the form

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ \phi_Y \downarrow & & \downarrow \phi_X \\ Y & \xrightarrow{i} & X \end{array}$$

in  $Sm_k$  in which  $i$  and (consequently)  $i'$  are closed embeddings of codimension  $c$ . This means that it is cartesian in  $Sm_k$  and the natural map  $N_{X'/Y'} \rightarrow \phi_Y^* N_{X/Y}$  is an isomorphism of vector bundles, see Def. 3.5 in [Ne1]. Let  $L$  be a line bundle on  $X$ . Let  $S$  and  $T$  be compatible closed subschemes in  $Y$  and  $X$ , see 2.1(ii), and  $S'$

and  $T'$  be their pullbacks to  $Y'$  and  $X'$  respectively. Then with the same notation, the diagram commutes

$$\begin{array}{ccc} W_S^q(Y; L_Y \otimes \det N) & \xrightarrow{i_*} & W_T^{q+c}(X; L) \\ \phi_Y^* \downarrow & & \downarrow \phi^* \\ W_{S'}^q(Y'; (f^*L)_{Y'} \otimes \det N') & \xrightarrow{i'_*} & W_{T'}^{q+c}(X'; f^*L) \end{array}$$

The condition of transversality  $\phi_Y^*N \cong N'$  guarantees that the twists in the left groups agree with respect to  $\phi_Y^*$ .

**2.5. Additivity formula.** (See [Ne1, Section 6.2].)

Let  $Y_1, Y_2$  and  $X$  be smooth varieties,  $Y = Y_1 \amalg Y_2$ , and let  $i : Y \hookrightarrow X$  be a closed embedding (equicodimensional, of codimension  $c$ ). Denote  $j_r : Y_r \rightarrow Y$  the natural embedding and let  $i_r = i \circ j_r : Y_r \hookrightarrow X$  for  $i = 1, 2$ . Let  $L$  be a line bundle on  $X$  and  $L_Y, L_{Y_i}$  denote its restrictions to  $Y$  and  $Y_i$ , respectively. Then the diagram commutes:

$$\begin{array}{ccc} W^q(Y; L_Y \otimes \det N_{X/Y}) & \xrightarrow{i_*} & W^{q+c}(X; L) \\ \searrow (j_1^*, j_2^*) & & \nearrow ((i_1)_*, (i_2)_*) \\ W^q(Y_1; L_{Y_1} \otimes \det N_{X/Y_1}) \oplus W^q(Y_2; L_{Y_2} \otimes \det N_{X/Y_2}) & & \end{array}$$

which can be expressed by the simple formula

$$i_* = (i_1)_* \circ j_1^* + (i_2)_* \circ j_2^*.$$

If furthermore  $S_1, S_2$ , and  $T$  are closed subvarieties in  $Y_1, Y_2$ , and  $X$ , respectively, not necessarily smooth,  $S = S_1 \amalg S_2$  and  $i(S) \subset T$ , then we have additivity with support:

$$i_*^{S,T} = (i_1)_*^{S_1,T} \circ j_1^* + (i_2)_*^{S_2,T} \circ j_2^*.$$

The case of several components follows by induction.

**2.6. Smooth divisor case.** (See [Ne1, Section 6.3].) Let  $X$  be a smooth variety,  $L$  a line bundle on  $X$ , and  $s : \mathcal{O}_X \rightarrow L$  a global section of  $L$  transversal to the zero section. Denote by  $D = D(s)$  the smooth divisor on  $X$  given by the zeros of  $s$  and  $i : D \hookrightarrow X$  the inclusion. Then  $N_{X/D} \cong L_D$  and we can consider the push-forward map  $i_* : W^0(D) \rightarrow W^1(X; L^\vee)$ , where  $W^0(D) = W^0(D; \mathcal{O}_D)$ . With this notation we have

$$i_*(1) = 0.$$

**2.7. Projection formulas.** (See [Ne1, Section 6.4] for the proof of the projection formulas and [GN] for the definition and properties of the product structure on the Witt groups introduced by S. Gille and the author.)

Let  $i : Y \hookrightarrow X$  be a codimension  $c$  closed embedding of smooth varieties. Let  $L$  and  $L'$  be line bundles over  $X$ , and let  $\alpha \in W^{q'}(X; L')$  and  $\beta \in W^q(Y; L_Y \otimes \det N)$ . Then

$$i_*(i^*\alpha \star \beta) = \alpha \star i_*\beta \text{ and } i_*(\beta \star i^*\alpha) = (-1)^{cq'} i_*\beta \star \alpha$$

in  $W^{q'+q+c}(X; L \otimes L')$ .

If, moreover,  $S$  and  $T$  are compatible closed subschemes in  $Y$  and  $X$ , see 4.1(ii),  $T'$  is another closed subscheme in  $X$ ,  $\alpha \in W_{T'}^{q'}(X; L')$  and  $\beta \in W_S^q(Y; L_Y \otimes \det N)$ , then the same formulas hold in  $W_{T \cap T'}^{q'+q+c}(X; L \otimes L')$ .

### 3. TRACES OF PROJECTIONS

In this section we define traces

$$p_* = p_*^T : W_{T \times \mathbb{P}^n}^{q+n}(X \times \mathbb{P}^n; L^{(n)}) \rightarrow W_T^q(X; L) \quad (3.1)$$

along projections of the form  $p = p_X^{(n)} : X \times \mathbb{P}^n \rightarrow X$  and prove their properties. Here  $T$  is a closed subscheme of  $X$ , not necessarily smooth, and

$$L^{(n)} = L^{p_X^{(n)}} = (p_X^{(n)})^* L \otimes \omega_{p_X^{(n)}}$$

is the twisted pull-back of  $L$  as defined in Section 1.1.

If  $S \subset T \times \mathbb{P}^n$  is another closed subscheme of  $X \times \mathbb{P}^n$ , we can combine (3.1) with the extension of support and get the trace operator

$$p_* = p_*^{S,T} : W_S^{q+n}(X \times \mathbb{P}^n; L^{(n)}) \rightarrow W_T^q(X; L).$$

The properties of the operators  $p_*^T$  can be easily generalized to the  $p_*^{S,T}$ , which is left to the reader.

**Definitions.** For  $a \in \mathbb{P}^n(k)$  denote  $i = i_{X,a}^{(n)} : X \hookrightarrow X \times \mathbb{P}^n$  the embedding  $x \mapsto (x, a)$ . As  $p \circ i = \text{id}_X$ , we have a canonical isomorphism  $(L^{(n)})^i \cong L$  and can consider the Gysin operator

$$i_* : W_T^q(X; L) \rightarrow W_{T \times \mathbb{P}^n}^{q+n}(X \times \mathbb{P}^n; L^{(n)}). \quad (3.2)$$

**Lemma 3.1.** *Let  $n$  be even.*

(i) *The operator (3.2) is an isomorphism.*

(ii) *It does not depend on the choice of a point  $a \in \mathbb{P}^n(k)$ , i.e., if  $a' \in \mathbb{P}^n(k)$  is another point, then  $(i_{X,a}^{(n)})_* = (i_{X,a'}^{(n)})_*$ .*

*Proof.* (i) By Definition 2.1(ii), a Gysin operator is the composition of three maps two of which are always isomorphisms. Thus it suffices to show that the extension of support map

$$W_{T \times a}^{q+n}(X \times \mathbb{P}^n; L^{(n)}) \rightarrow W_{T \times \mathbb{P}^n}^{q+n}(X \times \mathbb{P}^n; L^{(n)})$$

is an isomorphism. This map fits into the localization sequence

$$\begin{aligned} \dots \rightarrow W_{T \times a}^{q+n}(X \times \mathbb{P}^n; L^{(n)}) &\rightarrow W_{T \times \mathbb{P}^n}^{q+n}(X \times \mathbb{P}^n; L^{(n)}) \rightarrow \\ &\rightarrow W_{T \times (\mathbb{P}^n - a)}^{q+n}(X \times \mathbb{P}^n - T \times a; L^{(n)})|_{X \times \mathbb{P}^n - T \times a} \rightarrow \dots \end{aligned}$$

By excision

$$W_{T \times (\mathbb{P}^n - a)}^{q+n}(X \times \mathbb{P}^n - T \times a; L^{(n)})|_{X \times \mathbb{P}^n - T \times a} \cong W_{T \times (\mathbb{P}^n - a)}^{q+n}(X \times (\mathbb{P}^n - a); L^{(n)})|_{X \times (\mathbb{P}^n - a)}.$$

Considering  $\mathbb{P}^n - a$  as a line bundle over  $\mathbb{P}^{n-1}$ , we get by homotopy invariance

$$\begin{aligned} W_{T \times (\mathbb{P}^n - a)}^{q+n}(X \times (\mathbb{P}^n - a); L^{(n)})|_{X \times (\mathbb{P}^n - a)} &\cong \\ &W_{T \times \mathbb{P}^{n-1}}^{q+n}(X \times \mathbb{P}^{n-1}; L^{(n)})|_{X \times \mathbb{P}^{n-1}}. \end{aligned}$$

Here we consider  $\mathbb{P}^{n-1}$  as a linear subspace in  $\mathbb{P}^n$  not meeting  $a$ . Let  $U = X - T$  and consider the localization sequence

$$\begin{aligned} \dots \rightarrow W^{q+n-1}(U \times \mathbb{P}^{n-1}; L^{(n)}|_{U \times \mathbb{P}^{n-1}}) &\rightarrow W_{T \times \mathbb{P}^{n-1}}^{q+n}(X \times \mathbb{P}^{n-1}; L^{(n)}|_{X \times \mathbb{P}^{n-1}}) \rightarrow \\ &\rightarrow W^{q+n}(X \times \mathbb{P}^{n-1}; L^{(n)}|_{X \times \mathbb{P}^{n-1}}) \rightarrow \dots \end{aligned}$$

As  $L^{(n)} \cong (p_X^{(n)})^* L \otimes \mathcal{O}(-n-1)$  on  $X \times \mathbb{P}^n$ , we have

$$L^{(n)}|_{X \times \mathbb{P}^{n-1}} \cong (p_X^{(n-1)})^* L \otimes \mathcal{O}(-n-1) \text{ and } L^{(n)}|_{U \times \mathbb{P}^{n-1}} \cong (p_U^{(n-1)})^* L \otimes \mathcal{O}(-n-1).$$

Hence the side groups vanish by [Ne2, Cor. 4.2] since  $n$  is even. Thus the middle group vanishes as well, which proves (i).

(ii) There exists an automorphism  $\alpha$  of  $\mathbb{P}^n$  given by a matrix of  $SL_{n+1}(k)$  which takes  $a$  to  $a'$ . Thus  $(i_{X,a'}^{(n)})_* = (1_X \times \alpha)_* \circ (i_{X,a}^{(n)})_*$ , and it suffices to show that  $(1_X \times \alpha)_* = \text{id}$  on  $W^q(X \times \mathbb{P}^n; -)$  or, equivalently, that  $(1_X \times \alpha)^* = \text{id}$ . (Recall that  $(f^{-1})_* = f^*$  for an isomorphism  $f$ .) Clearly we can assume that  $\alpha$  is an elementary matrix. The following argument works for any theory satisfying homotopy invariance. Consider the automorphism  $\tilde{\alpha}$  of  $\mathbb{P}^n \times \mathbb{A}^1$  such that  $\tilde{\alpha}_0 = \text{id}_{\mathbb{P}^n}$  and  $\tilde{\alpha}_1 = \alpha$ . (If  $\alpha = e_{i,j}(\lambda)$ , then  $\tilde{\alpha}_t = e_{i,j}(t\lambda)$ .) Denote by  $i_0$  and  $i_1$  the embeddings  $\mathbb{P}^n \hookrightarrow \mathbb{P}^n \times \mathbb{A}^1$  given by  $y \mapsto (y, 0)$  and  $y \mapsto (y, 1)$  respectively, and  $\pi : \mathbb{P}^n \times \mathbb{A}^1 \rightarrow \mathbb{P}^n$  the projection. Since  $\pi \circ i_0 = \pi \circ i_1 = \text{id}_{\mathbb{P}^n}$  and  $\pi^*$  is an isomorphism by homotopy invariance, we have  $i_0^* = i_1^*$  which is an isomorphism as well.

Applying pull-backs to the diagram

$$\begin{array}{ccccc} \mathbb{P}^n & \xrightarrow{i_0} & \mathbb{P}^n \times \mathbb{A}^1 & \xleftarrow{i_1} & \mathbb{P}^n \\ id \downarrow & & \downarrow \tilde{\alpha} & & \downarrow \alpha \\ \mathbb{P}^n & \xrightarrow{i_0} & \mathbb{P}^n \times \mathbb{A}^1 & \xleftarrow{i_1} & \mathbb{P}^n \end{array}$$

proves the assertion. ■

**Definition 3.2.** Let  $X$  be a smooth quasi-projective variety over  $k$ ,  $T \hookrightarrow X$  be a closed subvariety, not necessarily smooth, and let  $p = p_X^{(n)} : X \times \mathbb{P}^n \rightarrow X$  denote the projection.

(i) If  $n$  is even, we define

$$p_* = (p_X^{(n)})_* : W_{T \times \mathbb{P}^n}^{q+n}(X \times \mathbb{P}^n; L^{(n)}) \rightarrow W_T^q(X; L)$$

as the inverse to the Gysin operator  $i_*$ ,

$$(p_X^{(n)})_* = (i_{X,a}^{(n)})_*^{-1}.$$

By Lemma 3.1, this does not depend on the choice of a point  $a \in \mathbb{P}^n(k)$ .

(ii) If  $n$  is odd,  $p_* = (p_X^{(n)})_*$  is defined as the composition

$$W_{T \times \mathbb{P}^n}^{q+n}(X \times \mathbb{P}^n; L^{(n)}) \xrightarrow{(j_X^{(n,n+1)})_*} W_{T \times \mathbb{P}^{n+1}}^{q+n+1}(X \times \mathbb{P}^{n+1}; L^{(n+1)}) \xrightarrow{(p_X^{(n+1)})_*} W_T^q(X; L).$$

Thus

$$(p_X^{(n)})_* = (p_X^{(n+1)})_* \circ (j_X^{(n,n+1)})_* = (i_{X,a}^{(n+1)})_*^{-1} \circ (j_X^{(n,n+1)})_* ,$$

where  $j^{(n,n+1)} : \mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1}$  is a  $k$ -linear embedding and  $j_X^{(n,n+1)} = 1_X \times j^{(n,n+1)}$ . Observe that  $(j_X^{(n,n+1)})_*$  does not depend on the choice of such a linear embedding. For, every two such embeddings can be connected by an  $SL$ -automorphism of  $\mathbb{P}^{n+1}$ , and the same argument as in Lemma 3.1, (ii) applies.

**Properties of push-forwards along projections.** As our definition of the traces of projections is stated in terms of Gysin maps, it is not a surprise that basic properties of the operators  $p_*$  can be deduced from the properties of Gysin operators.

**3.3. Composition.** Consider the diagram

$$\begin{array}{ccc} X \times \mathbb{P}^m \times \mathbb{P}^n & \xrightarrow{p_{X \times \mathbb{P}^m}^{(n)}} & X \times \mathbb{P}^m \\ p_{X \times \mathbb{P}^n}^{(m)} \downarrow & & \downarrow p_X^{(m)} \\ X \times \mathbb{P}^n & \xrightarrow{p_X^{(n)}} & X \end{array}$$

by means of which we introduce notation for the obvious projections, and also let  $p_X^{(m,n)} : X \times \mathbb{P}^m \times \mathbb{P}^n \rightarrow X$  denote the diagonal projection and let  $L^{(m,n)} = L^{p_X^{(m,n)}}$  for a line bundle  $L$  on  $X$ . Then for any  $q$  the following diagram commutes:

$$\begin{array}{ccc} W_{T \times \mathbb{P}^m \times \mathbb{P}^n}^{q+m+n}(X \times \mathbb{P}^m \times \mathbb{P}^n; L^{(m,n)}) & \xrightarrow{(p_{X \times \mathbb{P}^m}^{(n)})_*} & W_{T \times \mathbb{P}^m}^{q+m}(X \times \mathbb{P}^m; L^{(m)}) \\ (p_{X \times \mathbb{P}^n}^{(m)})_* \downarrow & & \downarrow (p_X^{(m)})_* \\ W_{T \times \mathbb{P}^n}^{q+n}(X \times \mathbb{P}^n; L^{(n)}) & \xrightarrow{(p_X^{(n)})_*} & W_T^q(X; L) \end{array} \quad (3.3)$$

*Proof.* Case 1:  $m, n$  even. Choose  $a \in \mathbb{P}^m(k), b \in \mathbb{P}^n(k)$ . All the four arrows in (3.3) are the isomorphisms inverse to the Gysin operators along the embeddings  $i_{X,a}^{(m)}, i_{X,b}^{(n)}, i_{X \times \mathbb{P}^n,a}^{(m)}, i_{X \times \mathbb{P}^m,b}^{(n)}$ , respectively. As Gysin maps are functorial, see 2.2, the diagram consisting of the  $i_*$ 's commutes, which proves the assertion.

Case 2:  $m$  odd,  $n$  even. Consider the diagram

$$\begin{array}{ccccc} & & W^{q+(m+1)+n} & \xrightarrow{(p_{X \times \mathbb{P}^{m+1}}^{(n)})_*} & W^{q+m+1} \\ & \nearrow (j^{m,m+1})_* & & \nearrow (j^{m,m+1})_* & \\ W^{q+m+n} & \xrightarrow{(p_{X \times \mathbb{P}^m}^{(n)})_*} & W^{q+m} & & \\ & \searrow (p_{X \times \mathbb{P}^n}^{(m)})_* & & \searrow (p_X^{(m)})_* & \\ & & W^{q+n} & \xrightarrow{(p_X^{(n)})_*} & W^q \\ & & & & \nearrow (p_X^{(m+1)})_* \end{array}$$

We leave it to the reader to add spaces, supports and twists accordingly. The side (triangular) faces commute by the definition of  $(p_-^{(m)})_*$  for odd  $m$ , see Definition 3.2(ii). The back face commutes by Case 1. The horizontal  $(p_-^{(n)})_*$ 's are the isomorphisms inverse to the respective  $(i_-^{(n)})_*$ 's. Thus the upper face commutes due to the compatibility of the  $i_*$ 's and  $j_*$ 's, see 2.2. This implies the commutativity of the front face.

Case 3:  $m$  and  $n$  odd - is left to the reader as an exercise of the same type.

**3.4. Base change.** Given  $\phi : X' \rightarrow X$  in  $Sm_k$ , denote

$$\phi^{(n)} = \phi \times 1_{\mathbb{P}^n} : X' \times \mathbb{P}^n \rightarrow X \times \mathbb{P}^n,$$

and let  $p_n^X : X \times \mathbb{P}^n \rightarrow \mathbb{P}^n$  and  $p_n^{X'} : X' \times \mathbb{P}^n \rightarrow \mathbb{P}^n$  denote the projections. As  $p_n^{X'} = p_n^X \circ \phi^{(n)}$ , we have

$$\omega_{X' \times \mathbb{P}^n / X'} \cong (p_n^{X'})^* \omega_{\mathbb{P}^n} \cong (\phi^{(n)})^* (p_n^X)^* \omega_{\mathbb{P}^n} \cong (\phi^{(n)})^* \omega_{X \times \mathbb{P}^n / X}, \quad (3.4)$$

all the isomorphisms being canonical. The resulting isomorphism  $\omega_{X' \times \mathbb{P}^n / X'} \cong (\phi^{(n)})^* \omega_{X \times \mathbb{P}^n / X}$  reflects the fact that the square

$$\begin{array}{ccc} X' \times \mathbb{P}^n & \xrightarrow{\phi^{(n)}} & X \times \mathbb{P}^n \\ p_{X'}^{(n)} \downarrow & & \downarrow p_X^{(n)} \\ X' & \xrightarrow{\phi} & X \end{array}$$

is transversal; we refer to 4.7 for a discussion of general transversal squares, which is unnecessary in this trivial case.

Given a line bundle  $L$  on  $X$ , it follows from (3.4) that

$$(\phi^* L)^{(n)} = (p_{X'}^{(n)})^* \phi^* L \otimes \omega_{X' \times \mathbb{P}^n / X'} \cong (\phi^{(n)})^* (p_X^{(n)})^* L \otimes (\phi^{(n)})^* \omega_{X \times \mathbb{P}^n / X} \cong (\phi^{(n)})^* L^{(n)},$$

where all the isomorphisms are canonical. Thus the upper arrow  $(\phi^{(n)})^*$  in the diagram

$$\begin{array}{ccc} W_{\phi^{-1}T \times \mathbb{P}^n}^{q+n}(X' \times \mathbb{P}^n; (\phi^* L)^{(n)}) & \xleftarrow{(\phi^{(n)})^*} & W_{T \times \mathbb{P}^n}^{q+n}(X \times \mathbb{P}^n; L^{(n)}) \\ (p_{X'}^{(n)})_* \downarrow & & \downarrow (p_X^{(n)})_* \\ W_{\phi^{-1}T}^q(X'; \phi^* L) & \xleftarrow{\phi^*} & W_T^q(X; L) \end{array}$$

has a correct target. We claim that the diagram commutes for any closed  $T$  in  $X$ .

*Proof.* Let  $i_{X'} : X' \rightarrow X' \times \mathbb{P}^n$ ,  $i_X : X \rightarrow X \times \mathbb{P}^n$  be constant sections given by the same  $k$ -point of  $\mathbb{P}^n$ . Then

$$\begin{array}{ccc} X' \times \mathbb{P}^n & \xrightarrow{\phi^{(n)}} & X \times \mathbb{P}^n \\ i_{X'} \uparrow & & \uparrow i_X \\ X' & \xrightarrow{\phi} & X \end{array}$$

is a transversal square in the sense of [Ne1; Def. 3.5], so  $(i_{X'})_* \circ \phi^* = (\phi^{(n)})^* \circ (i_X)_*$  by 2.4. Thus for  $n$  even,  $\phi^* \circ (i_X)_*^{-1} = (i_{X'})_*^{-1} \circ (\phi^{(n)})^*$ , which proves the property in this case.

If  $n$  is odd, then

$$\begin{aligned} \phi^* \circ (p_X^{(n)})_* &= \phi^* \circ (p_X^{(n+1)})_* \circ (j_X^{(n,n+1)})_* = (p_{X'}^{(n+1)})_* \circ (\phi^{(n+1)})^* \circ (j_X^{(n,n+1)})_* \\ &= (p_{X'}^{(n+1)})_* \circ (j_{X'}^{(n,n+1)})_* \circ (\phi^{(n)})^* = (p_{X'}^{(n)})_* \circ (\phi^{(n)})^*. \end{aligned}$$

Here the third equation is true by 2.4 since the square

$$\begin{array}{ccc} X' \times \mathbb{P}^n & \xrightarrow{\phi^{(n)}} & X \times \mathbb{P}^n \\ j_{X'}^{(n,n+1)} \downarrow & & \downarrow j_X^{(n,n+1)} \\ X' \times \mathbb{P}^{n+1} & \xrightarrow{\phi^{(n+1)}} & X \times \mathbb{P}^{n+1} \end{array}$$

is transversal.

**3.5. Compatibility with linear embeddings.** Let  $j = j^{(m,n)} : \mathbb{P}^m \rightarrow \mathbb{P}^n$  be a  $k$ -linear embedding and denote  $j_X^{(m,n)} = 1_X \times j^{(m,n)} : X \times \mathbb{P}^m \rightarrow X \times \mathbb{P}^n$ . Then  $(p_X^{(n)})_* \circ (j_X^{(m,n)})_* = (p_X^{(m)})_*$ , i.e., the following diagram commutes:

$$\begin{array}{ccc} W_{T \times \mathbb{P}^m}^{q+m}(X \times \mathbb{P}^m; L^{(m)}) & \xrightarrow{(j_X^{(m,n)})_*} & W_{T \times \mathbb{P}^n}^{q+n}(X \times \mathbb{P}^n; L^{(n)}) \\ & \searrow (p_X^{(m)})_* & \swarrow (p_X^{(n)})_* \\ & W_T^q(X; L) & \end{array}$$

*Proof.* Choose constant embeddings  $i_X^{(m)} : X \rightarrow X \times \mathbb{P}^m$  and  $i_X^{(n)} : X \rightarrow X \times \mathbb{P}^n$  compatible with  $j_X^{(m,n)}$ , i.e., so that  $i_X^{(n)} = j_X^{(m,n)} \circ i_X^{(m)}$ .

Case 1:  $m$  and  $n$  even. By 2.2 we have  $(i_X^{(n)})_* = (j_X^{(m,n)})_* \circ (i_X^{(m)})_*$ , the assertion follows.

Case 2:  $m$  even,  $n$  odd. Consider the diagram

$$\begin{array}{ccccc} X \times \mathbb{P}^m & \xrightarrow{j_X^{(m,n)}} & X \times \mathbb{P}^n & \xrightarrow{j_X^{(n,n+1)}} & X \times \mathbb{P}^{n+1} \\ & \searrow p_X^{(m)} & \downarrow p_X^{(n)} & \swarrow p_X^{(n+1)} & \\ & & X & & \end{array}$$

Take the respective Witt groups and get

$$(p_X^{(n)})_* \circ (j_X^{(m,n)})_* = (p_X^{(n+1)})_* \circ (j_X^{(n,n+1)})_* \circ (j_X^{(m,n)})_* = (p_X^{(n+1)})_* \circ (j_X^{(m,n+1)})_* = (p_X^{(m)})_* .$$

The latter is true by Case 1 and  $(j_X^{(m,n+1)})_* = (j_X^{(n,n+1)})_* \circ (j_X^{(m,n)})_*$  by 2.2.

Case 3:  $m$  odd,  $n$  even. Consider the diagram

$$\begin{array}{ccccc} X \times \mathbb{P}^m & \xrightarrow{j_X^{(m,m+1)}} & X \times \mathbb{P}^{m+1} & \xrightarrow{j_X^{(m+1,n)}} & X \times \mathbb{P}^n \\ & \searrow p_X^{(m)} & \downarrow p_X^{(m+1)} & \swarrow p_X^{(n)} & \\ & & X & & \end{array}$$

and get

$$(p_X^{(n)})_* \circ (j_X^{(m,n)})_* = (p_X^{(n)})_* \circ (j_X^{(m+1,n)})_* \circ (j_X^{(m,m+1)})_* = (p_X^{(m+1)})_* \circ (j_X^{(m,m+1)})_* = (p_X^{(m)})_* .$$

Case 4: If  $m$  and  $n$  are odd, then

$$\begin{aligned} (p_X^{(n)})_* \circ (j_X^{(m,n)})_* &= (p_X^{(n+1)})_* \circ (j_X^{(n,n+1)})_* \circ (j_X^{(m,n)})_* = (p_X^{(n+1)})_* \circ (j_X^{(m,n+1)})_* \\ &= (p_X^{(n+1)})_* \circ (j_X^{(m+1,n+1)})_* \circ (j_X^{(m,m+1)})_* = (p_X^{(m+1)})_* \circ (j_X^{(m,m+1)})_* = (p_X^{(m)})_* . \end{aligned}$$

**3.6. Compatibility with Gysin operators.** Let  $i : Y \hookrightarrow X$  be a codimension  $c$  closed embedding of smooth varieties. Let  $S$  and  $T$  be closed subschemes in  $Y$  and  $X$ , respectively, such that  $S \subset T \cap Y$ . Then the diagram commutes:

$$\begin{array}{ccc} W_{S \times \mathbb{P}^n}^{q+n}(Y \times \mathbb{P}^n; (L^i)^{(n)}) & \xrightarrow{(i^{(n)})_*} & W_{T \times \mathbb{P}^n}^{q+n+c}(X \times \mathbb{P}^n; L^{(n)}) \\ (p_Y^{(n)})_* \downarrow & & \downarrow (p_X^{(n)})_* \\ W_S^q(Y; L^i) & \xrightarrow{i_*} & W_T^{q+c}(X; L) \end{array}$$

*Proof.* (i)  $i_* \circ (p_Y^{(n)})_* = (p_X^{(n)})_* \circ (i^{(n)})_*$  amounts to  $(i_{X,a}^{(n)})_* \circ i_* = (i^{(n)})_* \circ (i_{Y,a}^{(n)})_*$  if  $n$  is even. The latter is true by 2.2.

(ii) If  $n$  is odd, then

$$\begin{aligned} i_* \circ (p_Y^{(n)})_* &= i_* \circ (p_Y^{(n+1)})_* \circ (j_Y^{(n,n+1)})_* = (p_X^{(n+1)})_* \circ (i^{(n+1)})_* \circ (j_Y^{(n,n+1)})_* \\ &= (p_X^{(n+1)})_* \circ (j_X^{(n,n+1)})_* \circ (i^{(n)})_* = (p_X^{(n)})_* \circ (i^{(n)})_* . \end{aligned}$$

**3.7. Projection formulas.** Let  $T$  and  $T'$  be closed subvarieties in a smooth  $X$  and let  $L$  and  $L'$  be line bundles on  $X$ . Then in  $W_{T' \cap T}^{q'+q}(X; L' \otimes L)$  we have

$$p_*(p^* \gamma \star \delta) = \gamma \star p_* \delta \quad \text{and} \quad p_*(\delta \star p^* \gamma) = (-1)^{nq'} p_* \delta \star \gamma$$

for any  $\gamma \in W_{T'}^{q'}(X; L')$  and  $\delta \in W_{T \times \mathbb{P}^n}^{q+n}(X \times \mathbb{P}^n; L^{(n)})$ . Here  $p = p_X^{(n)}$ . Observe that  $p^* L' \otimes L^{(n)} \cong (L' \otimes L)^{(n)}$ , *i.e.*, the twists agree; clearly  $f^* L' \otimes L^f \cong (L' \otimes L)^f$  canonically for any  $f : Y \rightarrow X$ .

*Proof.* Case 1:  $n$  even. Put  $\alpha = p^* \gamma \in W_{T' \times \mathbb{P}^n}^{q'}(X \times \mathbb{P}^n; p^* L')$  and  $\beta = p_* \delta = (i_*)^{-1} \delta \in W_t^q(X; L)$ , where  $i = i_{X,a}^{(n)}$ . Then  $\gamma = i^* \alpha$  and  $\delta = i_* \beta$ . Applying  $p_* = (i_*)^{-1}$  to the projection formulas of 2.7 proves the result.

Case 2:  $n$  odd. We have

$$\begin{aligned} (p_X^{(n)})_*((p_X^{(n)})^* \gamma \star \delta) &= (p_X^{(n+1)})_*(j_X^{(n,n+1)})_*((j_X^{(n,n+1)})^*(p_X^{(n+1)})^* \gamma \star \delta) && \text{Def. 3.2(ii)} \\ &= (p_X^{(n+1)})_*((p_X^{(n+1)})^* \gamma \star (j_X^{(n,n+1)})_* \delta) && \text{2.7 for } j_X^{(n,n+1)} \\ &= \gamma \star (p_X^{(n+1)})_*(j_X^{(n,n+1)})_* \delta = \gamma \star (p_X^{(n)})_* \delta && \text{Case 1.} \end{aligned}$$

The second formula can be reduced to Case 1 the same way.

**3.8. The section property.** Let  $s : X \rightarrow X \times \mathbb{P}^n$  be a  $k$ -rational section of  $p_X^{(n)}$ . Clearly,  $s = (1_X, f)$ , where  $f = p_n^X \circ s : X \rightarrow \mathbb{P}^n$  and  $p_n^X$  denotes the projection  $X \times \mathbb{P}^n \rightarrow \mathbb{P}^n$ . According to 1.1,  $(L^{(n)})^s \cong L$  canonically, and we can consider the Gysin operator

$$s_* : W_T^q(X; L) \rightarrow W_{T \times \mathbb{P}^n}^{q+n}(X \times \mathbb{P}^n; L^{(n)}) .$$

**Proposition 3.9.**  $(p_X^{(n)})_* \circ s_* = \text{id}$  on  $W_T^q(X; L)$ .

**Corollary 3.10.**  $s_* = (i_{X,a}^{(n)})_*$  if  $n$  is even.

**Lemma 3.11.**  $(p_X^{(n)})_* s_*(1) = 1$  in  $W^0(X; \mathcal{O}_X)$ .

*Proof of the proposition modulo the lemma.* Let  $\alpha \in W_T^q(X; L)$ . As  $s^* \circ (p_X^{(n)})^* = (p_X^{(n)} \circ s)^* = \text{id}$ , we have

$$\begin{aligned} (p_X^{(n)})_* s_*(\alpha) &= (p_X^{(n)})_* s_*(s^*(p_X^{(n)})^*(\alpha) \star 1) \\ &= (p_X^{(n)})_* ((p_X^{(n)})^*(\alpha) \star s_*(1)) && \text{by the projection formula 2.7} \\ &= \alpha \star (p_X^{(n)})_* s_*(1) && \text{by the projection formula 3.7} \\ &= \alpha \star 1 = \alpha && \text{by Lemma 3.11} \end{aligned}$$

The proof of the lemma occupies the rest of the section.

**3.12. The trace of diagonal.** Consider the diagonal embedding  $\Delta : \mathbb{P}^n \rightarrow \mathbb{P}^n \times \mathbb{P}^n$  and let  $p_i : \mathbb{P}^n \times \mathbb{P}^n \rightarrow \mathbb{P}^n$  denote the projection to the  $i$ th factor. As  $p_1 \circ \Delta = \text{id}_{\mathbb{P}^n}$ ,  $(L^{p_1})^\Delta \cong L$  canonically for a line bundle  $L$  on  $\mathbb{P}^n$ , and we can consider the trace

$$\Delta_* : W^q(\mathbb{P}^n; L) \rightarrow W^{q+n}(\mathbb{P}^n \times \mathbb{P}^n; L^{p_1}).$$

For  $a \in \mathbb{P}^n(k)$ , let  $i_a = i_{\mathbb{P}^n, a}^{(n)} : \mathbb{P}^n \rightarrow \mathbb{P}^n \times \mathbb{P}^n$ ,  $x \mapsto (x, a)$ . Then  $(L^{p_1})^{i_a} \cong L$  canonically, and we can equally consider the trace

$$(i_a)_* : W^q(\mathbb{P}^n; L) \rightarrow W^{q+n}(\mathbb{P}^n \times \mathbb{P}^n; L^{p_1}).$$

**Lemma 3.13.**  $\Delta_* = (i_a)_*$  if  $n$  is even.

*Proof.* We can assume that  $L \cong \mathcal{O}(l)$  for some  $l \in \mathbb{Z}$ .

Case 1:  $l$  is odd. Denote also by  $a$  the embedding  $pt \hookrightarrow \mathbb{P}^n$ ,  $pt \mapsto a$ , where  $pt = \text{Spec } k$ . Then  $\Delta \circ a = i_a \circ a$  and  $\Delta_* \circ a_* = (i_a)_* \circ a_*$  by 2.2. As  $l$  is odd, the trace

$$a_* : W^{q-n}(pt; L^a) \rightarrow W^q(\mathbb{P}^n; L)$$

is an isomorphism by the same argument as in the proof of Lemma 3.1(i), which proves the assertion in this case.

Case 2:  $l$  is even. Denote  $i'_a$  the embedding  $\mathbb{P}^n \rightarrow \mathbb{P}^n \times \mathbb{P}^n$ ,  $x \mapsto (a, x)$ , and consider the diagram of embeddings

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\Delta} & \mathbb{P}^n \times \mathbb{P}^n \\ a \uparrow & \xrightarrow{i_a} & \uparrow i'_a \\ pt & \xrightarrow{a} & \mathbb{P}^n \end{array}$$

which is transversal for either choice of the top arrow. By 2.4 we get the diagram

$$\begin{array}{ccc} W^q(\mathbb{P}^n; L) & \xrightarrow[\Delta_*]{(i_a)_*} & W^{q+n}(\mathbb{P}^n \times \mathbb{P}^n; L^{p_1}) \\ a^* \downarrow & & \downarrow (i'_a)^* \\ W^q(pt; a^* L) & \xrightarrow{a_*} & W^{q+n}(\mathbb{P}^n; L') \end{array}$$

which commutes for either choice of the top arrow; here  $L' = (i'_a)^* L^{p_1}$ . The pull-back  $(i'_a)^*$  is an isomorphism by [Ne2, Section 3] since  $L^{p_1} \cong p_2^* L' \otimes p_1^* \mathcal{O}(l)$  and  $l$  is even, whence  $\Delta_* = (i_a)_*$ . ■

By Definition 3.2(i) we get

**Corollary 3.14.**  $(p_1)_* \circ \Delta_* = \text{id}$  if  $n$  is even.

**Proof of Lemma 3.11 for even  $n$ .** Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{s} & X \times \mathbb{P}^n & \xrightarrow{p_X} & X \\ f \downarrow & & \downarrow f \times 1 & & \downarrow f \\ \mathbb{P}^n & \xrightarrow{\Delta} & \mathbb{P}^n \times \mathbb{P}^n & \xrightarrow{p_1} & \mathbb{P}^n \end{array}$$

The first square is transversal, thus we can apply the property 2.4 of Gysin operators. Using 3.4 for the second square, we get

$$(p_X)_* \circ s_* \circ f^* = f^* \circ (p_1)_* \circ \Delta_* = f^*$$

by Corollary 3.14. This equation is true on  $W^q(\mathbb{P}^n; L)$  with arbitrary  $q$  and  $L$ , the twists on  $X$ ,  $X \times \mathbb{P}^n$  and  $\mathbb{P}^n \times \mathbb{P}^n$  should be chosen accordingly. Applying it to the identity element in  $W^0(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n})$ , we get  $(p_X)_* s_*(1) = 1$ , which proves Lemma 3.11 in this case. ■

**Proof of Lemma 3.11 for odd  $n$ .** Let us now write  $s^{(n)}$  for  $s$  and let  $s^{(n+1)} = j_X^{(n, n+1)} \circ s^{(n)}$ . Clearly  $s^{(n+1)}$  is a section for  $p_X^{(n+1)}$ . Applying  $W$  to the diagram

$$\begin{array}{ccccc} X & \xrightarrow{s^{(n)}} & X \times \mathbb{P}^n & \xrightarrow{p_X^{(n)}} & X \\ 1 \downarrow & & \downarrow j_X^{(n, n+1)} & & \downarrow 1 \\ X & \xrightarrow{s^{(n+1)}} & X \times \mathbb{P}^{n+1} & \xrightarrow{p_X^{(n+1)}} & X \end{array}$$

completes the proof. Here we consider  $W^0(X; \mathcal{O}_X)$  in the corners with the respective shifts and twists in the middle and use 2.2 and Definition 3.2(ii). ■

#### 4. PUSH-FORWARDS ALONG PROJECTIVE MORPHISMS

**4.1. Definition of projective push-forwards.** Now that we have push-forwards along projections and closed embeddings, we are prepared to deal with arbitrary projective morphisms. Given such a morphism  $f : Y \rightarrow X$  of pure codimension  $c$ , one can represent it as a composition  $Y \xrightarrow{i} X \times \mathbb{P}^n \xrightarrow{p} X$ , where  $p = p_X^{(n)}$  and  $i$  is a closed embedding of codimension  $c + n$ . For a given line bundle  $L$  over  $X$  and closed  $S \subset Y$  and  $T \subset X$  with  $S \subset f^{-1}(T)$ , we define the push-forward

$$f_* = f_*^{S, T} : W_S^q(Y; L^f) \rightarrow W_T^{q+c}(X; L)$$

as the composition

$$W_S^q(Y; L^f) \xrightarrow{i_*} W_{T \times \mathbb{P}^n}^{q+c+n}(X \times \mathbb{P}^n; L^{(n)}) \xrightarrow{p_*} W_T^{q+c}(X; L). \quad (4.1)$$

**Proposition 4.2.** *The trace map  $f_* : W_S^q(Y; L^f) \rightarrow W_T^{q+c}(X; L)$  does not depend on the choice of a factorization  $f = p \circ i$ .*

*Proof.* We will first prove the assertion in the following special case. Let  $i : Y \hookrightarrow X$  be a closed embedding. We can write it as  $i = p_X^{(0)} \circ i^{(0)}$ , where  $i^{(0)} : Y \rightarrow X \times \mathbb{P}^0$  is given by  $y \mapsto (i(y), pt)$  and  $p_X^{(0)} : X \times \mathbb{P}^0 \rightarrow X$  is the (identity) projection. As  $(p_X^{(0)})_* = \text{id}$ , the composition in (4.1) yields the Gysin operator  $i_*$ . On the other hand, one can choose an embedding  $\tilde{i} : Y \hookrightarrow X \times \mathbb{P}^n$  covering  $i$ , i.e., satisfying  $p_X^{(n)} \circ \tilde{i} = i$ , and apply (4.1).

**Lemma 4.3.**  $i_* = (p_X^{(n)})_* \circ \tilde{i}_*$ .

*Proof of the lemma.* Consider the diagram

$$\begin{array}{ccc} Y \times \mathbb{P}^n & \xrightarrow{i^{(n)}} & X \times \mathbb{P}^n \\ p_Y^{(n)} \downarrow \uparrow s & \nearrow \tilde{i} & \downarrow p_X^{(n)} \\ Y & \xrightarrow{i} & X \end{array}$$

where  $i^{(n)} = i \times 1$ . Let  $s : Y \rightarrow Y \times \mathbb{P}^n$  be the section of  $p_Y^{(n)}$  determined by  $i^{(n)} \circ s = \tilde{i}$ . Then

$$\begin{aligned} (p_X^{(n)})_* \circ \tilde{i}_* &= (p_X^{(n)})_* \circ i_*^{(n)} \circ s_* && \text{functoriality of Gysin maps, Section 2.2} \\ &= i_* \circ (p_Y^{(n)})_* \circ s_* && \text{compatibility of Gysin's and } p_*\text{'s, Section 3.6} \\ &= i_* && \text{by Proposition 3.9.} \end{aligned}$$

The lemma is proved. ■

Now suppose we have  $f = p \circ i = p' \circ i'$  with  $Y \xrightarrow{i'} X \times \mathbb{P}^m \xrightarrow{p'} X$ , where  $p = p_X^{(n)}$  and  $p' = p_X^{(m)}$ . Let  $I : Y \rightarrow X \times \mathbb{P}^n \times \mathbb{P}^m$  be the unique embedding satisfying  $p_{X \times \mathbb{P}^m}^{(n)} \circ I = i'$  and  $p_{X \times \mathbb{P}^n}^{(m)} \circ I = i$ . We have

$$\begin{aligned} p'_* \circ i'_* &= p'_* \circ (p_{X \times \mathbb{P}^m}^{(n)})_* \circ I_* && \text{by Lemma 4.3} \\ &= p_* \circ (p_{X \times \mathbb{P}^n}^{(m)})_* \circ I_* && \text{by Section 3.3} \\ &= p_* \circ i_* && \text{by Lemma 4.3.} \end{aligned}$$

Thus we have proved that  $f_*$  is well defined. ■

## Properties of projective push-forwards.

### 4.4. The general definition agrees with special cases.

(i) If  $f$  is a closed embedding, then the trace  $f_*$  obtained as in Section 4.1 coincides with the Gysin operator defined in [Ne1], see Section 2.1.

(ii) If  $f$  is a projection of the form  $p_X^{(n)}$ , then the trace  $f_*$  defined in 4.1 agrees with the trace of  $p_X^{(n)}$  given by Definition 3.2. This is straightforward.

### 4.5. Functoriality. (i) $(\text{id}_X)_* = \text{id}_{W_T^q(X;L)}$ .

(ii) If  $Z \xrightarrow{g} Y \xrightarrow{f} X$  are projective morphisms,  $R \subset Z$ ,  $S \subset Y$  and  $T \subset X$  are compatible closed subvarieties, and  $L$  is a line bundle on  $X$ , then the diagram commutes:

$$\begin{array}{ccc} W_S^{q+r}(Y; L^f) & \xrightarrow{f_*^{S,T}} & W_T^{q+t}(X; L) \\ g_*^{R,S} \uparrow & \nearrow (fg)_*^{R,T} & \\ W_R^q(Z; L^{fg}) & & \end{array}$$

Here  $r = \text{codim } g$ ,  $s = \text{codim } f$ ,  $t = r + s$ .

*Proof.* The first assertion is straightforward.

Proof of (ii). We will write the proof in terms of the maps involved; we leave it to the reader to write down the Witt groups that are the domains and codomains of such maps. We won't write the supports in the notation for push-forwards.

Let  $Z \xrightarrow{i_g} Y \times \mathbb{P}^m \xrightarrow{p_Y^{(m)}} Y$  and  $Y \xrightarrow{i_f} X \times \mathbb{P}^n \xrightarrow{p_X^{(n)}} X$  be factorizations for  $g$  and  $f$ . Clearly we can assume that  $m$  and  $n$  are even. The diagram

$$\begin{array}{ccc} Y \times \mathbb{P}^m & \xrightarrow{i_f^{(m)}} & X \times \mathbb{P}^n \times \mathbb{P}^m \\ p_Y^{(m)} \downarrow & & \downarrow p_{X \times \mathbb{P}^n}^{(m)} \\ Y & \xrightarrow{i_f} & X \times \mathbb{P}^n \end{array}$$

is of the type considered in 3.6. Thus  $(i_f)_* \circ (p_Y^{(m)})_* = (p_{X \times \mathbb{P}^n}^{(m)})_* \circ (i_f^{(m)})_*$  and

$$\begin{aligned} f_*g_* &= (p_X^{(n)})_*(i_f)_*(p_Y^{(m)})_*(i_g)_* = (p_X^{(n)})_*(p_{X \times \mathbb{P}^n}^{(m)})_*(i_f^{(m)})_*(i_g)_* \\ &= (p_X^{(n)})_*(p_{X \times \mathbb{P}^n}^{(m)})_*(i_f^{(m)}i_g)_*, \end{aligned}$$

the latter by Section 2.2.

It now suffices to prove the following

**Lemma 4.6.** *Let  $h : Z \rightarrow X$  be a projective morphism decomposed as*

$$Z \xrightarrow{I} X \times \mathbb{P}^n \times \mathbb{P}^m \xrightarrow{p_X^{(n,m)}} X$$

with even  $m$  and  $n$ , where  $I$  is a closed embedding and  $p_X^{(n,m)} = p_X^{(n)} \circ p_{X \times \mathbb{P}^n}^{(m)} = p_X^{(m)} \circ p_{X \times \mathbb{P}^m}^{(n)}$ . Then  $h_* = (p_X^{(n,m)})_* \circ I_*$ , where  $(p_X^{(n,m)})_*$  is defined as either of the compositions  $(p_X^{(n)})_*(p_{X \times \mathbb{P}^n}^{(m)})_*$  or  $(p_X^{(m)})_*(p_{X \times \mathbb{P}^m}^{(n)})_*$ , which are equal according to Section 3.3.

Now put  $h = fg$ ,  $I = i_f^{(m)}i_g$  and get  $f_*g_* = (fg)_*$ .

*Proof of the lemma.* Choose a closed embedding  $j : \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^N$  with  $N$  sufficiently large even and let  $J = \text{id}_X \times j : X \times \mathbb{P}^n \times \mathbb{P}^m \rightarrow X \times \mathbb{P}^N$ . Choose  $a \in \mathbb{P}^n(k)$ ,  $b \in \mathbb{P}^m(k)$  and let  $c = j(a, b) \in \mathbb{P}^N(k)$ . Then  $J \circ i_{X \times \mathbb{P}^n, b}^{(m)} \circ i_{X, a}^{(n)} = i_{X, c}^{(N)}$ , hence by 2.2

$$J_* \circ (i_{X \times \mathbb{P}^n, b}^{(m)})_* \circ (i_{X, a}^{(n)})_* = (i_{X, c}^{(N)})_*.$$

As  $m$ ,  $n$  and  $N$  are even, all the  $i_*$ 's are invertible and we get by Definition 3.2(i)

$$(p_X^{(N)})_* \circ J_* = (p_X^{(n)})_* \circ (p_{X \times \mathbb{P}^n}^{(m)})_* = (p_X^{(n,m)})_*.$$

Thus

$$(p_X^{(n,m)})_* I_* = (p_X^{(N)})_* J_* I_* = (p_X^{(N)})_* (JI)_* = h_*,$$

where the last equation is true by Proposition 4.2 since  $J \circ I$  is a closed embedding.  $\blacksquare$

*Remark.* (a) Lemma 4.6 can be viewed as a generalization of Proposition 4.2, for we get the latter if  $m = 0$  or  $n = 0$ . It yields more flexibility in factoring projective

morphisms to obtain their push-forwards. The obvious generalization with multiple projective spaces is also true.

(b) The assumption that  $m$ ,  $n$  and  $N$  are even is not really necessary. For our purposes it suffices to prove the lemma under this assumption; the general case is left to the reader.

#### 4.7. Transversal base changes.

**Definition 4.8.** A square of the form

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \phi_Y \downarrow & & \downarrow \phi_X \\ Y & \xrightarrow{f} & X \end{array} \quad (4.2)$$

with  $f$  projective is called transversal in  $Sm_k$  if it is cartesian and the induced square of vector bundles

$$\begin{array}{ccc} T_{Y'} & \longrightarrow & (f')^*T_{X'} \\ \downarrow & & \downarrow \\ \phi_Y^*T_Y & \longrightarrow & (f\phi_Y)^*T_X \end{array}$$

is bicartesian in the category of vector bundles on  $Y'$ . Equivalently, one can require that the sequence

$$0 \rightarrow T_{Y'} \xrightarrow{(d\phi_Y, -df')} \phi_Y^*T_Y \oplus (f')^*T_{X'} \xrightarrow{(df, d\phi_X)} (f\phi_Y)^*T_X \rightarrow 0 \quad (4.3)$$

is exact, cf. [Me, Axiom (iv) in Section 2] or [LM].

If  $f$  is factored as  $Y \xrightarrow{i} X \times \mathbb{P}^n \xrightarrow{p_X^{(n)}} X$  with  $i$  a closed embedding, then pulling back along  $\phi_X$  we get a factorization of the entire diagram:

$$\begin{array}{ccccc} Y' & \xrightarrow{i'} & X' \times \mathbb{P}^n & \xrightarrow{p_X^{(n)'}} & X' \\ \phi_Y \downarrow & & \downarrow \phi_X^{(n)} & & \downarrow \phi_X \\ Y & \xrightarrow{i} & X \times \mathbb{P}^n & \xrightarrow{p_X^{(n)}} & X \end{array} \quad (4.4)$$

In these terms, the transversality of (4.2) is equivalent to the transversality of the left square in (4.4) in the sense of Section 2.4. (The right square is always transversal.) The latter is stated in terms of normal bundles: the natural map  $N_{i'} \rightarrow \phi_Y^*N_i$  is required to be an isomorphism.

**Lemma 4.9.** *If (4.2) is transversal and  $L$  is a line bundle on  $X$ , then the line bundles  $\phi_Y^*L^f$  and  $(\phi_X^*L)^{f'}$  on  $Y'$  can be identified canonically.*

*Proof.* The dual of (4.3) provides a natural isomorphism

$$\det(\phi_Y^*\Omega_Y \oplus (f')^*\Omega_{X'}) \cong \det \Omega_{Y'} \otimes \det (f\phi_Y)^*\Omega_X$$

which yields a natural isomorphism  $\phi_Y^* \omega_f \cong \omega_{f'}$ . (Compare to (3.4) where the same was verified in the case of projections by a more explicit computation.) It follows that

$$(\phi_X^* L)^{f'} = (f')^* \phi_X^* L \otimes \omega_{f'} \cong \phi_Y^* f^* L \otimes \phi_Y^* \omega_f \cong \phi_Y^* (f^* L \otimes \omega_f) = \phi_Y^* L^f.$$

The lemma is proved. ■

As we have transversal base changes for projections and closed embeddings, see Sections 2.4 and 3.4, we can derive the same property for arbitrary projective morphisms by using (4.4). Lemma 4.9 guarantees that the twists agree.

**Proposition 4.10.** *Suppose that  $f$  and (therefore)  $f'$  are projective equicodimensional morphisms of codimension  $c$  in a transversal square of the form (4.2). Let  $S, T, S', T'$  be compatible closed subschemes in  $Y, X, Y', X'$ , respectively. Then the diagram commutes:*

$$\begin{array}{ccc} W_{S'}^q(Y'; \phi_{Y'}^* L^f) & \xrightarrow{f'_*} & W_{T'}^{q+c}(X'; \phi_{X'}^* L) \\ \phi_Y^* \downarrow & & \downarrow \phi_X^* \\ W_S^q(Y; L^f) & \xrightarrow{f_*} & W_T^{q+c}(X; L) \end{array}$$

**4.11. Projection formulas.** Let  $f : Y \rightarrow X$  be a projective morphism of codimension  $c$ ,  $T$  and  $T'$  be closed subschemes in  $X$  and  $S \subset f^{-1}(T)$  a closed subscheme in  $Y$ , and let  $L$  and  $L'$  be line bundles on  $X$ . Then for any  $\alpha \in W_{T'}^{q'}(X; L')$  and  $\beta \in W_S^q(Y; L^f)$  we have

$$f_*(f^* \alpha \star \beta) = \alpha \star f_* \beta \text{ and } f_*(\beta \star f^* \alpha) = (-1)^{cq'} f_* \beta \star \alpha$$

in  $W_{T \cap T'}^{q'+q+c}(X; L \otimes L')$ . We get these formulas by factoring  $f$  and applying the respective formulas of Sections 2.7 and 3.7.

## REFERENCES

- [Ba1] P. Balmer, *Triangular Witt groups. Part I: The 12-term localization exact sequence*, *K-Theory* **19** (2000), 311-363.
- [Ba2] P. Balmer, *Triangular Witt groups. Part II: From usual to derived*, *Math. Zeitschr.* **236** (2001), 351-382.
- [Ba3] P. Balmer, *Witt cohomology, Mayer-Vietoris, homotopy invariance and the Gersten conjecture*, *K-Theory* **23** (2001), no. 1, 15-30.
- [CH] B. Calmès and J. Hornbostel, *Witt motives, transfers and dévissage*, preprint (2005), [www.math.jussieu.fr/~calmes/witt/witt.html](http://www.math.jussieu.fr/~calmes/witt/witt.html).
- [Fu] W. Fulton, *Intersection Theory*, Springer-Verlag, 1984.
- [G1] S. Gille, *A transfer morphism for Witt groups*, *J. reine angew. Math.* **564** (2003), 215-233.
- [G2] S. Gille, *Homotopy invariance of coherent Witt groups*, *Math. Zeitschrift* **244** (2003), 211-233.
- [G3] S. Gille, *The general dévissage theorem for Witt groups of schemes*, *Archiv Math.* **88** (2007), 333-343.
- [GN] S. Gille and A. Nenashev, *Pairings in triangular Witt theory*, *J. Algebra* **261** (2003), 292-309.
- [LM] M. Levine and F. Morel, *Cobordisme algébrique I*, *C. R. Acad. Sci. Paris, Sér. I Math.* **332** (2001), no. 8, 723-728.

- [Me] A. Merkurjev, *Algebraic oriented cohomology theories*, Algebraic Number Theory and Algebraic Geometry, Contemp. Math., Vol. 300, 171-194; American Math. Soc., Providence, R.I., 2002.
- [Ne1] A. Nenashev, *Gysin maps in Balmer-Witt theory*, J. Pure Appl. Algebra **211** (2007), 203-221.
- [Ne2] A. Nenashev, *On the Witt groups of projective bundles and split quadrics: geometric reasoning*, preprint at [www.math.uiuc.edu/K-theory/0696/](http://www.math.uiuc.edu/K-theory/0696/), to appear in K-Theory.
- [PS] I. Panin (after I. Panin and A. Smirnov), *Push-forwards in oriented cohomology theories of algebraic varieties: II*, preprint at <http://www.math.uiuc.edu/K-theory/0619/> (2003).
- [PS1] I. Panin (after I. Panin and A. Smirnov), *Oriented cohomology theories of algebraic varieties*, K-Theory **30** (2003), 265-314.
- [SP] A. Smirnov (after I. Panin and A. Smirnov), *Orientations and transfers in the cohomology of algebraic varieties*, Algebra i Analiz **18** (2006), no. 2, 167–224.
- [W] C. Walter, *Grothendieck-Witt groups of projective bundles*, preprint at [www.math.uiuc.edu/K-theory/0645/](http://www.math.uiuc.edu/K-theory/0645/) (2003).