

# Category of Noncommutative CW Complexes<sup>‡</sup>

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## Abstract

We expose the notion of noncommutative CW (NCCW) complexes, define noncommutative (NC) mapping cylinder and NC mapping cone, and prove the noncommutative Approximation Theorem. The long exact homotopy sequences associated with arbitrary morphisms are also deduced.

*Key Words:* *C\*-algebra, noncommutative CW complex, noncommutative mapping cylinder, noncommutative mapping cone.*

## 1 Introduction

Classical algebraic topology was fruitfully developed on the category of topological spaces with CW complex structure, see e.g. [W]. Our goal is to show that with the same success, theory can be developed in the framework of noncommutative topology.

In noncommutative geometry the notion of topological spaces is changed by the notion of  $C^*$ -algebras, motivating the spectra of  $C^*$ -algebras as some noncommutative spaces. In the works [ELP] and [P], it was introduced the notion of noncommutative CW (NCCW) complex and proved some elementary properties of NCCW complexes. We continue this line in proving some basic noncommutative results. In this paper we aim to explore the same properties of NCCW complexes, as the ones of CW complexes from algebraic topology. In particular, we prove

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some NC Cellular Approximation Theorem and the existence of homotopy exact sequences associated with morphisms. In the work [D3] we introduced the notion of NC Serre fibrations (NCSF) and studied cyclic theories for the (co)homology of these NCCW complexes. In [D4] we studied the Leray-Serre spectral sequences related with cyclic theories: periodic cyclic homology and KK-theory. In [DKT1] and [DKT2] we computed some noncommutative Chern characters. Some deep study should be related with the Busby invariant, studied in [D1], [D2].

Let us describe in more detail the content of the paper. In Section 2 we expose the pullback and pushout diagrams of G. Pedersen [P] on categories of  $C^*$ -algebras. In Section 3 we introduce NCCW complexes following S. Eilers, T.A. Loring and G. K. Pedersen, etc. We prove in Section 4 a noncommutative Cellular Approximation Theorem. We prove in Section 5 some long exact homotopy sequences associated with morphisms of  $C^*$ -algebras.

## 2 Constructions in categories of $C^*$ -algebras

In this section we expose the pullback and pushout constructions of S. Eilers, T.A. Loring and G. K. Pedersen [ELP] and of G. Pedersen [P] on categories of  $C^*$ -algebras, and after that we define mapping clinders and mapping cones associated with arbitrary morphisms.

Let introduce some general notations. By  $\mathbf{I} = [0, 1]$  denote the closed interval from 0 to 1 on the real line of real numbers. It is easy to construct a homeomorphism  $\mathbf{I}^n \approx \mathbf{B}^n$  between the  $n$ -cube and the  $n$  dimensional closed ball. Denote

also the interior of the cube  $\mathbf{I}^n$  by  $\mathbf{I}_0^n = (0, 1)^n = \overset{\circ}{\mathbf{I}^n}$ . It is easy to show that the boundary  $\partial\mathbf{I}^n = \mathbf{I}^n \setminus \mathbf{I}_0^n$  is homotopic to the  $(n - 1)$ -dimensional sphere  $\mathbf{S}^{n-1}$ . Denote the space of all the continuous functions on  $\mathbf{I}^n$  with values in a  $C^*$ -algebra  $A$  by  $\mathbf{I}^n = \mathbf{C}([0, 1]^n, A)$ , and by analogy by  $\mathbf{I}_0^n A := \mathbf{C}_0((0, 1)^n, A)$  the space of all continuous functions with compact support with values in  $A$ , and finally, by  $\mathbf{S}^n A = \mathbf{C}(\mathbf{S}^n, A)$  the space of all continuous maps from  $\mathbf{S}^n$  to  $A$ .

**Definition 2.1 (Pullback diagram)** A commutative diagram of  $C^*$ -algebras and  $*$ -homomorphisms

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & B \\ \downarrow \delta & & \downarrow \beta \\ A & \xrightarrow{\alpha} & C \end{array} \quad (2.1)$$



$$\begin{array}{ccc}
\mathbf{C} & \xrightarrow{\beta} & \mathbf{B} \\
\downarrow \alpha & & \downarrow \delta \\
\mathbf{A} & \xrightarrow{\gamma} & \mathbf{X} \\
& \searrow \varphi & \searrow \sigma \\
& & \mathbf{Y}
\end{array}
\tag{2.6}$$

**Definition 2.3 (NC cone)** For  $C^*$ -algebras the *NC cone* of  $A$  is defined as the tensor product with  $\mathbf{C}_0((0, 1])$ , i.e.

$$\text{Cone}(A) := \mathbf{C}_0((0, 1]) \otimes A. \tag{2.7}$$

**Definition 2.4 (NC suspension)** For  $C^*$ -algebras the *NC suspension* of  $A$  is defined as the tensor product with  $\mathbf{C}_0((0, 1))$ , i.e.

$$\mathbf{S}(A) := \mathbf{C}_0((0, 1)) \otimes A. \tag{2.8}$$

**Remark 2.5** If  $A$  admits a *NCCW complex structure*, the same have the cone  $\text{Cone}(A)$  of  $A$  and the suspension  $\mathbf{S}(A)$  of  $A$ .

**Definition 2.6 (NC mapping cylinder)** Consider a map  $f : A \rightarrow B$  between  $C^*$ -algebras. In the algebra  $\mathbf{C}(\mathbf{I}) \otimes A \oplus B$  consider the closed two-sided ideal  $\langle \{1\} \otimes a - f(a), \forall a \in A \rangle$ , generated by elements of type  $\{1\} \otimes a - f(a), \forall a \in A$ . The quotient algebra

$$\text{Cyl}(f) = \text{Cyl}(f : A \rightarrow B) := (\mathbf{C}(\mathbf{I}) \otimes A \oplus B) / \langle \{1\} \otimes a - f(a), \forall a \in A \rangle \tag{2.9}$$

is called the *NC mapping cylinder* and denote it by  $\text{Cyl}(f : A \rightarrow B)$ .

**Remark 2.7** It is easy to show that  $A$  is included in  $\text{Cyl}(f : A \rightarrow B)$  as  $\mathbf{C}\{0\} \otimes A \subset \text{Cyl}(f : A \rightarrow B)$  and  $B$  is included in also  $B \subset \text{Cyl}(f : A \rightarrow B)$ .

**Definition 2.8 (NC mapping cone)** In the algebra  $\mathbf{C}((0, 1]) \otimes A \oplus B$  consider the closed two-sided ideal  $\langle \{1\} \otimes a - f(a), \forall a \in A \rangle$ , generated by elements of type  $\{1\} \otimes a - f(a), \forall a \in A$ . We define the *mapping cone* as the quotient algebra

$$\text{Cone}(f) = \text{Cone}(f : A \rightarrow B) := (\mathbf{C}_0((0, 1]) \otimes A \oplus B) / \langle \{1\} \otimes a - f(a), \forall a \in A \rangle. \tag{2.10}$$

**Remark 2.9** *It is easy to show that  $B$  is included in  $\text{Cone}(f : A \rightarrow B)$ .*

**Proposition 2.10** *Both the mapping cylinder and mapping cone satisfy the pull-back diagrams*

$$\begin{array}{ccc}
\text{Cone}(\varphi) & \xrightarrow{pr_1} & \mathbf{C}_0(0, 1] \otimes A & \quad & \text{Cyl}(\varphi) & \xrightarrow{pr_1} & \mathbf{C}[0, 1] \otimes A \\
pr_2 \downarrow & & \downarrow \varphi \circ ev(1) & & pr_2 \downarrow & & \downarrow \varphi \circ ev(1) \\
B & \xrightarrow{id} & B & & B & \xrightarrow{id} & B
\end{array}$$

where  $ev(1)$  is the map of evaluation at the point  $1 \in [0, 1]$ .

**Remark 2.11** *The pullback diagrams in Proposition 2.10 can be used as the initial definition of mapping cylinder and mapping cone. The previous definitions are therefore the existence of those universal objects.*

**Remark 2.12** *It is reasonable to have that in the case of  $C^*$ -algebras of continuous functions  $A = \mathbf{C}(X)$ ,  $B = \mathbf{C}(Y)$  the  $C^*$ -algebras of continuous functions over the cone and the suspension of topological spaces are in general different from the cone and the suspension of  $C^*$ -algebras, we have just defined; the same is true that the mapping cylinder and the mapping cone of morphisms of  $C^*$ -algebras are different from the  $C^*$ -algebra of continuous functions on the mapping cylinder and the mapping cone of spaces.*

### 3 The category NCCW

In this section we introduce NCCW complexes following J. Cuntz and following S. Eilers, T. A. Loring and G. K. Pedersen, [ELP] etc.

**Definition 3.1** *A dimension 0 NCCW complex is defined, following [P] as a finite sum of  $C^*$  algebras of finite linear dimension, i.e. a sum of finite dimensional matrix algebras,*

$$A_0 = \bigoplus_k \mathbf{M}_{n(k)}. \quad (3.1)$$

*In dimension  $n$ , an NCCW complex is defined as a sequence  $\{A_0, A_1, \dots, A_n\}$  of  $C^*$ -algebras  $A_k$  obtained each from the previous one by the pullback construction*

$$\begin{array}{ccccccc}
0 & \longrightarrow & I_0^k F_k & \longrightarrow & A_k & \xrightarrow{\pi} & A_{k-1} & \longrightarrow & 0 \\
& & \parallel & & \downarrow \rho_k & & \downarrow \sigma_k & & \\
0 & \longrightarrow & I_0^k F_k & \longrightarrow & I^k F_k & \xrightarrow{\partial} & \mathbf{S}^{k-1} F_k & \longrightarrow & 0,
\end{array} \quad (3.2)$$

where  $F_k$  is some  $C^*$ -algebra of finite linear dimension,  $\partial$  the restriction morphism,  $\sigma_k$  the connecting morphism,  $\rho_k$  the projection on the first coordinates and  $\pi$  the projection on the second coordinates in the presentation

$$A_k = \mathbf{I}^k F_k \bigoplus_{\mathbf{S}^{k-1} F_k} A_{k-1} \quad (3.3)$$

**Proposition 3.2** *If the algebras  $A$  and  $B$  admit a NCCW complex structure, then the same has the NC mapping cylinder  $\text{Cyl}(f : A \rightarrow B)$ .*

PROOF. Let us remember from [P] that the interval  $\mathbf{I}$  admits a structure of an NCCW complex. Next, tensor product of two NCCW complex [P] is also an NCCW complex and finally the quotient of an NCCW complex is also an NCCW complex, loc. cit..  $\square$

**Proposition 3.3** *If the algebras  $A$  and  $B$  admit a NCCW complex structure, then the same has the NC mapping cone  $\text{Cone}(f : A \rightarrow B)$ .*

PROOF. The same argument as in Proof of Proposition 3.2.  $\square$

## 4 Approximation Theorem

We prove in this section a noncommutative analog of the well-known Cellular Approximation Theorem. First we introduce the so called noncommutative homotopy extension property (NC HEP).

**Definition 4.1 (NC HEP)** *For a given  $(f, \varphi_t)$  and a  $C^*$ -algebra  $C$ , we say that  $\tilde{h} = \tilde{\varphi}_t$  is a solution of the extension problem if we have the commutative homotopy extension diagram*

$$\begin{array}{ccc} & & C \\ & \nearrow (f, \varphi_t) & \downarrow \tilde{\varphi}_t \\ \text{Cyl}(i : A \hookrightarrow B) & \longleftarrow & C[0, 1] \otimes B \end{array}$$

**Definition 4.2 (NC NDR)** *We say that the pair of algebras  $(B, A)$  is a NCNDR pair, if there are continuous morphisms  $u : C[0, 1] \rightarrow B$  and  $\varphi : B \rightarrow C[0, 1] \otimes B \cong C(I, B)$  such that*

1.  $u^{-1}(A) = 0$ ;

2. If  $\varphi(b) = (x(t), b')$  and  $x(t) = 0 \in \mathbf{C}(\mathbf{I})$  then  $b' = b, \forall b \in B$ ;
3.  $\varphi(a) = (x(t), a), \forall a \in A, x(t) \in \mathbf{C}(\mathbf{I})$ ;
4.  $\varphi(b) = (x(t), b')$  and if  $x(t) = 1 \in \mathbf{C}(\mathbf{I})$  then  $b' \in A$  for all  $b \in B$  such that  $u(b) \neq 1$ .

The following proposition is easily to prove.

**Proposition 4.3** *The assertion that NC HEP has solution for every  $\varphi_t$  and  $C$  is equivalent to the the property that  $(B, A)$  is a NC NDR pair.*

PROOF. If  $(B, A)$  has NC HEP, we can for every  $C$ , construct  $\tilde{\varphi} : B \rightarrow \mathbf{C}[0, 1] \otimes B$  satisfying the NC HEP diagram. Choose  $C = B$  and  $f = \text{id}$  we have the function  $\varphi$  and then choose  $(D, C) = (\mathbf{C}[0, 1], 0)$  in the definition of NC NDR pair we have the function  $u$ .

Conversely, if  $(B, A)$  is a NC NDR pair, we can define

$$h = \varphi : B \rightarrow \mathbf{C}[0, 1] \otimes B, \quad (4.1)$$

the composition of which with  $f : C \rightarrow B$  satisfy the NC HEP diagram.  $\square$

**Theorem 4.4 (Extension)** *Suppose that  $B = \mathbf{I}^n F_n \oplus_{\mathbf{S}^{n-1} F_n} A$  and  $(C, D)$  is a NC NDR pair. Every relative morphism of pairs of  $C^*$ -algebras*

$$f : (D, C) \rightarrow (\text{Cyl}(i : A \hookrightarrow B), \mathbf{C}\{1\} \otimes A) \quad (4.2)$$

can be upto homotopy extended to a relative morphism of pairs of  $C^*$ -algebras

$$F : (D, C) \rightarrow (\mathbf{C}(\mathbf{I}) \otimes B, \mathbf{C}\{1\} \otimes B). \quad (4.3)$$

PROOF. The property that  $(D, C)$  is a NC NDR pair, there is a natural extension

$$f_1 : (D, C) \rightarrow (\mathbf{C}(\mathbf{I}) \otimes B, \mathbf{C}\{1\} \otimes A). \quad (4.4)$$

Composing  $f_1$  with the map, evaluating the value at 1 give a morphism

$$\text{ev}(1) \circ f_1 : (D, C) \rightarrow (\mathbf{C}\{1\} \otimes B, \mathbf{C}\{1\} \otimes A). \quad (4.5)$$

Therefore, there exists a natural extension  $f_2$  from the pair  $(D, C)$ . to the pair

$$(\mathbf{C}\{0\} \otimes \mathbf{C}(\mathbf{I}) \otimes B + \mathbf{C}(\mathbf{I}) \otimes \mathbf{C}\{1\} \otimes B, \text{Cyl}(\mathbf{C}\{1\} \otimes A \hookrightarrow \mathbf{C}\{1\} \otimes B)).$$

Once again, there is a natural extension  $f_3$  from the pair to the pair

$$\begin{aligned} & (\mathbf{C}\{0\} \otimes \mathbf{C}(\mathbf{I}) \otimes B + \mathbf{C}\{1\} \otimes \mathbf{C}(\mathbf{I}) \otimes B + \mathbf{C}(\mathbf{I}) \otimes \mathbf{C}(\mathbf{I}) \otimes A, \text{Cyl}(\mathbf{C}(\mathbf{I}) \otimes A \hookrightarrow \mathbf{C}(\mathbf{I}) \otimes B)) = \\ & = (\mathbf{C}\{0\} \otimes \text{Cyl}(\mathbf{C}(\mathbf{I}) \otimes A \hookrightarrow \mathbf{C}(\mathbf{I}) \otimes B) + \mathbf{C}\{1\} \otimes \text{Cyl}(\mathbf{C}(\mathbf{I}) \otimes A \hookrightarrow \mathbf{C}(\mathbf{I}) \otimes B), \\ & \quad , \text{Cyl}(\mathbf{C}(\mathbf{I}) \otimes A \hookrightarrow \mathbf{C}(\mathbf{I}) \otimes B)). \end{aligned}$$

And finally, there is a natural extension  $f_4$  from the pair  $(D, C)$  to the pair

$$\mathbf{C}(\mathbf{I}) \otimes \mathbf{C}(\mathbf{I}) \otimes B, \text{Cyl}(\mathbf{C}(\mathbf{I}) \otimes A \hookrightarrow \mathbf{C}(\mathbf{I}) \otimes B).$$

We define the desired extension

$$F : (D, C) \rightarrow (\mathbf{C}(\mathbf{I}) \otimes B, \mathbf{C}\{1\} \otimes B)$$

as

$$F(t, x) := f_4(t, 0, x). \quad (4.6)$$

□

**Theorem 4.5** *Let  $\{A_0, A_1, \dots, A_n\}$  and  $\{B_0, B_1, \dots, B_m\}$  be two NCCW complexes and  $f : A = A_n \rightarrow B_m = B$  an algebraic homomorphism (map). Then  $f$  is homotopical to a cellular NCCW complex map  $h : A \rightarrow B$ .*

PROOF. We construct a sequence of maps

$$g_p : \mathbf{A}_p \rightarrow \mathbf{C}(\mathbf{I}) \otimes B_p, \quad (4.7)$$

with 4 well-known properties:

1.  $g_p(x) = (0, f(x)), \forall x \in A_p$
2. If  $g_p(b) = (x(t), f(b))$  and if  $x(t) = 0 \in \mathbf{C}(\mathbf{I})$ , then  $f(b) = b, \forall b \in B$ .
3.  $\text{ev}(1) \circ g_p = g_{p-1}$ ,
4.  $g_p(A_p) \subset \mathbf{C}\{1\} \otimes B_p$ .

Indeed, following the definition of an NCCW complex structure, we have

$$A_0 = F_0 \otimes A, \quad F_0 = \bigoplus_{j_0} \mathbf{M}_n(j_0) \quad (4.8)$$

a finite system of quantum points, i.e. a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbf{I}_0^1 F_1 & \longrightarrow & A_1 & \longrightarrow & A_0 \longrightarrow 0 \\
& & \parallel & & \downarrow \rho_1 & & \downarrow \sigma_1 \\
0 & \longrightarrow & \mathbf{I}_0^1 F_1 & \longrightarrow & \mathbf{I}^1 F_1 & \xrightarrow{\alpha_1} & \mathbf{S}^0 F_1 \longrightarrow 0
\end{array} \tag{4.9}$$

in which the second square is a pullback diagram,

$$F_1 = \bigoplus_{j_1} \mathbf{M}_{n(j_1)} = \bigoplus_{j_1} \text{Mat}_{n(j_1)}, \tag{4.10}$$

and we can present  $A_1$  as

$$A_1 \approx \mathbf{I}^1 F_1 \bigoplus_{S^0 F_1} A_0. \tag{4.11}$$

Following the compressible theorems [W] and the previous Extension Theorem 4.4, the function  $g_0$  can be naturally extended to a function  $g_1$  with properties 1. - 4. and now we have again following the definition of an NCCW complex,

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbf{I}_0^2 F_2 & \longrightarrow & A_2 & \longrightarrow & A_1 \longrightarrow 0 \\
& & \parallel & & \downarrow \rho_2 & & \downarrow \sigma_2 \\
0 & \longrightarrow & \mathbf{I}_0^2 F_2 & \longrightarrow & \mathbf{I}^2 F_2 & \xrightarrow{\alpha_2} & \mathbf{S}^1 F_2 \longrightarrow 0
\end{array} \tag{4.12}$$

$$F_2 = \bigoplus_{j_2} \mathbf{M}_{n(j_2)} = \bigoplus_{j_2} \text{Mat}_{n(j_2)}, \tag{4.13}$$

and we can present  $A_2$  as

$$A_2 \approx \mathbf{I}^2 F_2 \bigoplus_{S^1 F_2} A_1. \tag{4.14}$$

Following the compressible theorems [W] and the previous Extension Theorem 4.4, the function  $g_1$  can be naturally extended to a function  $g_2$  with properties 1. - 4. The procedure is continued for all  $p$ . Once these functions  $g_p$  were defined, the function  $g : A \rightarrow \mathbf{C}(\mathbf{I}) \otimes B$  which is continuous and  $g$  is a homotopy of  $f$  to  $h$ , where

$$h(x) := \text{ev}(1) \circ g(x). \tag{4.15}$$

Because of 4. the function  $h : A \rightarrow B$  is a cellular NCCW complex map.  $\square$

## 5 Homotopy of NCCW complexes

We prove in this section the standard long exact homotopy sequences.

Let us first recall the definition of homotopical morphisms.

**Definition 5.1** A homotopy between two morphisms  $\varphi, \psi : A \rightarrow B$  is a morphism  $\Phi : A \rightarrow \mathbf{C}(\mathbf{I}) \otimes B$ , such that  $\Phi(0, \cdot) = \varphi$  and  $\Phi(1, \cdot) = \psi$ .

**Proposition 5.2** There is a natural homotopies  $\text{Cyl}(\varphi : A \rightarrow B) \simeq B$  and  $\text{Cone}(\varphi : A \rightarrow B) \simeq B/A$ , if the last one  $B/A$  is defined.

**Theorem 5.3** For every morphism  $\varphi : A \rightarrow B$ , there is a natural long exact homotopy sequence

$$\begin{aligned} \dots \longrightarrow \mathbf{S}^2(A) \longrightarrow \mathbf{S}(\text{Cone}(\varphi : A \rightarrow B)) \longrightarrow \mathbf{S}(\text{Cyl}(\varphi : A \rightarrow B)) \longrightarrow \\ \mathbf{S}(A) \longrightarrow \text{Cone}(\varphi : A \rightarrow B) \longrightarrow \text{Cyl}(\varphi : A \rightarrow B) \longrightarrow A \xrightarrow{\varphi} B \end{aligned} \quad (5.1)$$

PROOF. Put  $A_0 = B, A_1 = A$  and  $\varphi_0 = \varphi$  we have

$$A_0 \xrightarrow{\varphi_0 = \varphi} A_1. \quad (5.2)$$

Because of Proposition 5.2 we have

$$A_0 = B \xleftarrow{\varphi_0} A_1 = A \xleftarrow{\varphi_1} A_2 = \text{Cyl}(\varphi) \xleftarrow{\varphi_2} A_3 = \text{Cone}(\varphi). \quad (5.3)$$

Because of the exact sequence

$$\mathbf{S}(A) \longleftarrow \text{Cyl}(\varphi) \longleftarrow \text{Cone}(\varphi),$$

we have

$$\begin{aligned} A_0 = B \xleftarrow{\varphi_0} A_1 = A \xleftarrow{\varphi_1} A_2 = \text{Cyl}(\varphi) \xleftarrow{\varphi_2} A_3 = \text{Cone}(\varphi) \xleftarrow{\varphi_3} \\ \xleftarrow{\varphi_3} A_4 = \mathbf{S}(A) = \mathbf{C}_0((0, 1)) \otimes A. \end{aligned} \quad (5.4)$$

Because the tensor product  $\mathbf{C}_0((0, 1)) \otimes \cdot$  is a left exact functor and because of (5.3), we have

$$\begin{aligned} A_0 = B \xleftarrow{\varphi_0} A_1 = A \xleftarrow{\varphi_1} A_2 = \text{Cyl}(\varphi) \xleftarrow{\varphi_2} A_3 = \text{Cone}(\varphi) \xleftarrow{\varphi_3} \\ \xleftarrow{\varphi_3} \mathbf{S}(A) \xleftarrow{\varphi_4} \mathbf{S}(\text{Cyl}(\varphi)) \xleftarrow{\varphi_5} A_5 = \mathbf{S}(\text{Cone}(\varphi)) \xleftarrow{\varphi_5} \\ \xleftarrow{\varphi_5} A_6 = \mathbf{S}^2(A) \xleftarrow{\varphi_6} \dots, \end{aligned} \quad (5.5)$$

etc. □

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