

Higher Tits indices of linear algebraic groups

Viktor Petrov, Nikita Semenov

Abstract

Let G be a semisimple algebraic group over a field k . We introduce the higher Tits indices of G as the set of all Tits indices of G over all field extensions K/k . In the context of quadratic forms this notion coincides with the notion of the higher Witt indices introduced by M. Knebusch and classified by N. Karpenko and A. Vishik.

Next we classify the higher Tits indices for exceptional algebraic groups. Our main tools involve the Chow groups and the Chow motives of projective homogeneous varieties, Steenrod operations as well as the notion of the J -invariant introduced in [PSZ07].

1 Introduction

Let G denote a semisimple algebraic group of inner type defined over a field k . In his famous paper [Ti66] J. Tits defined the *Tits index* of G as a data consisting of the Dynkin diagram of G with some vertices being circled. Let K be an arbitrary field extension of k . In the present paper we investigate the following problem: What values can take the Tits index of the group G_K ?

In the theory of central simple algebras (i.e., when G is a group of type A_n) this problem is equivalent to the index reduction formula of A. Blanchet, A. Schofield, and M. Van den Bergh (see [SVB92]). Later their result was generalized by A. Merkurjev, I. Panin, and A. Wadsworth (see [MPW96] and [MPW98]).

In the theory of quadratic forms (i.e., when G is an orthogonal group) the problem above is equivalent to study of the *higher Witt indices* of quadratic forms. The higher Witt indices were introduced by M. Knebusch [Kn76]. They provide a nice discrete invariant of quadratic forms over any field k . Numerous results have been obtained so far. One of the main achievements

here is the celebrated result of N. Karpenko [Ka03], where he proves Hoffmann’s conjecture about all possible values of the first higher Witt index. One should also mention the interesting papers of N. Karpenko, A. Merkurjev and A. Vishik [Ka04], [KM03], [Vi04], [Vi06] concerning closely related problems in the theory of quadratic forms.

The main result of M. Knebusch in his paper [Kn76] asserts that one can always reduce the problem above to a certain finite number of field extensions K_i/k , $i = 0, \dots, h$ such that the Tits index of G over a field K/k equals one of the Tits indices of G_{K_i} . The fields K_i appearing in the Knebusch theory are the fields of rational functions on certain projective G -homogeneous varieties (see Section 2 below).

Moreover, the result of Karpenko [Ka04, Theorem 2.6] asserts that information on the higher Tits indices is hidden in a subring of the Chow ring of certain projective G -homogeneous varieties¹. In the present paper we exploit further this connection. Our main tools include the Steenrod operations in the Chow theory, Tits’ classification, Chow motives, and motivic invariants, like the *J-invariant* of algebraic groups introduced in [PSZ07].

It turns out, that in the most cases the splitting behaviour of an algebraic group G does not depend on the base field k , but on triviality and non-triviality of certain discrete invariant of G called the *J-invariant*. By definition this invariant measures the “size” of the subring of rational cycles in the Chow ring of the G -variety of complete flags (see Section 4). But usually the (non)-triviality of this invariant can be expressed in terms of (non)-triviality of the Tits algebras of G and/or of certain cohomological invariants of G , like the Rost invariant.

In particular, we prove that any anisotropic group of type E_7 over any field k with trivial Tits algebras or with a Tits algebra of index ≥ 4 has anisotropic kernel of type D_6 over a suitable field extension K/k of the base field. The main results of our paper are Theorems 5.3 and 6.1 that allow to compute all possible higher Tits indices for groups of type F_4 , 1E_6 , and E_7 with trivial Tits algebras, and to classify all generically cellular varieties of exceptional types.

Acknowledgements ...

¹His results concerns only quadrics, but can be straightforwardly generalized to arbitrary projective homogeneous varieties.

2 Tits' classification and Knebusch theory

We recall the definition and basic properties of the Tits indices of semisimple algebraic groups following J. Tits [Ti66] and [Ti71].

Let k be a field, k_s be a separable closure of k , G a semisimple algebraic group defined over k , S a maximal split torus of G defined over k , T a maximal torus containing S and defined over k , $\Delta = \Delta(G)$ the system of simple roots of G with respect to T and $\Delta_0 = \Delta_0(G)$ the subsystem of those roots which vanish on S .

In the present paper we consider only groups of *inner type*, i.e., groups that are twisted forms of a split group by means of a 1-cocycle in $H_{\text{et}}^1(k, G_0)$, where G_0 denotes the split adjoint group over k of the same type as G . Equivalently, this means that the $*$ -action (see [Ti66]) of the absolute Galois group on Δ is trivial. Therefore we don't define the $*$ -action and don't include it in the definition of the Tits index of G .

The *index* of G is a pair (Δ, Δ_0) . We represent the index as the Dynkin diagram of G with the vertices that don't belong to Δ_0 being circled.

There exists a certain subgroup of G called *semisimple anisotropic kernel*. We refer the reader to [Ti66] for its definition. Note that the index of the semisimple anisotropic kernel of G can be easily deduced from the index of G by removing the vertices of the Dynkin diagram which are circled.

To any semisimple group over k one can functorially associate certain central simple algebras, called the *Tits algebras*. We refer the reader to [Ti71] for a definition and description.

2.1 Examples. 1. Let A be a central simple k -algebra of degree $n + 1$ and $G = \text{SL}_1(A)$ the respective group of type A_n . Then the index of A equals $\frac{n+1}{r+1}$, where r is the number of circled vertices on the Tits diagram of G .

The Tits algebras of G are the λ -powers $\lambda^i A$, $i = 1, \dots, n$.

2. Let (V, q) be a regular odd-dimensional quadratic space over k and $G = \text{Spin}(V, q)$ be the respective group of type B_n . Then the number of circled vertices on its Tits diagram equals the Witt index of q .

The Tits algebra of G is the even Clifford algebra $C_0(V, q)$.

One can give similar descriptions for all semisimple algebraic groups over k .

Next we recall the construction of the generic splitting tower of Knebusch for semisimple algebraic groups (see [Kn76], [Kn77]).

Consider the set

$$\{\text{ind}(G_K)_{\text{an}} \mid K/k \text{ is a field extension}\}, \quad (1)$$

where $\text{ind}(G_K)_{\text{an}}$ stands for the Tits index of the semisimple anisotropic kernel of G_K .

2.2 Definition. Set (1) is called the set of the Higher Tits Indices of G .

This set can be obtained using the *generic splitting tower*, i.e., it suffices to consider not all field extensions K/k , but just a finite number of *generic* ones. The latter are defined inductively as follows.

First we set $K^0 = k$, $G^1 = G_{\text{an}}$ and consider the function fields $K_i^1 = K^0(X_i)$, $i \in \Delta(G^1) \subset \Delta(G)$, of the projective varieties of the maximal parabolic subgroups of G^1 of type i . Note that there are precisely $\text{rk}G_{\text{an}} = |\Delta(G_{\text{an}})|$ such varieties.

Next for each $j \in \Delta(G^1)$ we consider the group $G_j^2 = (G_{K_j^1}^1)_{\text{an}}$ over K_j^1 and apply the same procedure, i.e., consider the fields $K_i^2 = K_j^1(X_i)$, $i \in \Delta(G_j^2) \subset \Delta(G)$, where X_i stands for the projective variety over K_j^1 of the maximal parabolic subgroups of type i of the group G_j^2 . Proceeding further we obtain a set (a tower) of fields K_*^* . Its main property is that

$$\{\text{ind}(G_K)_{\text{an}} \mid K/k \text{ is a field extension}\} \quad (2)$$

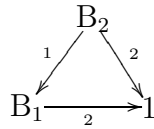
$$= \{\text{ind}(G_{K_*^j})_{\text{an}} \mid K_*^j/k \text{ is an element in the generic splitting tower of } G\}.$$

The maximal value of the upper index of K_*^* 's is called the *height* of G .

Conversely, given the higher Tits index of G , the Tits index of $G_{k(X_i)}$ can be restored as the minimal higher Tits index containing i . An overview of this ideas can be found in [KR94].

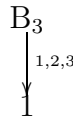
Sometimes it is convenient to represent the right hand side of identity (2) as an oriented labelled graph as following. First we define a graph whose vertices are the anisotropic groups G_j^s appearing in the construction above and which we denote by their Dynkin diagrams. There is an edge from G_j^s to $G_{j'}^{s'}$ with the label i if and only if $s' = s+1$, $i = j'$ and $G_i^{s+1} = ((G_j^s)_{K_j^{s-1}(X_i)})_{\text{an}}$. Next we identify the vertices of this graph which correspond to the same Dynkin types. Thus, the vertices of this graph represent all possible Tits indices of G_K without repetitions for all K/k . The height of G is the maximal length of the paths on the graph.

2.3 Examples. 1. Let G be an anisotropic group of type B_2 . Then its splitting graph is:

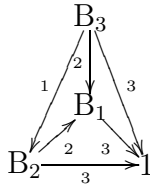


(enumeration of simple roots follows Bourbaki).

2. Let G be an anisotropic group of type B_3 and q the respective quadratic form of discriminant 1. Then its splitting graph is:



Clifford invariant and B_3 otherwise.



There exist generalizations for quadratic forms of bigger dimensions due to N. Karpenko and A. Vishik.

The pictures in the examples remind an automaton. Therefore one can call the graphs defined above the *Tits automata*.

2.4 Remark. Note that the first step, i.e., the groups G_*^2 are the most important ones. Indeed, the next groups $G_*^{\geq 3}$ involved in the construction are anisotropic kernels of the groups G_*^2 . Since the rank of G_*^2 is smaller than the rank of G^1 , we are deduced to the same situation but for groups of a smaller rank.

3 Cycles on projective homogeneous varieties and Chow motives

In this section we briefly describe the main properties of projective homogeneous varieties and their Chow rings (see [De74], [Hi82]).

Let G be a split semisimple algebraic group of rank n defined over a field k . We fix a split maximal torus T in G and a Borel subgroup B of G containing T and defined over k . We denote by Φ the root system of G , by $\Pi = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots of Φ with respect to B , by W the Weyl group, and by $S = \{s_1, \dots, s_n\}$ the corresponding set of fundamental reflections.

Let $P = P_\Theta$ be the (standard) parabolic subgroup corresponding to a subset $\Theta \subset \Pi$, i.e., $P = BW_\Theta B$, where $W_\Theta = \langle s_\theta, \theta \in \Theta \rangle$. Denote

$$W^\Theta = \{w \in W \mid \forall s \in \Theta \quad l(ws) = l(w) + 1\},$$

where l is the length function. It is easy to see that W^Θ consists of all representatives in the left cosets W/W_Θ which have minimal length.

As P_i we denote the maximal parabolic subgroup $P_{\Pi \setminus \{\alpha_i\}}$ of type i and as w_0 the longest element of W . Enumeration of simple roots follows Bourbaki.

It is well known that any projective G -homogeneous variety X is isomorphic to G/P_Θ for some subset Θ of the simple roots.

Now consider the Chow ring of the variety $X = G/P_\Theta$. It is known that $\text{CH}^*(G/P_\Theta)$ is a free abelian group with a basis given by varieties $[X_w]$ that correspond to the elements $w \in W^\Theta$. The degree (codimension) of the basis element $[X_w]$ equals $l(w_\theta) - l(w)$, where w_θ is the longest element of W_Θ .

Moreover, there exists a natural injective pull-back homomorphism

$$\begin{aligned} \text{CH}^*(G/P) &\rightarrow \text{CH}^*(G/B) \\ [X_w] &\mapsto [X_{ww_\theta}] \end{aligned}$$

The following results provide tools to perform computations in the Chow ring $\text{CH}(G/P_\Theta)$.

In order to multiply two basis elements $h = [X_w]$ and $g = [X_{w'}]$ of $\text{CH}^*(G/P_\Theta)$ such that $\deg h + \deg g = \dim G/P_\Theta$ we use the following formula (Poincaré duality):

$$[X_w] \cdot [X_{w'}] = \delta_{w, w_0 w' w_\theta} \cdot [X_1]. \quad (3)$$

In view of Poincaré duality we denote as $[Z_w]$ the cycle dual to $[X_w]$ with respect to the canonical basis. In other words, $[Z_w] = [X_{w_0 w w_\theta}]$.

In order to multiply two basis elements of $\text{CH}^*(G/B)$ one of which is of codimension 1 we use the following formula (Pieri formula):

$$[X_{w_0 s_\alpha}][X_w] = \sum_{\beta \in \Phi^+, l(ws_\beta)=l(w)-1} \langle \beta^\vee, \bar{\omega}_\alpha \rangle [X_{ws_\beta}], \quad (4)$$

where α is a simple root and the sum runs through the set of positive roots $\beta \in \Phi^+$, s_β denotes the reflection corresponding to β and $\bar{\omega}_\alpha$ is the fundamental weight corresponding to α . Here $[X_{w_0 s_\alpha}]$ is the element of codimension 1.

The *Poincaré polynomial* of a free \mathbb{Z} -graded finitely generated group A^* is, by definition, the polynomial $g(A^*, t) = \sum_{i=-\infty}^{+\infty} a_i t^i \in \mathbb{Z}[t, t^{-1}]$ with $a_i = \text{rk} A^i(X)$. The following formula (the Solomon theorem) allows to compute the Poincaré polynomial of $\text{CH}^*(X)$:

$$g(\text{CH}^*(X), t) = \frac{r(\Pi)}{r(\Theta)}, \quad r(-) = \prod_{i=1}^l \frac{t^{d_i(-)} - 1}{t - 1}, \quad (5)$$

where $d_i(\Theta)$ (resp. $d_i(\Pi)$) denote the degrees of the fundamental polynomial invariants of the root subsystem of Φ generated by Θ (resp. Π) and l its rank (see [Ca72]). The dimension of X equals $\text{deg} g(\text{CH}^*(X), t)$. There exists a Maple package [St] of J. Stembridge that provides tools to compute the Poincaré polynomials of projective G -homogeneous varieties.

Let $\mathbb{P} = \mathbb{P}(\Phi)$ denote the weight space. We denote as $\bar{\omega}_1, \dots, \bar{\omega}_n$ the basis of \mathbb{P} consisting of the fundamental weights. The symmetric algebra $S^*(\mathbb{P})$ is isomorphic to $\mathbb{Z}[\bar{\omega}_1, \dots, \bar{\omega}_n]$. The Weyl group W acts on \mathbb{P} , hence, on $S^*(\mathbb{P})$. Namely, for a simple root α_i

$$s_i(\bar{\omega}_j) = \begin{cases} \bar{\omega}_i - \alpha_i, & i = j; \\ \bar{\omega}_j, & \text{otherwise.} \end{cases}$$

We define a linear map $c: S^*(\mathbb{P})^{W_\Theta} \rightarrow \text{CH}^*(G/P_\Theta)$ as follows. For a homogeneous W_Θ -invariant $u \in \mathbb{Z}[\bar{\omega}_1, \dots, \bar{\omega}_n]$

$$c(u) = \sum_{w \in W^\Theta, l(w) = \text{deg}(u)} \Delta_w(u)[X_{w_0 w w_\theta}],$$

where for $w = s_{i_1} \dots s_{i_k}$ we denote by Δ_w the composition of derivations $\Delta_{s_{i_1}} \circ \dots \circ \Delta_{s_{i_k}}$ and the derivation $\Delta_{s_i}: S^*(\mathbb{P}) \rightarrow S^{*-1}(\mathbb{P})$ is defined by $\Delta_{s_i}(u) = \frac{u - s_i(u)}{\alpha_i}$.

Let $U = \Sigma_u(P_\Theta)$ denote the set of the (positive) roots lying in the unipotent radical of the parabolic subgroup P_Θ . Then the elementary symmetric polynomials $\sum_{u \in U} \sigma_i(u)$ are W_{P_Θ} -invariant and, in fact, coincide with the Chern classes of the tangent bundle T_X :

$$c(T_X) = c\left(\prod_{\gamma \in U} (1 + \gamma)\right). \quad (6)$$

The Maple package [map] provides efficient tools to compute the Chern classes of the tangent bundles.

To multiply two cycles $[X_{w_1}]$ and $[X_{w_2}]$ in $\text{CH}^*(X)$ we proceed as follows. First, we find preimages of $[X_{w_1}]$ and $[X_{w_2}]$ in $S^*(P) \otimes \mathbb{Q} = \mathbb{Q}[\bar{\omega}_1, \dots, \bar{\omega}_n]$ (the preimages always exist; see below), then we either expand the product in the polynomial ring $\mathbb{Q}[\bar{\omega}_1, \dots, \bar{\omega}_n]$ and apply the function c , or apply c directly using the Pieri formula (4), the Leibniz rule [Hi82, Ch. IV, Lemma 1.1(e)] or/and Poincaré duality (3).

To find a preimage of some $[X_w]$ we do the following. It is well known that the map $c \otimes \mathbb{Q}: \mathbb{Q}[\bar{\omega}_1, \dots, \bar{\omega}_n]^{W_\Theta} \rightarrow \text{CH}^*(G/P_\Theta) \otimes \mathbb{Q}$ defined above is a ring epimorphism, and the ring $\mathbb{Q}[\bar{\omega}_1, \dots, \bar{\omega}_n]^{W_\Theta}$ is generated by $\bar{\omega}_i$, $i \notin \Theta$, and by the W_Θ -invariant fundamental polynomials for the semisimple part of the Levi part of P_Θ , i.e., for the split group of type $\langle \Theta \rangle \subset \Phi$. The latter polynomials (as well as their degrees called degrees of fundamental polynomial invariants) are known. Explicit formulas for them are provided in [Meh88]. Now, since we know a generating set of $\mathbb{Q}[\bar{\omega}_1, \dots, \bar{\omega}_n]^{W_\Theta}$, we can compute its image in $\text{CH}^*(G/P_\Theta) \otimes \mathbb{Q}$ and, thus, find a set of generators of $\text{CH}^*(G/P_\Theta) \otimes \mathbb{Q}$ together with their preimages in $\mathbb{Q}[\bar{\omega}_1, \dots, \bar{\omega}_n]^{W_\Theta}$. Therefore we can compute a preimage of any element in $\text{CH}^*(G/P_\Theta) \otimes \mathbb{Q}$. Observe that we don't lose any information extending scalars to \mathbb{Q} , since the group $\text{CH}^*(G/P_\Theta)$ is free abelian.

The effective procedures to multiply cycles in the Chow rings of projective homogeneous varieties are implemented in the Maple package [map]².

Next we briefly describe Steenrod operations and motivic decompositions of projective G -homogeneous varieties with isotropic group G . The main reference to this account can be found in [CGM05] and [Br05]. We refer the reader to the book [EKM] of R. Elman, N. Karpenko, and A. Merkurjev or to the original paper [Ma68] of Yu. Manin for the definition and properties of the Chow motives.

The main result of papers [CGM05] and [Br05] asserts that the Chow motive of a projective G -homogeneous variety X with isotropic group G is decomposable as a direct sum of (twisted) motives of anisotropic projective G_{an} -homogeneous varieties Y_i . Moreover, one has an explicit algorithm to compute these motivic decompositions.

In the present section we give a combinatorial interpretation of these decompositions in terms of the Hasse diagrams of the weak Bruhat order.

²Created in collaboration with S. Nikolenko and K. Zainoulline.

Namely, consider an oriented labelled graph, called *Hasse diagram* of X , whose vertices are elements of W^Θ , where Θ denotes the type of the variety X , i.e., a graph whose vertices correspond to the free additive generators of $\mathrm{CH}^*(\overline{X})$, where $\overline{X} = X \times_{\mathrm{Spec} k} \mathrm{Spec} k_s$ and k_s stands for a separable closure of k . There is an edge from a vertex w to a vertex w' labelled with i if and only if $l(w) < l(w')$ and $w' = s_i w$.

Consider now the Chow motive of X and erase from the Hasse diagram of X all edges with labels not in Δ_0 (see Section 2). The Hasse diagram splits then into several non-connected components which correspond to the varieties Y_i . To illustrate this construction we give the following example.

3.1 Example. Let G be an isotropic group of type E_7 such that G_{an} has type D_4 . This means that the vertices 1, 6, and 7 on the Tits diagram of G are circled. Consider the projective G -homogeneous variety X of parabolic subgroups of type 7. Its Hasse diagram is provided in [PlSeVa, Figure 21]. Cutting the Hasse diagram along the edges with labels 1, 6, and 7 we see that the diagram splits into 14 components: 8 alone standing vertices which correspond to the elements of W^Θ of length 0, 1, 9, 10, 17, 18, 26, and 27, and therefore to the (twisted) Lefschetz motives \mathbb{Z} , $\mathbb{Z}(1)$, $\mathbb{Z}(9)$, $\mathbb{Z}(10)$, $\mathbb{Z}(17)$, $\mathbb{Z}(18)$, $\mathbb{Z}(26)$, $\mathbb{Z}(27)$, and 6 diagrams that correspond to different varieties of type D_4 . It is well known and easy to see that G_{an} corresponds to a(n anisotropic) 3-fold Pfister form φ and therefore by the celebrated result of M. Rost [Ro98] the Chow motives of a projective G_{an} -homogeneous variety splits as a direct sum of (twisted) Rost motives R which depend only on φ . The Rost motive R is indecomposable and over k_s (where φ splits) $R_s \simeq \mathbb{Z} \oplus \mathbb{Z}(3)$. Thus,

$$\mathcal{M}(X) \simeq (\oplus_{i=0,1,9,10,17,18,26,27} \mathbb{Z}(i)) \oplus (\oplus_{i=2}^{22} R(i)) \oplus R(11) \oplus R(12) \oplus R(13).$$

One should note that in the category of the Chow motives with finite coefficients of projective homogeneous varieties the Krull-Schmidt theorem holds (see [CM06, Theorem 9.6]). Therefore the motivic decompositions are unique.

Now we briefly recall following P. Brosnan [Br03] the basic properties of Steenrod operations constructed by V. Voevodsky.

Let X be a smooth projective variety over a field k with $\mathrm{char} k \neq 2$ and $p = 2$. For every $i \geq 0$ there exist certain homomorphisms $S^i = Sq^{2i}: \mathrm{Ch}^*(X) \rightarrow \mathrm{Ch}^{*+i}(X)$ called Steenrod operations. The total Steenrod operation is the sum $S = S_X = S^0 + S^1 + \dots: \mathrm{Ch}^*(X) \rightarrow \mathrm{Ch}^*(X)$. This

map is a ring homomorphism. The restriction $S^i|_{\mathrm{Ch}^n(X)}$ is 0 for $i > n$ and is the map $\alpha \mapsto \alpha^2$ for $n = i$. The map S^0 is the identity. Moreover, the total Steenrod operation commutes with pull-backs and, in particular, preserves rationality of cycles.

To compute the Steenrod operations on a projective G -homogeneous variety with a split group G we use an algorithm described in details in [DuZ07]. This algorithm is implemented in the Maple package [map].

4 J -invariant

In this section we recall the definition and the main properties of a motivic invariant of a semisimple algebraic group introduced in [PSZ07] and called the J -invariant. It was shown in paper [PSZ07] that this invariant determines the motivic behaviour of generically split projective homogeneous varieties (the definition of generically split varieties see below).

Let G_0 be a split semisimple algebraic group over k with a split maximal torus T and a Borel subgroup B containing T . Let $G = {}_\gamma G_0$ be the twisted form of G_0 given by a 1-cocycle $\gamma \in H^1(k, G_0)$.

Let X be a projective G -homogeneous variety and p a prime integer. To simplify the notation we denote $\mathrm{Ch}^*(X) = \mathrm{CH}^*(X) \otimes \mathbb{Z}/p$ and $\overline{X} = X \times_{\mathrm{Spec} k} \mathrm{Spec} k_s$, where k_s stands for a separable closure of k . We say that a cycle $J \in \mathrm{CH}^*(\overline{X})$ (resp. $J \in \mathrm{Ch}^*(\overline{X})$) is *rational* if it lies in the image of the natural restriction map $\mathrm{res}: \mathrm{CH}^*(X) \rightarrow \mathrm{CH}^*(\overline{X})$ (resp. $\mathrm{res}: \mathrm{Ch}^*(X) \rightarrow \mathrm{Ch}^*(\overline{X})$). We denote as $\overline{\mathrm{CH}}^*(X)$ (resp. as $\overline{\mathrm{Ch}}^*(X)$) the image of this map.

From now on and till the end of this section we consider the variety $X = {}_\gamma(G_0/B)$ of complete flags. Let \widehat{T} denote the group of characters of T and $S(\widehat{T}) \subset S^*(\mathbb{P})$ be the symmetric algebra (see Section 3). By R^* we denote the image of the characteristic map $c: S(\widehat{T}) \rightarrow \mathrm{Ch}^*(\overline{X})$ defined above. According to [KM05, Theorem 6.4] $R^* \subseteq \overline{\mathrm{Ch}}^*(X)$.

Let $\mathrm{Ch}^*(\overline{G})$ denote the Chow ring with \mathbb{Z}/p -coefficients of the group $(G_0)_{k_s}$. The explicit presentation of $\mathrm{Ch}^*(\overline{G})$ in terms of generators and relations is known for all groups and all primes p . Namely, by [Kc85, Theorem 3]

$$\mathrm{Ch}^*(\overline{G}) = (\mathbb{Z}/p)[x_1, \dots, x_r]/(x_1^{p^{k_1}}, \dots, x_r^{p^{k_r}}) \quad (7)$$

for certain numbers k_i , $i = 1, \dots, r$, and $\deg x_i = d_i$ for certain numbers $1 \leq d_1 \leq \dots \leq d_r$ coprime to p . A complete list of numbers $\{d_i p^{k_i}\}_{i=1, \dots, r}$, called

p -exceptional degrees of G_0 , is provided in [Kc85, Table II]. Taking the p -primary and p -coprimary parts of each p -exceptional degree one immediately restores the respective k_i 's and d_i 's.

Now we introduce an order on the set of additive generators of $\text{Ch}^*(\overline{G})$, i.e., on the monomials $x_1^{m_1} \dots x_r^{m_r}$. To simplify the notation, we denote the monomial $x_1^{m_1} \dots x_r^{m_r}$ by x^M , where M is an r -tuple of integers (m_1, \dots, m_r) . The codimension (in the Chow ring) of x^M is denoted by $|M|$. Observe that $|M| = \sum_{i=1}^r d_i m_i$.

Given two r -tuples $M = (m_1, \dots, m_r)$ and $N = (n_1, \dots, n_r)$ we say $x^M \leq x^N$ (or equivalently $M \leq N$) if either $|M| < |N|$, or $|M| = |N|$ and $m_i \leq n_i$ for the greatest i such that $m_i \neq n_i$. This gives a well-ordering on the set of all monomials (r -tuples) known also as *DegLex order*.

Consider the pull-back induced by the quotient map

$$\pi: \text{Ch}^*(\overline{X}) \rightarrow \text{Ch}^*(\overline{G})$$

According to [Gr58, Rem. 2°] π is surjective with the kernel generated by the subgroup of the non-constant elements of R^* .

Now we are ready to define the J -invariant of a group G .

4.1 Definition. Let $X = \gamma(G_0/B)$ be the twisted form of the variety of complete flags by means of a 1-cocycle $\gamma \in H^1(k, G_0)$. Denote as $\overline{\text{Ch}}^*(G)$ the image of the composite map

$$\text{Ch}^*(X) \xrightarrow{\text{res}} \text{Ch}^*(\overline{X}) \xrightarrow{\pi} \text{Ch}^*(\overline{G}).$$

Since both maps are ring homomorphisms, $\overline{\text{Ch}}^*(G)$ is a subring of $\text{Ch}^*(\overline{G})$.

For each $1 \leq i \leq r$ set j_i to be the smallest non-negative integer such that the subring $\overline{\text{Ch}}^*(G)$ contains an element a with the greatest monomial $x_i^{p^{j_i}}$ with respect to the DegLex order on $\text{Ch}^*(\overline{G})$, i.e., of the form

$$a = x_i^{p^{j_i}} + \sum_{x^M \leq x_i^{p^{j_i}}} c_M x^M, \quad c_M \in \mathbb{Z}/p.$$

The r -tuple of integers (j_1, \dots, j_r) is called the J -invariant of G modulo p and is denoted by $J_p(G)$. Note that $j_i \leq k_i$ for all i .

In case p is not a torsion prime of G we have $\text{Ch}^*(\overline{G}) = \mathbb{Z}/p$. Therefore the J -invariant is interesting only for torsion primes (see [Gr58, Definition 3] for

a definition of torsion primes). A table of possible values of the J -invariants is given in [PSZ07, Section 6].

To illustrate Definition 4.1 of the J -invariant we give the following example. For a prime integer p we denote as v_p the p -adic valuation.

4.2 Example. Let p be a prime integer and A and B be central simple k -algebras that generate the same subgroup in the Brauer group $\text{Br}(k)$. Set $G = \text{PGL}_1(A) \times \text{PGL}_1(B)$.

Then $J_p(G) = (v_p(\text{ind}A), 0)$. Indeed, the Chow ring

$$\text{Ch}^*(\overline{G}) = (\mathbb{Z}/p)[x_1, x_2]/(x_1^{p^{k_1}}, x_2^{p^{k_2}})$$

with $k_1 = v_p(\text{deg} A)$, $k_2 = v_p(\text{deg} B)$. Therefore r in the definition of the J -invariant equals 2. Denote $J_p(G) = (j_1, j_2)$ and consider the map

$$\text{res}: \text{Pic}(X_A \times X_B) \rightarrow \text{Pic}(\overline{X}_A \times \overline{X}_B),$$

where X_A (resp. X_B) denote the $\text{PGL}_1(A)$ - (resp. $\text{PGL}_1(B)$ -) variety of complete flags and Pic stands for the Picard group modulo p . Denote by h_A (resp. h_B) the image of $\bar{\omega}_1 \in S(\mathbb{P})$ in $\text{Pic}(\overline{X}_A)$ (resp. $\text{Pic}(\overline{X}_B)$) by means of the map c defined in Section 3.

Since A and B generate the same subgroup in the Brauer group, the cycle $1 \times h_B + \alpha h_A \times 1 \in \text{Pic}(\overline{X}_A \times \overline{X}_B)$ is rational for some $\alpha \in (\mathbb{Z}/p)^\times$ (see [MT95] for the description of the Picard groups of projective homogeneous varieties). The image of this cycle in $\text{Ch}^*(\overline{G})$ by means of π equals $x_2 + \alpha x_1$ (at least we can choose the generators x_1 and x_2 in such a way). Therefore, since $x_1 < x_2$ in the DegLex order, $j_2 = 0$. The proof that $j_1 = v_p(\text{ind}A)$ is the same as in [PSZ07, Section 7, case A_n] and we omit it.

Next we describe some useful properties of the J -invariant.

4.3 Proposition. *Let G be a semisimple algebraic group of inner type over k , p a prime integer and $J_p(G) = (j_1, \dots, j_r)$. Then*

1. *Let K/k be a field extension. Denote $J_p(G_K) = (j'_1, \dots, j'_r)$. Then $j'_i \leq j_i$, $i = 1, \dots, r$.*
2. *Fix an $i = 1, \dots, r$. Assume that in the presentation (7) for the semisimple anisotropic kernel G_{an} of G none of x_j has degree d_i . Then $j_i = 0$.*

3. Assume $d_i = 1$ for some $i = 1, \dots, r$.

Then $j_i \leq \max_A v_p(\text{ind}A)$ where A runs through all Tits algebras of G .
Conversely, if $j_i > 0$, then there exists a Tits algebra A of G with $v_p(\text{ind}A) > 0$.

4. Assume that the group G does not have simple components of type E_8 and for all primes p the J -invariant $J_p(G)$ is trivial. Then G is split.

Proof. 1. This is an obvious consequences of the definition of the J -invariant.

2. Let $\text{Ch}^*(G) = (\mathbb{Z}/p)[x_1, \dots, x_r]/(x_1^{p^{k_1}}, \dots, x_r^{p^{k_r}})$ with $\deg x_i = d_i$,
 $\text{Ch}^*(G_{\text{an}}) = (\mathbb{Z}/p)[x'_1, \dots, x'_{r'}]/(x_1^{p^{k'_1}}, \dots, x_{r'}^{p^{k'_{r'}}})$ with $\deg x'_i = d'_i$, and $J_p(G_{\text{an}}) = (j'_1, \dots, j'_{r'})$. It follows from the [Kc85, Table II] that $r' \leq r$ and $\{d'_i, i = 1, \dots, r'\} \subset \{d_i, i = 1, \dots, r\}$. On the other hand, by [PSZ07, Corollary 5.4], the polynomials $\prod_{i=1}^r \frac{1 - x^{d_i p^{j_i}}}{1 - x^{d_i}}$ and $\prod_{i=1}^{r'} \frac{1 - x^{d'_i p^{j'_i}}}{1 - x^{d'_i}}$ are equal. This implies the claim.

3. Assume that $j_i > 0$ and all Tits algebras of G are trivial. Then by [MT95] the group $\text{Pic}(\overline{X})$, where X denotes the G -variety of complete flags, is rational. Therefore by the very definition of the J -invariant $j_i = 0$. A contradiction.

Let A be a Tits algebra of G corresponding to a vertex t of the Dynkin diagram such that $\pi(h_t) = x_i \in \text{Pic}(\overline{G})$, where $h_t \in \text{Pic}(\overline{X})$ is the image of $\bar{\omega}_t \in S(\mathbb{P})$ by means of the map c constructed above. We show now that $j_i \leq v_p(\text{ind}A) =: s$, where A is the Tits algebra corresponding to the vertex t .

Consider the projective homogeneous variety $X \times \text{SB}(A)$, where $\text{SB}(A)$ denotes the Severi-Brauer variety of right ideals of A of reduced dimension 1. Denote by $h_A \in \text{Pic}(\overline{\text{SB}(A)}) = \text{Pic}(\mathbb{P}^{\deg A - 1})$ the canonical generator as in Section 3.

By the results of A. Merkurjev and J.-P. Tignol [MT95] the cycle $\alpha = h_t \times 1 - 1 \times h_A \in \text{Pic}(\overline{X} \times \overline{\text{SB}(A)})$ is rational. Since the cycles $\alpha^{p^s} = h_t^{p^s} \times 1 - 1 \times h_A^{p^s} \in \text{Ch}^*(\overline{X} \times \overline{\text{SB}(A)})$ and $h_A^{p^s} \in \text{Ch}^*(\overline{\text{SB}(A)})$ are rational, the cycle $h_t^{p^s} \times 1 \in \text{Ch}^*(\overline{X} \times \overline{\text{SB}(A)})$ is rational as well.

The projection morphism $\text{pr}: X \times \text{SB}(A) \rightarrow X$ is a projective bundle by [PSZ07, Corollary 3.4]. In particular, $\text{CH}^*(X \times \text{SB}(A)) = \bigoplus_{j=0}^{\deg A - 1} \text{CH}^{*-j}(X)$. Therefore the pull-back pr^* has a section δ . By the construction of this section it is compatible with a base change. Passing to the splitting field k_s we

obtain that the cycle $\bar{\delta}(h_t^{p^s} \times 1) = h_t^{p^s} \in \text{Ch}^*(\overline{X})$ is rational and the image $\pi(h_t^{p^s}) = x_i^{p^s}$. By the definition of the J -invariant, $j_i \leq s$.

4. The statement follows from [PSZ07, Corollary 6.10] and [Gi97, Theorem C]. \square

4.4 Remark. The fact that j_i (with $d_i = 1$) provides an upper bound for $v_p(\text{ind}A)$, where A runs through the Tits algebras of G , is not true. A counter-example is e.g. a group of type E_7 with a Tits algebra of index more than 2.

5 Generically split varieties

In this section we begin to study the higher Tits indices of semisimple algebraic groups over k . First, we would like to understand under what conditions our group G splits over the field of rational functions of a projective G -homogeneous variety X .

5.1 Definition. Let G be a semisimple algebraic group over k and X a projective G -homogeneous variety. We say that X is *generically split*, if the group G splits (i.e., contains a split maximal torus) over $k(X)$.

5.2 Remark. If X is generically split, then the Chow motive of X splits over $k(X)$ as a direct sum of Lefschetz motives. This explains the terminology “generically split”. One can also call such varieties *generically cellular*, since over $k(X)$ they are cellular via the Bruhat decomposition.

5.3 Theorem. *Let G_0 be a split semisimple algebraic group over k , $G = {}_\gamma G_0$ the twisted form of G_0 given by a 1-cocycle $\gamma \in H^1(k, G_0)$, and X a projective G -homogeneous variety. If X is generically split, then for all primes p the following identity on the Poincaré polynomials holds:*

$$\frac{g(\text{Ch}^*(\overline{X}), t)}{g(\overline{\text{Ch}}^*(X), t)} = \prod_{i=1}^r \frac{t^{d_i p^{j_i}} - 1}{t^{d_i} - 1}, \quad (8)$$

where $J_p(G) = (j_1, \dots, j_r)$ and d_i 's are the p -coprimary parts of the p -exceptional degrees of G_0 .

Proof. In the proof of this theorem we use results established in our paper [PSZ07].

Let p be a prime integer. We fix preimages e_i of $x_i \in \text{Ch}^*(\overline{G})$. For an r -tuple $M = (m_1, \dots, m_r)$ we set $e^M = \prod_{i=1}^r e_i^{m_i}$.

First we recall the definition of filtrations on $\text{Ch}^*(\overline{X})$ and $\overline{\text{Ch}}^*(X)$ (see [PSZ07, Definition 4.13]). Given two pairs (L, l) and (M, m) , where L and M are r -tuples and l and m are integers, we say that $(L, l) \leq (M, m)$ if either $L < M$, or $L = M$ and $l \leq m$.

The (M, m) -th term of a filtration on $\text{Ch}^*(\overline{X})$ is the subring of $\text{Ch}^*(\overline{X})$ generated by the elements $e^I \alpha$, $I \leq M$, $\alpha \in R^{\leq m}$. We denote as $A^{*,*}$ the graded ring associated to this filtration. As $A_{\text{rat}}^{*,*}$ we denote the graded subring of $A^{*,*}$ associated to the subring $\overline{\text{Ch}}^*(X) \subset \text{Ch}^*(\overline{X})$ of rational cycles with the induced filtration.

Consider the Poincaré polynomial of A_{rat} with respect to the grading induced by the usual grading of $\text{Ch}^*(\overline{X})$. Proposition 4.18 of [PSZ07] which explicitly describes a \mathbb{Z}/p -basis of $A_{\text{rat}}^{*,*}$ implies that the Poincaré polynomial $g(A_{\text{rat}}, t) =: \sum_{i=0}^{\dim X} a_i t^i$ ($a_i \in \mathbb{Z}$) of A_{rat} equals the right hand side of formula (8).

On the other hand, $\dim \overline{\text{Ch}}^*(X) = \dim A_{\text{rat}}$ and the coefficients b_i of the Poincaré polynomial $g(\overline{\text{Ch}}^*(X), t) =: \sum_{i=0}^{\dim X} b_i t^i$ are obviously bigger than or equal to a_i for all i . Therefore $g(\overline{\text{Ch}}^*(X), t) = g(A_{\text{rat}}, t)$. This finishes the proof of the theorem. \square

The right hand side of formula (8) depends only on the value of the J -invariant of G . In turn, the left hand side depends on the rationality of cycles on X . Available information on cycles that are rational as sure as fate allows to establish the following result.

5.4 Theorem. *Let G be a group of type $\Phi = F_4, E_6, E_7$ or E_8 given by a 1-cocycle from $H^1(k, G_0)$, where G_0 stands for the split adjoint group of the same type as G , and X be the variety of the parabolic subgroups of G of type i . The variety X is not generically split if and only if*

1	$\Phi = F_4, i = 4, J_2(G) = (1)$
2	$\Phi = E_6, i = 1, 6, J_2(G) = (1)$
3	$\Phi = E_6, i = 2, 4, J_3(G) = (j_1, *), j_1 \neq 0$
4	$\Phi = E_7, i = 1, 3, 4, 6, J_2(G) = (j_1, *, *, *), j_1 \neq 0$
5	$\Phi = E_7, i = 1, 6, 7, J_2(G) = (*, j_2, *, *), j_2 \neq 0$
6	$\Phi = E_7, i = 7, J_3(G) = (1)$
7	$\Phi = E_8, i = 1, 6, 7, 8, J_2(G) = (j_1, *, *, *), j_1 \neq 0$
8	$\Phi = E_8, i = 7, 8, J_3(G) = (1, *)$

(“*” means any value).

Proof. First we prove using Theorem 5.3 that the cases listed in the table are not generically split. Indeed, assume the contrary. Then in cases 3) and 4) the right hand side of formula (8) does not have a term of degree 1. On the other hand, the Picard group $\text{Pic}(\overline{X})$ is rational.

In cases 1), 5), 7) the right hand side of formula (8) does not have a term of degree 3. On the other hand, the group $\text{Ch}^3(\overline{X})$ is rational: in all these cases it is contained in the subring generated by (rational) $\text{Ch}^1(\overline{X})$, $\text{Ch}^2(\overline{X})$, and by the Chern classes of the tangent bundle $T_{\overline{X}}$ (which are rational).

In case 8) one comes to the same contradiction considering $\text{Ch}^4(\overline{X})$. In case 2) one easily comes to a contradiction, since in this case the right hand side of formula (8) does not divide the Poincaré polynomial $g(\text{Ch}^*(\overline{X}), t)$.

Next we show that all other varieties not listed in the table are generically split. Let G and X be an exceptional group and a G -variety not listed in the table. Consider $G_{k(X)}$. Using Proposition 4.3(2) one immediately sees case-by-case that for all primes p the J -invariant of the anisotropic kernel of $G_{k(X)}$ is trivial. Therefore this anisotropic kernel is trivial by Proposition 4.3(4) and the group G splits over $k(X)$. \square

5.5 Remark. The cases 3) and 4) in the table above also follow from the index reduction formula for exceptional groups [MPW98].

5.6 Remark. In view of the results obtained in [PSZ07] and in the present paper the following holds:

1. $\Phi = F_4, E_6, J_2(G) = (1)$ if and only if G has a non-trivial cohomological invariant f_3 .
2. $\Phi = E_6, J_3(G) = (j_1, *)$ or $\Phi = E_7, J_2(G) = (j_1, *, *, *)$, $j_1 \neq 0$, if and only if G has a non-trivial Tits algebra.
3. $\Phi = E_7, J_3(G) = (1)$ if and only if G has a non-trivial cohomological invariant g_3 .

6 Index of groups of type E_7

In this section we prove an index reduction formula for groups of type E_7 . Our result can be considered as a generalization of the usual index reduction formula for central simple algebras.

Another variations on this theme are the Main Tool Lemma of A. Vishik [Vi07, Theorem 3.1] and an application of the Rost degree formula [Me03, Theorem 7.2].

To prove the main result of this section we use Chow motives and Steenrod operation and a relation between rational cycles on projective homogeneous varieties and their splitting properties.

To simplify the notation we will denote the Lefschetz motives in the category of Chow motives with \mathbb{Z}/p -coefficient not as $(\mathbb{Z}/p)(i)$, but still as $\mathbb{Z}(i)$.

6.1 Theorem. *Let G be an anisotropic group of type E_7 and X (resp. Y) be the projective G -homogeneous variety of parabolic subgroups of type P_1 (resp. P_7). Then Y does not have a $k(X)$ -rational point.*

Proof. Before proving this theorem we discuss briefly the plan of the proof. First, assuming that Y has a $k(X)$ -rational point we find a rational cycle, say α , in codimension 17 on the product $X \times X$. Using Steenrod operations we produce certain projector-like cycle, say β , on $X \times X$. Applying duality arguments to β we find a sub-cycle, say γ , in the cycle α . On the other hand, multiplying α with certain cycle known to be always rational, we obtain again a projector-like cycle, but of another shape because of existence of γ inside of α . Applying duality arguments again we come to a contradiction.

From now on until the end of the proof of this theorem we set $p = 2$. To simplify the notation we denote as $\text{pt} = [X_1]$ the class of a rational point on \overline{X} (or \overline{Y}).

Assume that Y has a $k(X)$ -rational point. All claims below are proved under this assumption.

6.2 Claim. *If $J_2(G)$ is trivial, then G is isotropic.*

Indeed, assume that $J_2(G)$ is trivial and G is anisotropic. This means that there exists an odd-degree field extension K/k such that G_K is split. Now we distinguish two cases. Assume that G has a non-trivial Tits algebra A . It is well known that $2|\text{ind}A$, in particular $\text{ind}A_K \neq 1$. A contradiction.

Assume now that G has the trivial Tits algebras. Now we follow the conventions and notation of paper [Ga01]. Let \mathcal{B} be the Brown algebra associated to G . Since G is anisotropic, the group $H = \text{Aut}^+(\mathcal{B})$ is a group of outer type E_6 . Since G_K is split, the algebra \mathcal{B}_K has a singular element and, in particular, H_K is of inner type E_6 . This leads to a contradiction, since $[K : k]$ is odd.

6.3 Claim. *The varieties X and Y are not generically split.*

Indeed, the variety X is not generically split by Corollary 5.4, since $J_2(G)$ is not trivial. On the other hand, the variety Y is not generically split, since it has a rational point over $k(X)$ and G is not split over $k(X)$.

In the following claims we use the Sweedler notation for Hopf algebras to denote the cycles in the Chow rings of projective homogeneous varieties, i.e., we do not write the sums and the indices.

6.4 Claim. *Some power of any element in a finite monoid is an idempotent. In particular, for $q = 1 \times \text{pt} + x_{(1)} \times x_{(2)} \in \text{Ch}^{\dim X}(\overline{X} \times \overline{X})$ with $x_{(2)} \in \text{Ch}^{<\dim X}(\overline{X})$ there exists $n \in \mathbb{N}$ such that q^{on} is a non-trivial projector.*

The first statement of the claim is well known. For completeness we give its proof. Let q be an element of our monoid. Since the monoid is finite, we can find in the sequence $\{q^i\}_{i \in \mathbb{N}}$ two equal cycles q^{n_1} and q^{n_2} with $n_2 \geq 2n_1$. Define $n = (n_2 - n_1)n_1$. Then $q^{2n} = q^n$, i.e., q^n is an idempotent.

6.5 Lemma. *Let X and Y be arbitrary projective homogeneous varieties such that $X_{k(Y)}$ and $Y_{k(X)}$ are isotropic. Then the Chow motives $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ have a common non-trivial direct summand R such that $R_{k_s} \simeq \mathbb{Z} \oplus M$ for some motive M .*

Proof. Since X has a $k(Y)$ -rational point and Y has a $k(X)$ -rational point, we can apply [PSZ07, Lemma 1.6] (generic point argument) and get two cycles $\alpha \in \text{Ch}^{\dim Y}(X \times Y)$ and $\beta \in \text{Ch}^{\dim X}(Y \times X)$ such that $\overline{\alpha} = 1 \times \text{pt} + x_{(1)} \times x_{(2)}$ and $\overline{\beta} = 1 \times \text{pt} + x'_{(1)} \times x'_{(2)}$. The compositions $\alpha \circ \beta$ and $\beta \circ \alpha$ give cycles on $Y \times Y$ and $X \times X$ as in the previous claim. Therefore some powers $(\alpha \circ \beta)^{on}$ and $(\beta \circ \alpha)^{on}$ define projectors over k_s . The mutually inverse isomorphisms between the motives corresponding to these projectors are given by the rational maps α and $\beta \circ (\alpha \circ \beta)^{on-1}$. Applying the Rost nilpotence theorem [CGM05, Section 8] we finish the proof of the lemma. \square

6.6 Claim. *$R_{k(X)} \simeq \mathbb{Z} \oplus \mathbb{Z}(17) \oplus R'$ for some motive R' .*

Consider the motive $R_{k(X)}$. We claim first that $R_{k(X)} \simeq \mathbb{Z} \oplus \mathbb{Z}(l) \oplus R'$ for some motive R' and some $l \in \mathbb{Z}$. Indeed, let $q \in \text{Ch}_{\dim X}(X \times X)$ be a projector corresponding to R . Consider \overline{q} over k_s . The cycle $\overline{q} \cdot \overline{q}^t$, where \overline{q}^t denotes the transposed cycle, equals $n \text{pt} \times \text{pt}$, where n is the dimension of the realization $\text{Ch}^*(X, \overline{q})$. The number n is, of course, even, since otherwise

we get a rational zero-cycle on X of odd degree, and, in particular, a contradiction with Claim 6.2. On the other hand, the Krull-Schmidt theorem [CM06, Theorem 9.6] in the category of Chow motives implies that $R_{k(X)}$ is a direct sum of the Lefschetz motives and the indecomposable Rost motives. (The Rost motives are indecomposable, since $(G_{k(X)})_{\text{an}}$ has type D_4 by our assumptions and by Claim 6.3). Therefore $R_{k(X)}$ must contain as a direct summand some Lefschetz motive $\mathbb{Z}(l)$ pair to \mathbb{Z} .

Next we compute l . By assumptions the motive R is a common motive of X and Y . Therefore the number l has the property that $\mathbb{Z}(l)$ is a direct summand of the motives $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ over $k(X)$ (or over $k(Y)$). Using the Hasse diagrams of X and Y (see [PlSeVa, Figures 21 and 23]) one can easily see as in Example 3.1 that there is only one such common dimension, namely $l = 17$.

In the following claim we put all computations that we need to prove our theorem. The computations were done using algorithms described in Section 3. As $[i_1, \dots, i_l]$ we denote the product $s_{i_1} \dots s_{i_l}$ in the Weyl group.

6.7 Claim. *a) The 16-th Steenrod operation (modulo 2) of*

$$f := Z_{[7,6,5,4,3,2,4,5,6,1,3,4,5,2,4,3,1]} \in \text{Ch}^{17}(\overline{X})$$

equals $S^{16}(f) = X_{[1]} = \text{pt}$.

b) The 16-th Chern class of the tangent bundle $T_{\overline{X}}$ of \overline{X} equals

$$c := Z_{[6,5,4,2,3,1,4,3,5,4,2,6,5,4,3,1]} + Z_{[4,2,3,1,4,3,6,5,4,2,7,6,5,4,3,1]} \\ + Z_{[4,3,1,5,4,3,6,5,4,2,7,6,5,4,3,1]} \in \text{Ch}^{16}(\overline{X}).$$

c) For any $g \in \text{Ch}^9(\overline{X})$ and $h \in \text{Ch}^8(\overline{X})$ $S^8(g)S^8(h) = cgh$.

d) $c \cdot \text{Ch}^6(\overline{X}) = c \cdot \text{Ch}^{12}(\overline{X}) = 0$.

e) $S^8(\text{Ch}^8(\overline{X})) = (\mathbb{Z}/2)c$.

Proof of Theorem 6.1: Consider the cycle $f = Z_{[7,6,5,4,3,2,4,5,6,1,3,4,5,2,4,3,1]} \in \text{Ch}^{17}(\overline{X})$. This cycle is rational over $k(X)$ by [CGM05, Proposition 6.1]. Indeed, the Hasse diagram for \overline{X} is represented in [PlSeVa, Figure 23] (That figure contains the left half of the Hasse diagram which is too big to be represented at whole. One should symmetrically reflect that diagram to get a complete picture). Since by assumption the group $(G_{k(X)})_{\text{an}}$ has type D_4 , one should erase from the Hasse diagram of \overline{X} all edges with labels 1, 6, and

7. One immediately sees that the vertex corresponding to f splits from the diagram. Thus, f is defined over $k(X)$.

By [PSZ07, Lemma 1.6] the cycle $a := 1 \times f + x_{(1)} \times x_{(2)} \in \text{Ch}^{17}(\overline{X} \times \overline{X})$, $x_{(2)} \in \text{Ch}^{<17}(\overline{X})$, is rational (i.e., defined over k). By Claim 6.7(a) the cycle $S^{16}(a) = 1 \times \text{pt} + x'_{(1)} \times x'_{(2)} \in \text{Ch}^{33}(\overline{X} \times \overline{X})$, $x'_{(2)} \in \text{Ch}^{<33}(\overline{X})$ ($\dim \overline{X} = 33$). Therefore using Claim 6.4 one obtains a (rational) projector on $\overline{X} \times \overline{X}$. We denote this projector as q .

We claim that this projector contains a summand of the form $r \times s$ with $r \in \text{Ch}^{17}(\overline{X})$, $s \in \text{Ch}^{16}(\overline{X})$ and $rs = \text{pt}$. Indeed, by Claim 6.6 the motive $\mathcal{M}(X)$ has an indecomposable direct summand R such the projector corresponding to R_{k_s} contains the sum $1 \times \text{pt} + r' \times s'$ with $r' \in \text{Ch}^{17}(\overline{X})$, $s' \in \text{Ch}^{16}(\overline{X})$ and $r's' = \text{pt}$. Our projector q contains the summand $1 \times \text{pt}$. Therefore the Krull-Schmidt theorem for Chow motives with finite coefficients [CM06, Theorem 9.6] implies that q must also contain a summand $r \times s$ with $r \in \text{Ch}^{17}(\overline{X})$, $s \in \text{Ch}^{16}(\overline{X})$ and $rs = \text{pt}$.

Since q comes from the cycle a by means of S^{16} , the cycle a contains a summand $g \times h$ with $g \in \text{Ch}^9(\overline{X})$ and $h \in \text{Ch}^8(\overline{X})$. Indeed, otherwise we can't get the cycle $r \times s$ in q , since $S^i(\alpha) = 0$ for all $\alpha \in \text{Ch}^{<i}(\overline{X})$. Moreover, a contains a summand $g \times h$ ($g \in \text{Ch}^9(\overline{X})$, $h \in \text{Ch}^8(\overline{X})$) with $S^8(g)S^8(h) = \text{pt}$, since $rs = \text{pt}$.

Consider now the (rational) product $(1 \times c) \cdot a \in \text{Ch}^{33}(\overline{X} \times \overline{X})$. This product contains the sum $1 \times cf + g \times ch = 1 \times \text{pt} + g \times ch$. As in Claim 6.4 we may assume that $(1 \times c) \cdot a$ is a projector which contains the summands $1 \times \text{pt}$ and $g \times ch$ by Claim 6.7(c). The motive corresponding to this sum has the Poincaré polynomial $1 + t^9$. Consider now the motive of X over $k(X)$. As in Example 3.1 one can see that $\mathcal{M}(X)$ splits as a direct sum of (twisted) Lefschetz motives and (twisted) Rost motives corresponding to the anisotropic kernel of $G_{k(X)}$ which has strongly inner type D_4 by our assumptions and by Claim 6.3. The Poincaré polynomial of the Rost motive appearing in the motivic decomposition is $1 + t^3$. Moreover, the Lefschetz motive $\mathbb{Z}(9)$ does not appear in the motivic decomposition. Therefore by the Krull-Schmidt theorem [CM06, Theorem 9.6] the summand $g \times ch$ of the product $(1 \times c) \cdot a$ is a part of a twisted Rost motive. Therefore the product $(1 \times c) \cdot a$ must contain a pair summand for $g \times ch$. Since the Poincaré polynomial of our Rost motive is $1 + t^3$ the pair to $g \times ch$ summand has the form $\tilde{g} \times \tilde{ch}$ with $\tilde{g} \in \text{Ch}^{9 \pm 3}(\overline{X})$. This leads to a contradiction with Claim 6.7(d). The theorem is proved. \square

6.8 Remark. The index reduction formula for central simple algebras [MPW98, Section 8, Type E₇] implies that Theorem 6.1 holds if the group G has a non-trivial Tits algebra. The kernel of our proof does not depend on the (non)-triviality of the Tits algebras.

6.9 Corollary. *Let G be an anisotropic group over k of type E₇ with trivial Tits algebras or with a Tits algebra of index ≥ 4 . Then there exists a field extension K/k such that the anisotropic kernel of G_K has type D₆.*

Proof. Let X be the projective G -homogeneous variety of parabolic subgroups of type P_1 .

Assume first that all Tits algebras of G are trivial. Let K/k be a field extension. It follows from the Tits classification [Ti66] that the anisotropic kernel of G_K has type E₇, D₆, E₆, D₄, or 1.

Set now $K = k(X)$. Then obviously the anisotropic kernel of G_K can't have types E₇ and E₆ (see the diagrams [Ti66, p. 59]). On the other hand, G_K can't have types D₄ and 1 by the previous theorem. Therefore G_K has type D₆.

Assume now that G has a Tits algebra A of index ≥ 4 . It follows from the index reduction formula [MPW98, Section 8, Type E₇] that $\text{ind}A_{k(X)} = \min\{\text{ind}A, 4\} = 4$. Therefore $(G_{k(X)})_{\text{an}}$ has type D₆, since otherwise the anisotropic kernel of $G_{k(X)}$ has the Tits algebra of index at most 2 by [Ti71, 6.5.5]. \square

6.10 Remark. The corollary does not hold if the Tits algebra of G has index 2. A counter-example already exists over the field \mathbb{R} of real numbers.

Now we summarize the results of the present paper and provide a complete list of the the higher Tits indices for anisotropic groups of type F₄, ¹E₆, and E₇ with trivial Tits algebras.

F ₄	$J_2 = (0)$	$\{1, F_4\}$
F ₄	$J_2 = (1)$	$\{1, B_3, F_4\}$
E ₆	$J_2 = (0), J_3 = (0, 1)$	$\{1, E_6\}$
E ₆	$J_2 = (1), J_3 = (0, *)$	$\{1, D_4, E_6\}$
E ₆	$J_2 = (0), J_3 = (j_1, *), j_1 \neq 0$	$\{1, 2A_2, E_6\}$
E ₆	$J_2 = (1), J_3 = (j_1, *), j_1 \neq 0$	$\{1, 2A_2, D_4, E_6\}$
E ₇	$J_2 = (0, *, *, *), J_3 = (0)$	$\{1, D_4, D_6, E_7\}$
E ₇	$J_2 = (0, *, *, *), J_3 = (1)$	$\{1, D_4, D_6, E_6, E_7\}$

References

- [Br03] P. Brosnan. Steenrod operations in Chow theory. *Trans. Amer. Math. Soc.* **355** (2003), no. 5, 1869–1903.
- [Br05] P. Brosnan. On motivic decompositions arising from the method of Białynicki-Birula. *Invent. Math.* **161** (2005), no. 1, 91–111.
- [Ca72] R. Carter. Simple groups of Lie type. *Pure and Appl. Math.*, vol. 28, John Wiley & Sons, 1972.
- [CGM05] V. Chernousov, S. Gille, A. Merkurjev. Motivic decomposition of isotropic projective homogeneous varieties. *Duke Math. J.* **126** (2005), no. 1, 137–159.
- [CM06] V. Chernousov, A. Merkurjev. Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem. *Transform. Groups* **11** (2006), no. 3, 371–386.
- [De74] M. Demazure. Désingularisation des variétés de Schubert généralisées. *Ann. Sci. Éc. Norm. Sup.* **7** (1974), 53–88.
- [DuZ07] Haibao Duan, Xuezhong Zhao. A unified formula for Steenrod operations in flag manifolds. *Comp. Math.* **143** (2007), 257–270.
- [EKM] R. Elman, N. Karpenko, A. Merkurjev. Algebraic and geometric theory of quadratic forms. A book in progress, available from <http://www.institut.math.jussieu.fr/~karpenko/>
- [Ga01] S. Garibaldi. Structurable algebras and groups of type E_6 and E_7 . *J. Algebra* **236** (2001), no. 2, 651–691.
- [Gi97] Ph. Gille. La R -équivalence sur les groupes algébriques réductifs définis sur un corps global. *Inst. Hautes Études Sci. Publ. Math.* **86** (1997), 199–235.
- [Gr58] A. Grothendieck. La torsion homologique et les sections rationnelles. Exposé 5 in *Anneaux de Chow et applications*, Séminaire C. Chevalley, 2e année, 1958.
- [Hi82] H. Hiller. Geometry of Coxeter groups. *Res. Notes in Math.* 54, Pitman Adv. Publ. Prog., 1982.

- [Ka03] N. Karpenko. On the first Witt index of quadratic forms. *Invent. Math.* **153** (2003), no. 2, 455–462.
- [Ka04] N. Karpenko. Holes in I^n . *Ann. Scient. Éc. Norm. Sup.* **37** (2004), 973–1002.
- [Kc85] V. Kac. Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups. *Invent. Math.* **80** (1985), 69–79.
- [KM03] N. Karpenko, A. Merkurjev. Essential dimension of quadrics. *Invent. Math.* **153** (2003), no. 2, 361–372.
- [KM05] N. Karpenko, A. Merkurjev. Canonical p -dimension of algebraic groups. *Adv. Math.* **205** (2006) no. 2, 410–433.
- [Kn76] M. Knebusch. Generic splitting of quadratic forms. I. *Proc. London Math. Soc.* **33** (1976), no. 1, 65–93.
- [Kn77] M. Knebusch. Generic splitting of quadratic forms. II. *Proc. London Math. Soc.* **34** (1977), no. 1, 1–31.
- [KR94] I. Kersten, U. Rehmann. Generic splitting of reductive groups. *Tohoku Math. J. (2)* **46** (1994), no. 1, 35–70.
- [Ma68] Y. Manin. Correspondences, motives and monoidal transformations. *Math. USSR Sbornik* **6** (1968), 439–470.
- [map] Chern classes of the tangent bundles, Steenrod operations, and the multiplicative structure of the Chow rings of projective homogeneous varieties. Maple package. <http://www.mathematik.uni-muenchen.de/~semenov/>
- [Me03] A. Merkurjev. Steenrod operations and degree formulas. *J. Reine Angew. Math.* **565** (2003), 13–26.
- [Meh88] M. L. Mehta. Basic sets of invariant polynomials for finite reflection groups. *Comm. Algebra* **16** (1988), no. 5, 1083–1098.
- [MPW96] A. Merkurjev, I. Panin, A. Wadsworth. Index reduction formulas for twisted flag varieties. I. *K-Theory J.* **10** (1996), no. 6, 517–596.

- [MPW98] A. Merkurjev, I. Panin, A. Wadsworth. Index reduction formulas for twisted flag varieties. II. *K-Theory J.* **14** (1998), no. 2, 101–196.
- [MT95] A. Merkurjev, J.-P. Tignol. The multipliers of similitudes and the Brauer group of homogeneous varieties. *J. Reine Angew. Math.* **461** (1995), 13–47.
- [PlSeVa] E. Plotkin, A. Semenov, N. Vavilov. Visual basic representations: an atlas. *Internat. J. Algebra Comput.* **8** (1998), no. 1, 61–95.
- [PSZ07] V. Petrov, N. Semenov, K. Zainoulline. *J*-invariant of linear algebraic groups. Preprint 2007. Available from <http://www.arxiv.org>
- [Ro98] M. Rost. The motive of a Pfister form. Preprint 1998. Available from <http://www.math.uni-bielefeld.de/~rost/>
- [St] J. Stembridge. The coxeter and weyl maple packages. <http://www.math.lsa.umich.edu/~jrs/maple.html>
- [SVB92] A. Schofield, M. Van den Bergh. The index of a Brauer class on a Severi-Brauer variety. *Trans. Amer. Math. Soc.* **333** (1992), no. 2, 729–739.
- [Ti66] J. Tits. Classification of algebraic semisimple groups. In *Algebraic Groups and Discontinuous Subgroups*, Amer. Math. Soc., Providence, RI, 1966, 33–62.
- [Ti71] J. Tits. Représentations linéaires irréductibles d’un groupe réductif sur un corps quelconque. *J. Reine Angew. Math.* **247** (1971), 196–220.
- [Vi04] A. Vishik. Motives of quadrics with application to the theory of quadratic forms. In *Geometric methods in the algebraic theory of quadratic forms*. Lect. Notes in Math. 1835, Springer, Berlin, 2004, 25–101.
- [Vi06] A. Vishik. Fields of u -invariant $2^r + 1$. Preprint 2006. Available from <http://www.math.uni-bielefeld.de/LAG>
- [Vi07] A. Vishik. Generic points of quadrics and Chow groups. *Manuscr. Math.* **122** (2007), 365–374.