

NIVEAU SPECTRAL SEQUENCE AND FAILURE OF GERSTEN-TYPE CONJECTURE, ON SINGULAR SCHEMES

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ABSTRACT. Using the niveau filtration of the category of perfect complexes, we construct a spectral sequence for Thomason’s non-connective K -theory, giving in particular a Gersten-type complex for singular schemes. Our spectral sequence coincides with Thomason’s hypercohomology spectral sequence exactly when this Gersten complex is locally exact, a condition which fails for general singular schemes.

Our main result is the following application of abstract triangular geometry [2].

Theorem 1. *Let X be a (topologically) noetherian scheme of finite Krull dimension. Then there exists a spectral sequence whose first page is*

$$(1) \quad E_1^{p,q} = \bigoplus_{x \in X^{(p)}} K_{-p-q}(\mathcal{O}_{X,x} \text{ on } \{x\}) \quad \text{for } p, q \in \mathbb{Z},$$

converging towards $K_{-n}(X)$ for $n \in \mathbb{Z}$, “along $n = p + q$ ”, that is, the indexing in the spectral sequence is such that $d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ for $r \geq 1$. Here, $K_*(X \text{ on } Y)$ stands for Thomason’s non-connective (or Bass) K -theory of those perfect complexes of \mathcal{O}_X -modules which are acyclic on $X - Y$, see [12, § 6].

Note the presence of negative K -groups, which is crucial throughout the paper.

For X regular, this coniveau spectral sequence is due to Quillen [10, Thm. 5.4], who used *dévissage* to replace the local terms by K -groups of residue fields. Ever since, it has been an interesting problem to extend this result to singular schemes. Important progress appeared even before Thomason, e.g. in Levine [8], but under some restrictions. The above Theorem seems to provide the most general coniveau spectral sequence one could wish for. As we shall see, its proof is a direct application of two results: First, the long exact sequence including negative K -groups due to Schlichting [11] and secondly, the author’s description of the *idempotent completion* of the successive quotients of the coniveau filtration of $D^{\text{perf}}(X)$, see [2]. These two results appeared only very recently but as two new branches of the same trunk, namely Thomason’s insightful paper [12]. The reader might even wonder how Thomason “missed” such a result. One of the reasons might be that he gave another spectral sequence in [12, Thm. 10.3], the *hypercohomology spectral sequence*:

$$(2) \quad E_2^{p,q} = H_{\text{Zar}}^p(X, \mathcal{K}_{-q}) \xrightarrow{p+q=n} K_{-n}(X),$$

where \mathcal{K}_m is the sheafification of the presheaf $U \mapsto K_m(U)$, for $m \in \mathbb{Z}$. It is legitimate to ask whether the E_2 -page of our spectral sequence (1) coincides with that of (2). This fails in general, see Remark 6.

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Actually, we can also construct a *niveau* spectral sequence, involving dimension instead of codimension. To synthesize, let us consider a general *dimension function*

$$(3) \quad \dim : \text{Irred}(X) \longrightarrow \mathbb{Z}$$

on the set $\text{Irred}(X)$ of irreducible closed subsets of X ; i.e. we assume $\dim(Y) < \dim(Y')$ for every two $Y, Y' \in \text{Irred}(X)$ such that $Y \subset Y'$ and $Y \neq Y'$. The Krull dimension $\dim(Y) := \dim_{\text{Krull}}(Y)$ is an example, and so is the opposite of the Krull codimension $\dim(Y) := -\text{codim}_{\text{Krull}}(Y)$; but there are other dimension functions, e.g. in intersection theory. As usual, we extend our dimension function to any non-empty closed $Z \subset X$ by $\dim(Z) := \max \{ \dim(Y) \mid Y \in \text{Irred}(X) \text{ and } Y \subset Z \} \in \mathbb{Z}$ (and $\dim(\emptyset) := -\infty$). We moreover assume that there exists $d \in \mathbb{N}$ such that

$$(4) \quad -d \leq \dim(Z) \leq d$$

for every non-empty closed subset $Z \subset X$. This obviously holds for \dim_{Krull} and for $-\text{codim}_{\text{Krull}}$ when X has finite Krull dimension.

For every $i \in \mathbb{Z}$, we define $X_{(i)} := \{ x \in X \mid \dim(\overline{\{x\}}) = i \}$. For the dimension function $-\text{codim}_{\text{Krull}}$, this $X_{(-i)}$ is the set $X^{(i)} = \{ x \in X \mid \dim_{\text{Krull}}(\mathcal{O}_{X,x}) = i \}$ of Theorem 1, which therefore becomes a special case of the following.

Theorem 2. *Let X be a topologically noetherian scheme with a dimension function (3) satisfying (4). Then there exists a converging spectral sequence*

$$E_1^{p,q} = \bigoplus_{x \in X_{(-p)}} K_{-p-q}(\mathcal{O}_{X,x} \text{ on } \{x\}) \xrightarrow{p+q=n} K_{-n}(X).$$

Proof. We define for each $p \in \mathbb{Z}$ the following subcategory of $\text{D}^{\text{perf}}(X)$

$$\text{D}_p = \text{D}_{(\leq p)}^{\text{perf}}(X) := \{ E \in \text{D}^{\text{perf}}(X) \mid \dim(\text{supph}(E)) \leq p \},$$

where the closed subset $\text{supph}(E) \subset X$ is the support of the total homology of the perfect complex E . Because of (4), we have a finite filtration

$$(5) \quad 0 = \text{D}_{-d-1} \subset \text{D}_{-d} \subset \cdots \subset \text{D}_{p-1} \subset \text{D}_p \subset \cdots \subset \text{D}_d = \text{D}^{\text{perf}}(X).$$

Since this filtration can be realized at the level of Waldhausen models, there is a spectral sequence starting with the Waldhausen K -theory of (some model of) $\text{D}_p / \text{D}_{p-1}$ and converging to Thomason's connective K -theory of X . This would lead to Quillen's classical spectral sequence for X regular. Unfortunately, there is no good description of $\text{D}_p / \text{D}_{p-1}$ when X is singular. Still, following Thomason's insight of [12, § 5], we can instead describe the *idempotent completion* of this quotient. Recall from [3] that the idempotent completion $\widetilde{\mathcal{K}}$ of a triangulated category \mathcal{K} inherits a unique triangulation such that $\mathcal{K} \hookrightarrow \widetilde{\mathcal{K}}$ is exact. Then, by [2, Thm. 3.24 and § 4.1], we know that for every $p \in \mathbb{Z}$, localization induces an equivalence

$$(6) \quad \widetilde{\text{D}_p / \text{D}_{p-1}} \xrightarrow{\sim} \coprod_{x \in X_{(p)}} \text{D}_{\{x\}}^{\text{perf}}(\mathcal{O}_{X,x})$$

between the idempotent completion of the quotient $\text{D}_p / \text{D}_{p-1}$ and the coproduct over $x \in X_{(p)}$ of the derived categories of perfect complexes of $\mathcal{O}_{X,x}$ -modules with homology supported on the closed point $x \in \text{Spec}(\mathcal{O}_{X,x})$.

The K -theoretic price to pay for introducing idempotent completions is the appearance of negative K -groups in the localization long exact sequence. To explain

this, we introduce some notations and results from Schlichting [11]. Let \mathfrak{M} be a collection of “models” and let $D(\mathcal{A})$ be a triangulated category functorially associated to $\mathcal{A} \in \mathfrak{M}$. Here \mathfrak{M} will be the category of complicial bi-Waldhausen categories and $D(\mathcal{A})$ the derived category of \mathcal{A} . We loosely speak of the non-connective K -theory of $D(\mathcal{A})$ to mean the non-connective K -theory of \mathcal{A} , as defined in [11].

We say that a sequence $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$ is exact in \mathfrak{M} if $D(\mathcal{A}) \rightarrow D(\mathcal{B}) \rightarrow D(\mathcal{C})$ is *exact up to direct summands*, meaning that $D(\mathcal{A})$ is a thick triangulated subcategory of $D(\mathcal{B})$, that the composite $D(\mathcal{A}) \rightarrow D(\mathcal{C})$ is zero and that the induced functor $D(\mathcal{B})/D(\mathcal{A}) \rightarrow D(\mathcal{C})$ is fully faithful and cofinal, i.e. is an equivalence after idempotent completion, which also means that any object of $D(\mathcal{C})$ is a direct summand of an object of $D(\mathcal{B})/D(\mathcal{A})$. Given such an exact sequence $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$, we have by [11, Thm. 1, Thm. 6 and § 6.5] a long exact sequence of K -groups:

$$\cdots \rightarrow K_1(\mathcal{B}) \rightarrow K_1(\mathcal{C}) \rightarrow K_0(\mathcal{A}) \rightarrow K_0(\mathcal{B}) \rightarrow K_0(\mathcal{C}) \rightarrow K_{-1}(\mathcal{A}) \rightarrow K_{-1}(\mathcal{B}) \rightarrow \cdots$$

In our situation, since the filtration (5) can be defined on the level of models, and using our abuse of notation $K_*(D(\mathcal{A})) = K_*(\mathcal{A})$, we get a long exact sequence

$$\cdots \rightarrow K_n(D_{p-1}) \rightarrow K_n(D_p) \rightarrow K_n(\widetilde{D_p/D_{p-1}}) \rightarrow K_{n-1}(D_{p-1}) \rightarrow \cdots \quad (n \in \mathbb{Z}).$$

So, we obtain as usual an exact couple and the associated spectral sequence, which looks as follows in cohomological indexing: $E_1^{p,q} = K_{-p-q}(\widetilde{D_{-p}/D_{-p-1}})$. This spectral sequence has the wanted form by (6) and by agreement of the negative K -theory of [11] with Thomason’s, see [11, Thm. 5]. Convergence is clear since $E_1^{p,q}$ is concentrated in the band $-d \leq p \leq d$ by hypothesis (3). \square

Remark 3. We can replace everywhere K -theory by any cohomology theory which associates a long exact sequence to every short sequence of triangulated categories $\mathcal{J} \rightarrow \mathcal{K} \rightarrow \mathcal{L}$ which is exact up to direct summands (see above). This is true in particular for triangular Witt groups tensored with $\mathbb{Z}[\frac{1}{2}]$ by the localization theorem [1, 6.2 & 6.8] and the cofinality theorem of Hornbostel–Schlichting [7, App. A].

Definition 4. Let X be a (topologically) noetherian scheme of finite Krull dimension d . For any $q \in \mathbb{Z}$, we define the q^{th} *augmented weak Gersten complex*

$$\begin{aligned} 0 \rightarrow K_q(X) \rightarrow \bigoplus_{x \in X^{(0)}} K_q(\mathcal{O}_{X,x}) \rightarrow \bigoplus_{x \in X^{(1)}} K_{q-1}(\mathcal{O}_{X,x} \text{ on } \{x\}) \rightarrow \cdots \\ \cdots \rightarrow \bigoplus_{x \in X^{(p)}} K_{q-p}(\mathcal{O}_{X,x} \text{ on } \{x\}) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(d)}} K_{q-d}(\mathcal{O}_{X,x} \text{ on } \{x\}) \rightarrow 0. \end{aligned}$$

to be the $-q^{\text{th}}$ line of the E_1 -page of our coniveau spectral sequence (1), augmented by the edge homomorphism $K_q(X) \rightarrow \bigoplus_{x \in X^{(0)}} K_q(\mathcal{O}_{X,x})$. We dropped the “on $\{x\}$ ” when $x \in X^{(0)}$, since there $\{x\} = \text{Spec}(\mathcal{O}_{X,x})$. A similar complex exists with dimension instead of codimension and differs from the above in general. We call these complexes *weak Gersten complexes* because they differ from Gersten complexes for regular schemes in that we cannot replace the local terms by the K -groups of the residue fields, since we do not have *dévissage* in this context.

Remark 5. After Mochizuki [9], one might say that a local noetherian ring R *satisfies the generalized Gersten conjecture* if the augmented weak Gersten complexes are exact for $X = \text{Spec}(R)$, for all $q \in \mathbb{Z}$. This is equivalent to the vanishing of $K_n(D^{p+1}) \rightarrow K_n(D^p)$ for all $n \in \mathbb{Z}$ and $p \geq 0$, where D^p is D_{-p} from (5) for the dimension $- \text{codim}_{\text{Krull}}$, that is, $D^p = \{ E \in D^{\text{perf}}(X) \mid \text{codim}(\text{supph}(E)) \geq p \}$.

Remark 6. For $q \geq 2$ this conjecture is known to fail in general. For instance, for the local domain R obtained by localization of a cusp at its singularity, say $R = k[t^2, t^3]_{(t^2, t^3)}$, the (edge) homomorphism $K_2(R) \rightarrow K_2(k(t))$ is not injective as proven in Dennis–Sherman [6, Appendix], where more examples are provided.

Proposition 7. *If $\mathcal{O}_{X,x}$ satisfies the generalized Gersten conjecture for every $x \in X$ then the E_2 -page of our spectral sequence (1) agrees with that of Thomason (2). Conversely, for R local, if the E_2 -page of our spectral sequence agrees with Thomason’s on $X = \text{Spec}(R)$ then the generalized Gersten conjecture holds for R .*

Proof. The argument is standard. If the weak Gersten complex is locally exact on X , the weak Gersten complex defines a flasque (skyscraper) resolution of the sheaf \mathcal{K}_q . Hence our E_2 -page contains the Zariski cohomology of \mathcal{K}_q . The converse is obvious since Zariski cohomology H^p on a local scheme vanishes for $p \neq 0$. \square

Remark 8. By Proposition 7 and Remark 6, our coniveau spectral sequence (1) differs from Thomason’s (2) in general. Note that ours can be interesting even for local schemes. We now give an example of a local ring R for which the inclusion $D^{p+1} \hookrightarrow D^p$ fails to induce a monomorphism on K_n even for $n = 0$ (see Remark 5).

Let k be an infinite field and define the ring R as the localization of $k[x, y, z, w] = k[X, Y, Z, W]/\langle XW - YZ \rangle$ at the point $\mathfrak{m} = \langle x, y, z, w \rangle$. In [5], Dutta, Hochster and McLaughlin provide examples of finite length modules M which admit a finite projective resolution (that is $M \in D_{\{\mathfrak{m}\}}^{\text{perf}}(R) = D^3$) but with the property that

$$(7) \quad \chi(M, R/\mathfrak{p}) \neq 0 \quad (\text{and even } \chi(M, R/\mathfrak{p}) < 0)$$

for some $\mathfrak{p} \in \text{Spec}(R)^{(1)}$, for instance for $\mathfrak{p} = \langle x, y \rangle$, see [5, middle of p. 254]. Of course, $\chi(M_1, M_2) = \sum (-1)^i \text{length}(\text{Tor}_i^R(M_1, M_2))$ is Serre’s intersection multiplicity. Note that this $\chi(M_1, M_2) \in \mathbb{Z}$ is well-defined for any perfect complex $M_1 \in D^{\text{perf}}(R) = K^{\text{b}}(R\text{-free})$ and any finitely generated R -module M_2 satisfying $\text{supph}(M_1) \cap \text{supp}(M_2) \subset \{\mathfrak{m}\}$; in this case $\text{Tor}_i^R(M_1, M_2)$ simply is the homology in degree i of the (naive) tensor product $M_1 \otimes M_2$; these homology groups are of finite length since $\text{supph}(M_1 \otimes M_2) \subset \text{supph}(M_1) \cap \text{supp}(M_2) \subset \{\mathfrak{m}\}$.

Proposition 9 (Levine). *With the above notations, $[M] \in K_0(D^3)$ does not map to zero under the canonical homomorphism $K_0(D^3) \rightarrow K_0(D^2)$. In particular, this local ring R does not satisfy the generalized Gersten Conjecture of Remark 5.*

Proof. For each $a \in k$ consider the prime $\mathfrak{p}_a := \langle x - az, y - aw \rangle \in \text{Spec}(R)^{(1)}$. Note that for $a \neq b$, we have $\mathfrak{p}_a + \mathfrak{p}_b = \langle x, y, z, w \rangle = \mathfrak{m}$. Hence, for each $\mathfrak{q} \in \text{Spec}(R)$ with $\mathfrak{q} \neq \mathfrak{m}$, we have at most one $a \in k$ such that $\mathfrak{p}_a \subset \mathfrak{q}$. Consider now a closed $C \subset \text{Spec}(R)$ such that $\text{codim}(C) = 2$. Then $C = V(\mathfrak{q}_1) \cup \dots \cup V(\mathfrak{q}_n)$ for prime ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_n \in \text{Spec}(R)^{(2)}$. By the above consideration, since the field k is infinite, there exists an element $a \in k$ such that $\mathfrak{p}_a \not\subset \mathfrak{q}_i$ for all $i = 1, \dots, n$. Therefore, $C \cap V(\mathfrak{p}_a) = V(\mathfrak{q}_1 + \mathfrak{p}_a) \cup \dots \cup V(\mathfrak{q}_n + \mathfrak{p}_a) = \{\mathfrak{m}\}$.

Let us assume *ab absurdo* that $[M] \mapsto 0 \in K_0(D^2)$. Recall the notation $D_C^{\text{perf}}(R) := \{E \in D^{\text{perf}}(R) \mid \text{supph}(E) \subset C\}$. Since we have

$$K_0(D^2) = \text{colim}_{\substack{C \subset \text{Spec}(R) \\ \text{codim}(C) \geq 2}} K_0(D_C^{\text{perf}}(R)),$$

we can assume that $[M] \mapsto 0$ in $K_0(D_C^{\text{perf}}(R))$ for some closed $C \subset \text{Spec}(R)$ with $\text{codim}(C) = 2$. By the first part of the proof above, there exists $a \in k$ such

that $C \cap V(\mathfrak{p}_a) = \{\mathfrak{m}\}$. So, $\chi(E, R/\mathfrak{p}_a) \in \mathbb{Z}$ is defined for any $E \in D_C^{\text{perf}}(R)$. This expression is obviously additive on distinguished triangles in $D_C^{\text{perf}}(R)$. Hence $\chi(E, R/\mathfrak{p}_a)$ only depends on the class of E in $K_0(D_C^{\text{perf}}(R))$. Therefore, our absurd assumption $[M] = 0$ in $K_0(D_C^{\text{perf}}(R))$ implies

$$(8) \quad \chi(M, R/\mathfrak{p}_a) = 0.$$

Finally, since $M \in D_{\{\mathfrak{m}\}}^{\text{perf}}(R)$, the number $\chi(M, M_2)$ is defined for any finitely generated R -module M_2 and, as before, induces a well-defined homomorphism $\chi(M, -)$ on $K_0(R\text{-mod}) = G_0(R)$. To get a contradiction between (7) and (8), hence to finish the proof, it now suffices to show that $[R/\mathfrak{p}_a] = [R/\mathfrak{p}]$ in $G_0(R)$ for all $a \in k$. To see this, let us check that the following sequence is exact :

$$(9) \quad 0 \rightarrow \text{Ker}(\beta) \rightarrow R^2 \xrightarrow{\beta} R^2 \xrightarrow{\alpha_a} R \rightarrow R/\mathfrak{p}_a \rightarrow 0$$

where $\alpha_a = \begin{pmatrix} x - az & y - aw \\ -z & x \end{pmatrix}$ and $\beta = \begin{pmatrix} w & -y \\ -z & x \end{pmatrix}$, for this will imply that $[R/\mathfrak{p}_a] = [R] - [\text{Ker}(\beta)]$ in $G_0(R)$ and the latter is independent of $a \in k$. In fact, only exactness at the middle R^2 needs a proof, and this can be given in the spirit of [5, Lemma 2.2]. Since localization is exact, we prove exactness of (9) with R replaced by $R = k[x, y, z, w] = k[X, Y, Z, W]/\langle XW - YZ \rangle$. Let $\begin{pmatrix} u \\ v \end{pmatrix} \in R^2$ in the kernel of α_a . Let $U, V \in k[X, Y, Z, W]$ such that $U \mapsto u$ and $V \mapsto v$ in R . Then $\alpha_a \cdot \begin{pmatrix} u \\ v \end{pmatrix} = 0$ reads

$$(X - aZ)U + (Y - aW)V = (XW - YZ)G$$

for some $G \in k[X, Y, Z, W]$. A direct computation from the above gives

$$(X - aZ)(U - WG) = -(Y - aW)(V + ZG).$$

This implies that $(Y - aW) \mid (U - WG)$, i.e. $U = WG + (Y - aW)H$ for some $H \in k[X, Y, Z, W]$. Replacing U in the last equation and cancelling by $(Y - aW)$ gives $V = -ZG - (X - aZ)H$. Defining g and h in R to be the classes of G and H respectively, we obtain $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} w & -y \\ -z & x \end{pmatrix} \cdot \begin{pmatrix} g - ah \\ -h \end{pmatrix}$, as was to be shown. \square

Finally, we discuss a condition for (1) to be a fourth quadrant spectral sequence.

Conjecture 10 (Weibel [13, 2.9]). *For X a noetherian scheme of Krull dimension d , we have $K_m(X) = 0$ for $m < -d$.*

This Conjecture has recently been proven for X essentially of finite type over a field of characteristic zero, by Cortiñas–Haesemeyer–Schlichting–Weibel [4]. Let us check that this Conjecture is of local nature.

Proposition 11. *Let X be a noetherian scheme of Krull dimension d . Then the following are equivalent :*

- (i) *Conjecture 10 holds for the local rings $\mathcal{O}_{X,x}$ for all $x \in X$.*
- (ii) *For all $p \geq 0$, all $x \in X^{(p)}$ and all $q > 0$ we have $K_{-p-q}(\mathcal{O}_{X,x} \text{ on } \{x\}) = 0$.*

When (i) and (ii) hold true, then X satisfies Conjecture 10 as well.

Proof. Claim : For X as above, if $K_{-p-q}(\mathcal{O}_{X,x} \text{ on } \{x\}) = 0$ for all $0 \leq p \leq d - 1$ and for all $q > 0$, then for $n > d$, we have $\bigoplus_{x \in X^{(d)}} K_{-n}(\mathcal{O}_{X,x} \text{ on } \{x\}) \cong K_{-n}(X)$.

The hypothesis of the Claim means that our $E_1^{p,q}$ is zero for $p \neq d$ and $q > 0$. So, for $q > 0$, we have $E_1^{p,q} = E_\infty^{p,q}$ and this group vanishes except for $p = d$ where it gives the above direct sum. The Claim now follows from Theorem 2.

Assume (i) and let us prove (ii) by induction on $p \geq 0$. For $p = 0$, we have of course $K_*(\mathcal{O}_{X,x} \text{ on } \{x\}) = K_*(\mathcal{O}_{X,x})$, so (ii) is exactly Conjecture 10. For $p \geq 1$, apply the Claim to the scheme $\text{Spec}(\mathcal{O}_{X,x})$. The hypothesis of the Claim holds by induction hypothesis. The Claim then says that for $n > p$ we have $K_{-n}(\mathcal{O}_{X,x} \text{ on } \{x\}) = K_{-n}(\mathcal{O}_{X,x})$ and the latter vanishes by (i). This proves (ii).

Combining the Claim and the convergence of (1), we see that (ii) implies Conjecture 10 for X . The latter implication for $X = \text{Spec}(\mathcal{O}_{X,x})$ gives (i) as well. \square

Corollary 12. *Conjecture 10 is equivalent to: For every noetherian local ring (R, \mathfrak{m}) of Krull dimension d and all $n > d$, we have $K_{-n}(\text{Spec}(R) \text{ on } \{\mathfrak{m}\}) = 0$. This is also equivalent to our coniveau spectral sequence (1) being concentrated in the fourth quadrant ($p \geq 0$ and $q \leq 0$) for every finite dimensional noetherian scheme X .*

Proof. Contemplate condition (ii) in Proposition 11 and the definition of $E_1^{p,q}$. \square

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