

# HIGHER $K$ -THEORY VIA UNIVERSAL INVARIANTS

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ABSTRACT. Using the formalism of Grothendieck’s derivators, we construct ‘the universal localizing invariant of dg categories’. By this, we mean a morphism  $\mathcal{U}_l$  from the pointed derivator  $\mathrm{HO}(\mathrm{dgc}\mathrm{at})$  associated with the Morita homotopy theory of dg categories to a triangulated strong derivator  $\mathcal{M}_{\mathrm{dg}}^{\mathrm{loc}}$  such that  $\mathcal{U}_l$  commutes with filtered homotopy colimits, preserves the point, sends each exact sequence of dg categories to a triangle and is universal for these properties.

Similarly, we construct the ‘universal additive invariant of dg categories’, i.e. the universal morphism of derivators  $\mathcal{U}_a$  from  $\mathrm{HO}(\mathrm{dgc}\mathrm{at})$  to a strong triangulated derivator  $\mathcal{M}_{\mathrm{dg}}^{\mathrm{add}}$  which satisfies the first two properties but the third one only for split exact sequences. We prove that Waldhausen  $K$ -theory appears as a mapping space in the target of the universal additive invariant.

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*Key words and phrases.* Grothendieck derivator, Quillen model structure, Bousfield localization, simplicial presheaves, stabilization, Drinfeld’s dg quotient, cyclic homology, (non)-connective  $K$ -theory, Additivity, presheaves of spectra, upper triangular dg category.

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## 1. INTRODUCTION

Differential graded categories (=dg categories) enhance our understanding of triangulated categories appearing in algebra and geometry, see [19].

They are considered as non-commutative schemes by Drinfeld [9] [10] and Kontsevich [23] [24] in their program of non-commutative algebraic geometry, i.e. the study of dg categories and their homological invariants.

In this article, using the formalism of Grothendieck’s derivators, we construct ‘the universal localizing invariant of dg categories’ *cf.* [21]. By this, we mean a morphism  $\mathcal{U}_l$  from the pointed derivator  $\mathrm{HO}(\mathrm{dgc}\mathrm{at})$  associated with the Morita homotopy theory of dg categories, see [36] [35] [34], to a triangulated strong derivator  $\mathcal{M}_{dg}^{loc}$  such that  $\mathcal{U}_l$  commutes with filtered homotopy colimits, preserves the point, sends each exact sequence of dg categories to a triangle and is universal for these properties. Because of its universality property reminiscent of motives, *cf.* section 4.1 of [25], we call  $\mathcal{M}_{dg}^{loc}$  the (stable) *localizing motivator* of dg categories.

Similarly, we construct ‘the universal additive invariant of dg categories’, i.e. the universal morphism of derivators  $\mathcal{U}_a$  from  $\mathrm{HO}(\mathrm{dgc}\mathrm{at})$  to a strong triangulated derivator  $\mathcal{M}_{dg}^{add}$  which satisfies the first two properties but the third one only for split exact sequences. We call  $\mathcal{M}_{dg}^{add}$  the *additive motivator* of dg categories.

We prove that Waldhausen  $K$ -theory appears as a mapping space in the base category  $\mathcal{M}_{dg}^{add}(e)$  of the additive motivator. To the best of the author’s knowledge, this is the first conceptual characterization of Quillen-Waldhausen  $K$ -theory [31] [41].

The representation of  $K$ -theory as a mapping space extends our results in [35] [34], where we represented  $K_0$  using functors with values in *additive categories* rather than morphisms of derivators with values in strong *triangulated derivators*.

For example, the mixed complex construction [22], from which all variants of cyclic homology are deduced, and the non-connective algebraic  $K$ -theory [32] are localizing invariants and factor through  $\mathcal{U}_l$  and through  $\mathcal{U}_a$ . The connective  $K$ -theory [41] is an example of an additive invariant which is not a localizing one. We prove that it becomes corepresentable in  $\mathcal{M}_{dg}^{add}$ , see theorem 16.10.

Our construction is similar in spirit to those of Meyer-Nest [29], Cortiñas-Thom [8] and Garkusha [12]. It splits into several general steps and also offers some insight into the relationship between the theory of derivators [14] [13] [20] [28] [3] and the classical theory of Quillen model categories [31]. Derivators allow us to state and prove precise universal properties and to dispense with many of the technical problems one faces in using model categories.

In chapter 3 we point out the connection between the notions of Grothendieck derivator and that of small homotopy theory in the sense of Heller [14]. In chapter 4, we recall Cisinski’s theory of derived Kan extensions [4] and in chapter 5, we develop his ideas on the Bousfield localization of derivators [7]. In particular, we characterize the derivator associated with a left Bousfield localization of a Quillen model category by a universal property, see theorem 5.4. This is based on a constructive description of the local weak equivalences.

In chapter 6, starting from a Quillen model category  $\mathcal{M}$  satisfying some finiteness conditions, we construct a morphism of prederivators

$$\mathrm{HO}(\mathcal{M}) \xrightarrow{\mathbb{R}h} \mathrm{L}_{\Sigma}\mathrm{Hot}_{\mathcal{M}_f}$$

which commutes with filtered homotopy colimits, has a derivator as target and is universal for these properties. In chapter 7 we study morphisms of pointed derivators and in chapter 8 we prove a general result which guarantees that small weak generators are preserved under left Bousfield localizations. In chapter 9, we recall Heller's stabilization construction [14] and we prove that this construction takes 'finitely generated' unstable theories to compactly generated stable ones. We establish the connection between Heller's stabilization and Hovey/Schwede's stabilization [17] [33] by proving that if we start with a pointed Quillen model category which satisfies some mild 'generation' hypotheses, then the two stabilization procedures yield equivalent results. This allows us to characterize Hovey/Schwede's construction by a universal property and in particular to give a very simple characterisation of the classical category of spectra in the sense of Bousfield-Friedlander [2]. In chapter 10, by applying the general arguments of the previous chapters to the Morita homotopy theory of dg categories [36] [35] [34], we construct the universal morphism of derivators

$$\mathcal{U}_t : \mathrm{HO}(\mathrm{dgc}at) \longrightarrow \mathrm{St}(\mathrm{L}_{\Sigma, P}\mathrm{Hot}_{\mathrm{dgc}at_f})$$

which commutes with filtered homotopy colimits, preserves the point and has a strong triangulated derivator as target. For every inclusion  $\mathcal{A} \xrightarrow{K} \mathcal{B}$  of a full dg subcategory, we have an induced morphism

$$S_K : \mathrm{cone}(\mathcal{U}_t(\mathcal{A} \xrightarrow{K} \mathcal{B})) \rightarrow \mathcal{U}_t(\mathcal{B}/\mathcal{A}),$$

where  $\mathcal{B}/\mathcal{A}$  denotes Drinfeld's dg quotient. By applying the localization techniques of section 5 and using the fact that the derivator  $\mathrm{St}(\mathrm{L}_{\Sigma, P}\mathrm{Hot}_{\mathrm{dgc}at_f})$  admits a stable Quillen model, we inverse the morphisms  $S_K$  and finally obtain the universal localizing invariant of dg categories

$$\mathcal{U}_l : \mathrm{HO}(\mathrm{dgc}at) \longrightarrow \mathcal{M}_{dg}^{loc}.$$

We establish a connection between the triangulated category  $\mathcal{M}_{dg}^{loc}(e)$  and Waldhausen's  $K$ -theory by showing that Waldhausen's  $S_\bullet$ -construction corresponds to the suspension functor in  $\mathcal{M}_{dg}^{loc}(e)$ . In section 12, we prove that the derivator  $\mathcal{M}_{dg}^{loc}$  admits a stable Quillen model given by a left Bousfield localization of a category of presheaves of spectra. In section 13, we introduce the concept of upper triangular dg category and construct a Quillen model structure on this class of dg categories, which satisfies strong compactness conditions. In section 14, we establish the connection between upper triangular dg categories and split short exact sequences and use the Quillen model structure of section 13 to prove an 'approximation result', see proposition 14.2. In section 15, by applying the techniques of section 5, we construct the universal morphism of derivators

$$\mathcal{U}_u : \mathrm{HO}(\mathrm{dgc}at) \longrightarrow \mathcal{M}_{dg}^{unst}$$

which commutes with filtered homotopy colimits, preserves the point and sends each split short exact sequence to a homotopy cofiber sequence. We prove that Waldhausen's  $K$ -theory space construction appears as a fibrant object, see proposition 15.9. In section 16 we stabilize the derivator  $\mathcal{M}_{dg}^{unst}$ , using the fact that it admits a Quillen model and finally obtain the universal additive invariant of dg categories

$$\mathcal{U}_a : \mathrm{HO}(\mathrm{dgc}at) \longrightarrow \mathcal{M}_{dg}^{add}.$$

Connective algebraic  $K$ -theory is additive and so factors through  $\mathcal{U}_a$ . We prove that it corresponds to a fibrant resolution of  $\mathcal{U}_a(\mathcal{A})[1]$ , see theorem 16.9, which allows us to obtain the equivalence

$$\mathrm{Map}(\mathcal{U}_a(k), \mathcal{U}_a(\mathcal{A})[1]) \xrightarrow{\sim} |N.w.S.\mathcal{A}_f|$$

and the isomorphisms

$$\pi_{i+1} \mathrm{Map}(\mathcal{U}_a(k), \mathcal{U}_a(\mathcal{A})[1]) \xrightarrow{\sim} K_i(\mathcal{A}), \forall i \geq 0.$$

In section 17, we point out some questions that deserve further investigation.

## 2. ACKNOWLEDGMENTS

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## 3. PRELIMINARIES

Our reference for the language of derivators is [3]. The derivators in this article are always considered over the category  $cat$  of small categories. Let  $\mathbb{D}$  and  $\mathbb{D}'$  be derivators. We denote by  $\underline{\mathrm{Hom}}(\mathbb{D}, \mathbb{D}')$  the category of all morphisms of derivators, by  $\underline{\mathrm{Hom}}_c(\mathbb{D}, \mathbb{D}')$  the category of morphisms of derivators which commute with homotopy colimits and by  $\underline{\mathrm{Hom}}_{flt}(\mathbb{D}, \mathbb{D}')$  the category of morphisms of derivators which commute with filtered homotopy colimits. We denote the terminal category by  $e$  and by  $i : \Gamma \rightarrow \square$  the inclusion of categories

$$\begin{array}{ccc} (0, 0) & \longleftarrow & (0, 1) \\ \uparrow & & \\ (1, 0) & & \end{array} \qquad \begin{array}{ccc} (0, 0) & \longleftarrow & (0, 1) \\ \uparrow & & \uparrow \\ (1, 0) & \longleftarrow & (1, 1) \end{array}.$$

Associated with a small category  $L$  and a point  $x : e \rightarrow L$  in  $L$  we have the following adjunction:

$$\begin{array}{ccc} \mathbb{D}(L) & & \\ \uparrow & & \downarrow \\ x_! & \mathbb{D}(e) & x^* \\ \downarrow & & \uparrow \end{array}$$

**Definition 3.1.** A derivator  $\mathbb{D}$  is strong if for every finite free category  $A$  and every small category  $B$ , the natural functor

$$\mathbb{D}(A \times B) \longrightarrow \mathrm{Fun}(A^o, \mathbb{D}(B))$$

is full and essentially surjective.

Notice that a strong derivator is the same thing as a small homotopy theory as defined in [14].

**Definition 3.2.** A derivator  $\mathbb{D}$  is regular if in  $\mathbb{D}$ , sequential homotopy colimits commute with finite products and homotopy pullbacks.

Let  $\mathcal{M}$  be a Quillen model category. By [5], it is known that  $\mathcal{M}$  gives rise to a derivator which we denote by  $\mathrm{HO}(\mathcal{M})$ . Observe that proposition 2.15 in [6] implies that  $\mathrm{HO}(\mathcal{M})$  is a strong derivator.

We denote by  $\mathrm{Ho}(\mathcal{M})$  the homotopy category of  $\mathcal{M}$ . By definition, it equals  $\mathrm{HO}(\mathcal{M})(e)$ .

Finally, for a small category  $A$ , we denote by  $\underline{A}$  the pre-derivator naturally associated to  $A$ .

#### 4. DERIVED KAN EXTENSIONS

Let  $A$  be a small category and  $\mathrm{Fun}(A^o, Sset)$  the Quillen model category of simplicial pre-sheaves on  $A$ , endowed with the projective model structure, see [37]. We have at our disposal the functor

$$\begin{array}{ccc} A & \xrightarrow{h} & \mathrm{Fun}(A^o, Sset) \\ X & \longmapsto & \mathrm{Hom}_A(?, X), \end{array}$$

where  $\mathrm{Hom}_A(?, X)$  is considered as a discrete simplicial set.

The functor  $h$  gives rise to a morphism of pre-derivators

$$\underline{A} \xrightarrow{h} \mathrm{HO}(\mathrm{Fun}(A^o, Sset)).$$

Using the notation of [4], we denote by  $\mathrm{Hot}_A$  the derivator  $\mathrm{HO}(\mathrm{Fun}(A^o, Sset))$ . The following results are proven in [4].

Let  $\mathbb{D}$  be a derivator.

**Theorem 4.1.** *The morphism  $h$  induces an equivalence of categories*

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(\underline{A}, \mathbb{D}) & & \\ \varphi \downarrow & \uparrow h^* & \\ \underline{\mathrm{Hom}}_1(\mathrm{Hot}_A, \mathbb{D}) & & \end{array}$$

*Proof.* This theorem is equivalent to corollary 3.26 in [4], since we have

$$\underline{\mathrm{Hom}}(\underline{A}, \mathbb{D}) \simeq \mathbb{D}(A^o).$$

✓

**Lemma 4.2.** *We have an adjunction*

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(\mathrm{Hot}_A, \mathbb{D}) & & \\ \mathrm{inc} \uparrow & \downarrow \Psi & \\ \underline{\mathrm{Hom}}_1(\mathrm{Hot}_A, \mathbb{D}) & & \end{array}$$

where

$$\Psi(F) := \varphi(F \circ h).$$

*Proof.* We construct a universal 2-morphism of functors

$$\epsilon : \mathrm{inc} \circ \Psi \longrightarrow \mathrm{Id}.$$

Let  $F$  be a morphism of derivators belonging to  $\underline{\mathbf{Hom}}(\mathbf{Hot}_A, \mathbb{D})$ . Let  $L$  be a small category and  $X$  an object of  $\mathbf{Hot}_A(L)$ . Recall from [4] that we have the diagram

$$\begin{array}{ccc} & \nabla \int X & \\ \pi \swarrow & & \searrow \varpi \\ L^{op} & & A. \end{array}$$

Now, let  $p$  be the functor  $\pi^{op}$  and  $q$  the functor  $\varpi^{op}$ . By the dual of proposition 1.15 in [4], we have the following functorial isomorphism

$$p_! q^*(h) \xrightarrow{\sim} X.$$

Now let  $\epsilon_L(X)$  be the composed morphism.

$$\epsilon_L(X) : \Psi(F)(X) = p_! q^* F(h) = p_! F(q^* h) \rightarrow F(p_! q^* h) \xrightarrow{\sim} F(X).$$

Now, notice using theorem 4.1 that  $\epsilon$  induces an adjunction. √

## 5. LOCALIZATION: MODEL CATEGORIES VERSUS DERIVATORS

Let  $\mathcal{M}$  be a left proper, cellular Quillen model category, see [15].

We fix a frame on  $\mathcal{M}$ , see definition 16.6.21 in [15]. Let  $D$  be a small category and  $F$  a functor from  $D$  to  $\mathcal{M}$ . We denote by  $\mathbf{hocolim} F$  the object of  $\mathcal{M}$ , as in definition 19.1.2 of [15]. Let  $S$  be a set of morphisms in  $\mathcal{M}$  and denote by  $\mathbf{L}_S \mathcal{M}$  the left Bousfield localization of  $\mathcal{M}$  by  $S$ .

Notice that the Quillen adjunction

$$\begin{array}{ccc} & \mathcal{M} & \\ Id \downarrow & & \uparrow Id \\ & \mathbf{L}_S \mathcal{M} & \end{array},$$

induces a morphism of derivators

$$\gamma : \mathbf{HO}(\mathcal{M}) \xrightarrow{\mathbb{L}Id} \mathbf{HO}(\mathbf{L}_S \mathcal{M})$$

which commutes with homotopy colimits.

**Proposition 5.1.** *Let  $\mathcal{W}_S$  be the smallest class of morphisms in  $\mathcal{M}$  satisfying the following properties:*

- a) *Every element in  $S$  belongs to  $\mathcal{W}_S$ .*
- b) *Every weak equivalence of  $\mathcal{M}$  belongs to  $\mathcal{W}_S$ .*
- c) *If in a commutative triangle, two out of three morphisms belong to  $\mathcal{W}_S$ , then so does the third one. The class  $\mathcal{W}_S$  is stable under retractions.*
- d) *Let  $D$  be a small category and  $F$  and  $G$  functors from  $D$  to  $\mathcal{M}$ . If  $\eta$  is a morphism of functors from  $F$  to  $G$  such that for every object  $d$  in  $D$ ,  $F(d)$  and  $G(d)$  are cofibrant objects and the morphism  $\eta(d)$  belongs to  $\mathcal{W}_S$ , then so does the morphism*

$$\mathbf{hocolim} F \longrightarrow \mathbf{hocolim} G.$$

*Then the class  $\mathcal{W}_S$  equals the class of  $S$ -local equivalences in  $\mathcal{M}$ , see [15].*

*Proof.* The class of  $S$ -local equivalences satisfies properties  $a)$ ,  $b)$ ,  $c)$  and  $d)$ : Properties  $a)$  and  $b)$  are satisfied by definition, proposition 3.2.3 and 3.2.4 in [15] imply property  $c)$  and proposition 3.2.5 in [15] implies property  $d)$ .

Let us now show that conversely, each  $S$ -local equivalence is in  $\mathcal{W}_S$ . Let

$$X \xrightarrow{g} Y$$

be an  $S$ -local equivalence in  $\mathcal{M}$ . Without loss of generality, we can suppose that  $X$  is cofibrant. Indeed, let  $Q(X)$  be a cofibrant resolution of  $X$  and consider the diagram

$$\begin{array}{ccc} Q(X) & & \\ \pi \downarrow \sim & \searrow^{g \circ \pi} & \\ X & \xrightarrow{g} & Y. \end{array}$$

Notice that since  $\pi$  is a weak equivalence,  $g$  is an  $S$ -local equivalence if and only if  $g \circ \pi$  is one.

By theorem 4.3.6 in [15],  $g$  is an  $S$ -local equivalence if and only if the morphism  $L_S(g)$  appearing in the diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ j(X) \downarrow & & \downarrow j(Y) \\ L_S X & \xrightarrow{L_S(g)} & L_S Y \end{array}$$

is a weak equivalence in  $\mathcal{M}$ . This shows that it is enough to prove that  $j(X)$  and  $j(Y)$  belong to  $\mathcal{W}_S$ . Apply the small object argument to the morphism

$$X \longrightarrow *$$

using the set  $\widetilde{\Lambda}(S)$ , see proposition 4.2.5 in [15]. We have the factorization

$$\begin{array}{ccc} X & \xrightarrow{\quad} & * \\ & \searrow^{j(X)} & \nearrow \\ & L_S(X) & \end{array},$$

where  $j(X)$  is a relative  $\widetilde{\Lambda}(S)$ -cell complex.

We will now prove two stability conditions concerning the class  $\mathcal{W}_S$ .

S1) Consider the following push-out

$$\begin{array}{ccc} W_0 & \longrightarrow & W_2 \\ f \downarrow & \lrcorner & \downarrow f_* \\ W_1 & \longrightarrow & W_3, \end{array}$$

where  $W_0$ ,  $W_1$  and  $W_2$  are cofibrant objects in  $\mathcal{M}$  and  $f$  is a cofibration which belongs to  $\mathcal{W}_S$ . Observe that  $f_*$  corresponds to the colimit of the

morphism of diagrams

$$\begin{array}{ccccc} W_0 & \xlongequal{\quad} & W_0 & \longrightarrow & W_2 \\ \downarrow f & & \parallel & & \parallel \\ W_1 & \xleftarrow{f} & W_0 & \longrightarrow & W_2 . \end{array}$$

Now, proposition 19.9.4 in [15] and property *d*) imply that  $f_*$  belongs to  $\mathcal{W}_S$ .

S2) Consider the following diagram

$$\mathbf{X} : X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{\quad} \dots$$

in  $\mathcal{M}$ , where the objects are cofibrant and the morphisms are cofibrations which belong to the class  $\mathcal{W}_S$ . Observe that the transfinite composition of  $X$  corresponds to the colimit of the morphism of diagrams

$$\begin{array}{ccccccc} X_0 & \xlongequal{\quad} & X_0 & \xlongequal{\quad} & X_0 & \xlongequal{\quad} & \dots \\ \parallel & & \downarrow f_0 & & \downarrow f_1 \circ f_0 & & \\ X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{\quad} & \dots \end{array}$$

Now, since  $\mathbf{X}$  is a Reedy cofibrant diagram on category with fibrant constants, see definition 15.10.1 in [15], theorem 19.9.1 and property *d*) imply that the transfinite composition of  $\mathbf{X}$  belongs to  $\mathcal{W}_S$ . Notice that the above argument can be immediatly generalized to a transfinite composition of a  $\lambda$ -sequence, where  $\lambda$  denotes an ordinal, see section 10.2 in [15].

Now, the construction of the morphism  $j(X)$  and the stability conditions *S1*) and *S2*) show us that it is enough to prove that the elements of  $\widetilde{\Lambda}(S)$  belong to  $\mathcal{W}_S$ . By proposition 4.2.5 in [15], it is sufficient to show that the set

$$\Lambda(S) = \{ \tilde{\mathbf{A}} \otimes \Delta[n] \}_{\tilde{\mathbf{A}} \otimes \partial \Delta[n]} \amalg \{ \tilde{\mathbf{B}} \otimes \Delta[n] \xrightarrow{\Lambda(g)} \tilde{\mathbf{B}} \otimes \Delta[n] \mid (A \xrightarrow{g} B) \in S, n \geq 0 \},$$

of horns in  $S$  is contained in  $\mathcal{W}_S$ . Recall from definition 4.2.1 in [15] that  $\tilde{g} : \tilde{\mathbf{A}} \rightarrow \tilde{\mathbf{B}}$  denotes a cosimplicial resolution of  $g : A \rightarrow B$  and  $\tilde{g}$  is a Reedy cofibration. We have the diagram

$$\begin{array}{ccc} \tilde{\mathbf{A}} \otimes \partial \Delta[n] & \xrightarrow{\tilde{g} \otimes 1} & \tilde{\mathbf{B}} \otimes \partial \Delta[n] \\ \downarrow 1 \otimes i & & \downarrow 1 \otimes i \\ \tilde{\mathbf{A}} \otimes \Delta[n] & \xrightarrow{\tilde{g} \otimes 1} & \tilde{\mathbf{B}} \otimes \Delta[n] . \end{array}$$

Observe that the morphism

$$\tilde{\mathbf{A}} \otimes \Delta[n] \xrightarrow{\tilde{g} \otimes 1} \tilde{\mathbf{B}} \otimes \Delta[n]$$

identifies with

$$\tilde{\mathbf{A}}^n \xrightarrow{\tilde{g}^n} \tilde{\mathbf{B}}^n ,$$

and so belongs to  $\mathcal{W}_S$ . Now, the morphism

$$\tilde{\mathbf{A}} \otimes \partial\Delta[n] \xrightarrow{\tilde{g} \otimes 1} \tilde{\mathbf{B}} \otimes \partial\Delta[n]$$

corresponds to the induced map of latching objects

$$\mathbf{L}_n \tilde{\mathbf{A}} \longrightarrow \mathbf{L}_n \tilde{\mathbf{B}},$$

which is a cofibration in  $\mathcal{M}$  by proposition 15.3.11 in [15].

Now by propositions 15.10.4, 16.3.12 and theorem 19.9.1 of [15], we have the following commutative diagram

$$\begin{array}{ccc} \text{hocolim}_{\partial(\vec{\Delta}\downarrow[n])} \tilde{\mathbf{A}} & \longrightarrow & \text{hocolim}_{\partial(\vec{\Delta}\downarrow[n])} \tilde{\mathbf{B}} \\ \sim \downarrow & & \downarrow \sim \\ \mathbf{L}_n \tilde{\mathbf{A}} & \longrightarrow & \mathbf{L}_n \tilde{\mathbf{B}}, \end{array}$$

where the vertical arrows are weak equivalences and  $\partial(\vec{\Delta}\downarrow[n])$  denotes the category of strictly increasing maps with target  $[n]$ . By property  $d$ ) of the class  $\mathcal{W}_S$ , we conclude that  $\tilde{g} \otimes 1_{\partial\Delta[n]}$  belongs to  $\mathcal{W}_S$ .

We have the following diagram

$$\begin{array}{ccc} \tilde{\mathbf{A}} \otimes \Delta[n] & \xrightarrow{\Pi} & \tilde{\mathbf{B}} \otimes \Delta[n] \\ \uparrow I & & \searrow \Lambda(g) \\ \tilde{\mathbf{A}} \otimes \Delta[n] & \xrightarrow{\tilde{g} \otimes 1} & \tilde{\mathbf{B}} \otimes \Delta[n]. \end{array}$$

Notice that the morphism  $I$  belongs to  $\mathcal{W}_S$  by the stability condition  $S1$ ) applied to the morphism

$$\tilde{\mathbf{A}} \otimes \partial\Delta[n] \xrightarrow{\tilde{g} \otimes 1} \tilde{\mathbf{B}} \otimes \partial\Delta[n],$$

which is a cofibration and belongs to  $\mathcal{W}_S$ . Since the morphism  $\tilde{g} \otimes 1$  belongs to  $\mathcal{W}_S$  so does  $\Lambda(g)$ .

This proves the proposition.  $\checkmark$

Let  $\mathbb{D}$  be derivator and  $S$  a class of morphisms in  $\mathbb{D}(e)$ .

**Definition 5.2** (Cisinski [7]). *The derivator  $\mathbb{D}$  admits a left Bousfield localization by  $S$  if there exists a morphism of derivators*

$$\gamma : \mathbb{D} \rightarrow \mathbf{L}_S \mathbb{D},$$

which commutes with homotopy colimits, sends the elements of  $S$  to isomorphisms in  $\mathbf{L}_S \mathbb{D}(e)$  and satisfies the universal property: for every derivator  $\mathbb{D}'$  the morphism  $\gamma$  induces an equivalence of categories

$$\underline{\mathbf{Hom}}_1(\mathbf{L}_S \mathbb{D}, \mathbb{D}') \xrightarrow{\gamma^*} \underline{\mathbf{Hom}}_{1,S}(\mathbb{D}, \mathbb{D}'),$$

where  $\underline{\mathbf{Hom}}_{1,S}(\mathbb{D}, \mathbb{D}')$  denotes the category of morphisms of derivators which commute with homotopy colimits and send the elements of  $S$  to isomorphisms in  $\mathbb{D}'(e)$ .

**Lemma 5.3.** *Suppose that  $\mathbb{D}$  is a triangulated derivator,  $S$  is stable under the loop space functor  $\Omega(e) : \mathbb{D}(e) \rightarrow \mathbb{D}(e)$ , see [3], and  $\mathbb{D}$  admits a left Bousfield localization  $\mathbb{L}_S \mathbb{D}$  by  $S$ .*

*Then  $\mathbb{L}_S \mathbb{D}$  is also a triangulated derivator.*

*Proof.* Recall from [3] that since  $\mathbb{D}$  is a triangulated derivator, we have the following equivalence

$$\begin{array}{c} \mathbb{D} \\ \Sigma \uparrow \downarrow \Omega \\ \mathbb{D} \end{array}$$

Notice that both morphisms of derivators,  $\Sigma$  and  $\Omega$ , commute with homotopy colimits. Since  $S$  is stable under the functor  $\Omega(e) : \mathbb{D}(e) \rightarrow \mathbb{D}(e)$  and  $\mathbb{D}$  admits a left Bousfield localization  $\mathbb{L}_S \mathbb{D}$  by  $S$ , we have an induced morphism

$$\Omega : \mathbb{L}_S \mathbb{D} \rightarrow \mathbb{L}_S \mathbb{D}.$$

Let  $s$  be an element of  $S$ . We now show that the image of  $s$  by the functor  $\gamma \circ \Sigma$  is an isomorphism in  $\mathbb{L}_S \mathbb{D}(e)$ . For this consider the category  $\Gamma$ , see section 3, and the functors

$$(0, 0) : e \rightarrow \Gamma \quad \text{and} \quad p : \Gamma \rightarrow e.$$

Now recall from section 7 from [14] that

$$\Omega(e) := p_! \circ (0, 0)^*.$$

This description show us that the image of  $s$  under the functor  $\gamma \circ \Sigma$  is an isomorphism in  $\mathbb{L}_S \mathbb{D}(e)$  because  $\gamma$  commutes with homotopy colimits. In conclusion, we have an induced adjunction

$$\begin{array}{c} \mathbb{L}_S \mathbb{D} \\ \Sigma \uparrow \downarrow \Omega \\ \mathbb{L}_S \mathbb{D} \end{array}$$

which is clearly an equivalence. This proves the lemma. ✓

**Theorem 5.4** (Cisinski [7]). *The morphism of derivators*

$$\gamma : \mathbf{HO}(\mathcal{M}) \xrightarrow{\mathbb{L}Id} \mathbf{HO}(\mathbb{L}_S \mathcal{M})$$

*is a left Bousfield localization of  $\mathbf{HO}(\mathcal{M})$  by the image of the set  $S$  in  $\mathbf{Ho}(\mathcal{M})$ .*

*Proof.* Let  $\mathbb{D}$  be a derivator.

The morphism  $\gamma$  admits a fully faithful right adjoint

$$\sigma : \mathbf{HO}(\mathbb{L}_S \mathcal{M}) \longrightarrow \mathbf{HO}(\mathcal{M}).$$

Therefore, the induced functor

$$\gamma^* : \underline{\mathbf{Hom}}_1(\mathbf{HO}(\mathbb{L}_S \mathcal{M}), \mathbb{D}) \longrightarrow \underline{\mathbf{Hom}}_{1,S}(\mathbf{HO}(\mathcal{M}), \mathbb{D}),$$

admits a left adjoint  $\sigma^*$  and  $\sigma^* \gamma^* = (\gamma \sigma)^*$  is isomorphic to the identity. Therefore  $\gamma^*$  is fully faithful. We now show that  $\gamma^*$  is essentially surjective. Let  $F$  be an object of  $\underline{\mathbf{Hom}}_{1,S}(\mathbf{HO}(\mathcal{M}), \mathbb{D})$ . Notice that since  $\mathbb{D}$  satisfies the conservativity axiom, it is sufficient to show that the functor

$$F(e) : \mathbf{Ho}(\mathcal{M}) \rightarrow \mathbb{D}(e)$$

sends the images in  $\mathrm{Ho}(\mathcal{M})$  of the  $S$ -local equivalences of  $\mathcal{M}$  to isomorphisms in  $\mathbb{D}(e)$ . The morphism  $F$  then becomes naturally a morphism of derivators

$$\overline{F} : \mathrm{HO}(\mathbb{L}_S \mathcal{M}) \rightarrow \mathbb{D}$$

such that  $\gamma^*(\overline{F}) = F$ . Now, since  $F$  commutes with homotopy colimits, the functor

$$\mathcal{M} \rightarrow \mathrm{Ho}(\mathcal{M}) \rightarrow \mathbb{D}(e)$$

sends the elements of  $\mathcal{W}_S$  to isomorphisms. This proves the theorem since by proposition 5.1 the class  $\mathcal{W}_S$  equals the class of  $S$ -local equivalences in  $\mathcal{M}$ .  $\checkmark$

## 6. FILTERED HOMOTOPY COLIMITS

Let  $\mathcal{M}$  be a cellular Quillen model category, with  $I$  the set of generating cofibrations. Suppose that the domains and codomains of the elements of  $I$  are cofibrant,  $\aleph_0$ -compact,  $\aleph_0$ -small and homotopically finitely presented, see definition 2.1.1 in [38].

**Example 6.1.** *Consider the quasi-equivalent, resp. quasi-equiconic, resp. Morita, Quillen model structure on  $\mathrm{dgc}at$  constructed in [34] [35] [36].*

*Recall that a dg functor  $F$  from  $\mathcal{C}$  to  $\mathcal{E}$  is a quasi-equivalence, resp. quasi-equiconic, resp. a Morita dg functor, if it satisfies one of the following conditions C1) or C2):*

- C1) *The dg category  $\mathcal{C}$  is empty and all the objects of  $\mathcal{E}$  are contractible.*
- C2) *For every object  $c_1$  and  $c_2$  in  $\mathcal{C}$ , the morphism of complexes from  $\mathrm{Hom}_{\mathcal{C}}(c_1, c_2)$  to  $\mathrm{Hom}_{\mathcal{E}}(F(c_1), F(c_2))$  is a quasi-isomorphism and the functor  $H^0(F)$ , resp.  $H^0(\mathrm{pre}\text{-}\mathrm{tr}(F))$ , resp.  $H^0(\mathrm{pre}\text{-}\mathrm{tr}(F))^{\mathbb{P}}$ , is essentially surjective.*

*Observe that the domains and codomains of the set  $I$  of generating cofibrations in  $\mathrm{dgc}at$  satisfy the conditions above for all the Quillen model structures.*

The following proposition is a simplification of proposition 2.2 in [38].

**Proposition 6.2.** *Let  $\mathcal{M}$  be a Quillen model category which satisfies the conditions above. Then*

- 1) *A filtered colimit of trivial fibrations is a trivial fibration.*
- 2) *For any filtered diagram  $X_i$  in  $\mathcal{M}$ , the natural morphism*

$$\mathrm{hocolim}_{i \in I} X_i \longrightarrow \mathrm{colim}_{i \in I} X_i$$

*is an isomorphism in  $\mathrm{Ho}(\mathcal{M})$ .*

- 3) *Any object  $X$  in  $\mathcal{M}$  is equivalent to a filtered colimit of strict finite  $I$ -cell objects.*
- 4) *An object  $X$  in  $\mathcal{M}$  is homotopically finitely presented if and only if it is equivalent to a retract of a strict finite  $I$ -cell object.*

*Proof.* The proof of 1), 2) and 3) is exactly the same as that of proposition 2.2 in [38]. The proof of 4) is also the same once we observe that the domains and codomains of the elements of the set  $I$  are already homotopically finitely presented by hypothesis.  $\checkmark$

In everything that follows, we fix:

- A co-simplicial resolution functor

$$(\Gamma(-) : \mathcal{M} \rightarrow \mathcal{M}^\Delta, i)$$

in the model category  $\mathcal{M}$ , see definition 16.1.8 in [15]. This means that for every object  $X$  in  $\mathcal{M}$ ,  $\Gamma(X)$  is cofibrant in the Reedy model structure on  $\mathcal{M}^\Delta$  and

$$i(X) : \Gamma(X) \xrightarrow{\sim} c^*(X)$$

is a weak equivalence on  $\mathcal{M}^\Delta$ , where  $c^*(X)$  denotes the constant co-simplicial object associated with  $X$ .

- A fibrant resolution functor

$$((-)_f : \mathcal{M} \rightarrow \mathcal{M}, \epsilon)$$

in the model category  $\mathcal{M}$ , see [15].

**Definition 6.3.** Let  $\mathcal{M}_f$  be the smallest full subcategory of  $\mathcal{M}$  such that

- $\mathcal{M}_f$  contains (a representative of the isomorphism class of) each strictly finite  $I$ -cell object of  $\mathcal{M}$  and
- the category  $\mathcal{M}_f$  is stable under the functors  $(-)_f$  and  $\Gamma(-)^n$ ,  $n \geq 0$ .

*Remark 6.4.* Notice that  $\mathcal{M}_f$  is a small category and that every object in  $\mathcal{M}_f$  is weakly equivalent to a strict finite  $I$ -cell.

We have the inclusion

$$\mathcal{M}_f \xhookrightarrow{i} \mathcal{M}.$$

**Definition 6.5.** Let  $S$  be the set of pre-images of the weak equivalences in  $\mathcal{M}$  under the functor  $i$ .

**Lemma 6.6.** The induced functor

$$\mathcal{M}_f[S^{-1}] \xrightarrow{\text{Ho}(i)} \text{Ho}(\mathcal{M})$$

is fully faithful, where  $\mathcal{M}_f[S^{-1}]$  denotes the localization of  $\mathcal{M}$  by the set  $S$ .

*Proof.* Let  $X, Y$  be objects of  $\mathcal{M}_f$ . Notice that  $(Y)_f$  is a fibrant resolution of  $Y$  in  $\mathcal{M}$  which belongs to  $\mathcal{M}_f$  and

$$\begin{array}{ccc} \Gamma(X)^0 \amalg \Gamma(X)^0 & \longrightarrow & \Gamma(X)^0 \\ d^0 \amalg d^1 \downarrow & \nearrow s^0 & \\ \Gamma(X)^1 & & \end{array}$$

is a cylinder object for  $\Gamma(X)^0$ , see proposition 16.1.6. from [15]. Since this cylinder object also belongs to  $\mathcal{M}_f$ , the localized category  $\mathcal{M}_f[S^{-1}]$  can be constructed inside  $\mathcal{M}$ .

This implies the lemma. ✓

We denote by  $\text{Fun}(\mathcal{M}_f^o, Sset)$  the Quillen model category of simplicial pre-sheaves on  $\mathcal{M}_f$  endowed with the projective model structure, see section 4. Let  $\Sigma$  be the image in  $\text{Fun}(\mathcal{M}_f^o, Sset)$  by the functor  $h$ , see section 4, of the set  $S$  in  $\mathcal{M}_f$ . Since the category  $\text{Fun}(\mathcal{M}_f^o, Sset)$  is cellular and left proper, its left Bousfield localization by

the set  $\Sigma$  exists, see [15]. We denote it by  $\mathbf{L}_\Sigma \mathbf{Fun}(\mathcal{M}_f^o, Sset)$ . We have a composed functor that we still denote by  $h$

$$h : \mathcal{M}_f \rightarrow \mathbf{Fun}(\mathcal{M}_f^o, Sset) \xrightarrow{Id} \mathbf{L}_\Sigma \mathbf{Fun}(\mathcal{M}_f^o, Sset).$$

Now, consider the functor

$$\begin{array}{ccc} \underline{h} : \mathcal{M} & \longrightarrow & \mathbf{Fun}(\mathcal{M}_f^o, Sset) \\ X & \longmapsto & \mathbf{Hom}(\Gamma(-), X)|_{\mathcal{M}_f}. \end{array}$$

We also have a composed functor that we still denote by  $\underline{h}$

$$\underline{h} : \mathcal{M} \rightarrow \mathbf{Fun}(\mathcal{M}_f^o, Sset) \xrightarrow{Id} \mathbf{L}_\Sigma \mathbf{Fun}(\mathcal{M}_f^o, Sset).$$

Now, observe that the natural equivalence

$$i(-) : \Gamma(-) \longrightarrow c^*(-),$$

induces, for every object  $X$  in  $\mathcal{M}_f$ , a morphism  $\Psi(X)$  in  $\mathbf{L}_\Sigma \mathbf{Fun}(\mathcal{M}_f^o, Sset)$

$$\Psi(X) : h(X) = \mathbf{Hom}(c^*(-), X) \longrightarrow \mathbf{Hom}(\Gamma(-), X) =: (\underline{h} \circ I)(X),$$

which is functorial in  $X$ .

**Lemma 6.7.** *The functor  $\underline{h}$  preserves weak equivalences between fibrant objects.*

*Proof.* Let  $X$  be a fibrant object in  $\mathcal{M}$ . We have an equivalence

$$\mathbf{Hom}(\Gamma(Y), X) \xrightarrow{\sim} \mathbf{Map}_{\mathcal{M}}(Y, X),$$

see [15]. This implies the lemma. ✓

*Remark 6.8.* The previous lemma implies that the functor  $\underline{h}$  admits a right derived functor

$$\begin{array}{ccc} \mathbb{R}\underline{h} : \mathbf{Ho}(\mathcal{M}) & \longrightarrow & \mathbf{Ho}(\mathbf{L}_\Sigma \mathbf{Fun}(\mathcal{M}_f^o, Sset)) \\ X & \longmapsto & \mathbf{Hom}(\Gamma(-), X_f)|_{\mathcal{M}_f}. \end{array}$$

Since the functor

$$h : \mathcal{M}_f \rightarrow \mathbf{L}_\Sigma \mathbf{Fun}(\mathcal{M}_f^o, Sset),$$

sends, by definition, the elements of  $S$  to weak equivalences, we have an induced morphism

$$\mathbf{Ho}(h) : \mathcal{M}_f[S^{-1}] \rightarrow \mathbf{Ho}(\mathbf{L}_\Sigma \mathbf{Fun}(\mathcal{M}_f^o, Sset)).$$

*Remark 6.9.* Notice that lemma 4.2.2 from [37] implies that for every  $X$  in  $\mathcal{M}_f$ , the morphism  $\Psi(X)$

$$\Psi(X) : \mathbf{Ho}(h)(X) \longrightarrow (\mathbb{R}\underline{h} \circ \mathbf{Ho}(I))(X)$$

is an isomorphism in  $\mathbf{Ho}(\mathbf{L}_\Sigma \mathbf{Fun}(\mathcal{M}_f^o, Sset))$ .

This shows that the functors

$$\mathbf{Ho}(h), \mathbb{R}\underline{h} \circ \mathbf{Ho}(I) : \mathcal{M}_f[S^{-1}] \rightarrow \mathbf{Ho}(\mathbf{L}_\Sigma \mathbf{Fun}(\mathcal{M}_f^o, Sset))$$

are canonically isomorphic and so we have the following diagram

$$\begin{array}{ccc} \mathcal{M}_f[S^{-1}] & \xrightarrow{\mathbf{Ho}(i)} & \mathbf{Ho}(\mathcal{M}), \\ \mathbf{Ho}(h) \downarrow & & \swarrow \mathbb{R}\underline{h} \\ \mathbf{Ho}(\mathbf{L}_\Sigma \mathbf{Fun}(\mathcal{M}_f^o, Sset)) & & \end{array}$$

which is commutative up to isomorphism.

**Lemma 6.10.** *The functor  $\mathbb{R}\underline{h}$  commutes with filtered homotopy colimits.*

*Proof.* Let  $\{Y_i\}_{i \in I}$  be a filtered diagram in  $\mathcal{M}$ . We can suppose, without loss of generality, that  $Y_i$  is fibrant in  $\mathcal{M}$ . By proposition 6.2, the natural morphism

$$\operatorname{hocolim}_{i \in I} Y_i \longrightarrow \operatorname{colim}_{i \in I} Y_i$$

is an isomorphism in  $\operatorname{Ho}(\mathcal{M})$  and  $\operatorname{colim}_{i \in I} Y_i$  is also fibrant. Since the functor

$$\operatorname{Ho}(\operatorname{Fun}(\mathcal{M}_f^o, Sset)) \xrightarrow{\mathbb{L}Id} \operatorname{Ho}(\operatorname{L}_\Sigma \operatorname{Fun}(\mathcal{M}_f^o, Sset)),$$

commutes with homotopy colimits and in  $\operatorname{Ho}(\operatorname{Fun}(\mathcal{M}_f^o, Sset))$  they are calculated objectwise, it is sufficient to show that the morphism

$$\operatorname{hocolim}_{i \in I} \mathbb{R}\underline{h}(Y_i)(X) \longrightarrow \mathbb{R}\underline{h}(\operatorname{colim}_{i \in I} Y_i)(X)$$

is an isomorphism in  $\operatorname{Ho}(Sset)$ , for every object  $X$  in  $\mathcal{M}_f$ . Now, since every object  $X$  in  $\mathcal{M}_f$  is homotopically finitely presented, see proposition 6.2, we have the following equivalences:

$$\begin{aligned} \mathbb{R}\underline{h}(\operatorname{colim}_{i \in I} Y_i)(X) &= \operatorname{Hom}(\Gamma(X), \operatorname{colim}_{i \in I} Y_i) \\ &\simeq \operatorname{Map}(\Gamma(X), \operatorname{colim}_{i \in I} Y_i) \\ &\simeq \operatorname{colim}_{i \in I} \operatorname{Map}(X, Y_i) \\ &\simeq \operatorname{hocolim}_{i \in I} \mathbb{R}\underline{h}(Y_i)(X) \end{aligned}$$

This proves the lemma. ✓

We now denote by  $\operatorname{L}_\Sigma \operatorname{Hot}_{\mathcal{M}_f}$  the derivator associated with  $\operatorname{L}_\Sigma \operatorname{Fun}(\mathcal{M}_f^o, Sset)$  and by  $\underline{\mathcal{M}}_f[S^{-1}]$  the pre-derivator  $\underline{\mathcal{M}}_f$  localized at the set  $S$ , see [4].

Observe that the morphism of functors

$$\Psi : h \longrightarrow \underline{h} \circ i$$

induces a 2-morphism of derivators

$$\bar{\Psi} : \operatorname{Ho}(h) \longrightarrow \mathbb{R}\underline{h} \circ \operatorname{Ho}(i).$$

**Lemma 6.11.** *The 2-morphism  $\bar{\Psi}$  is an isomorphism.*

*Proof.* For the terminal category  $e$ , the 2-morphism  $\bar{\Psi}$  coincides with the morphism of functors of remark 6.9. Since this one is an isomorphism, so is  $\bar{\Psi}$  by conservativity. This proves the lemma. ✓

As before, we have the following diagram

$$\begin{array}{ccc} \underline{\mathcal{M}}_f[S^{-1}] & \xrightarrow{\operatorname{Ho}(i)} & \operatorname{HO}(\mathcal{M}), \\ \operatorname{Ho}(h) \downarrow & \swarrow \mathbb{R}\underline{h} & \\ \operatorname{L}_\Sigma \operatorname{Hot}_{\mathcal{M}_f} & & \end{array}$$

which is commutative up to isomorphism in the 2-category of pre-derivators. Notice that by lemma 6.10,  $\mathbb{R}\underline{h}$  commutes with filtered homotopy colimits.

Let  $\mathbb{D}$  be a derivator.

**Lemma 6.12.** *The morphism of pre-derivators*

$$\underline{\mathcal{M}}_f[S^{-1}] \xrightarrow{\text{Ho}(h)} \text{L}_\Sigma \text{Hot}_{\mathcal{M}_f},$$

induces an equivalence of categories

$$\underline{\text{Hom}}_!(\text{L}_\Sigma \text{Hot}_{\mathcal{M}_f}, \mathbb{D}) \xrightarrow{\text{Ho}(h)^*} \underline{\text{Hom}}(\underline{\mathcal{M}}_f[S^{-1}], \mathbb{D}).$$

*Proof.* The category  $\underline{\text{Hom}}_!(\text{L}_\Sigma \text{Hot}_{\mathcal{M}_f}, \mathbb{D})$  is equivalent, by theorem 5.4, to the category  $\underline{\text{Hom}}_{!,\Sigma}(\text{Hot}_{\mathcal{M}_f}, \mathbb{D})$ . This last category identifies under the equivalence

$$\underline{\text{Hom}}_!(\text{Hot}_{\mathcal{M}_f}, \mathbb{D}) \rightarrow \underline{\text{Hom}}(\underline{\mathcal{M}}_f, \mathbb{D})$$

given by theorem 4.1 with the full subcategory of  $\underline{\text{Hom}}(\underline{\mathcal{M}}_f, \mathbb{D})$  consisting of the morphisms of pre-derivators which send the elements of  $S$  to isomorphisms in  $\mathbb{D}(e)$ . Now observe that this last category identifies with  $\underline{\text{Hom}}(\underline{\mathcal{M}}_f[S^{-1}], \mathbb{D})$ , by definition of the localized pre-derivator  $\underline{\mathcal{M}}_f[S^{-1}]$ . This proves the lemma.  $\checkmark$

Recall from section 9.5 in [11] that the co-simplicial resolution functor  $\Gamma(-)$  that we have fixed in the beginning of this section allows us to construct a Quillen adjunction:

$$\begin{array}{ccc} & \mathcal{M} & \\ & \uparrow \text{Re} & \downarrow \underline{h} = \text{sing} \\ & \text{Fun}(\mathcal{M}_f^o, S\text{set}) & \end{array}.$$

Since the functor  $\text{Re}$  sends the elements of  $\Sigma$  to weak equivalences in  $\mathcal{M}$ , we have the following Quillen adjunction

$$\begin{array}{ccc} & \mathcal{M} & \\ & \uparrow \text{Re} & \downarrow \underline{h} \\ & \text{L}_\Sigma \text{Fun}(\mathcal{M}_f^o, S\text{set}) & \end{array},$$

and a natural weak equivalence

$$\eta : \text{Re} \circ h \xrightarrow{\sim} i,$$

see [11].

This implies that we have the following diagram

$$\begin{array}{ccc} \mathcal{M}_f[S^{-1}] & \xrightarrow{\text{Ho}(i)} & \text{Ho}(\mathcal{M}), \\ \text{Ho}(h) \downarrow & \nearrow \text{LRe} & \\ \text{Ho}(\text{L}_\Sigma \text{Fun}(\mathcal{M}_f^o, S\text{set})) & \xleftarrow{\mathbb{R}\underline{h}} & \end{array}$$

which is commutative up to isomorphism.

We now claim that  $\text{LRe} \circ \mathbb{R}\underline{h}$  is naturally isomorphic to the identity. Indeed, by proposition 6.2, each object of  $\mathcal{M}$  is isomorphic in  $\text{Ho}(\mathcal{M})$ , to a filtered colimit of strict finite  $I$ -cell objects. Since  $\mathbb{R}\underline{h}$  and  $\text{LRe}$  commute with filtered homotopy colimits and  $\text{LRe} \circ \text{Ho}(h) \simeq \text{Id}$ , we conclude that  $\text{LRe} \circ \mathbb{R}\underline{h}$  is naturally isomorphic to the identity. This implies that the morphism  $\mathbb{R}\underline{h}$  is fully faithful.

Now, observe that the natural weak equivalence  $\eta$  induces a 2-isomorphism and so we obtain the following diagram

$$\begin{array}{ccc} \underline{\mathcal{M}}_f[S^{-1}] & \xrightarrow{\text{Ho}(i)} & \text{HO}(\mathcal{M}), \\ \text{Ho}(h) \downarrow & \nearrow \mathbb{L}Re & \\ \text{L}_\Sigma \text{Hot}_{\mathcal{M}_f} & \xleftarrow{\mathbb{R}\underline{h}} & \end{array}$$

which is commutative up to isomorphism in the 2-category of pre-derivators. Notice that  $\mathbb{L}Re \circ \mathbb{R}\underline{h}$  is naturally isomorphic to the identity (by conservativity) and so the morphism of derivators  $\mathbb{R}\underline{h}$  is fully faithful.

Let  $\mathbb{D}$  be a derivator.

**Theorem 6.13.** *The morphism of derivators*

$$\text{HO}(\mathcal{M}) \xrightarrow{\mathbb{R}\underline{h}} \text{L}_\Sigma \text{Hot}_{\mathcal{M}_f},$$

induces an equivalence of categories

$$\underline{\text{Hom}}_! (\text{L}_\Sigma \text{Hot}_{\mathcal{M}_f}, \mathbb{D}) \xrightarrow{\mathbb{R}\underline{h}^*} \underline{\text{Hom}}_{flt} (\text{HO}(\mathcal{M}), \mathbb{D}),$$

where  $\underline{\text{Hom}}_{flt} (\text{HO}(\mathcal{M}), \mathbb{D})$  denotes the category of morphisms of derivators which commute with filtered homotopy colimits.

*Proof.* We have the following adjunction

$$\begin{array}{ccc} \underline{\text{Hom}}(\text{HO}(\mathcal{M}), \mathbb{D}) & & \\ \mathbb{R}\underline{h}^* \uparrow & \Downarrow \mathbb{L}Re^* & \\ \underline{\text{Hom}}(\text{L}_\Sigma \text{Hot}_{\mathcal{M}_f}, \mathbb{D}) & & \end{array}$$

with  $\mathbb{R}\underline{h}^*$  a fully faithful functor.

Now notice that the adjunction of lemma 4.2 induces naturally an adjunction

$$\begin{array}{ccc} \underline{\text{Hom}}(\text{L}_\Sigma \text{Hot}_{\mathcal{M}_f}, \mathbb{D}) & & \\ \uparrow \downarrow \Psi & & \\ \underline{\text{Hom}}_! (\text{L}_\Sigma \text{Hot}_{\mathcal{M}_f}, \mathbb{D}) & & \end{array}$$

This implies that the composed functor

$$\mathbb{R}\underline{h}^* : \underline{\text{Hom}}_! (\text{L}_\Sigma \text{Hot}_{\mathcal{M}_f}, \mathbb{D}) \longrightarrow \underline{\text{Hom}}_{flt} (\text{HO}(\mathcal{M}), \mathbb{D})$$

is fully faithful.

We now show that this functor is essentially surjective.

Let  $F$  be an object of  $\underline{\text{Hom}}_{flt} (\text{HO}(\mathcal{M}), \mathbb{D})$ . Consider the morphism

$$\mathbb{L}Re^*(F) := F \circ \mathbb{L}Re.$$

Notice that this morphism does not necessarily commute with homotopy colimits. Now, by the above adjunction, we have a universal 2-morphism

$$\varphi : \Psi(\mathbb{L}Re^*(F)) \longrightarrow \mathbb{L}Re^*(F).$$

Now, consider the 2-morphism

$$\mathbb{R}\underline{h}^* : \mathbb{R}\underline{h}^*(\Psi(\mathbb{L}Re^*(F))) \longrightarrow (\mathbb{R}\underline{h}^* \circ \mathbb{L}Re^*)(F) \simeq F.$$

We will now show that this 2-morphism is a 2-isomorphism. By conservativity, it is sufficient to show this for the case of the terminal category  $e$ . For this, observe that  $\mathbb{R}\underline{h}^*(\varphi)$  induces an isomorphism

$$\Psi(\mathbb{L}Re^*(F)) \circ \mathbb{R}\underline{h} \circ \mathrm{Ho}(i) \longrightarrow F \circ \mathrm{Ho}(i).$$

Now each object of  $\mathcal{M}$  is isomorphic, in  $\mathrm{Ho}(\mathcal{M})$ , to a filtered colimit of strict finite  $I$ -cell objects. Since  $F$  and  $\Psi(\mathbb{L}Re^*(F))$  commute with filtered homotopy colimits,  $\mathbb{R}\underline{h}^*(\varphi)$  induces an isomorphism. This shows that the functor  $\mathbb{R}\underline{h}^*$  is essentially surjective.

This proves the theorem.  $\checkmark$

## 7. POINTED DERIVATORS

Recall from the previous section that we have constructed a derivator  $\mathbb{L}_\Sigma \mathrm{Hot}_{\mathcal{M}_f}$  associated with a Quillen model category  $\mathcal{M}$  satisfying suitable finiteness assumptions.

Now suppose that  $\mathrm{Ho}(\mathcal{M})$  is pointed, i.e. that the morphism

$$\emptyset \longrightarrow *,$$

in  $\mathcal{M}$ , where  $\emptyset$  denotes the initial object and  $*$  the terminal one, is a weak equivalence. Consider the morphism

$$P : \tilde{\emptyset} \longrightarrow h(\emptyset),$$

where  $\tilde{\emptyset}$  denotes the initial object in  $\mathbb{L}_\Sigma \mathrm{Fun}(\mathcal{M}_f^o, Sset)$ .

Observe that, since  $\mathbb{R}\underline{h}$  admits a left adjoint,  $h(\emptyset)$  identifies with the terminal object in

$$\mathrm{Ho}(\mathbb{L}_\Sigma \mathrm{Fun}(\mathcal{M}_f^o, Sset)),$$

because

$$h(\emptyset) = \mathrm{Ho}(h)(\emptyset) \xrightarrow{\sim} \mathbb{R}\underline{h} \circ \mathrm{Ho}(i)(\emptyset) \xrightarrow{\sim} \mathbb{R}\underline{h}(*).$$

We denote by

$$\mathbb{L}_{\Sigma, P} \mathrm{Fun}(\mathcal{M}_f^o, Sset),$$

the left Bousfield localization of  $\mathbb{L}_\Sigma \mathrm{Fun}(\mathcal{M}_f^o, Sset)$  at the morphism  $P$ .

Notice that the category

$$\mathrm{Ho}(\mathbb{L}_{\Sigma, P} \mathrm{Fun}(\mathcal{M}_f^o, Sset)),$$

is now a pointed one.

We have the following morphisms of derivators

$$\begin{array}{c} \mathrm{Ho}(\mathcal{M}) \\ \mathbb{L}Re \uparrow \quad \downarrow \mathbb{R}\underline{h} \\ \mathbb{L}_\Sigma \mathrm{Hot}_{\mathcal{M}_f} \\ \Phi \downarrow \\ \mathbb{L}_{\Sigma, P} \mathrm{Hot}_{\mathcal{M}_f}. \end{array}$$

By construction, we have a pointed morphism of derivators

$$\mathrm{HO}(\mathcal{M}) \xrightarrow{\Phi \circ \mathbb{R}\underline{h}} \mathbb{L}_{\Sigma, P} \mathrm{Hot}_{\mathcal{M}_f},$$

which commutes with filtered homotopy colimits and preserves the point.

Let  $\mathbb{D}$  be a pointed derivator.

**Proposition 7.1.** *The morphism of derivators  $\Phi \circ \mathbb{R}\underline{h}$  induces an equivalence of categories*

$$\underline{\mathbf{Hom}}_!(\mathbb{L}_{\Sigma, P}\mathbf{Hot}_{\mathcal{M}_f}, \mathbb{D}) \xrightarrow{(\Phi \circ \mathbb{R}\underline{h})^*} \underline{\mathbf{Hom}}_{flt, p}(\mathbf{HO}(\mathcal{M}), \mathbb{D}),$$

where  $\underline{\mathbf{Hom}}_{flt, p}(\mathbf{HO}(\mathcal{M}), \mathbb{D})$  denotes the category of morphisms of derivators which commute with filtered homotopy colimits and preserve the point.

*Proof.* By theorem 5.4, we have an equivalence of categories

$$\underline{\mathbf{Hom}}_!(\mathbb{L}_{\Sigma, P}\mathbf{Hot}_{\mathcal{M}_f}, \mathbb{D}) \xrightarrow{\Phi^*} \underline{\mathbf{Hom}}_{!, P}(\mathbb{L}_{\Sigma}\mathbf{Hot}_{\mathcal{M}_f}, \mathbb{D}).$$

By theorem 6.13, we have an equivalence of categories

$$\underline{\mathbf{Hom}}_!(\mathbb{L}_{\Sigma}\mathbf{Hot}_{\mathcal{M}_f}, \mathbb{D}) \xrightarrow{\mathbb{R}\underline{h}^*} \underline{\mathbf{Hom}}_{flt}(\mathbf{HO}(\mathcal{M}), \mathbb{D}).$$

We now show that under this last equivalence, the category  $\underline{\mathbf{Hom}}_{!, P}(\mathbb{L}_{\Sigma}\mathbf{Hot}_{\mathcal{M}_f}, \mathbb{D})$  identifies with  $\underline{\mathbf{Hom}}_{flt, p}(\mathbf{HO}(\mathcal{M}), \mathbb{D})$ . Let  $F$  be an object of  $\underline{\mathbf{Hom}}_{!, P}(\mathbb{L}_{\Sigma}\mathbf{Hot}_{\mathcal{M}_f}, \mathbb{D})$ . Since  $F$  commutes with homotopy colimits, it preserves the initial object. This implies that  $F \circ \mathbb{R}\underline{h}$  belongs to  $\underline{\mathbf{Hom}}_{flt, p}(\mathbf{HO}(\mathcal{M}), \mathbb{D})$ .

Let now  $G$  be an object of  $\underline{\mathbf{Hom}}_{flt, p}(\mathbf{HO}(\mathcal{M}), \mathbb{D})$ . Consider, as in the proof of theorem 6.13, the morphism

$$\Psi(\mathbb{L}Re^*(G)) : \mathbb{L}_{\Sigma}\mathbf{Hot}_{\mathcal{M}_f} \longrightarrow \mathbb{D}.$$

Since  $\Psi(\mathbb{L}Re^*(G))$  commutes with homotopy colimits, by construction, it sends  $\tilde{\emptyset}$  to the point of  $\mathbb{D}$ . Observe also that  $h(\emptyset)$  is also sent to the point of  $\mathbb{D}$  because

$$\Psi(\mathbb{L}Re^*(G))(h(\emptyset)) \simeq G(\emptyset).$$

This proves the proposition. ✓

## 8. SMALL WEAK GENERATORS

Let  $\mathcal{N}$  be a pointed, left proper, compactly generated Quillen model category. Observe that in particular this implies that  $\mathcal{N}$  is finitely generated, as in section 7.4 in [16]. We denote by  $\mathcal{G}$  the set of cofibers of the generating cofibrations  $I$  in  $\mathcal{N}$ . By corollary 7.4.4 in [16], the set  $\mathcal{G}$  is a set of small weak generators for  $\mathbf{Ho}(\mathcal{N})$ , see definitions 7.2.1 and 7.2.2 in [16]. Let  $S$  be a set of morphisms in  $\mathcal{N}$  between objects which are homotopically finitely presented, see [38], and  $\mathbb{L}_S\mathcal{N}$  the left Bousfield localization of  $\mathcal{N}$  by  $S$ . We have an adjunction

$$\begin{array}{ccc} \mathbf{Ho}(\mathcal{N}) & & \\ \mathbb{L}Id \downarrow & \uparrow \mathbb{R}Id & \\ \mathbf{Ho}(\mathbb{L}_S\mathcal{N}) & & \end{array}$$

**Lemma 8.1.** *The image of the set  $\mathcal{G}$  under the functor  $\mathbb{L}Id$  is a set of small weak generators in  $\mathbf{Ho}(\mathbb{L}_S\mathcal{N})$ .*

*Proof.* The previous adjunction is equivalent to

$$\begin{array}{c} \mathrm{Ho}(\mathcal{N}) \\ \downarrow \uparrow \\ (-)_f \downarrow \uparrow \\ \mathrm{Ho}(\mathcal{N})_S \end{array}$$

where  $\mathrm{Ho}(\mathcal{N})_S$  denotes the full subcategory of  $\mathrm{Ho}(\mathcal{M})$  formed by the  $S$ -local objects of  $\mathcal{N}$  and  $(-)_f$  denotes a fibrant resolution functor in  $\mathbf{L}_S\mathcal{N}$ , see [15]. Clearly, this implies that the image of the set  $\mathcal{G}$  under the functor  $(-)_f$  is a set of weak generators in  $\mathrm{Ho}(\mathbf{L}_S\mathcal{N})$ .

We now show that the  $S$ -local objects in  $\mathcal{N}$  are stable under filtered homotopy colimits. Let  $\{X_i\}_{i \in I}$  be a filtered diagram of  $S$ -local objects. By proposition 6.2, we have an isomorphism

$$\mathrm{hocolim}_{i \in I} X_i \xrightarrow{\sim} \mathrm{colim}_{i \in I} X_i$$

in  $\mathrm{Ho}(\mathcal{N})$ . We now show that  $\mathrm{colim}_{i \in I} X_i$  is an  $S$ -local object. Let  $g : A \rightarrow B$  be an element of  $S$ . We have at our disposal the following commutative diagram

$$\begin{array}{ccc} \mathrm{Map}(B, \mathrm{colim}_{i \in I} X_i) & \xrightarrow{g^*} & \mathrm{Map}(A, \mathrm{colim}_{i \in I} X_i) \\ \uparrow \sim & & \uparrow \sim \\ \mathrm{colim}_{i \in I} \mathrm{Map}(B, X_i) & \xrightarrow{\mathrm{colim}_{i \in I} g_i^*} & \mathrm{colim}_{i \in I} \mathrm{Map}(A, X_i) \end{array}$$

Now observe that since  $A$  and  $B$  are homotopically finitely presented objects, the vertical arrows in the diagram are isomorphisms in  $\mathrm{Ho}(S\mathrm{set})$ . Since each object  $X_i$  is  $S$ -local, the morphism  $g_i^*$  is an isomorphism in  $\mathrm{Ho}(S\mathrm{set})$  and so is  $\mathrm{colim}_{i \in I} g_i^*$ . This implies that  $\mathrm{colim}_{i \in I} X_i$  is an  $S$ -local object. This shows that the inclusion

$$\mathrm{Ho}(\mathcal{M})_S \hookrightarrow \mathrm{Ho}(\mathcal{M}),$$

commutes with filtered homotopy colimits and so the image of the set  $\mathcal{G}$  under the functor  $(-)_f$  consists of small objects of  $\mathrm{Ho}(\mathbf{L}_S\mathcal{N})$ .

This proves the lemma. ✓

Recall from the previous chapter that we have constructed a pointed derivator  $\mathbf{L}_{\Sigma, \mathcal{P}}\mathrm{Hot}_{\mathcal{M}_f}$ . We will now construct a strictly pointed Quillen model category whose associated derivator is equivalent to  $\mathbf{L}_{\Sigma, \mathcal{P}}\mathrm{Hot}_{\mathcal{M}_f}$ . Consider the pointed Quillen model category

$$* \downarrow \mathrm{Fun}(\mathcal{M}_f^o, S\mathrm{set}) = \mathrm{Fun}(\mathcal{M}_f^o, S\mathrm{set}_\bullet).$$

We have the following Quillen adjunction

$$\begin{array}{c} \mathrm{Fun}(\mathcal{M}_f^o, S\mathrm{set}_\bullet) \\ \uparrow \downarrow U \\ (-)_+ \uparrow \downarrow \\ \mathrm{Fun}(\mathcal{M}_f^o, S\mathrm{set}), \end{array}$$

where  $U$  denotes the forgetful functor.

We denote by  $L_{\Sigma,P}\text{Fun}(\mathcal{M}_f^o, Sset_\bullet)$  the left Bousfield localization of  $\text{Fun}(\mathcal{M}_f^o, Sset_\bullet)$  by the image of the set  $\Sigma \cup \{P\}$  under the functor  $(-)_+$ . We denote by  $L_{\Sigma,P}\text{Hot}_{\mathcal{M}_f^\bullet}$  the derivator associated with  $L_{\Sigma,P}\text{Fun}(\mathcal{M}_f^o, Sset_\bullet)$ .

*Remark 8.2.* Since the derivators associated with  $L_{\Sigma,P}\text{Fun}(\mathcal{M}_f^o, Sset)$  and  $L_{\Sigma,P}\text{Fun}(\mathcal{M}_f^o, Sset_\bullet)$  are characterized by the same universal property we have a canonical equivalence of pointed derivators

$$L_{\Sigma,P}\text{Hot}_{\mathcal{M}_f} \xrightarrow{\sim} L_{\Sigma,P}\text{Hot}_{\mathcal{M}_f^\bullet}$$

Notice also that the category  $\text{Fun}(\mathcal{M}_f^o, Sset_\bullet)$  endowed with the projective model structure is pointed, left proper, compactly generated and that the domains and codomains of the elements of the set  $(\Sigma \cup \{P\})_+$  are homotopically finitely presented objects. Therefore by lemma 8.1, the set

$$\mathcal{G} = \{\mathbf{F}_{\Delta[n]_+/\partial\Delta[n]_+}^X \mid X \in \mathcal{M}_f, n \geq 0\},$$

of cofibers of the generating cofibrations in  $\text{Fun}(\mathcal{M}_f^o, Sset_\bullet)$  is a set of small weak generators in  $\text{Ho}(L_{\Sigma,P}\text{Fun}(\mathcal{M}_f^o, Sset_\bullet))$ .

## 9. STABILIZATION

Let  $\mathbb{D}$  be a regular pointed strong derivator.

In [14], Heller constructs a universal morphism to a triangulated strong derivator

$$\mathbb{D} \xrightarrow{\text{stab}} \text{St}(\mathbb{D}),$$

which commutes with homotopy colimits.

This construction is done in two steps. First consider the following ordered set

$$\mathbf{V} := \{(i, j) \mid |i - j| \leq 1\} \subset \mathbb{Z} \times \mathbb{Z}$$

naturally as a small category. We denote by

$$\dot{\mathbf{V}} := \{(i, j) \mid |i - j| = 1\} \subset \mathbf{V},$$

the full subcategory of ‘points on the boundary’.

Now, let  $\text{Spec}(\mathbb{D})$  be the full subderivator of  $\mathbb{D}_{\mathbf{V}}$ , see definition 3.4 in [3], formed by the objects  $X$  in  $\mathbb{D}_{\mathbf{V}}(L)$ , whose image under the functor

$$\mathbb{D}_{\mathbf{V}}(L) = \mathbb{D}(\mathbf{V} \times L) \longrightarrow \text{Fun}(\mathbf{V}^{op}, \mathbb{D}(L))$$

is of the form

$$\begin{array}{ccccc} & & * & \longrightarrow & X_{(1,1)} \cdots \\ & & \uparrow & & \uparrow \\ & & * & \longrightarrow & X_{(0,0)} \longrightarrow * \\ & \uparrow & & & \uparrow \\ \cdots & X_{(-1,-1)} & \longrightarrow & * & , \end{array}$$

see section 8 in [14]. We have an evaluation functor  $ev_{(0,0)} : \text{Spec}(\mathbb{D}) \rightarrow \mathbb{D}$ , which admits a left adjoint  $L[0, 0]$ .

Finally, let  $\text{St}(\mathbb{D})$  be the full reflexive subderivator of  $\text{Spec}(\mathbb{D})$  consisting of the  $\Omega$ -spectra, as defined in [14].

We have the following adjunctions

$$\begin{array}{ccc}
 & \mathbb{D} & \\
 & \downarrow L[0,0] & \uparrow ev_{(0,0)} \\
 stab & \text{Spec}(\mathbb{D}) & \\
 & \downarrow loc & \uparrow \\
 & \text{St}(\mathbb{D}) & 
 \end{array}$$

in the 2-category of derivators.

Let  $\mathbb{T}$  be a triangulated strong derivator. The following theorem is proved in [14].

**Theorem 9.1.** *The morphism  $stab$  induces an equivalence of categories*

$$\underline{\text{Hom}}_1(\text{St}(\mathbb{D}), \mathbb{T}) \xrightarrow{stab^*} \underline{\text{Hom}}_1(\mathbb{D}, \mathbb{T}).$$

**Lemma 9.2.** *Let  $\mathcal{G}$  be a set of objects in  $\mathbb{D}(e)$  which satisfies the following conditions:*

A1) *If, for each  $g$  in  $\mathcal{G}$ , we have*

$$\text{Hom}_{\mathbb{D}(e)}(g, X) = \{*\},$$

*then  $X$  is isomorphic to  $*$ , where  $*$  denotes the terminal and initial object in  $\mathbb{D}(e)$ .*

A2) *For every set  $K$  and each  $g$  in  $\mathcal{G}$  the canonical map*

$$\underset{\substack{S \subseteq K \\ S \text{ finite}}}{\text{colim}} \text{Hom}_{\mathbb{D}(e)}(g, \coprod_{\alpha \in S} X_\alpha) \xrightarrow{\sim} \text{Hom}_{\mathbb{D}(e)}(g, \coprod_{\alpha \in S} X_\alpha)$$

*is bijective.*

Then the set

$$\{\Sigma^n stab(g) \mid g \in \mathcal{G}, n \in \mathbb{Z}\}$$

*of objects in  $\text{St}(\mathbb{D})(e)$ , where  $\Sigma$  denotes the suspension functor in  $\text{St}(\mathbb{D})(e)$ , satisfies A1) and A2).*

*Proof.* Let  $\underline{X}$  be an object of  $\text{St}(\mathbb{D})(e)$ . Suppose that for each  $g$  in  $\mathcal{G}$  and  $n$  in  $\mathbb{Z}$ , we have

$$\text{Hom}_{\text{St}(\mathbb{D})(e)}(\Sigma^n stab(g), \underline{X}) = \{*\}.$$

Then by the following isomorphisms

$$\begin{aligned}
 \text{Hom}_{\text{St}(\mathbb{D})(e)}(\Sigma^n stab(g), \underline{X}) &\simeq \text{Hom}_{\text{St}(\mathbb{D})(e)}(stab(g), \Omega^n \underline{X}) \\
 &\simeq \text{Hom}_{\mathbb{D}(e)}(g, ev_{(0,0)} \Omega^n \underline{X}) \\
 &\simeq \text{Hom}_{\mathbb{D}(e)}(g, ev_{(n,n)} \underline{X}),
 \end{aligned}$$

we conclude that, for all  $n$  in  $\mathbb{Z}$ , we have

$$ev_{(n,n)} \underline{X} = *.$$

By the conservativity axiom,  $\underline{X}$  is isomorphic to  $*$  in  $\text{St}(\mathbb{D})(e)$ . This shows condition A1). Now observe that condition A2) follows from the following isomorphisms

$$\begin{aligned}
\text{Hom}_{\text{St}(\mathbb{D})(e)}(\Sigma^n \text{stab}(g), \bigoplus_{\alpha \in K} \underline{X}_\alpha) &\simeq \text{Hom}_{\text{St}(\mathbb{D})(e)}(\text{stab}(g), \Omega^n \bigoplus_{\alpha \in K} \underline{X}_\alpha) \\
&\simeq \text{Hom}_{\text{St}(\mathbb{D})(e)}(\text{stab}(g), \bigoplus_{\alpha \in K} \Omega^n \underline{X}_\alpha) \\
&\simeq \text{Hom}_{\mathbb{D}(e)}(g, \text{ev}_{(0,0)} \coprod_{\alpha \in K} \Omega^n \underline{X}_\alpha) \\
&\simeq \text{Hom}_{\mathbb{D}(e)}(g, \coprod_{\alpha \in K} \text{ev}_{(0,0)} \Omega^n \underline{X}_\alpha) \\
&\simeq \text{colim}_{\substack{S \subseteq K \\ S \text{ finite}}} \text{Hom}_{\mathbb{D}(e)}(g, \coprod_{\alpha \in S} \text{ev}_{(0,0)} \Omega^n \underline{X}_\alpha) \\
&\simeq \text{colim}_{\substack{S \subseteq K \\ S \text{ finite}}} \text{Hom}_{\mathbb{D}(e)}(g, \text{ev}_{(0,0)} \coprod_{\alpha \in S} \Omega^n \underline{X}_\alpha) \\
&\simeq \text{colim}_{\substack{S \subseteq K \\ S \text{ finite}}} \text{Hom}_{\text{St}(\mathbb{D})(e)}(\text{stab}(g), \bigoplus_{\alpha \in S} \Omega^n \underline{X}_\alpha) \\
&\simeq \text{colim}_{\substack{S \subseteq K \\ S \text{ finite}}} \text{Hom}_{\text{St}(\mathbb{D})(e)}(\Sigma^n \text{stab}(g), \bigoplus_{\alpha \in S} \underline{X}_\alpha)
\end{aligned}$$

√

**Lemma 9.3.** *Let  $\mathbb{T}$  be a triangulated derivator and  $\mathcal{G}$  a set of objects in  $\mathbb{T}(e)$  which satisfies conditions A1) and A2) of lemma 9.2.*

*Then for every small category  $L$  and every point  $x : e \rightarrow L$  in  $L$ , the set*

$$\{x_!(g) \mid g \in \mathcal{G}, x : e \rightarrow L\}$$

*satisfies conditions A1) and A2) in the category  $\mathbb{T}(L)$ .*

*Proof.* Suppose that

$$\text{Hom}_{\mathbb{T}(L)}(x_!(g), M) = \{*\},$$

for every  $g \in \mathcal{G}$  and every point  $x$  in  $L$ . Then by adjunction  $x^*M$  is isomorphic to  $*$  in  $\mathbb{T}(e)$  and so by the conservativity axiom,  $M$  is isomorphic to  $*$  in  $\mathbb{T}(L)$ . This shows condition A1). Condition A2) follows from the following isomorphisms

$$\begin{aligned}
\text{Hom}_{\mathbb{T}(L)}(x_!(g), \bigoplus_{\alpha \in K} M_\alpha) &\simeq \text{Hom}_{\mathbb{T}(e)}(g, x^* \bigoplus_{\alpha \in K} M_\alpha) \\
&\simeq \text{Hom}_{\mathbb{T}(e)}(g, \bigoplus_{\alpha \in K} x^* M_\alpha) \\
&\simeq \bigoplus_{\alpha \in K} \text{Hom}_{\mathbb{T}(e)}(g, x^* M_\alpha) \\
&\simeq \bigoplus_{\alpha \in K} \text{Hom}_{\mathbb{T}(L)}(x_!(g), M_\alpha).
\end{aligned}$$

√

*Remark 9.4.* Notice that if  $\mathbb{D}$  is a regular pointed strong derivator and we have at our disposal of a set  $\mathcal{G}$  of objects in  $\mathbb{D}(e)$  which satisfies conditions A1) and A2), then lemma 8.1 and lemma 9.3 imply that  $\text{St}(\mathbb{D})(L)$  is a compactly generated triangulated category, for every small category  $L$ .

**Relation with Hovey/Schwede's stabilization.** We will now relate Heller's construction with the construction of spectra as it is done by Hovey in [17] and Schwede in [33].

Let  $\mathcal{M}$  be a pointed, simplicial, left proper, cellular, almost finitely generated Quillen model category, see definition 4.1 in [17], where sequential colimits commute

with finite products and homotopy pullbacks. This implies that the associated derivator  $\mathrm{HO}(\mathcal{M})$  will be regular.

**Example 9.5.** Consider the category  $\mathbf{L}_{\Sigma, P}\mathrm{Fun}(\mathcal{M}_f^o, \mathbf{Sset}_\bullet)$  defined in section 8. Notice that the category of pointed simplicial pre-sheaves  $\mathrm{Fun}(\mathcal{M}_f^o, \mathbf{Sset}_\bullet)$  is pointed, simplicial, left proper, cellular and even finitely generated, see definition 4.1 in [17]. Since limits and colimits in  $\mathrm{Fun}(\mathcal{M}_f^o, \mathbf{Sset}_\bullet)$  are calculated objectwise, we conclude that sequential colimits commute with finite products. Now, by theorem 4.1.1 in [15] the category  $\mathbf{L}_{\Sigma, P}\mathrm{Fun}(\mathcal{M}_f^o, \mathbf{Sset}_\bullet)$  is also pointed, simplicial, left proper and cellular.

Now observe that the domains and codomains of each morphism in  $\Lambda((\Sigma \cup \{P\})_+)$ , see definition 4.2.1 in [15], are finitely presented, since the forgetful functor

$$\mathrm{Fun}(\mathcal{M}_f^o, \mathbf{Sset}_\bullet) \rightarrow \mathrm{Fun}(\mathcal{M}^o, \mathbf{Sset})$$

commutes with filtered colimits and homotopy pullbacks. Now, by proposition 4.2.4 in [15], we conclude that a morphism  $A \xrightarrow{f} B$  in  $\mathbf{L}_{\Sigma, P}\mathrm{Fun}(\mathcal{M}_f^o, \mathbf{Sset}_\bullet)$ , with  $B$  a local object, is a local fibration if and only if it has the right lifting property with respect to the set

$$J \cup \Lambda((\Sigma \cup \{P\})_+),$$

where  $J$  denotes the set of generating acyclic cofibrations in  $\mathrm{Fun}(\mathcal{M}_f^o, \mathbf{Sset}_\bullet)$ . This shows that  $\mathbf{L}_{\Sigma, P}\mathrm{Fun}(\mathcal{M}_f^o, \mathbf{Sset}_\bullet)$  is almost finitely generated.

Recall from section 1.2 in [33] that since  $\mathcal{M}$  is a pointed, simplicial model category, we have a Quillen adjunction

$$\begin{array}{ccc} & \mathcal{M} & \\ \Sigma(-) \downarrow & & \uparrow \Omega(-) \\ & \mathcal{M} & \end{array}$$

where  $\Sigma(X)$  denotes the suspension of an object  $X$ , i.e. the pushout of the diagram

$$\begin{array}{ccc} X \otimes \partial\Delta^1 & \longrightarrow & X \otimes \Delta^1 \\ \downarrow & & \\ * & & \end{array}$$

Recall also that in [17] and [33] the authors construct a stable Quillen model category  $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M})$  of spectra associated with  $\mathcal{M}$  and the left Quillen functor  $\Sigma(-)$ . We have the following Quillen adjunction, see [17],

$$\begin{array}{ccc} & \mathcal{M} & \\ \Sigma^\infty \downarrow & & \uparrow \mathrm{ev}_0 \\ & \mathrm{Sp}^{\mathbb{N}}(\mathcal{M}) & \end{array}$$

and thus a morphism to a strong triangulated derivator

$$\mathrm{HO}(\mathcal{M}) \xrightarrow{\mathbb{L}\Sigma^\infty} \mathrm{HO}(\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}))$$

which commutes with homotopy colimits.

By theorem 9.1, we have at our disposal a diagram

$$\begin{array}{ccc} \mathrm{HO}(\mathcal{M}) & & \\ \mathrm{stab} \downarrow & \searrow \mathbb{L}\Sigma^\infty & \\ \mathrm{St}(\mathrm{HO}(\mathcal{M})) & \xrightarrow{\varphi} & \mathrm{HO}(\mathrm{Sp}^{\mathbb{N}}(\mathcal{M})), \end{array}$$

which is commutative up to isomorphism in the 2-category of derivators.

Now suppose also that we have a set  $\mathcal{G}$  of small weak generators in  $\mathrm{Ho}(\mathcal{M})$ , as in definitions 7.2.1 and 7.2.2 in [17]. Suppose also that each object of  $\mathcal{G}$  considered in  $\mathcal{M}$  is cofibrant, finitely presented, homotopy finitely presented and has a finitely presented cylinder object.

**Example 9.6.** *Observe that the category  $\mathrm{Fun}(\mathcal{M}_f^o, \mathrm{Sset}_\bullet)$  is pointed and finitely generated. By corollary 7.4.4 in [16], the set*

$$\mathcal{G} = \{\mathbf{F}_{\Delta[n]_+/\partial\Delta[n]_+}^X \mid X \in \mathcal{M}_f, n \geq 0\},$$

*is a set of small weak generators in  $\mathrm{Ho}(\mathrm{Fun}(\mathcal{M}_f^o, \mathrm{Sset}_\bullet))$ . Since the domains and codomains of the set*

$$(\Sigma \cup \{P\})_+$$

*are homotopically finitely presented objects, lemma 8.1 implies that  $\mathcal{G}$  is a set of small weak generators in  $\mathrm{Ho}(\mathrm{L}_{\Sigma, P}\mathrm{Fun}(\mathcal{M}_f^o, \mathrm{Sset}_\bullet))$ . Clearly the elements of  $\mathcal{G}$  are cofibrant, finitely presented and have a finitely presented cylinder object. They are also homotopically finitely presented.*

Under the hypotheses above on the category  $\mathcal{M}$ , we have the following comparison theorem

**Theorem 9.7.** *The induced morphism of triangulated derivators*

$$\varphi : \mathrm{St}(\mathrm{HO}(\mathcal{M})) \longrightarrow \mathrm{HO}(\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}))$$

*is an equivalence.*

The proof of theorem 9.7 will consist in verifying the conditions of the following general proposition.

**Proposition 9.8.** *Let  $F : \mathbb{T}_1 \rightarrow \mathbb{T}_2$  be a morphism of strong triangulated derivators. Suppose that the triangulated categories  $\mathbb{T}_1(e)$  and  $\mathbb{T}_2(e)$  are compactly generated and that there is a set  $\mathcal{G} \subset \mathbb{T}_1(e)$  of compact generators, which is stable under suspensions and satisfies the following conditions :*

a)  *$F(e)$  induces bijections*

$$\mathrm{Hom}_{\mathbb{T}_1(e)}(g_1, g_2) \rightarrow \mathrm{Hom}_{\mathbb{T}_2(e)}(Fg_1, Fg_2), \forall g_1, g_2 \in \mathcal{G}$$

*and*

b) *the set of objects  $\{Fg \mid g \in \mathcal{G}\}$  is a set of compact generators in  $\mathbb{T}_2(e)$ .*

*Then the morphism  $F$  is an equivalence of derivators.*

*Proof.* Conditions a) and b) imply that  $F(e)$  is an equivalence of triangulated category, see [30].

Now, let  $L$  be a small category. We show that conditions a) and b) are also verified by  $F(L)$ ,  $\mathbb{T}_1(L)$  and  $\mathbb{T}_2(L)$ . By lemma 9.3 the sets

$$\{x_!(g) \mid g \in \mathcal{G}, x : e \rightarrow L\} \quad \text{and} \quad \{x_!(Fg) \mid g \in \mathcal{G}, x : e \rightarrow L\}$$

consist of compact generators for  $\mathbb{T}_1(L)$ , resp.  $\mathbb{T}_2(L)$ , which are stable under suspensions. Since  $F$  commutes with homotopy colimits  $F(x_!(g)) = x_!(Fg)$  and so the following isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbb{T}_1(L)}(x_!(g_1), x_!(g_2)) &\simeq \mathrm{Hom}_{\mathbb{T}_1(e)}(g_1, x^*x_!(g_2)) \\ &\simeq \mathrm{Hom}_{\mathbb{T}_2(e)}(F(g_1), F(x^*x_!(g_2))) \\ &\simeq \mathrm{Hom}_{\mathbb{T}_2(e)}(Fg_1, x^*F(x_!(g_2))) \\ &\simeq \mathrm{Hom}_{\mathbb{T}_2(L)}(x_!F(g_1), x_!F(g_2)) \end{aligned}$$

imply the proposition. √

Let us now prove theorem 9.7 :

*Proof.* Let us first prove condition b) of proposition 9.8. Since the set  $\mathcal{G}$  of small generators in  $\mathrm{Ho}(\mathcal{M})$  satisfies the conditions of lemma 9.2, we have a set

$$\{\Sigma^n \mathrm{stab}(g) \mid g \in \mathcal{G}, n \in \mathbb{Z}\}$$

of compact generators in  $\mathrm{St}(\mathrm{HO}(\mathcal{M}))(e)$ , which is stable under suspensions. We now show that the set

$$\{\Sigma^n \mathbb{L}\Sigma^\infty(g) \mid g \in \mathcal{G}, n \in \mathbb{Z}\}$$

is a set of compact generators in  $\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}))$ . These objects are compact because the functor  $\mathbb{R}ev_0$  in the adjunction

$$\begin{array}{ccc} & \mathrm{Ho}(\mathcal{M}) & \\ & \downarrow \mathbb{L}\Sigma^\infty \quad \uparrow \mathbb{R}ev_0 & \\ & \mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}(\mathcal{M})) & \end{array}$$

commutes with filtered homotopy colimits. We now show that they form a set of generators. Let  $Y$  be an object in  $\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}))$ , that we can suppose, without loss of generality, to be an  $\Omega$ -spectrum, see [17]. Suppose that

$$\mathrm{Hom}(\Sigma^n \mathbb{L}\Sigma^\infty(g_i), Y) \simeq \mathrm{colim}_m \mathrm{Hom}(g_i, \Omega^m Y_{m+p}) = \{*\}, \quad n \geq 0.$$

Since  $Y$  is an  $\Omega$ -spectrum, we have

$$Y_p = *, \quad \forall p \geq 0.$$

This implies that  $Y$  is isomorphic to  $*$  in  $\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}))$ .

We now show condition a). Let  $g_1$  and  $g_2$  be objects in  $\mathcal{G}$ . Observe that we have the following isomorphisms, see [14]

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{St}(\mathrm{HO}(\mathcal{M}))(e)}(\mathrm{stab}(g_1), \mathrm{stab}(g_2)) \\ &\simeq \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(g_1, (ev_{(0,0)} \circ loc \circ L[0,0])(g_2)) \\ &\simeq \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(g_1, ev_{(0,0)}(\mathrm{hocolim}(L[0,0](g_2) \rightarrow \Omega\sigma L[0,0](g_2) \rightarrow \dots))) \\ &\simeq \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(g_1, \mathrm{hocolim} ev_{(0,0)}(L[0,0](g_2) \rightarrow \Omega\sigma L[0,0](g_2) \rightarrow \dots)) \\ &\simeq \mathrm{colim}_j \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(g_1, \Omega^j \Sigma^j(g_2)). \end{aligned}$$

Now, by corollary 4.13 in [17], we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}))}(\mathbb{L}\Sigma^\infty(g_1), \mathbb{L}\Sigma^\infty(g_2)) &\simeq \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}))}(\Sigma^\infty(g_1), (\Sigma^\infty(g_2))_f) \\ &\simeq \mathrm{colim}_m \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(g_1, \Omega^m (\Sigma^m(g_2))_f), \end{aligned}$$

where  $(\Sigma^\infty(g_2))_f$  denotes a levelwise fibrant resolution of  $\Sigma^\infty(g_2)$  in the category  $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M})$ .

Now, notice that since  $g_2$  is cofibrant, so is  $\Sigma^m(g_2)$  and so we have the following isomorphism

$$\Omega^m(\Sigma^m(g_2))_f \xrightarrow{\sim} (\mathbb{R}\Omega)^m \circ (\mathbb{L}\Sigma)^m(g_2)$$

in  $\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}))$ . This implies that for  $j \geq 0$ , we have an isomorphism

$$\Omega^j \Sigma^j(g_2) \xrightarrow{\sim} \Omega^j(\Sigma^j(g_2))_f$$

in  $\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}))$  and so

$$\mathrm{Hom}_{\mathrm{St}(\mathrm{HO}(\mathcal{M}))_{(e)}}(\mathrm{stab}(g_1), \mathrm{stab}(g_2)) = \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}))}(\mathbb{L}\Sigma^\infty(g_1), \mathbb{L}\Sigma^\infty(g_2)).$$

Let now  $p$  be an integer. Notice that

$$\mathrm{Hom}_{\mathrm{St}(\mathrm{HO}(\mathcal{M}))_{(e)}}(\mathrm{stab}(g_1), \Sigma^p \mathrm{stab}(g_2)) = \mathrm{colim}_j \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(g_1, \Omega^j \Sigma^{j+p}(g_2))$$

and that

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}))}(\mathbb{L}\Sigma^\infty(g_1), \Sigma^p \mathbb{L}\Sigma^\infty(g_2)) = \mathrm{colim}_m \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(g_1, \Omega^m(\Sigma^{m+p}(g_2))_f).$$

This proves condition *a*) and so the theorem is proven.  $\checkmark$

*Remark 9.9.* If we consider for  $\mathcal{M}$  the category  $\mathrm{L}_{\Sigma, P}\mathrm{Fun}(\mathcal{M}_f^o, Sset_\bullet)$ , we have equivalences of derivators

$$\varphi : \mathrm{St}(\mathrm{L}_{\Sigma, P}\mathrm{Hot}_{\mathcal{M}_f^\bullet}) \xrightarrow{\sim} \mathrm{HO}(\mathrm{Sp}^{\mathbb{N}}(\mathrm{L}_{\Sigma, P}\mathrm{Fun}(\mathcal{M}_f^o, Sset_\bullet))) \xleftarrow{\sim} \mathrm{St}(\mathrm{L}_{\Sigma, P}\mathrm{Hot}_{\mathcal{M}_f}).$$

Let  $\mathbb{D}$  be a strong triangulated derivator.

Now, by theorem 9.1 and proposition 7.1, we have the following proposition

**Proposition 9.10.** *We have an equivalence of categories*

$$\underline{\mathrm{Hom}}_! (\mathrm{St}(\mathrm{L}_{\Sigma, P}\mathrm{Hot}_{\mathrm{dgc}at_f}), \mathbb{D}) \xrightarrow{(\mathrm{stab} \circ \Phi \circ \mathbb{R}\underline{h})^*} \underline{\mathrm{Hom}}_{\mathrm{flt}, p} (\mathrm{HO}(\mathrm{dgc}at), \mathbb{D}).$$

Since the category  $Sset_\bullet$  satisfies all the conditions of theorem 9.7, we have the following characterization of the classical category of spectra, after Bousfield-Friedlander [2], by a universal property.

**Proposition 9.11.** *We have an equivalence of categories*

$$\underline{\mathrm{Hom}}_! (\mathrm{HO}(\mathrm{Sp}^{\mathbb{N}}(Sset_\bullet)), \mathbb{D}) \xrightarrow{\sim} \mathbb{D}(e).$$

*Proof.* By theorems 9.7 and 4.1, we have the following equivalences

$$\begin{aligned} \underline{\mathrm{Hom}}_! (\mathrm{HO}(\mathrm{Sp}^{\mathbb{N}}(Sset_\bullet)), \mathbb{D}) &\simeq \underline{\mathrm{Hom}}_! (\mathrm{HO}(Sset_\bullet), \mathbb{D}) \\ &= \underline{\mathrm{Hom}}_! (\mathrm{Hot}_\bullet, \mathbb{D}) \\ &\simeq \underline{\mathrm{Hom}}_! (\mathrm{Hot}, \mathbb{D}) \\ &\simeq \mathbb{D}(e). \end{aligned}$$

This proves the proposition.  $\checkmark$

*Remark 9.12.* An analogous characterization of the category of spectra, but in the context of stable  $\infty$ -categories is proved in [26, 17.6].

## 10. DG QUOTIENTS

Recall from [34] [35] that we have at our disposal a Morita Quillen model structure on the category of small dg categories  $\mathbf{dgc}at$ , see example 6.1. As shown in [34] [35] the homotopy category  $\mathbf{Ho}(\mathbf{dgc}at)$  is pointed. In the following, we will be considering this Quillen model structure. We denote by  $I$  the set of generating cofibrations.

**Notation 10.1.** We denote by  $\mathcal{E}$  the set of inclusions of full dg subcategories

$$\mathcal{G} \hookrightarrow \mathcal{H},$$

where  $\mathcal{H}$  is a strict finite  $I$ -cell.

Recall that we have a morphism of derivators

$$\mathcal{U}_t := \text{stab} \circ \Phi \circ \mathbb{R}\underline{h} : \mathbf{HO}(\mathbf{dgc}at) \rightarrow \mathbf{St}(\mathbf{L}_{\Sigma, P}\mathbf{Hot}_{\mathbf{dgc}at_f})$$

which commutes with filtered homotopy colimits and preserves the point.

Let us now make some general arguments.

Let  $\mathbb{D}$  be a pointed derivator. We denote by  $I$  the category associated to the graph

$$0 \leftarrow 1.$$

Consider the functor  $t = 1 : e \rightarrow I$ . Since the functor  $t$  is an open immersion and the derivator  $\mathbb{D}$  is pointed, the functor

$$t_! : \mathbb{D}(e) \rightarrow \mathbb{D}(I)$$

has a left adjoint

$$t^? : \mathbb{D}(I) \rightarrow \mathbb{D}(e),$$

see [3]. We denote it by

$$\text{cone} : \mathbb{D}(I) \rightarrow \mathbb{D}(e).$$

Let  $F : \mathbb{D} \rightarrow \mathbb{D}'$  be a morphism of pointed derivators. Notice that we have a natural transformation of functors.

$$S : \text{cone} \circ F(I) \rightarrow F(e) \circ \text{cone}.$$

**Proposition 10.2.** Let  $\mathcal{A} \xrightarrow{R} \mathcal{B}$  be an inclusion of a full dg subcategory and  $\Gamma_R$

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{R} & \mathcal{B} \\ \downarrow & & \\ 0 & & \end{array}$$

the associated object in  $\mathbf{HO}(\mathbf{dgc}at)(\Gamma)$ , where  $0$  denotes the terminal object in  $\mathbf{Ho}(\mathbf{dgc}at)$ . Then there exists a filtered category  $J$  and an object  $D_R$  in  $\mathbf{HO}(\mathbf{dgc}at)(\Gamma \times J)$ , such that

$$p_!(D_R) \xrightarrow{\sim} \Gamma_R,$$

where  $p : \Gamma \times J \rightarrow \Gamma$  denotes the projection functor. Moreover, for every point  $j : e \rightarrow J$  in  $J$  the object  $(1 \times j)^*$  in  $\mathbf{HO}(\mathbf{dgc}at)(\Gamma)$  is of the form

$$0 \leftarrow Y_j \xrightarrow{L_j} X_j,$$

where  $Y_j \xrightarrow{L_j} X_j$ , belongs to the set  $\mathcal{E}$ .

*Proof.* Apply the small object argument to the morphism

$$\emptyset \longrightarrow \mathcal{B}$$

using the set of generating cofibrations  $I$  and obtain the factorization

$$\begin{array}{ccc} \emptyset & \xrightarrow{\quad} & \mathcal{B} \\ & \searrow i & \nearrow p \\ & & Q(\mathcal{B}) \end{array},$$

where  $i$  is an  $I$ -cell. Now consider the following fiber product

$$\begin{array}{ccc} p^{-1}(\mathcal{A}) & \hookrightarrow & Q(\mathcal{B}) \\ \sim \downarrow & \lrcorner & \downarrow p \\ \mathcal{A} & \xrightarrow{J} & \mathcal{B}. \end{array}$$

Notice that  $p^{-1}(\mathcal{A})$  is a full dg subcategory of  $Q(\mathcal{B})$ .

Now, by proposition 6.2, we have an isomorphism

$$\operatorname{colim}_{j \in J} X_j \xrightarrow{\sim} Q(\mathcal{B}),$$

where  $J$  is the filtered category of inclusions of strict finite sub- $I$ -cells  $X_j$  into  $Q(\mathcal{B})$ .

For each  $j \in J$ , consider the fiber product

$$\begin{array}{ccc} Y_j & \hookrightarrow & X_j \\ \downarrow & \lrcorner & \downarrow \\ p^{-1}(\mathcal{A}) & \hookrightarrow & Q(\mathcal{B}). \end{array}$$

In this way, we obtain a morphism of diagrams

$$\{Y_j\}_{j \in J} \hookrightarrow \{X_j\}_{j \in J},$$

such that for each  $j$  in  $J$ , the inclusion

$$Y_j \hookrightarrow X_j$$

belongs to the set  $\mathcal{E}$  and  $J$  is filtered.

Consider now the diagram  $D_I$

$$\{0 \leftarrow Y_j \hookrightarrow X_j\}_{j \in J}$$

in the category  $\operatorname{Fun}(\Gamma \times J, \operatorname{dgc}at)$ . Now, notice that we have the isomorphism

$$\operatorname{colim}_{j \in J} \{0 \leftarrow Y_j \hookrightarrow X_j\} \xrightarrow{\sim} \{0 \leftarrow p^{-1}(\mathcal{A}) \hookrightarrow Q(\mathcal{B})\}$$

in  $\operatorname{Fun}(\Gamma, \operatorname{dgc}at)$  and the weak equivalence

$$\begin{array}{ccc} \{0 \leftarrow p^{-1}(\mathcal{A}) \hookrightarrow Q(\mathcal{B})\} & & \\ \parallel & \downarrow \sim & \downarrow \sim \\ \{0 \leftarrow \mathcal{A} \hookrightarrow \mathcal{B}\} & & \end{array}$$

in  $\text{Fun}(\Gamma, \text{dgcats})$ , when endowed with the projective model structure, see [15]. Since  $\text{Fun}(\Gamma, \text{dgcats})$  is clearly also compactly generated, we have an isomorphism

$$\text{hocolim}_{j \in J} (0 \leftarrow Y_j \rightarrow X_j) \xrightarrow{\sim} \text{colim}_{j \in J} (0 \leftarrow Y_j \rightarrow X_j).$$

Finally, notice that  $D_R$  is an object of  $\text{HO}(\text{dgcats})(\Gamma \times J)$  and that  $p_!(D_R)$ , where  $p : \Gamma \times J \rightarrow J$  denotes the projection functor, identifies with

$$\text{hocolim}_{i \in J} (0 \leftarrow Y_j \rightarrow X_j).$$

This proves the proposition.  $\checkmark$

**Notation 10.3.** We denote by  $\mathcal{E}_{st}$  the set of morphisms  $S_L$ , where  $L$  belongs to the set  $\mathcal{E}$ .

Let  $\mathbb{D}$  be a strong triangulated derivator.

**Theorem 10.4.** *If*

$$G : \text{St}(\mathbb{L}_{\Sigma, P} \text{Hot}_{\text{dgcats}_f}) \rightarrow \mathbb{D}$$

*is a morphism of triangulated derivators commuting with arbitrary homotopy colimits and such that  $G(e)(S_L)$  is invertible for each  $L$  in  $\mathcal{E}$ , then  $G(e)(S_K)$  is invertible for each inclusion  $K : \mathcal{A} \hookrightarrow \mathcal{B}$  of a full dg subcategory.*

*Proof.* Let  $\mathcal{A} \xrightarrow{K} \mathcal{B}$  be an inclusion of a full dg subcategory. Consider the morphism

$$\varphi_K := \varphi(\Gamma_K) : (i_! \circ \mathcal{U}_T)(\Gamma_K) \longrightarrow (\mathcal{U}_T \circ i_!)(\Gamma_K)$$

in  $\text{St}(\mathbb{L}_{\Sigma, P} \text{Hot}_{\text{dgcats}_f})(\square)$ .

Let  $D_K$  be the object of  $\text{HO}(\text{dgcats})(\Gamma \times J)$  constructed in proposition 10.2. In particular  $p_!(D_K) \xrightarrow{\sim} \Gamma_K$ , where  $p' : \Gamma \times J \rightarrow \Gamma$  denotes the projection functor.

The inclusion  $i : \Gamma \hookrightarrow \square$ , induces a commutative square

$$\begin{array}{ccc} \text{HO}(\text{dgcats})(\square \times J) & \xrightarrow{\mathcal{U}_T(\square \times J)} & \text{St}(\mathbb{L}_{\Sigma, P} \text{Hot}_{\text{dgcats}_f})(\square \times J) \\ \downarrow (i \times 1)^* & & \downarrow (i \times 1)^* \\ \text{HO}(\text{dgcats})(\Gamma \times J) & \xrightarrow{\mathcal{U}_T(\Gamma \times J)} & \text{St}(\mathbb{L}_{\Sigma, P} \text{Hot}_{\text{dgcats}_f})(\Gamma \times J) \end{array}$$

and a morphism

$$\Psi : ((i \times 1)_! \circ \mathcal{U}_T(\Gamma \times J))(D_K) \longrightarrow (\mathcal{U}_T(\square \times J) \circ (i \times 1)_!)(D_K).$$

We will now show that

$$p_! \Psi \xrightarrow{\sim} \varphi_K,$$

where  $p : \square \times J \rightarrow \square$ , denotes the projection functor.

The fact that we have the following commutative square

$$\begin{array}{ccc} \square & \xleftarrow{p} & \square \times J \\ \uparrow i & & \uparrow i \times 1 \\ \Gamma & \xleftarrow{p'} & \Gamma \times J \end{array}$$

and that the morphism of derivators  $\mathcal{U}_T$  commutes with filtered homotopy colimits implies the following equivalences

$$\begin{aligned}
& p_! \Psi \\
&= p_! \circ (i \times 1)_! \circ \mathcal{U}_T(\Gamma \times j)(D_K) \quad \rightarrow \quad p_! \circ \mathcal{U}_T(\square \times J) \circ (i \times 1)_!(D_K) \\
&\simeq i_! \circ p'_! \circ \mathcal{U}_T(\Gamma \times J)(D_K) \quad \rightarrow \quad \mathcal{U}_T(\square \times J) \circ p_! \circ (i \times 1)_!(D_K) \\
&\simeq i_! \circ \mathcal{U}_T(\Gamma) \circ p'_!(D_K) \quad \rightarrow \quad \mathcal{U}_T(\square) \circ i_! \circ p'_!(D_K) \\
&\simeq (i_! \circ \mathcal{U}_T(\Gamma))(\Gamma_K) \quad \rightarrow \quad (\mathcal{U}_T(\square) \circ i_!)(\Gamma_K) \\
&= \varphi_J
\end{aligned}$$

This shows that

$$p_!(\Psi) \xrightarrow{\sim} \varphi_K.$$

We now show that  $\Psi$  is an isomorphism. For this, by conservativity, it is enough to show that for every object  $j : e \rightarrow J$  in  $J$ , the morphism

$$(1 \times j)^*(\Psi),$$

is an isomorphism in  $\mathbf{St}(\mathbf{L}_{\Sigma, P} \mathbf{Hot}_{\mathbf{dgc}at_f})(\square)$ . Recall from proposition 10.2 that  $(1 \times j)^*(D_K)$  identifies with

$$\{0 \leftarrow Y_j \xrightarrow{L_j} X_j\},$$

where  $L_j$  belongs to  $\mathcal{E}$ . We now show that  $(1 \times j)^*(\Psi)$  identifies with  $\varphi_{L_j}$ , which by hypotheses, is an isomorphism.

Now, the following commutative diagram

$$\begin{array}{ccc}
\square & \xrightarrow{1 \times j} & \square \times J \\
i \uparrow & & \uparrow i \times i \\
\Gamma & \xrightarrow{1 \times j} & \Gamma \times J
\end{array}$$

and the dual of proposition 2.8 in [4] imply that we have the following equivalences

$$\begin{aligned}
& (1 \times j)^* \Psi \\
&= ((1 \times j)^* \circ (i \times 1)_! \circ \mathcal{U}_T(\Gamma \times J))(D_K) \quad \rightarrow \quad ((1 \times j)^* \circ \mathcal{U}_T(\square \times J) \circ (i \times 1)_!)(D_K) \\
&\simeq (i_! \circ (1 \times j)^* \circ \mathcal{U}_T(\Gamma \times J))(D_K) \quad \rightarrow \quad (\mathcal{U}_T(\square \times J) \circ (1 \times j)^* \circ (i \times 1)_!)(D_K) \\
&\simeq (i_! \circ \mathcal{U}_T(\Gamma) \circ (1 \times j)^*)(D_K) \quad \rightarrow \quad (\mathcal{U}_T(\square) \circ i_! \circ (1 \times j)^*)(D_K) \\
&\simeq i_! \circ \mathcal{U}_T(\Gamma)(\Gamma_{L_j}) \quad \rightarrow \quad \mathcal{U}_T(\square) \circ i_!(\Gamma_{L_j}) \\
&= \varphi_{L_j}
\end{aligned}$$

Since by hypotheses  $\varphi_{L_j}$  is an isomorphism and the morphism  $G$  commutes with homotopy colimits the theorem is proven.  $\checkmark$

## 11. THE UNIVERSAL LOCALIZING INVARIANT

Recall from theorem 9.7 and remark 9.9 that if we consider for the category  $\mathcal{M}$  the category  $\mathbf{L}_{\Sigma, P} \mathbf{Fun}(\mathbf{dgc}at_f^o, Sset_{\bullet})$ , see example 9.6, we have an equivalence of triangulated derivators

$$\varphi : \mathbf{St}(\mathbf{L}_{\Sigma, P} \mathbf{Hot}_{\mathbf{dgc}at_f}) \xrightarrow{\sim} \mathbf{HO}(\mathbf{Sp}^{\mathbb{N}}(\mathbf{L}_{\Sigma, P} \mathbf{Fun}(\mathbf{dgc}at_f^o, Sset_{\bullet}))).$$

Now, stabilize the set  $\mathcal{E}_{st}$  defined in the previous section under the functor loop space and choose for each element of this stabilized set a representative in the category  $\mathbf{Sp}^{\mathbb{N}}(\mathbf{L}_{\Sigma, P} \mathbf{Fun}(\mathbf{dgc}at_f^o, Sset_{\bullet}))$ . We denote the set of these representatives by  $\widetilde{\mathcal{E}}_{st}$ . Since  $\mathbf{Sp}^{\mathbb{N}}(\mathbf{L}_{\Sigma, P} \mathbf{Fun}(\mathbf{dgc}at_f^o, Sset_{\bullet}))$  is a left proper, cellular Quillen model

category, see [17], its left Bousfield localization by  $\widetilde{\mathcal{E}}_{st}$  exists. We denote it by  $L_{\widetilde{\mathcal{E}}_{st}} \mathrm{Sp}^{\mathbb{N}}(\mathrm{L}_{\Sigma, P} \mathrm{Fun}(\mathrm{dgc}at_f^o, Sset_{\bullet}))$ . By lemma 5.3 it is a stable Quillen model category.

*Remark 11.1.* Since the localization morphism

$$\gamma : \mathrm{St}(\mathrm{L}_{\Sigma, P} \mathrm{Hot}_{\mathrm{dgc}at_f}) \xrightarrow{\mathbb{L}Id} \mathrm{HO}(L_{\widetilde{\mathcal{E}}_{st}} \mathrm{Sp}^{\mathbb{N}}(\mathrm{L}_{\Sigma, P} \mathrm{Fun}(\mathrm{dgc}at_f^o, Sset_{\bullet})))$$

commutes with homotopy colimits and inverts the set of morphisms  $\mathcal{E}_{st}$ , theorem 10.4 allows us to conclude that it inverts all morphisms  $S_K$  for each inclusion  $\mathcal{A} \hookrightarrow \mathcal{B}$  of a full dg subcategory.

**Definition 11.2.** - *The Localizing motivator of dg categories  $\mathcal{M}_{dg}^{loc}$  is the triangulated derivator associated with the stable Quillen model category*

$$L_{\widetilde{\mathcal{E}}_{st}} \mathrm{Sp}^{\mathbb{N}}(\mathrm{L}_{\Sigma, P} \mathrm{Fun}(\mathrm{dgc}at_f^o, Sset_{\bullet})).$$

- *The Universal localizing invariant of dg categories is the canonical morphism of derivators*

$$\mathcal{U}_l : \mathrm{HO}(\mathrm{dgc}at) \rightarrow \mathcal{M}_{dg}^{loc}.$$

We sum up the construction of  $\mathcal{M}_{dg}^{loc}$  in the following diagram

$$\begin{array}{ccc}
 \mathrm{dgc}at_f[S^{-1}] & \longrightarrow & \mathrm{HO}(\mathrm{dgc}at) \\
 \mathrm{Ho}(h) \downarrow & \nearrow \mathbb{L}Re & \nearrow \mathbb{R}h \\
 \mathrm{L}_{\Sigma} \mathrm{Hot}_{\mathrm{dgc}at_f} & & \\
 \Phi \downarrow & & \\
 \mathrm{L}_{\Sigma, P} \mathrm{Hot}_{\mathrm{dgc}at_f} & & \\
 \mathrm{stab} \downarrow & & \\
 \mathrm{St}(\mathrm{L}_{\Sigma, P} \mathrm{Hot}_{\mathrm{dgc}at_f}) & & \\
 \gamma \downarrow & \searrow \mathcal{U}_l & \\
 \mathcal{M}_{dg}^{loc} & & 
 \end{array}$$

Observe that the morphism of derivators  $\mathcal{U}_l$  is pointed, commutes with filtered homotopy colimits and satisfies the following condition:

Dr) For every inclusion  $\mathcal{A} \xrightarrow{K} \mathcal{B}$  of a full dg subcategory the canonical morphism

$$S_K : \mathrm{cone}(\mathcal{U}_l(\mathcal{A} \xrightarrow{K} \mathcal{B})) \rightarrow \mathcal{U}_l(\mathcal{B}/\mathcal{A})$$

is invertible in  $\mathcal{M}_{dg}^{loc}(e)$ .

We now give a conceptual characterization of condition Dr). Let  $I$  be the category associated with the graph  $0 \leftarrow 1$ .

**Lemma 11.3.** *The isomorphism classes in  $\mathrm{HO}(\mathrm{dgc}at)(I)$  associated with the inclusions  $\mathcal{A} \xrightarrow{K} \mathcal{B}$  of full dg subcategories coincide with the classe of homotopy monomorphisms in  $\mathrm{dgc}at$ , see section 2 in [39].*

*Proof.* Recall from section 2 in [39] that in a model category  $\mathcal{M}$  a morphism  $X \xrightarrow{f} Y$  is a homotopy monomorphism if for every object  $Z$  in  $\mathcal{M}$ , the induced morphism of simplicial sets

$$\mathrm{Map}(Z, X) \xrightarrow{f_*} \mathrm{Map}(Z, Y)$$

induces an injection on  $\pi_0$  and isomorphisms on all  $\pi_i$  for  $i > 0$  (for all base points).

Now, by lemma 2.4 of [39] a dg functor  $\mathcal{A} \xrightarrow{F} \mathcal{B}$  is an homotopy monomorphism on the quasi-equivalent Quillen model category on  $\mathrm{dgc}at$  if and only if it is quasi-fully faithful, i.e. for any two objects  $X$  and  $Y$  in  $\mathcal{A}$  the morphism of complexes  $\mathrm{Hom}_{\mathcal{A}}(X, Y) \rightarrow \mathrm{Hom}_{\mathcal{B}}(FX, FY)$  is a quasi-isomorphism.

Recall that by corollary 5.10 of [35] [34] the mapping space functor  $\mathrm{Map}(\mathcal{A}, \mathcal{B})$  in the Morita Quillen model category identifies with the mapping space  $\mathrm{Map}(\mathcal{A}, \mathcal{B}_f)$  in the quasi-equivalent Quillen model category, where  $\mathcal{B}_f$  denotes a Morita fibrant resolution of  $\mathcal{B}$ . This implies that a dg functor  $\mathcal{A} \xrightarrow{F} \mathcal{B}$  is a homotopy monomorphism if and only if  $\mathcal{A}_f \xrightarrow{F_f} \mathcal{B}_f$  is a quasi-fully faithful dg functor.

Now, notice that an inclusion  $\mathcal{A} \hookrightarrow \mathcal{B}$  of a full dg subcategory is a homotopy monomorphism. Conversely, let  $\mathcal{A} \xrightarrow{F} \mathcal{B}$  be a homotopy monomorphism. Consider the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{A}}_f & \hookrightarrow & \mathcal{B}_f \\ \pi \uparrow & & \parallel \\ \mathcal{A}_f & \xrightarrow{F_f} & \mathcal{B}_f \\ \sim \uparrow & & \uparrow \sim \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B}, \end{array}$$

where  $\widetilde{\mathcal{A}}_f$  denotes the full dg subcategory of  $\mathcal{B}_f$  whose objects are those in the image by the dg functor  $F_f$ . Since  $F_f$  is a quasi-fully faithful dg functor, the dg functor  $\pi$  is a quasi-equivalence. This proves the lemma.  $\checkmark$

*Remark 11.4.* Lemma 11.3 shows that condition  $Dr$ ) is equivalent to

$Dr')$  For every homotopy monomorphism  $\mathcal{A} \xrightarrow{F} \mathcal{B}$  in  $\mathrm{HO}(\mathrm{dgc}at)(I)$  the canonical morphism

$$\mathrm{cone}(\mathcal{U}_I(\mathcal{A} \xrightarrow{F} \mathcal{B})) \rightarrow \mathcal{U}_I(\mathrm{cone}(F))$$

is invertible in  $\mathcal{M}_{dg}^{loc}(e)$ .

Let  $\mathbb{D}$  be a strong triangulated derivator.

**Theorem 11.5.** *The morphism  $\mathcal{U}_I$  induces an equivalence of categories*

$$\underline{\mathrm{Hom}}_I(\mathcal{M}_{dg}^{loc}, \mathbb{D}) \xrightarrow{\mathcal{U}_I^*} \underline{\mathrm{Hom}}_{flt, Dr, p}(\mathrm{HO}(\mathrm{dgc}at), \mathbb{D}),$$

where  $\underline{\mathrm{Hom}}_{flt, Dr, p}(\mathrm{HO}(\mathrm{dgc}at), \mathbb{D})$  denotes the category of morphisms of derivators which commute with filtered homotopy colimits, satisfy condition  $Dr$ ) and preserve the point.

*Proof.* By theorem 5.4, we have the following equivalence of categories

$$\underline{\mathrm{Hom}}_I(\mathcal{M}_{dg}^{loc}, \mathbb{D}) \xrightarrow{\gamma^*} \underline{\mathrm{Hom}}_{I, \widetilde{est}}(\mathrm{St}(\mathrm{L}_{\Sigma, P}\mathrm{Fun}(\mathrm{dgc}at_f^o, \mathcal{S}set_{\bullet})), \mathbb{D}).$$

We now show that we have the following equivalence of categories

$$\underline{\mathrm{Hom}}_{\mathbb{I}, \widetilde{\mathcal{E}}_{st}}(\mathrm{St}(\mathrm{L}_{\Sigma, P}\mathrm{Fun}(\mathrm{dgc}at_f^o, Sset_{\bullet})), \mathbb{D}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathbb{I}, \mathcal{E}_{st}}(\mathrm{St}(\mathrm{L}_{\Sigma, P}\mathrm{Fun}(\mathrm{dgc}at_f^o, Sset_{\bullet})), \mathbb{D}).$$

Let  $G$  be an element of  $\underline{\mathrm{Hom}}_{\mathbb{I}, \mathcal{E}_{st}}(\mathrm{St}(\mathrm{L}_{\Sigma, P}\mathrm{Fun}(\mathrm{dgc}at_f^o, Sset_{\bullet})), \mathbb{D})$  and  $s$  an element of  $\mathcal{E}_{st}$ . We show that the image of  $s$  under the functor  $G(e) \circ \Omega(e)$  is an isomorphism in  $\mathbb{D}(e)$ . Recall from the proof of lemma 5.3 that the functor  $G(e)$  commutes with  $\Sigma(e)$ . Since the suspension and loop space functors in  $\mathbb{D}(e)$  are inverse of each other we conclude that the image of  $s$  under the functor  $G(e) \circ \Omega(e)$  is an isomorphism in  $\mathbb{D}(e)$ . Now, simply observe that the category on the right hand side of the above equivalence identifies with  $\underline{\mathrm{Hom}}_{flt, Dr, p}(\mathrm{HO}(\mathrm{dgc}at), \mathbb{D})$  under the equivalence

$$\underline{\mathrm{Hom}}_{\mathbb{I}}(\mathrm{St}(\mathrm{L}_{\Sigma, P}\mathrm{Hot}_{\mathrm{dgc}at_f}), \mathbb{D}) \xrightarrow{(\mathrm{stabo}\Phi \circ \mathbb{R}h)^*} \underline{\mathrm{Hom}}_{flt, p}(\mathrm{HO}(\mathrm{dgc}at), \mathbb{D}),$$

of proposition 9.10.

This proves the theorem. ✓

**Terminology 11.6.** *We call an object of the right hand side category of theorem 11.5 a localizing invariant of dg categories.*

We now present some examples.

**Hochschild and cyclic homology.** Let  $\mathcal{A}$  be a small  $k$ -flat  $k$ -category. The *Hochschild chain complex* of  $\mathcal{A}$  is the complex concentrated in homological degrees  $p \geq 0$  whose  $p$ th component is the sum of the

$$\mathcal{A}(X_p, X_0) \otimes \mathcal{A}(X_p, X_{p-1}) \otimes \mathcal{A}(X_{p-1}, X_{p-2}) \otimes \cdots \otimes \mathcal{A}(X_0, X_1),$$

where  $X_0, \dots, X_p$  range through the objects of  $\mathcal{A}$ , endowed with the differential

$$d(f_p \otimes \cdots \otimes f_0) = f_{p-1} \otimes \cdots \otimes f_0 f_p + \sum_{i=1}^p (-1)^i f_p \otimes \cdots \otimes f_i f_{i-1} \otimes \cdots \otimes f_0.$$

Via the cyclic permutations

$$t_p(f_{p-1} \otimes \cdots \otimes f_0) = (-1)^p f_0 \otimes f_{p-1} \otimes \cdots \otimes f_1$$

this complex becomes a precyclic chain complex and thus gives rise to a *mixed complex*  $C(\mathcal{A})$ , i.e. a dg module over the dg algebra  $\Lambda = k[B]/(B^2)$ , where  $B$  is of degree  $-1$  and  $dB = 0$ . All variants of cyclic homology only depend on  $C(\mathcal{A})$  considered in  $\mathcal{D}(\Lambda)$ . For example, the cyclic homology of  $\mathcal{A}$  is the homology of the complex  $C(\mathcal{A}) \underset{\Lambda}{\otimes} k$ , cf. [18].

If  $\mathcal{A}$  is a  $k$ -flat differential graded category, its mixed complex is the sum-total complex of the bicomplex obtained as the natural re-interpretation of the above complex. If  $\mathcal{A}$  is an arbitrary dg  $k$ -category, its Hochschild chain complex is defined as the one of a  $k$ -flat (e.g. a cofibrant) resolution of  $\mathcal{A}$ . The following theorem is proved in [22].

**Theorem 11.7.** *The map  $\mathcal{A} \mapsto C(\mathcal{A})$  yields a morphism of derivators*

$$\mathrm{HO}(\mathrm{dgc}at) \rightarrow \mathrm{HO}(\Lambda - \mathrm{Mod}),$$

*which commutes with filtered homotopy colimits, preserves the point and satisfies condition Dr).*

*Remark 11.8.* By theorem 11.5 the morphism of derivators  $C$  factors through  $\mathcal{U}_l$  and so gives rise to a morphism

$$C : \mathcal{M}_{dg}^{loc} \rightarrow \mathrm{HO}(\Lambda - \mathrm{Mod}).$$

**Non-connective  $K$ -theory.** Let  $\mathcal{A}$  be a small dg category. Its non-connective  $K$ -theory spectrum  $K(\mathcal{A})$  is defined by applying Schlichting's construction [32] to the Frobenius pair associated with the category of cofibrant perfect  $\mathcal{A}$ -modules (to the empty dg category we associate 0). Recall that the conflations in the Frobenius category of cofibrant perfect  $\mathcal{A}$ -modules are the short exact sequences which split in the category of graded  $\mathcal{A}$ -modules.

**Theorem 11.9.** *The map  $\mathcal{A} \mapsto K(\mathcal{A})$  yields a morphism of derivators*

$$\mathrm{HO}(\mathrm{dgc}at) \rightarrow \mathrm{HO}(\mathrm{Spt}),$$

*to the derivator associated with the category of spectra, which commutes with filtered homotopy colimits, preserves the point and satisfies condition Dr).*

*Proof.* Proposition 11.15 in [32], which an adaption of theorem 1.9.8 in [40], implies that we have a well defined morphism of derivators

$$\mathrm{HO}(\mathrm{dgc}at) \rightarrow \mathrm{HO}(\mathrm{Spt}).$$

Lemma 6.3 in [32] implies that this morphism commutes with filtered homotopy colimits and theorem 11.10 implies that condition Dr) is satisfied.  $\checkmark$

*Remark 11.10.* By theorem 11.5, the morphism of derivators  $K$  factors through  $\mathcal{U}_l$  and so gives rise to a morphism

$$K : \mathcal{M}_{dg}^{loc} \rightarrow \mathrm{HO}(\mathrm{Spt}).$$

We now establish a connection between Waldhausen's  $S_\bullet$ -construction, see [41] and the suspension functor in the triangulated category  $\mathcal{M}_{dg}^{loc}(e)$ . Let  $\mathcal{A}$  be a Morita fibrant dg category, see [34] [35]. Notice that  $Z^0(\mathcal{A})$  carries a natural exact category structure obtained by pulling back the garded-split structure on  $\mathcal{C}_{dg}(\mathcal{A})$  along the Yoneda functor

$$\begin{array}{ccc} h : Z^0(\mathcal{A}) & \longrightarrow & \mathcal{C}_{dg}(\mathcal{A}) \\ A & \mapsto & \mathrm{Hom}^\bullet(?, A). \end{array}$$

We denote by  $S_\bullet \mathcal{A}$  the Waldhausen  $S_\bullet$ -construction of  $\mathcal{A}$ , see [41]. Observe that we obtain in this way a simplicial Morita fibrant dg category. We denote by  $\Delta$  the simplicial category, see [27], and by  $p : \Delta \rightarrow e$  the projection functor.

**Proposition 11.11.** *There is a canonical isomorphism in  $\mathcal{M}_{dg}^{loc}(e)$*

$$p_! \mathcal{U}_l(S_\bullet \mathcal{A}) \xrightarrow{\sim} \mathcal{U}_l(\mathcal{A})[1].$$

*Proof.* As in [27, 3.3], we consider the sequence in  $\mathrm{HO}(\mathrm{dgc}at)(\Delta)$

$$0 \rightarrow \mathcal{A}_\bullet \rightarrow PS_\bullet \mathcal{A} \rightarrow S_\bullet \mathcal{A} \rightarrow 0,$$

where  $\mathcal{A}_\bullet$  denotes the constant simplicial dg category with value  $\mathcal{A}$  and  $PS_\bullet \mathcal{A}$  the path object of  $S_\bullet \mathcal{A}$ . For each point  $n : e \rightarrow \Delta$ , the  $n$ th component of the above sequence is the following short exact sequence in  $\mathrm{Ho}(\mathrm{dgc}at)$

$$0 \rightarrow \mathcal{A} \xrightarrow{I} PS_n \mathcal{A} = S_{n+1} \mathcal{A} \xrightarrow{Q} S_n \mathcal{A} \rightarrow 0,$$

where  $I$  maps  $A \in \mathcal{A}$  to the constant sequence

$$0 \rightarrow A \xrightarrow{Id} A \xrightarrow{Id} \dots \xrightarrow{Id} A$$

and  $Q$  maps a sequence

$$0 \rightarrow A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_n$$

to

$$A_1/A_0 \rightarrow \dots \rightarrow A_n/A_0.$$

Since the morphism of derivators  $\mathcal{U}_l$  satisfies condition Dr), the conservativity axiom implies that we obtain a triangle

$$\mathcal{U}_l(\mathcal{A}_\bullet) \rightarrow \mathcal{U}_l(PS_\bullet) \rightarrow \mathcal{U}_l(S_\bullet) \rightarrow \mathcal{U}_l(\mathcal{A}_\bullet)[1]$$

in  $\mathcal{M}_{dg}^{loc}(\Delta)$ . By applying the functor  $p_!$ , we obtain the following triangle

$$p_!\mathcal{U}_l(\mathcal{A}_\bullet) \rightarrow p_!\mathcal{U}_l(PS_\bullet\mathcal{A}) \rightarrow p_!\mathcal{U}_l(S_\bullet\mathcal{A}) \rightarrow p_!\mathcal{U}_l(\mathcal{A}_\bullet)[1]$$

in  $\mathcal{M}_{dg}^{loc}(e)$ . We now show that we have natural isomorphisms

$$p_!\mathcal{U}_l(\mathcal{A}_\bullet) \xrightarrow{\sim} \mathcal{U}_l(\mathcal{A})$$

and

$$p_!\mathcal{U}_l(PS_\bullet\mathcal{A}) \xrightarrow{\sim} 0,$$

in  $\mathcal{M}_{dg}^{loc}(e)$ , where 0 denotes the zero object in the triangulated category  $\mathcal{M}_{dg}^{loc}(e)$ . This clearly implies the proposition. Since the morphisms of derivators  $\Phi$ ,  $stab$  and  $\gamma$  commute with homotopy colimits it is enough to show that we have isomorphisms

$$p_!\mathbb{R}\underline{h}(\mathcal{A}_\bullet) \xrightarrow{\sim} \mathbb{R}\underline{h}(\mathcal{A})$$

and

$$p_!\mathbb{R}\underline{h}(PS_\bullet\mathcal{A}) \xrightarrow{\sim} *$$

in  $\mathbf{Hot}_{\mathrm{dgc}at_f}(e)$ , where  $*$  denotes the terminal object in  $\mathbf{Hot}_{\mathrm{dgc}at_f}(e)$ . Notice that since  $\mathcal{A}$  and  $PS_n\mathcal{A}$ ,  $n \geq 0$  are Morita fibrant dg categories, we have natural isomorphisms

$$\underline{h}(\mathcal{A}_\bullet) \xrightarrow{\sim} \mathbb{R}\underline{h}(\mathcal{A}_\bullet)$$

and

$$\underline{h}(PS_\bullet\mathcal{A}) \xrightarrow{\sim} \mathbb{R}\underline{h}(PS_\bullet\mathcal{A})$$

in  $\mathbf{Hot}_{\mathrm{dgc}at_f}(\Delta)$ .

Now, since homotopy colimits in  $\mathbf{Fun}(\mathrm{dgc}at_f, Sset)$  are calculated objectwise and since  $\underline{h}(\mathcal{A}_\bullet)$  is a constant simplicial object in  $\mathbf{Fun}(\mathrm{dgc}at_f, Sset)$ , corollary 18.7.7 in [15] implies that we have an isomorphism

$$p_!\mathbb{R}\underline{h}(\mathcal{A}_\bullet) \xrightarrow{\sim} \mathbb{R}\underline{h}(\mathcal{A})$$

in  $\mathcal{M}_{dg}^{loc}(e)$ .

Notice also that since  $PS_\bullet\mathcal{A}$  is a contractible simplicial object, see [27], so is  $\underline{h}(PS_\bullet\mathcal{A})$ . Since homotopy colimits in  $\mathbf{Fun}(\mathrm{dgc}at_f, Sset)$  are calculated objectwise, we have an isomorphism

$$p_!\mathbb{R}\underline{h}(PS_\bullet\mathcal{A}) \xrightarrow{\sim} *$$

in  $\mathbf{Hot}_{\mathrm{dgc}at_f}(e)$ .

This proves the proposition. ✓

## 12. A QUILLEN MODEL IN TERMS OF PRESHEAVES OF SPECTRA

In this section, we construct another Quillen model category whose associated derivator is  $\mathcal{M}_{dg}^{loc}$ .

Consider the Quillen adjunction

$$\begin{array}{c} \text{Fun}(\text{dgc}at_f^o, Sset_\bullet) \\ \Sigma^\infty \downarrow \uparrow ev_0 \\ \text{Sp}^{\mathbb{N}}(\text{Fun}(\text{dgc}at_f^o, Sset_\bullet)). \end{array}$$

Recall from section 8 that we have a set of morphisms  $(\Sigma \cup \{P\})_+$  in the category  $\text{Fun}(\text{dgc}at_f^o, Sset_\bullet)$ . Now stabilize the image of this set by the derived functor  $\mathbb{L}\Sigma^\infty$ , under the functor loop space in  $\text{Ho}(\text{Sp}^{\mathbb{N}}(\text{Fun}(\text{dgc}at_f^o, Sset_\bullet)))$ . For each one of the morphisms thus obtained, choose a representative in the model category  $\text{Sp}^{\mathbb{N}}(\text{Fun}(\text{dgc}at_f^o, Sset_\bullet))$ .

**Notation 12.1.** *Let us denote this set by  $G$  and by  $L_G \text{Sp}^{\mathbb{N}}(\text{Fun}(\text{dgc}at_f^o, Sset_\bullet))$  the associated left Bousfield localization.*

**Proposition 12.2.** *We have an equivalence of triangulated strong derivators*

$$\text{HO}(\text{Sp}^{\mathbb{N}}(L_{\Sigma, P} \text{Fun}(\mathcal{M}_f^o, Sset_\bullet))) \xrightarrow{\sim} \text{HO}(L_G \text{Sp}^{\mathbb{N}}(\text{Fun}(\mathcal{M}_f^o, Sset_\bullet))).$$

*Proof.* Observe that theorems 5.4 and 9.7 imply that both derivators have the same universal property.

This proves the proposition. ✓

*Remark 12.3.* Notice that the stable Quillen model category

$$\text{Sp}^{\mathbb{N}}(\text{Fun}(\mathcal{M}_f^o, Sset_\bullet))$$

identifies with

$$\text{Fun}(\mathcal{M}_f^o, \text{Sp}^{\mathbb{N}}(Sset_\bullet))$$

endowed with the projective model structure.

The above considerations imply the following proposition.

**Proposition 12.4.** *We have an equivalence of derivators*

$$\text{HO}(L_{\widetilde{\mathcal{E}}_{st}, G} \text{Fun}(\text{dgc}at_f^o, \text{Sp}^{\mathbb{N}}(Sset_\bullet))) \xrightarrow{\sim} \mathcal{M}_{dg}^{loc}.$$

## 13. UPPER TRIANGULAR DG CATEGORIES

In this section we study upper triangular dg categories using the formalism of Quillen's homotopical algebra. In the next section, we will relate this important class of dg categories with split short exact sequences in  $\text{Ho}(\text{dgc}at)$ .

**Definition 13.1.** *An upper triangular dg category  $\underline{\mathcal{B}}$  is given by an upper triangular matrix*

$$\underline{\mathcal{B}} := \begin{pmatrix} \mathcal{A} & X \\ 0 & \mathcal{C} \end{pmatrix},$$

where  $\mathcal{A}$  and  $\mathcal{C}$  are small dg categories and  $X$  is a  $\mathcal{A}$ - $\mathcal{C}$ -bimodule.

A morphism  $\underline{F} : \underline{\mathcal{B}} \rightarrow \underline{\mathcal{B}'}$  of upper triangular dg categories is given by a triple  $\underline{F} := (F_A, F_C, F_X)$ , where  $F_A$ , resp.  $F_C$ , is a dg functor from  $\mathcal{A}$  to  $\mathcal{A}'$ , resp. from  $\mathcal{C}$  to  $\mathcal{C}'$ , and  $F_X$  is a morphism of  $\mathcal{A}$ - $\mathcal{C}$ -bimodules from  $X$  to  $X'$  (we consider  $X'$

endowed with the action induced by  $F_{\mathcal{A}}$  and  $F_{\mathcal{C}}$ ). The composition is the natural one.

**Notation 13.2.** We denote by  $\text{dgcat}^{tr}$  the category of upper triangular dg categories.

Let  $\underline{\mathcal{B}} \in \text{dgcat}^{tr}$ .

**Definition 13.3.** Let  $|\underline{\mathcal{B}}|$  be the totalization of  $\underline{\mathcal{B}}$ , i.e. the small dg category whose set of objects is the disjoint union of the set of objects of  $\mathcal{A}$  and  $\mathcal{C}$  and whose morphisms are given by

$$\text{Hom}_{|\underline{\mathcal{B}}|}(x, x') := \begin{cases} \text{Hom}_{\mathcal{A}}(x, x') & \text{if } x, x' \in \mathcal{A} \\ \text{Hom}_{\mathcal{C}}(x, x') & \text{if } x, x' \in \mathcal{C} \\ X(x, x') & \text{if } x \in \mathcal{A}, x' \in \mathcal{C} \\ 0 & \text{if } x \in \mathcal{C}, x' \in \mathcal{A} \end{cases}.$$

We have the following adjunction

$$\begin{array}{c} \text{dgcat}^{tr} \\ \begin{array}{c} \downarrow \uparrow \\ \text{dgcat} \end{array} \\ \text{dgcat} \end{array},$$

where

$$I(\mathcal{B}') := \begin{pmatrix} \mathcal{B}' & \text{Hom}_{\mathcal{B}'}(-, -) \\ 0 & \mathcal{B}' \end{pmatrix}.$$

**Lemma 13.4.** The category  $\text{dgcat}^{tr}$  is complete and cocomplete.

*Proof.* Let  $\{\underline{\mathcal{B}}_j\}_{j \in J}$  be a diagram in  $\text{dgcat}^{tr}$ . Observe that the upper triangular dg category

$$\begin{pmatrix} \text{colim}_{j \in J} \mathcal{A}_j & \text{colim}_{j \in J} |\underline{\mathcal{B}}_j|(-, -) \\ 0 & \text{colim}_{j \in J} \mathcal{C}_j \end{pmatrix},$$

where  $\text{colim}_{j \in J} |\underline{\mathcal{B}}_j|(-, -)$  is the  $\text{colim}_{j \in J} \mathcal{A}_j$ - $\text{colim}_{j \in J} \mathcal{C}_j$ -bimodule naturally associated with the dg category  $\text{colim}_{j \in J} |\underline{\mathcal{B}}_j|$ , corresponds to  $\text{colim}_{j \in J} \underline{\mathcal{B}}_j$ . Observe also that the upper triangular dg category

$$\begin{pmatrix} \lim_{j \in J} \mathcal{A}_j & \lim_{j \in J} X_j \\ 0 & \lim_{j \in J} \mathcal{C}_j \end{pmatrix},$$

corresponds to  $\lim_{j \in J} \underline{\mathcal{B}}_j$ . This proves the lemma. ✓

**Notation 13.5.** Let  $p_1(\underline{\mathcal{B}}) := \mathcal{A}$  and  $p_2(\underline{\mathcal{B}}) := \mathcal{C}$ .

We have at our disposal the following adjunction

$$\begin{array}{c} \text{dgcat}^{tr} \\ \begin{array}{c} \uparrow \downarrow \\ \text{dgcat} \times \text{dgcat} \end{array} \\ \text{dgcat} \times \text{dgcat} \end{array},$$

where

$$E(\mathcal{B}', \mathcal{B}'') := \begin{pmatrix} \mathcal{B}' & 0 \\ 0 & \mathcal{B}'' \end{pmatrix}.$$

Recall from [36][35][34] that  $\mathbf{dgc}at$  admits a structure of cofibrantly generated Quillen model category whose weak equivalences are the Morita dg functors. This structure clearly induces a componentwise model structure on  $\mathbf{dgc}at \times \mathbf{dgc}at$  which is also cofibrantly generated.

**Proposition 13.6.** *The category  $\mathbf{dgc}at^{tr}$  admits a structure of cofibrantly generated Quillen model category whose weak equivalences, resp. fibration, are the morphisms  $\underline{F} : \underline{\mathcal{B}} \rightarrow \underline{\mathcal{B}'}$  such that  $(p_1 \times p_2)(\underline{F})$  are quasi-equivalences, resp. fibrations, in  $\mathbf{dgc}at \times \mathbf{dgc}at$ .*

*Proof.* We show that the previous adjunction  $(E, p_1 \times p_2)$  verifies conditions (1) and (2) of theorem 11.3.2 from [15].

- (1) Since the functor  $E$  is also a right adjoint to  $p_1 \times p_2$ , the functor  $p_1 \times p_2$  commutes with colimits and so condition (1) is verified.
- (2) Let  $J$ , resp.  $J \times J$ , be the set of generating trivial cofibrations in  $\mathbf{dgc}at$ , resp. in  $\mathbf{dgc}at \times \mathbf{dgc}at$ . Since the functor  $p_1 \times p_2$  commutes with filtered colimits it is enough to prove the following: let  $\underline{G} : \underline{\mathcal{B}'} \rightarrow \underline{\mathcal{B}''}$  be an element of the set  $E(J \times J)$ ,  $\underline{\mathcal{B}}$  an object in  $\mathbf{dgc}at^{tr}$  and  $\underline{\mathcal{B}'} \rightarrow \underline{\mathcal{B}}$  a morphism in  $\mathbf{dgc}at^{tr}$ . Consider the following push-out in  $\mathbf{dgc}at^{tr}$  :

$$\begin{array}{ccc} \underline{\mathcal{B}'} & \longrightarrow & \underline{\mathcal{B}} \\ \underline{G} \downarrow & \lrcorner & \downarrow \underline{G}_* \\ \underline{\mathcal{B}''} & \longrightarrow & \underline{\mathcal{B}'} \amalg_{\underline{\mathcal{B}'}} \underline{\mathcal{B}}. \end{array}$$

We now prove that  $(p_1 \times p_2)(\underline{G}_*)$  is a weak-equivalence in  $\mathbf{dgc}at \times \mathbf{dgc}at$ . Observe that the image of the previous push-out under the functors  $p_1$  and  $p_2$  correspond to the following two push-outs in  $\mathbf{dgc}at$  :

$$\begin{array}{ccc} \mathcal{A}' & \longrightarrow & \mathcal{A} \\ \downarrow G_{\mathcal{A}'} \sim & \lrcorner & \downarrow G_{\mathcal{A}'_*} \\ \mathcal{A}'' & \longrightarrow & \mathcal{A}'' \amalg_{\mathcal{A}'} \mathcal{A} \end{array} \quad \begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow G_{\mathcal{C}'} \sim & \lrcorner & \downarrow G_{\mathcal{C}'_*} \\ \mathcal{C}'' & \longrightarrow & \mathcal{C}'' \amalg_{\mathcal{C}'} \mathcal{C}. \end{array}$$

Since  $G_{\mathcal{A}'_*}$  and  $G_{\mathcal{C}'_*}$  belong to  $J$  the morphism

$$(p_1 \times p_2)(\underline{G}_*) = (G_{\mathcal{A}'_*}, G_{\mathcal{C}'_*})$$

is a weak-equivalence in  $\mathbf{dgc}at \times \mathbf{dgc}at$ . This proves condition (2). √

The proposition is proven.

Let  $\underline{\mathcal{B}}, \underline{\mathcal{B}'} \in \mathbf{dgc}at^{tr}$ .

**Definition 13.7.** *A morphism  $\underline{F} : \underline{\mathcal{B}} \rightarrow \underline{\mathcal{B}'}$  is a total Morita dg functor if  $F_{\mathcal{A}}$  and  $F_{\mathcal{C}}$  are Morita dg functors, see [36][35][34], and  $F_X$  is a quasi-isomorphism of  $\mathcal{A}$ - $\mathcal{C}$ -bimodules.*

*Remark 13.8.* Notice that if  $\underline{F}$  is a total Morita dg functor then  $|\underline{F}|$  is a Morita dg functor in  $\mathbf{dgc}at$  but the converse is not true.

**Theorem 13.9.** *The category  $\mathrm{dgc}at^{tr}$  admits a structure of cofibrantly generated Quillen model category whose weak equivalences  $\mathcal{W}$  are the total Morita dg functors and whose fibrations are the morphisms  $\underline{F} : \underline{\mathcal{B}} \rightarrow \underline{\mathcal{B}}'$  such that  $F_{\mathcal{A}}$  and  $F_{\mathcal{C}}$  are Morita fibrations, see [35] [34], and  $F_X$  is a componentwise surjective morphism of bimodules.*

*Proof.* The proof is based on enlarging the set  $E(I \times I)$ , resp.  $E(J \times J)$ , of generating cofibrations, resp. generating trivial cofibrations, of the Quillen model structure of proposition 13.6.

Let  $\tilde{I}$  be the set of morphisms in  $\mathrm{dgc}at^{tr}$

$$\begin{pmatrix} k & S^{n-1} \\ 0 & k \end{pmatrix} \hookrightarrow \begin{pmatrix} k & D^n \\ 0 & k \end{pmatrix}, n \in \mathbb{Z},$$

where  $S^{n-1}$  is the complex  $k[n-1]$  and  $D^n$  the mapping cone on the identity of  $S^{n-1}$ . The  $k$ - $k$ -bimodule  $S^{n-1}$  is sent to  $D^n$  by the identity on  $k$  in degree  $n-1$ .

Consider also the set  $\tilde{J}$  of morphisms in  $\mathrm{dgc}at^{tr}$

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \hookrightarrow \begin{pmatrix} k & D^n \\ 0 & k \end{pmatrix}, n \in \mathbb{Z}.$$

Observe that a morphism  $\underline{F} : \underline{\mathcal{B}} \rightarrow \underline{\mathcal{B}}'$  in  $\mathrm{dgc}at^{tr}$  has the right lifting property (=R.L.P.) with respect to the set  $\tilde{J}$ , resp.  $\tilde{I}$ , if and only if  $F_X$  is a componentwise surjective morphism, resp. surjective quasi-isomorphism, of  $\mathcal{A}$ - $\mathcal{C}$ -bimodules.

Define  $I := E(I \times I) \cup \tilde{I}$  as the set of *generating cofibrations* in  $\mathrm{dgc}at^{tr}$  and  $J := E(J \times J) \cup \tilde{J}$  as the set of *generating trivial cofibrations*. We now prove that conditions (1)-(6) of theorem 2.1.19 from [16] are satisfied. This is clearly the case for conditions (1)-(3).

- (4) We now prove that  $J - cell \subset \mathcal{W}$ , see [15]. Since by proposition 13.6 we have  $E(J \times J) - cell \subset \mathcal{W}'$ , where  $\mathcal{W}'$  denotes the weak equivalences of proposition 13.6, it is enough to prove that pushouts with respect to any morphism in  $\tilde{J}$  belong to  $\mathcal{W}$ . Let  $n$  be an integer and  $\underline{\mathcal{B}}$  an object in  $\mathrm{dgc}at^{tr}$ . Consider the following push-out in  $\mathrm{dgc}at^{tr}$  :

$$\begin{array}{ccc} \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} & \xrightarrow{\underline{T}} & \underline{\mathcal{B}} \\ \downarrow & \lrcorner & \downarrow \underline{R} \\ \begin{pmatrix} k & D^n \\ 0 & k \end{pmatrix} & \longrightarrow & \underline{\mathcal{B}}'. \end{array}$$

Notice that the morphism  $\underline{T}$  corresponds to specifying an object  $A$  in  $\mathcal{A}$  and an object  $C$  in  $\mathcal{C}$ . The upper triangular dg category  $\underline{\mathcal{B}}'$  is then obtained from  $\underline{\mathcal{B}}$  by gluing a new morphism of degree  $n$  from  $A$  to  $C$ . Observe that  $R_{\mathcal{A}}$  and  $R_{\mathcal{C}}$  are the identity dg functors and that  $R_X$  is a quasi-isomorphism of bimodules. This shows that  $\underline{R}$  belongs to  $\mathcal{W}$  and so condition (4) is proved.

- (5-6) We now show that  $\mathrm{R.L.P.}(I) = \mathrm{R.L.P.}(J) \cap \mathcal{W}$ . The proof of proposition 13.6 implies that  $\mathrm{R.L.P.}(E(I \times I)) = \mathrm{R.L.P.}(E(J \times J)) \cap \mathcal{W}'$ . Let  $\underline{F} : \underline{\mathcal{B}} \rightarrow \underline{\mathcal{B}}'$  be a morphism in  $\mathrm{R.L.P.}(\tilde{I})$ . Clearly  $\underline{F}$  belongs to  $\mathrm{R.L.P.}(\tilde{J})$  and  $F_X$  is a quasi-isomorphism of bimodules. This shows that  $\mathrm{R.L.P.}(I) \subset \mathrm{R.L.P.}(J) \cap \mathcal{W}$ .

Let now  $\underline{F} : \underline{\mathcal{B}} \rightarrow \underline{\mathcal{B}}'$  be a morphism in  $\text{R.L.P.}(\tilde{J}) \cap \mathcal{W}$ . Clearly  $\underline{F}$  belongs to  $\text{R.L.P.}(\tilde{I})$  and so  $\text{R.L.P.}(J) \cap \mathcal{W} \subset \text{R.L.P.}(I)$ . This proves conditions (5) and (6).

This proves the theorem.  $\checkmark$

*Remark 13.10.* Notice that the Quillen model structure of theorem 13.9 is cellular, see [15] and that the domains and codomains of  $I$  (the set of generating cofibrations) are cofibrant,  $\aleph_0$ -compact,  $\aleph_0$ -small and homotopically finitely presented, see definition 2.1.1. from [38]. This implies that we are in the conditions of proposition 6.2 and so any object  $\underline{\mathcal{B}}$  in  $\text{dgc}at^{tr}$  is weakly equivalent to a filtered colimit of strict finite  $I$ -cell objects.

**Proposition 13.11.** *If  $\underline{\mathcal{B}}$  be a strict finite  $I$ -cell object in  $\text{dgc}at^{tr}$ , then  $p_1(\underline{\mathcal{B}})$ ,  $p_2(\underline{\mathcal{B}})$  and  $|\underline{\mathcal{B}}|$  are strict finite  $I$ -cell objects in  $\text{dgc}at$ .*

*Proof.* We consider the following inductive argument :

- notice that the initial object in  $\text{dgc}at^{tr}$  is

$$\begin{pmatrix} \emptyset & 0 \\ 0 & \emptyset \end{pmatrix}$$

and it is sent to  $\emptyset$  (the initial object in  $\text{dgc}at$ ) by the functors  $p_1$ ,  $p_2$  and  $|\cdot|$ .

- suppose that  $\underline{\mathcal{B}}$  is an upper triangular dg category such that  $p_1(\underline{\mathcal{B}})$ ,  $p_2(\underline{\mathcal{B}})$  and  $|\underline{\mathcal{B}}|$  are strict finite  $I$ -cell objects in  $\text{dgc}at$ . Let  $\underline{G} : \underline{\mathcal{B}}' \rightarrow \underline{\mathcal{B}}''$  be an element of the set  $I$  in  $\text{dgc}at^{tr}$ , see the proof of theorem 13.9, and  $\underline{\mathcal{B}}' \rightarrow \underline{\mathcal{B}}$  a morphism. Consider the following push-out in  $\text{dgc}at^{tr}$  :

$$\begin{array}{ccc} \underline{\mathcal{B}}' & \longrightarrow & \underline{\mathcal{B}} \\ \underline{G} \downarrow & \lrcorner & \downarrow \underline{G}_* \\ \underline{\mathcal{B}}'' & \longrightarrow & \text{PO} . \end{array}$$

We now prove that  $p_1(\text{PO})$ ,  $p_2(\text{PO})$  and  $|\text{PO}|$  are strict finite  $I$ -cell objects in  $\text{dgc}at$ . We consider the following two cases :

- 1)  $\underline{G}$  belongs to  $E(I \times I)$ : observe that  $p_1(\text{PO})$ ,  $p_2(\text{PO})$  and  $|\text{PO}|$  correspond exactly to the following push-outs in  $\text{dgc}at$  :

$$\begin{array}{ccccc} \begin{array}{ccc} \mathcal{A}' & \longrightarrow & \mathcal{A} \\ \downarrow G_{\mathcal{A}'} \sim & \lrcorner & \downarrow \\ \mathcal{A}'' & \longrightarrow & p_1(\text{PO}) \end{array} & \begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow G_{\mathcal{C}'} \sim & \lrcorner & \downarrow \\ \mathcal{C}'' & \longrightarrow & p_2(\text{PO}) \end{array} & \begin{array}{ccc} \mathcal{A}' \amalg \mathcal{C}' & \longrightarrow & |\underline{\mathcal{B}}| \\ \downarrow G_{\mathcal{A}'} \amalg G_{\mathcal{C}'} & \lrcorner & \downarrow \\ \mathcal{A}'' \amalg \mathcal{C}'' & \longrightarrow & |\text{PO}| . \end{array} \end{array}$$

Since  $G_{\mathcal{A}'}$  and  $G_{\mathcal{C}'}$  belong to  $I$  this case is proved.

- 2)  $\underline{G}$  belongs to  $\tilde{I}$ : observe that  $p_1(\text{PO})$  identifies with  $\mathcal{A}$ ,  $p_2(\text{PO})$  identifies with  $\mathcal{C}$  and  $|\text{PO}|$  corresponds to the following push-out in  $\text{dgc}at$  :

$$\begin{array}{ccc} \mathcal{C}(n) & \longrightarrow & |\underline{\mathcal{B}}| \\ S(n) \downarrow & \lrcorner & \downarrow \\ \mathcal{P}(n) & \longrightarrow & |\text{PO}| , \end{array}$$

where  $S(n)$  is a generating cofibration in  $\mathbf{dgc}at$ , see section 2 in [36]. This proves this case.

The proposition is proven.  $\checkmark$

#### 14. SPLIT SHORT EXACT SEQUENCES

In this section, we establish the connection between split short exact sequences of dg categories and upper triangular dg categories.

**Definition 14.1.** *A split short exact sequence of dg categories is a short exact sequence of dg categories, see [19], which is Morita equivalent to one of the form*

$$0 \longrightarrow \mathcal{A} \begin{array}{c} \xleftarrow{R} \\ \xrightarrow{i_{\mathcal{A}}} \end{array} \mathcal{B} \begin{array}{c} \xleftarrow{i_{\mathcal{C}}} \\ \xrightarrow{P} \end{array} \mathcal{C} \longrightarrow 0,$$

where the identities  $P \circ i_{\mathcal{A}} = 0$ ,  $P \circ i_{\mathcal{C}} = Id_{\mathcal{C}}$ ,  $R \circ i_{\mathcal{A}} = Id_{\mathcal{A}}$  are satisfied.

To a split short exact sequence, we can naturally associate the upper triangular dg category

$$\underline{\mathcal{B}} := \begin{pmatrix} \mathcal{A} & \text{Hom}_{\mathcal{B}}(i_{\mathcal{C}}(-), i_{\mathcal{A}}(-)) \\ 0 & \mathcal{C} \end{pmatrix}.$$

Conversely to an upper triangular dg category  $\underline{\mathcal{B}}$  such that  $\mathcal{C}$  admits a zero object (for instance if  $\mathcal{C}$  is Morita fibrant), we can associate a split short exact sequence

$$0 \longrightarrow \mathcal{A} \begin{array}{c} \xleftarrow{R} \\ \xrightarrow{i_{\mathcal{A}}} \end{array} \underline{\mathcal{B}} \begin{array}{c} \xleftarrow{i_{\mathcal{C}}} \\ \xrightarrow{P} \end{array} \mathcal{C} \longrightarrow 0,$$

where  $P$  and  $R$  are the projection dg functors. Moreover, this construction is functorial in  $\underline{\mathcal{B}}$  and sends total Morita equivalences to Morita equivalent split short exact sequences. Notice also that by lemma 13.4 this functor preserves colimits.

**Proposition 14.2.** *Every split short exact sequence of dg categories is weakly equivalent to a filtered homotopy colimit of split short exact sequences whose components are strict finite  $I$ -cell objects in  $\mathbf{dgc}at$ .*

*Proof.* Let

$$0 \longrightarrow \mathcal{A} \begin{array}{c} \xleftarrow{R} \\ \xrightarrow{i_{\mathcal{A}}} \end{array} \mathcal{B} \begin{array}{c} \xleftarrow{i_{\mathcal{C}}} \\ \xrightarrow{P} \end{array} \mathcal{C} \longrightarrow 0,$$

be a split short exact sequence of dg categories. We can suppose that  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  are Morita fibrant dg categories, see [35] [34]. Consider the upper triangular dg category

$$\underline{\mathcal{B}} := \begin{pmatrix} \mathcal{A} & \text{Hom}_{\mathcal{B}}(i_{\mathcal{C}}(-), i_{\mathcal{A}}(-)) \\ 0 & \mathcal{C} \end{pmatrix}.$$

Now by remark 13.10,  $\underline{\mathcal{B}}$  is equivalent to a filtered colimit of strict finite  $I$ -cell objects in  $\mathbf{dgc}at^{tr}$ . Consider the image of this diagram by the functor, described above, which sends an upper triangular dg category to a split short exact sequence. By proposition 13.11 the components of each split short exact sequence of this diagram are strict finite  $I$ -cell objects in  $\mathbf{dgc}at$ . Since the category  $\mathbf{dgc}at$  satisfies the conditions of proposition 6.2, filtered homotopy colimits are equivalent to filtered colimits and so the proposition is proven.  $\checkmark$

## 15. QUASI-ADDITIVITY

Recall from section 11 that we have at our disposal the Quillen model category  $\mathbf{L}_{\Sigma, P}\mathbf{Fun}(\mathbf{dgc}at_f^o, Sset)$  which is *homotopically pointed*, i.e. the morphism  $\emptyset \rightarrow *$ , from the initial object  $\emptyset$  to the terminal one  $*$ , is a weak equivalence. We now consider a strictly pointed Quillen model.

**Proposition 15.1.** *We have a Quillen equivalence*

$$\begin{array}{c} * \downarrow \mathbf{L}_{\Sigma, P}\mathbf{Fun}(\mathcal{M}_f^o, Sset) \\ \quad \uparrow \downarrow U \\ (-)_+ \quad \uparrow \downarrow U \\ \mathbf{L}_{\Sigma, P}\mathbf{Fun}(\mathcal{M}_f^o, Sset), \end{array}$$

where  $U$  denotes the forgetful functor.

This follows from the fact that the category  $\mathbf{L}_{\Sigma, P}\mathbf{Fun}(\mathbf{dgc}at_f^o, Sset)$  is homotopically pointed and from the following general argument.

**Proposition 15.2.** *Let  $\mathcal{M}$  be a homotopically pointed Quillen model category. We have a Quillen equivalence.*

$$\begin{array}{c} * \downarrow \mathcal{M} \\ \quad \uparrow \downarrow U \\ (-)_+ \quad \uparrow \downarrow U \\ \mathcal{M}, \end{array}$$

where  $U$  denotes the forgetful functor.

*Proof.* Clearly the functor  $U$  preserves cofibrations, fibrations and weak equivalences, by construction. Let now  $N \in \mathcal{M}$  and  $M \in * \downarrow \mathcal{M}$ . Consider the following commutative diagram in  $\mathcal{M}$

$$\begin{array}{ccc} N \simeq \emptyset \amalg N & \xrightarrow{f} & U(M) \\ & \searrow \sim & \nearrow f^\# \\ & i \amalg \mathbf{1} & * \amalg N \end{array},$$

where  $f^\#$  is the morphism which corresponds to  $f$ , considered as a morphism in  $\mathcal{M}$ , under the adjunction and  $i : \emptyset \xrightarrow{\sim} *$ . Since the morphism  $i \amalg \mathbf{1}$  corresponds to the homotopy colimit of  $i$  and  $\mathbf{1}$ , which are both weak equivalences, proposition 5.1 implies that  $i \amalg \mathbf{1}$  is a weak equivalence. Now, by the property ‘2 out of 3’, the morphism  $f$  is a weak equivalence if and only if  $f^\#$  is one. This proves the proposition.  $\checkmark$

**Notation 15.3.** *We denote by  $w\mathcal{A}$  the category of homotopy equivalences of a dg category  $\mathcal{A}$  and by  $N.w\mathcal{A}$  its nerve.*

Now, consider the morphism

$$\begin{array}{ccc} \mathbf{Ho}(\mathbf{dgc}at) & \rightarrow & \mathbf{Ho}(* \downarrow \mathbf{L}_{\Sigma, P}\mathbf{Fun}(\mathcal{M}_f^o, Sset)) \\ \mathcal{A} & \mapsto & \begin{cases} \mathbf{Hom}_{\mathbf{dgc}at}(\Gamma(?), \mathcal{A}_f)_+ \\ \simeq \mathbf{Map}_{\mathbf{dgc}at}(?, \mathcal{A})_+ \\ \simeq N.w\mathbf{rep}_{dg}(?, \mathcal{A})_+ \end{cases} \end{array}$$

which by sections 6, 7 and proposition 15.1 corresponds to the component  $(\Phi \circ \mathbb{R}\underline{h})(e)$  of the morphism of derivators

$$\Phi \circ \mathbb{R}\underline{h} : \mathrm{HO}(\mathrm{dgc}at) \longrightarrow \mathrm{L}_{\Sigma, P} \mathrm{Hot}_{\mathrm{dgc}at_f},$$

see proposition 7.1. Observe that the simplicial presheaf  $N.\mathrm{wrep}_{dg}(\?, \mathcal{A})$  is already canonically pointed.

**Proposition 15.4.** *The canonical morphism*

$$\Psi : N.\mathrm{wrep}_{dg}(\?, \mathcal{A})_+ \rightarrow N.\mathrm{wrep}_{dg}(\?, \mathcal{A})$$

*is a weak equivalence in  $* \downarrow \mathrm{L}_{\Sigma, P} \mathrm{Fun}(\mathcal{M}_f^o, Sset)$ .*

*Proof.* Observe that  $N.\mathrm{wrep}_{dg}(\?, \mathcal{A})$  is a fibrant object in  $* \downarrow \mathrm{L}_{\Sigma, P} \mathrm{Fun}(\mathcal{M}_f^o, Sset)$  and that the canonical morphism  $\Psi$  corresponds to the co-unit of the adjunction of proposition 15.1. Since this adjunction is a Quillen equivalence, the proposition is proved.  $\checkmark$

Recall now from remark 8.2 that we have a canonical equivalence of pointed derivators

$$\mathrm{L}_{\Sigma, P} \mathrm{Hot}_{\mathrm{dgc}at_f} \xrightarrow{\sim} \mathrm{L}_{\Sigma, P} \mathrm{Hot}_{\mathrm{dgc}at_f \bullet},$$

where  $\mathrm{L}_{\Sigma, P} \mathrm{Hot}_{\mathrm{dgc}at_f \bullet}$  is the derivator associated with the Quillen model category  $\mathrm{L}_{\Sigma, P} \mathrm{Fun}(\mathrm{dgc}at_f^o, Sset \bullet)$ . From now on, we will consider this Quillen model. We have the following morphism of derivators

$$\Phi \circ \mathbb{R}\underline{h} : \mathrm{HO}(\mathrm{dgc}at) \longrightarrow \mathrm{L}_{\Sigma, P} \mathrm{Hot}_{\mathrm{dgc}at_f \bullet},$$

which commutes with filtered homotopy colimits and preserves the point.

**Notation 15.5.** - We denote by  $\mathcal{E}^s$  the set of retractions of dg categories

$$\mathcal{G} \xrightleftharpoons[i_{\mathcal{G}}]{R} \mathcal{H},$$

where  $\mathcal{G}$  and  $\mathcal{H}$  are strict finite I-cell objects in  $\mathrm{dgc}at$  and  $R \circ i_{\mathcal{G}} = \mathrm{Id}_{\mathcal{G}}$ .

- We denote by  $\mathcal{E}_{un}^s$  the set of morphisms  $S_L$  in  $\mathrm{Ho}(\mathrm{L}_{\Sigma, P} \mathrm{Hot}_{\mathrm{dgc}at_f \bullet})$ , see section 10, where  $L$  belongs to the set  $\mathcal{E}^s$ .

Now choose for each element of the set  $\mathcal{E}_{un}^s$  a representative in the category  $\mathrm{L}_{\Sigma, P} \mathrm{Fun}(\mathrm{dgc}at_f^o, Sset \bullet)$ . We denote this set of representatives by  $\widetilde{\mathcal{E}_{un}^s}$ . Since  $\mathrm{L}_{\Sigma, P} \mathrm{Fun}(\mathrm{dgc}at_f^o, Sset \bullet)$  is a left proper, cellular Quillen model category, see [15], its left Bousfield localization by  $\widetilde{\mathcal{E}_{un}^s}$  exists. We denote it by  $\mathrm{L}_{\widetilde{\mathcal{E}_{un}^s}} \mathrm{L}_{\Sigma, P} \mathrm{Fun}(\mathrm{dgc}at_f^o, Sset \bullet)$  and by  $\mathrm{L}_{\widetilde{\mathcal{E}_{un}^s}} \mathrm{L}_{\Sigma, P} \mathrm{Hot}_{\mathrm{dgc}at_f \bullet}$  the associated derivator. We have the following morphism of derivators

$$\Psi : \mathrm{L}_{\Sigma, P} \mathrm{FunHot}_{\mathrm{dgc}at_f \bullet} \rightarrow \mathrm{L}_{\widetilde{\mathcal{E}_{un}^s}} \mathrm{L}_{\Sigma, P} \mathrm{FunHot}_{\mathrm{dgc}at_f \bullet}.$$

*Remark 15.6.* - Notice that by construction the domains and codomains of the set  $\widetilde{\mathcal{E}_{un}^s}$  are homotopically finitely presented objects. Therefore by lemma 8.1 the set

$$\mathcal{G} = \{\mathbf{F}_{\Delta[n]_+ / \partial \Delta[n]_+}^X \mid X \in \mathcal{M}_f, n \geq 0\},$$

of cofibers of the generating cofibrations in  $\mathrm{Fun}(\mathcal{M}_f^o, Sset \bullet)$  is a set of small weak generators in  $\mathrm{Ho}(\mathrm{L}_{\widetilde{\mathcal{E}_{un}^s}} \mathrm{L}_{\Sigma, P} \mathrm{Fun}(\mathcal{M}_f^o, Sset \bullet))$ .

- Notice also that proposition 14.2 implies that variants of proposition 10.2 and theorem 10.4 are also verified: simply consider the set  $\mathcal{E}^s$  instead of  $\mathcal{E}$  and a retraction of dg categories instead of an inclusion of a full dg subcategory. The proofs are exactly the same.

Let  $\mathcal{M}$  be a left proper cellular model category,  $S$  a set of maps in  $\mathcal{M}$  and  $L_S\mathcal{M}$  the left Bousfield localization of  $\mathcal{M}$  with respect to  $S$ , see [15]. Recall from [15, 4.1.1.] that an object  $X$  in  $L_S\mathcal{M}$  is fibrant if  $X$  is fibrant in  $\mathcal{M}$  and for every element  $f : A \rightarrow B$  of  $S$  the induced map of homotopy function complexes  $f^* : \text{Map}(B, X) \rightarrow \text{Map}(A, X)$  is a weak equivalence.

**Proposition 15.7.** *An object  $F \in L_{\widetilde{\mathcal{E}}_{un}^s} L_{\Sigma, P} \text{Fun}(\text{dgc}at_f^o, Sset_\bullet)$  is fibrant if and only if it satisfies the following conditions*

- 1)  $F(\mathcal{B}) \in Sset_\bullet$  is fibrant, for all  $\mathcal{B} \in \text{dgc}at_f$ .
- 2)  $F(\emptyset) \in Sset_\bullet$  is contractible.
- 3) For every Morita equivalence  $\mathcal{B} \xrightarrow{\sim} \mathcal{B}'$  in  $\text{dgc}at_f$  the morphism  $F(\mathcal{B}') \xrightarrow{\sim} F(\mathcal{B})$  is a weak equivalence in  $Sset_\bullet$ .
- 4) Every split short exact sequence

$$0 \longrightarrow \mathcal{B}' \begin{array}{c} \xleftarrow{R} \\ \xrightarrow{i_{\mathcal{B}'}} \end{array} \mathcal{B} \begin{array}{c} \xleftarrow{i_{\mathcal{B}''}} \\ \xrightarrow{P} \end{array} \mathcal{B}'' \longrightarrow 0$$

in  $\text{dgc}at_f$  induces a homotopy fiber sequence

$$F(\mathcal{B}'') \rightarrow F(\mathcal{B}) \rightarrow F(\mathcal{B}')$$

in  $Sset$ .

*Proof.* Clearly condition 1) corresponds to the fact that  $F$  is fibrant in  $\text{Fun}(\text{dgc}at_f^o, Sset_\bullet)$ . Now observe that  $\text{Fun}(\text{dgc}at_f^o, Sset_\bullet)$  is a simplicial Quillen model category with the simplicial action given by

$$\begin{array}{ccc} Sset \times \text{Fun}(\text{dgc}at_f^o, Sset_\bullet) & \rightarrow & \text{Fun}(\text{dgc}at_f^o, Sset_\bullet) \\ (K, F) & \mapsto & K_+ \wedge F, \end{array}$$

where  $K_+ \wedge F$  denotes the componentwise smash product. This simplicial structure and the construction of the localized Quillen model category  $L_{\widetilde{\mathcal{E}}_{un}^s} L_{\Sigma, P} \text{Fun}(\text{dgc}at_f^o, Sset_\bullet)$ , see section 8, allow us to recover conditions 2) and 3). Condition 4) follows from the construction of the set  $\widetilde{\mathcal{E}}_{un}^s$  and from the fact that the functor

$$\text{Map}(?, F) : \text{Fun}(\text{dgc}at_f^o, Sset_\bullet)^{op} \rightarrow Sset$$

transforms homotopy cofiber sequences into homotopy fiber sequences.

This proves the proposition. ✓

Let  $\mathcal{A}$  be a dg category.

**Notation 15.8.** *We denote by  $K(\mathcal{A}) \in \text{Fun}(\text{dgc}at_f^o, Sset_\bullet)$  the following simplicial presheaf*

$$\mathcal{B} \mapsto |N.wS.\text{rep}_{dg}(\mathcal{B}, \mathcal{A})|,$$

where  $|-|$  denotes the fibrant realization functor of bisimplicial sets.

**Proposition 15.9.** *The simplicial presheaf  $K(\mathcal{A})$  is fibrant in  $L_{\widetilde{\mathcal{E}}_{un}^s} L_{\Sigma, P} \text{Fun}(\text{dgc}at_f^o, Sset_\bullet)$ .*

*Proof.* Observe that  $K(\mathcal{A})$  satisfies conditions (1)-(3). We now prove that Waldhausen's fibration theorem [41, 1.6.4] implies condition (4). Apply the contravariant functor  $\text{rep}_{dg}(\cdot, \mathcal{A})$  to the split short exact sequence

$$0 \longrightarrow \mathcal{B}' \begin{array}{c} \xleftarrow{R} \\ \xrightarrow{i_{\mathcal{B}'}} \end{array} \mathcal{B} \begin{array}{c} \xleftarrow{i_{\mathcal{B}''}} \\ \xrightarrow{P} \end{array} \mathcal{B}'' \longrightarrow 0$$

and obtain a split short exact sequence

$$0 \longrightarrow \text{rep}_{dg}(\mathcal{B}'', \mathcal{A}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \text{rep}_{dg}(\mathcal{B}, \mathcal{A}) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \text{rep}_{dg}(\mathcal{B}', \mathcal{A}) \longrightarrow 0.$$

Now consider the Waldhausen category  $v\text{rep}_{dg}(\mathcal{B}, \mathcal{A}) := Z^0(\text{rep}_{dg}(\mathcal{B}, \mathcal{A}))$ , where the weak equivalences are the morphisms  $f$  such that  $\text{cone}(f)$  is contractible. Consider also the Waldhausen category  $w\text{rep}_{dg}(\mathcal{B}, \mathcal{A})$ , which has the same cofibrations as  $v\text{rep}_{dg}(\mathcal{B}, \mathcal{A})$  but the weak equivalences are the morphisms  $f$  such that  $\text{cone}(f)$  belongs to  $\text{rep}_{dg}(\mathcal{B}'', \mathcal{A})$ . Observe that we have the inclusion  $v\text{rep}_{dg}(\mathcal{B}, \mathcal{A}) \subset w\text{rep}_{dg}(\mathcal{B}, \mathcal{A})$  and an equivalence  $\text{rep}_{dg}(\mathcal{B}, \mathcal{A})^w \simeq Z^0(\text{rep}_{dg}(\mathcal{B}', \mathcal{A}))$ , see section 1.6 from [41]. The conditions of theorem 1.6.4 from [41] are satisfied and so we have a homotopy fiber sequence

$$|N.wS.\text{rep}_{dg}(\mathcal{B}'', \mathcal{A})| \rightarrow |N.wS.\text{rep}_{dg}(\mathcal{B}, \mathcal{A})| \rightarrow |N.wS.\text{rep}_{dg}(\mathcal{B}', \mathcal{A})|$$

in  $Sset$ . This proves the proposition.  $\checkmark$

Let  $p : \Delta \rightarrow e$  be the projection functor.

**Proposition 15.10.** *The canonical morphism*

$$S^1 \wedge N.w\text{rep}_{dg}(\cdot, \mathcal{A}) \rightarrow |N.wS.\text{rep}_{dg}(\cdot, \mathcal{A})| = K(\mathcal{A})$$

is a weak equivalence in  $L_{\mathcal{E}_{un}^s} L_{\Sigma, P} \text{Fun}(\text{dgc}at_f^o, Sset_{\bullet})$ .

*Proof.* Observe that the same conceptual proof as in proposition 11.11 allows us to construct the following homotopy cartesian square

$$\begin{array}{ccc} N.w\text{rep}_{dg}(\cdot, \mathcal{A}) & \longrightarrow & p_!(N.w\text{rep}_{dg}(\cdot, PS.\mathcal{A})) \simeq * \\ \downarrow & & \downarrow \\ * & \longrightarrow & p_!(N.w\text{rep}_{dg}(\cdot, S.\mathcal{A})). \end{array}$$

Observe also that we have the following equivalence

$$p_!(N.w\text{rep}_{dg}(\cdot, S.\mathcal{A})) \xrightarrow{\simeq} |N.wS.\text{rep}_{dg}(\cdot, \mathcal{A})|.$$

This proves the proposition.  $\checkmark$

*Remark 15.11.* By considering inductively the argument of proposition 15.10, we obtain weak equivalences

$$S^1 \wedge N.wS.^{(n)}\text{rep}_{dg}(\cdot, \mathcal{A}) \xrightarrow{\simeq} |N.wS.^{(n+1)}\text{rep}_{dg}(\cdot, \mathcal{A})|, n \geq 0$$

in  $L_{\mathcal{E}_{un}^s} L_{\Sigma, P} \text{Fun}(\text{dgc}at_f^o, Sset_{\bullet})$ .

**Definition 15.12.** - *The Unstable motivator of dg categories  $\mathcal{M}_{dg}^{unst}$  is the derivator associated with the Quillen model category*

$$L_{\mathcal{E}_{un}^s} L_{\Sigma, P} \text{Fun}(\text{dgc}at_f^o, Sset_{\bullet}).$$

- The Universal unstable invariant of dg categories is the canonical morphism of derivators

$$\mathcal{U}_u : \text{HO}(\text{dgcats}) \rightarrow \mathcal{M}_{dg}^{unst}.$$

**Proposition 15.13.** *We have the following weak equivalence of simplicial sets*

$$\text{Map}(\mathcal{U}_u(k), S^1 \wedge \mathcal{U}_u(\mathcal{A})) \xrightarrow{\sim} |N.w.S.\mathcal{A}_f|$$

in  $L_{\mathcal{E}_{un}^s} L_{\Sigma, P} \text{Fun}(\text{dgcats}_f^o, Sset_\bullet)$ . In particular, we have the following isomorphism

$$\pi_{i+1} \text{Map}(\mathcal{U}_u(k), S^1 \wedge \mathcal{U}_u(\mathcal{A})) \xrightarrow{\sim} K_i(\mathcal{A}), \forall i \geq 0.$$

*Proof.* This follows from propositions 15.9, 15.10 and from the following weak equivalences

$$\begin{aligned} \text{Map}(\mathcal{U}_u(k), S^1 \wedge \mathcal{U}_u(\mathcal{A})) &\simeq \text{Map}(\mathbb{R}\underline{h}(k), K(\mathcal{A})) \\ &\simeq (K(\mathcal{A}))(k) \\ &\simeq |N.w.S.\mathcal{A}_f|. \end{aligned}$$

✓

## 16. THE UNIVERSAL ADDITIVE INVARIANT

Consider the Quillen model category  $L_{\mathcal{E}_{un}^s} L_{\Sigma, P} \text{Fun}(\text{dgcats}_f^o, Sset_\bullet)$  constructed in the previous section. The definition of the set  $\mathcal{E}_{un}^s$  and the same arguments as those of example 9.5 and example 9.6 allows us to conclude that  $L_{\mathcal{E}_{un}^s} L_{\Sigma, P} \text{Fun}(\text{dgcats}_f^o, Sset_\bullet)$  satisfies the conditions of theorem 9.7. In particular we have an equivalence of triangulated derivators

$$\text{St}(\mathcal{M}_{dg}^{unst}) \xrightarrow{\sim} \text{HO}(\text{Sp}^{\mathbb{N}}(L_{\mathcal{E}_{un}^s} L_{\Sigma, P} \text{Fun}(\text{dgcats}_f^o, Sset_\bullet))).$$

**Definition 16.1.** - The Additive motivator of dg categories  $\mathcal{M}_{dg}^s$  is the triangulated derivator associated with the stable Quillen model category

$$\text{Sp}^{\mathbb{N}}(L_{\mathcal{E}_{un}^s} L_{\Sigma, P} \text{Fun}(\text{dgcats}_f^o, Sset_\bullet)).$$

- The Universal additive invariant of dg categories is the canonical morphism of derivators

$$\mathcal{U}_a : \text{HO}(\text{dgcats}) \rightarrow \mathcal{M}_{dg}^{add}.$$

*Remark 16.2.* Observe that remark 15.6 and remark 9.4 imply that  $\mathcal{M}_{dg}^{add}$  is a compactly generated triangulated derivator.

We sum up the construction of  $\mathcal{M}_{dg}^{add}$  in the following diagram

$$\begin{array}{ccc}
 \underline{\mathrm{dgc}}_f[S^{-1}] & \longrightarrow & \mathrm{HO}(\mathrm{dgc}) \\
 \mathrm{Ho}(h) \downarrow & \swarrow \mathrm{LRe} & \nearrow \mathbb{R}h \\
 \mathrm{L}_\Sigma \mathrm{Hot}_{\mathrm{dgc}_f} & & \\
 \Phi \downarrow & & \searrow \mathcal{U}_u \\
 \mathrm{L}_{\Sigma, P} \mathrm{Hot}_{\mathrm{dgc}_f \bullet} & & \\
 \Psi \downarrow & & \searrow \mathcal{U}_a \\
 \mathcal{M}_{dg}^{unst} & & \\
 \varphi \downarrow & & \\
 \mathcal{M}_{dg}^{add} & & 
 \end{array}$$

Observe that the morphism of derivators  $\mathcal{U}_a$  is pointed, commutes with filtered homotopy colimits and satisfies the following condition :

A) For every split short exact sequence

$$0 \longrightarrow \mathcal{A} \begin{array}{c} \xleftarrow{R} \\ \xrightarrow{i_{\mathcal{A}}} \end{array} \mathcal{B} \begin{array}{c} \xleftarrow{i_{\mathcal{C}}} \\ \xrightarrow{P} \end{array} \mathcal{C} \longrightarrow 0$$

in  $\mathrm{Ho}(\mathrm{dgc})$ , we have a split triangle

$$\mathcal{U}_a(\mathcal{A}) \begin{array}{c} \xleftarrow{R} \\ \xrightarrow{i_{\mathcal{A}}} \end{array} \mathcal{U}_a(\mathcal{B}) \begin{array}{c} \xleftarrow{i_{\mathcal{C}}} \\ \xrightarrow{P} \end{array} \mathcal{U}_a(\mathcal{C}) \longrightarrow \mathcal{U}_a(\mathcal{A})[1]$$

in  $\mathcal{M}_{dg}^{add}(e)$ . (This implies that the dg functors  $i_{\mathcal{A}}$  and  $i_{\mathcal{C}}$  induce an isomorphism

$$\mathcal{U}_a(\mathcal{A}) \oplus \mathcal{U}_a(\mathcal{C}) \xrightarrow{\sim} \mathcal{U}_a(\mathcal{B})$$

in  $\mathcal{M}_{dg}^{add}(e)$ ).

*Remark 16.3.* Since the dg category  $\mathcal{B}$  in the above split short exact sequence is Morita equivalent to the dg category  $E(\mathcal{A}, \mathcal{B}, \mathcal{C})$ , see section 1.1 from [41], condition A) is equivalent to the additivity property stated by Waldhausen in [41, 1.3.2].

Let  $\mathbb{D}$  be a strong triangulated derivator.

**Theorem 16.4.** *The morphism  $\mathcal{U}_a$  induces an equivalence of categories*

$$\underline{\mathrm{Hom}}_! (\mathcal{M}_{dg}^{add}, \mathbb{D}) \xrightarrow{\mathcal{U}_a^*} \underline{\mathrm{Hom}}_{\mathrm{flt}, A, P} (\mathrm{HO}(\mathrm{dgc}), \mathbb{D}),$$

where  $\underline{\mathrm{Hom}}_{\mathrm{flt}, A, P} (\mathrm{HO}(\mathrm{dgc}), \mathbb{D})$  denotes the category of morphisms of derivators which commute with filtered homotopy colimits, satisfy condition A) and preserve the point.

*Proof.* By theorem 9.1, we have an equivalence of categories

$$\underline{\mathrm{Hom}}_! (\mathcal{M}_{dg}^{add}, \mathbb{D}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_! (\mathcal{M}_{dg}^{unst}, \mathbb{D}).$$

By theorem 5.4, we have an equivalence of categories

$$\underline{\mathrm{Hom}}_! (\mathcal{M}_{dg}^{unst}, \mathbb{D}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{!, \mathcal{E}_{un}^s} (\mathrm{L}_{\Sigma, P} \mathrm{Hot}_{\mathrm{dgc}_f \bullet}, \mathbb{D}).$$

Now, we observe that since  $\mathbb{D}$  is a strong triangulated derivator, the category  $\underline{\text{Hom}}_{\mathcal{E}_{un}^s}(\mathbb{L}_{\Sigma, P} \text{Hot}_{\text{dgc}at_f}, \mathbb{D})$  identifies  $\underline{\text{Hom}}_{\text{flt}, \mathcal{A}, P}(\text{HO}(\text{dgc}at), \mathbb{D})$ . This proves the theorem.  $\checkmark$

**Terminology 16.5.** *We call an object of the right hand side category of theorem 16.4 an additive invariant of dg categories.*

**Example 16.6.** - *The Hochschild and cyclic homology and the non-connective K-theory defined in section 11 are examples of additive invariants.*  
 - *Another example is given by Waldhausen's connective K-theory spectrum*

$$K^c : \text{HO}(\text{dgc}at) \rightarrow \text{HO}(Spt),$$

see [41].

*Remark 16.7.* By theorem 16.4, the morphism of derivators  $K^c$  factors through  $\mathcal{U}_a$  and so gives rise to a morphism

$$K^c : \mathcal{M}_{dg}^{add} \rightarrow \text{HO}(Spt).$$

We now will prove that this morphism of derivators is co-representable in  $\mathcal{M}_{dg}^{add}$ . Let  $\mathcal{A}$  be a small dg category.

**Notation 16.8.** *We denote by  $K(\mathcal{A})^c \in \text{Sp}^{\mathbb{N}}(\mathbb{L}_{\mathcal{E}_{un}^s} \mathbb{L}_{\Sigma, P} \text{Fun}(\text{dgc}at_f^o, Sset_{\bullet}))$  the spectrum such that*

$$K(\mathcal{A})_n^c := |N.wS^{(n+1)} \text{rep}_{dg}(\mathcal{A})|, \quad n \geq 0,$$

*endowed with the natural structure morphisms, see [41].*

Notice that  $\mathcal{U}_a(\mathcal{A})$  identifies in  $\text{Ho}(\mathcal{M}_{dg}^{add})$  with the suspension spectrum given by

$$(\Sigma^{\infty} |N.w\text{rep}_{dg}(\mathcal{A})|)_n := S^n \wedge |N.w\text{rep}_{dg}(\mathcal{A})|.$$

Now proposition 15.10 and the fact that the morphism of derivators  $\varphi$  commutes with homotopy colimits implies that we have an isomorphism

$$\mathcal{U}_a(\mathcal{A})[1] \xrightarrow{\sim} p\mathcal{U}_a(S.\mathcal{A}).$$

In particular, we have a natural morphism

$$\eta : \mathcal{U}_a(\mathcal{A})[1] \rightarrow K(\mathcal{A})^c$$

in  $\text{Sp}^{\mathbb{N}}(\mathbb{L}_{\mathcal{E}_{un}^s} \mathbb{L}_{\Sigma, P} \text{Fun}(\text{dgc}at_f^o, Sset_{\bullet}))$  induced by the identity in degree 0.

**Theorem 16.9.** *The morphism  $\eta$  is a fibrant resolution of  $\mathcal{U}_a(\mathcal{A})[1]$ .*

*Proof.* We prove first that  $K(\mathcal{A})^c$  is a fibrant object in  $\text{Sp}^{\mathbb{N}}(\mathbb{L}_{\mathcal{E}_{un}^s} \mathbb{L}_{\Sigma, P} \text{Fun}(\text{dgc}at_f^o, Sset_{\bullet}))$ .

By [16][33] we need to show that  $K(\mathcal{A})^c$  is an  $\Omega$ -spectrum, i.e. that  $K(\mathcal{A})_n^c$  is a fibrant object  $\mathbb{L}_{\mathcal{E}_{un}^s} \mathbb{L}_{\Sigma, P} \text{Fun}(\text{dgc}at_f^o, Sset_{\bullet})$  and that the induced map

$$K(\mathcal{A})_n^c \rightarrow \Omega K(\mathcal{A})_{n+1}^c$$

is a weak equivalence. By Waldhausen's additivity theorem, see [41], we have weak equivalences

$$K(\mathcal{A})_n^c \xrightarrow{\sim} \Omega K(\mathcal{A})_{n+1}^c$$

in  $\text{Fun}(\text{dgc}at_f^o, Sset_{\bullet})$ . Now observe that for every integer  $n$ ,  $K^c(\mathcal{A})_n$  satisfies conditions (1)-(3) of proposition 15.9. Condition (4) follows from Waldhausen's fibration theorem, as in the proof of proposition 15.10, applied to the  $S$ -construction. This shows that  $K(\mathcal{A})^c$  is an  $\Omega$ -spectrum.

We now prove that  $\eta$  is a (componentwise) weak equivalence in  $\mathrm{Sp}^{\mathbb{N}}(\mathcal{L}_{\mathcal{E}_{un}^s} \mathcal{L}_{\Sigma, P} \mathrm{Fun}(\mathrm{dgc}at_f^o, Sset_{\bullet}))$ . Observe that the  $n$ -component of the morphism  $\eta$  is the composition of the  $n$ -structural morphism of the spectrum  $K(\mathcal{A})^c$ , which by proposition 15.10 and remark 15.11, is a weak equivalence, with the suspension of a weak equivalence which by proposition 5.1 is also a weak equivalence.

This proves the theorem. ✓

**Theorem 16.10.** *We have the following weak equivalence of simplicial sets*

$$\mathrm{Map}(\mathcal{U}_a(k), \mathcal{U}_a(\mathcal{A})[1]) \xrightarrow{\simeq} |N.w.S.\mathcal{A}_f|$$

in  $\mathrm{Sp}^{\mathbb{N}}(\mathcal{L}_{\mathcal{E}_{un}^s} \mathcal{L}_{\Sigma, P} \mathrm{Fun}(\mathrm{dgc}at_f^o, Sset_{\bullet}))$ . In particular, we have the following isomorphism

$$\pi_{i+1} \mathrm{Map}(\mathcal{U}_a(k), \mathcal{U}_a(\mathcal{A})[1]) \xrightarrow{\simeq} K_i(\mathcal{A}), \forall i \geq 0.$$

*Proof.* Since  $\mathcal{U}_a(k)$  identifies with the suspension spectrum

$$\Sigma^{\infty} |N.wrep_{dg}(\cdot, k)|$$

which is cofibrant, the simplicial structure in  $\mathrm{Sp}^{\mathbb{N}}(\mathcal{L}_{\mathcal{E}_{un}^s} \mathcal{L}_{\Sigma, P} \mathrm{Fun}(\mathrm{dgc}at_f^o, Sset_{\bullet}))$  yields the following equivalences

$$\begin{aligned} \mathrm{Map}(\mathcal{U}_a(k), \mathcal{U}_a(\mathcal{A})[1]) &\simeq \mathrm{Map}(\mathbb{R}\underline{h}(k), K(\mathcal{A})_0^c) \\ &\simeq (K(\mathcal{A})_0^c)(k) \\ &\simeq |N.w.S.\mathcal{A}_f|. \end{aligned}$$

This proves the theorem. ✓

## 17. CONCLUDING REMARKS

By the universal properties of  $\mathcal{U}_u$ ,  $\mathcal{U}_a$  and  $\mathcal{U}_l$ , we obtain the following diagram :

$$\begin{array}{ccc} \mathrm{HO}(\mathrm{dgc}at) & \xrightarrow{\mathcal{U}_u} & \mathcal{M}_{dg}^{unst} \\ & \searrow^{\mathcal{U}_a} & \downarrow \\ & & \mathcal{M}_{dg}^{add} \\ & \searrow_{\mathcal{U}_l} & \downarrow^{\Phi} \\ & & \mathcal{M}_{dg}^{loc}. \end{array}$$

Notice that Waldhausen's connective  $K$ -theory is an example of an additive invariant which is NOT a derived one, see [19]. Waldhausen's connective  $K$ -theory becomes co-representable in  $\mathcal{M}_{dg}^{add}$  by theorem 16.10.

To the best of the author's knowledge, this is the first conceptual characterization of Quillen-Waldhausen  $K$ -theory [31] [41].

An analogous result should be true for non-connective  $K$ -theory and the morphism  $\Phi$  should be thought of as co-representing 'the passage from additivity to localization'.

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