

# Pure motives, mixed motives and extensions of motives associated to singular surfaces

by

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## Abstract

We first recall the construction of the Chow motive modelling intersection cohomology of a proper surface  $\overline{X}$  and study its fundamental properties. Using Voevodsky's category of effective geometrical motives, we then study the motive of the exceptional divisor  $D$  in a non-singular blow-up of  $\overline{X}$ . If all geometric irreducible components of  $D$  are of genus zero, then Voevodsky's formalism allows us to construct certain one-extensions of Chow motives, as canonical sub-quotients of the motive with compact support of the smooth part of  $\overline{X}$ . Specializing to Hilbert–Blumenthal surfaces, we recover a motivic interpretation of a recent construction of A. Caspar.

Keywords: intersection cohomology, intersection motives, Künneth filtration, motives à la Voevodsky, Kummer–Chern–Eisenstein extensions.

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## 0 Introduction

The modest aim of this article is to construct non-trivial extensions of two Chow motives in Voevodsky’s category of effective geometrical motives, by studying a very special and concrete geometric situation, namely that of the (singular) Baily–Borel compactification of a Hilbert–Blumenthal surface.

This example illustrates a much more general principle: motives of complete varieties  $Y$  that are singular, can provide interesting extensions of two Chow motives. The cohomological theories of mixed sheaves suggest where to look for these Chow motives: one should come from the open smooth part  $Y_{\text{reg}}$  of  $Y$  — the *intersection motive* of  $Y$  — the other should be constructed out of the complement of  $Y_{\text{reg}}$  in  $Y$  — the *boundary motive* of  $Y_{\text{reg}}$ . I decided to discuss this principle (for which no originality is claimed, since it has been part of the mathematical culture for some time) in more detail separately [Wi], in order to preserve the structure of the present article. It is intended as a research article with a large instructional component.

The geometric object of interest is a proper surface  $\overline{X}$  over an arbitrary base field  $k$ .

The first three sections contain nothing fundamentally new, except maybe for the systematic use of Künneth filtrations (which are canonical) instead of Künneth decompositions (which in general are not). Section 1 reviews a special case of a result of Borho and MacPherson [BoMp], computing the

intersection cohomology of  $\overline{X}$  in terms of the cohomology of a desingularization  $\tilde{X}$ . The result (Theorem 1.1), predicted by the Decomposition Theorem of [BBD], is that the former is a direct factor of the latter. Its complement is given by the second cohomology of the exceptional divisor  $D$  of  $\tilde{X}$ . This follows from the well-known non-degeneracy of the intersection pairing on the components  $D_m$  of  $D$ . As remarked already by de Cataldo and Migliorini [CtMi], this latter observation allows to directly translate the construction into the motivic world, and to construct the intersection motive  $h_{l*}(\overline{X})$  of  $\overline{X}$ . This is done in Section 2. We get a canonical decomposition

$$h(\tilde{X}) = h_{l*}(\overline{X}) \oplus \bigoplus_m h^2(D_m)$$

in the category of Chow motives over  $k$ . Recall that this category is pseudo-Abelian. The above decomposition should be considered as remarkable: to construct a sub-motive of  $h(\tilde{X})$  does not *a priori* necessitate the *identification*, but only the *existence* of a complement. In our situation, the complement *is* canonical, thanks to the very special geometrical situation. This point is reflected by the rather subtle functoriality properties of  $h_{l*}(\overline{X})$  (Proposition 2.5): viewed as a sub-motive of  $h(\tilde{X})$ , it is respected by pull-backs, viewed as a quotient, it is respected by push-forwards under dominant morphisms of surfaces. Section 3 is devoted to the existence and the study of the Künneth filtration of  $h_{l*}(\overline{X})$ . The main ingredient is of course Murre's construction of Künneth projectors for the motive  $h(\tilde{X})$  [Mr1]. Theorem 3.8 shows how to adapt these to our construction.

As suggested by one of the fundamental properties of intersection cohomology [BBD], the intersection motive of  $\overline{X}$  satisfies the Hard Lefschetz Theorem for ample line bundles on  $\overline{X}$ . We prove this result (Theorem 4.1) in Section 4. In fact, we give a slight generalization (Variant 4.2), which will turn out to be useful for the example we shall study in the last section.

Section 5 is concerned with the motive of the boundary  $D$  of the desingularization  $\tilde{X}$  of  $\overline{X}$ . This boundary being singular in general, the right language for the study of its motive is given by Voevodsky's triangulated category of effective geometrical motives. The section starts with a review of the definition of this category, and of its relation to Chow motives. It is then easy to define motivic analogues of  $H^0$  and  $H^2$  of  $D$ , and to see that they are Chow motives. The most interesting part is the motivic analogue of the part of degree one  $H^1$ , which will be seen as a canonical sub-quotient of the motive of  $D$ .

In Section 6, we unite everything said before, and give our main result (Theorem 6.6). Assuming that all geometric irreducible components  $D_m$  of  $D$  are curves of genus zero, we define a map associating to any line bundle on  $X$  a one-extension of the degree zero-part of the intersection motive of  $\overline{X}$  by

the degree one-part of the motive of  $D$ . We have no difficulty to admit that without the main result of a recent article of Caspar [Cs], it would have been difficult to guess this statement. It thus appeared appropriate to conclude this article by a discussion of his result. This is what is done in Section 7, where we show that in the geometric setting considered in [loc. cit.], Theorem 6.6 specializes to a construction which, *modulo* an assumption on the compatibility of the  $\ell$ -adic regulator with the localization sequences, yields a motivic interpretation of Caspar's construction.

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**Notations and convention:**  $k$  denotes a fixed base field, and  $CH$  stands for the tensor product with  $\mathbb{Q}$  of the Chow group. The  $\mathbb{Q}$ -linear category of Chow motives over  $k$  is denoted by  $CHM(k)$ . Our standard reference for Chow motives is Scholl's survey article [S].

## 1 Intersection cohomology of surfaces

In order to motivate the construction of the intersection motive, to be given in the next section, we shall recall the computation of the *intersection cohomology* of a complex surface.

Thus, throughout this section, our base field  $k$  will be equal to  $\mathbb{C}$ . We consider the following situation:

$$X \xhookrightarrow{j} X^* \xleftarrow{i} Z$$

The morphism  $i$  is a closed immersion of a sub-scheme  $Z$ , with complement  $j$ . The scheme  $X^*$  is a surface over  $\mathbb{C}$ , all of whose singularities are contained in  $Z$ . Thus, the surface  $X$  is smooth.

Our aim is to identify the intersection cohomology groups  $H_{!*}^n(X^*(\mathbb{C}), \mathbb{Q})$ . Note that since  $X$  is smooth, the complex  $\mathbb{Q}_X[2]$  consisting of the constant local system  $\mathbb{Q}$ , placed in degree  $-2$ , can be viewed as a *perverse sheaf* (for the middle perversity) on  $X(\mathbb{C})$  [BBD, Sect. 2.2.1]. Hence its *intermediate extension*  $j_{!*}\mathbb{Q}_X[2]$  [BBD, (2.2.3.1)] is defined as a perverse sheaf on  $X^*(\mathbb{C})$ . By definition,

$$H_{!*}^n(X^*(\mathbb{C}), \mathbb{Q}) = H^{n-2}(X^*(\mathbb{C}), j_{!*}\mathbb{Q}_X[2]), \forall n \in \mathbb{Z}.$$

In order to identify  $H_{!*}^n(X^*(\mathbb{C}), \mathbb{Q})$ , note first that the normalization of  $X^*$  is finite over  $X^*$ , and the direct image under finite morphisms is exact for the perverse  $t$ -structure [BBD, Cor. 2.2.6 (i)]. Therefore, intersection cohomology is invariant under passage to the normalization. In the sequel, we therefore assume that  $X^*$  is normal. In particular, its singularities are isolated.

Next, note that if  $X^*$  is smooth, then the complex  $j_{!*}\mathbb{Q}_X[2]$  equals  $\mathbb{Q}_{X^*}[2]$ . Transitivity of  $j_{!*}$  [BBD, (2.1.7.1)] shows that we may enlarge  $X$ , and hence assume that the closed sub-scheme  $Z$  is finite.

Choose a resolution of singularities. More precisely, consider in addition the following diagram, assumed to be cartesian:

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{j}} & \tilde{X} & \xleftarrow{\tilde{i}} & D \\ \parallel & & \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{j} & X^* & \xleftarrow{i} & Z \end{array}$$

The morphism  $\pi$  is assumed proper (and birational) and the surface  $\tilde{X}$ , smooth. We then have the following special case of [BoMp, Thm. 1.7].

**Theorem 1.1.** (i) For  $n \neq 2$ ,

$$H_{!*}^n(X^*(\mathbb{C}), \mathbb{Q}) = H^n(\tilde{X}(\mathbb{C}), \mathbb{Q}) .$$

(ii) The group  $H_{!*}^2(X^*(\mathbb{C}), \mathbb{Q})$  is a direct factor of  $H^2(\tilde{X}(\mathbb{C}), \mathbb{Q})$ , with a canonical complement. As a sub-group, this complement is given by the map

$$\tilde{i}_* : H_{D(\mathbb{C})}^2(\tilde{X}(\mathbb{C}), \mathbb{Q}) \longrightarrow H^2(\tilde{X}(\mathbb{C}), \mathbb{Q})$$

from cohomology with support in  $D(\mathbb{C})$ ; this map is injective. As a quotient, the complement is given by the restriction

$$\tilde{r}^* : H^2(\tilde{X}(\mathbb{C}), \mathbb{Q}) \longrightarrow H^2(D(\mathbb{C}), \mathbb{Q}) ;$$

this map is surjective.

Note that this result is compatible with further blow-up of  $\tilde{X}$  in points belonging to  $D$ .

Let us construct the maps between  $H_{!*}^n(X^*(\mathbb{C}), \mathbb{Q})$  and  $H^n(\tilde{X}(\mathbb{C}), \mathbb{Q})$  leading to the above identifications. Consider the total direct image  $\pi_*\mathbb{Q}_{\tilde{X}}$ ; following the convention used in [BBD], we drop the letter “ $R$ ” from our notation.

**Lemma 1.2.** The complex  $\pi_*\mathbb{Q}_{\tilde{X}}[2]$  is a perverse sheaf on  $X^*$ .

*Proof.* Let  $P$  be a point (of  $Z$ ) over which  $\pi$  is not an isomorphism, and denote by  $i_P$  its inclusion into  $X^*$ . By definition [BBD, Déf. 2.1.2], we

need to check that (a) the higher inverse images  $H^n i_P^* \pi_* \mathbb{Q}_{\tilde{X}}$  vanish for  $n > 2$ , (b) the higher exceptional inverse images  $H^n i_P^! \pi_* \mathbb{Q}_{\tilde{X}}$  vanish for  $n < 2$ .

(a) By proper base change, the group in question equals  $H^n(\pi^{-1}(P), \mathbb{Q})$ . Since  $\pi^{-1}(P)$  is of dimension at most one, there is no cohomology above degree two.

(b) The surface  $\tilde{X}$  is smooth. Duality and proper base change imply that the group in question is abstractly isomorphic to the dual of  $H^{4-n}(\pi^{-1}(P), \mathbb{Q})$ . This group vanishes if  $4 - n$  is strictly larger than two. **q.e.d.**

For  $a \in \mathbb{Z}$ , denote by  $\tau_{\leq a}$  the functor associating to a complex the  $a$ -th step of its canonical filtration (with respect to the classical  $t$ -structure). Recall that  $j_{!*} \mathbb{Q}_X[2]$  equals  $\tau_{\leq -1}(j_* \mathbb{Q}_X[2])$  [BBD, Prop. 2.1.11]. We now see how to relate it to  $\pi_* \mathbb{Q}_{\tilde{X}}[2]$ : apply  $\pi_*$  to the exact triangle

$$\tilde{i}_* \tilde{i}^! \mathbb{Q}_{\tilde{X}} \longrightarrow \mathbb{Q}_{\tilde{X}} \longrightarrow \tilde{j}_* \mathbb{Q}_X \longrightarrow \tilde{i}_* \tilde{i}^! \mathbb{Q}_{\tilde{X}}[1] .$$

This gives an exact triangle

$$i_* F[0] \longrightarrow \tau_{\leq -1}(\pi_* \mathbb{Q}_{\tilde{X}}[2]) \longrightarrow j_{!*} \mathbb{Q}_X[2] \longrightarrow i_* F[1] ;$$

in fact, as in the proof of Lemma 1.2, one sees that  $F$  is a sheaf concentrated in  $Z$ . More precisely, the restriction to any point  $P$  of  $Z$  of this sheaf equals the kernel of the composition

$$\tilde{i}^* \tilde{i}_* : H_{\pi^{-1}(P)}^2(\tilde{X}(\mathbb{C}), \mathbb{Q}) \longrightarrow H^2(\tilde{X}(\mathbb{C}), \mathbb{Q}) \longrightarrow H^2(\pi^{-1}(P), \mathbb{Q}) .$$

We thus get the following.

**Lemma 1.3.** *There is a canonical exact sequence*

$$0 \longrightarrow i_* F[0] \longrightarrow \tau_{\leq -1}(\pi_* \mathbb{Q}_{\tilde{X}}[2]) \longrightarrow j_{!*} \mathbb{Q}_X[2] \longrightarrow 0$$

*of perverse sheaves on  $X^*$ .*

*Proof of Theorem 1.1.* We shall show that the composition

$$\tilde{i}^* \tilde{i}_* : H_{D(\mathbb{C})}^2(\tilde{X}(\mathbb{C}), \mathbb{Q}) \longrightarrow H^2(D(\mathbb{C}), \mathbb{Q})$$

is in fact an isomorphism. This implies that the sheaf  $F$  is zero. It also implies injectivity of

$$\tilde{i}_* : H_{D(\mathbb{C})}^2(\tilde{X}(\mathbb{C}), \mathbb{Q}) \longrightarrow H^2(\tilde{X}(\mathbb{C}), \mathbb{Q}) ,$$

as well as surjectivity of

$$\tilde{i}^* : H^2(\tilde{X}(\mathbb{C}), \mathbb{Q}) \longrightarrow H^2(D(\mathbb{C}), \mathbb{Q}) .$$

Hence the statement of our theorem.

In order to prove bijectivity of  $\tilde{i}^* \tilde{i}_*$ , note that we may assume that  $D$  is a divisor, whose irreducible components are smooth. Indeed, if  $f : \tilde{X}' \rightarrow \tilde{X}$  is a further blow-up, such that  $f^{-1}(D)$  has the required property [H, Thm.  $I_2^{N,n}$ ], then the push-forward  $f_*$  is a left inverse of the pull-back  $f^*$ , and the diagrams involving cohomology of  $D(\mathbb{C})$  and  $f^{-1}(D(\mathbb{C}))$ , and cohomology with support

in  $D(\mathbb{C})$  and  $f^{-1}(D(\mathbb{C}))$ , respectively, commute thanks to proper base change. Therefore, bijectivity on the level of  $\tilde{X}$  follows from bijectivity on the level of  $\tilde{X}'$ .

If  $D_m$  are the irreducible components of  $D$ , then the closed covering  $D = \cup_m D_m$  induces canonical isomorphisms

$$\bigoplus_m H_{D_m(\mathbb{C})}^2(\tilde{X}(\mathbb{C}), \mathbb{Q}) \xrightarrow{\sim} H_{D(\mathbb{C})}^2(\tilde{X}(\mathbb{C}), \mathbb{Q})$$

and

$$H^2(D(\mathbb{C}), \mathbb{Q}) \xrightarrow{\sim} \bigoplus_m H^2(D_m(\mathbb{C}), \mathbb{Q}) .$$

Purity identifies each  $H_{D_m(\mathbb{C})}^2(\tilde{X}(\mathbb{C}), \mathbb{Q})$  with  $H^0(D_m(\mathbb{C}), \mathbb{Q})(-1)$  (it is here that we use that the  $D_m$  are smooth). The induced morphism

$$\tilde{v}^* \tilde{i}_* : \bigoplus_m H^0(D_m(\mathbb{C}), \mathbb{Q}) \longrightarrow \bigoplus_m H^2(D_m(\mathbb{C}), \mathbb{Q})(1)$$

corresponds to the intersection pairing on the components of  $D$ . This pairing is well known to be negative definite [Mm, p. 6]. In particular, it is non-degenerate. **q.e.d.**

**Remark 1.4.** The analogue of Theorem 1.1 holds for  $\ell$ -adic cohomology, and when  $k$  is a finite field of characteristic unequal to  $\ell$ . The proof is exactly the same. Note that by Abhyankar's result on resolution of singularities in dimension two [L2, Theorem],  $X^*$  can be desingularized for *any* base field  $k$ . In addition (see the discussion in [L1, pp. 191–194]), by further blowing up possible singularities of (the components of) the pre-image  $D$  of  $Z$ , it can be assumed to be a divisor with normal crossings, whose irreducible components are smooth. This discussion also shows that the system of such resolutions is filtering.

## 2 Construction of the intersection motive

Fix a base field  $k$ , and assume given a proper surface  $\overline{X}$  over  $k$ . The aim of this section is to recall the construction of the *Chow motive* modelling intersection cohomology of  $\overline{X}$ , and to study its functoriality properties. The discussion preceding Theorem 1.1 showed that intersection cohomology is invariant under passage to the normalization  $X^*$  of  $\overline{X}$ ; the same should thus

be expected from the motive we intend to construct. <sup>1</sup> Fix

$$X \hookrightarrow X^* \xleftarrow{i} Z$$

where  $i$  is a closed immersion of a finite sub-scheme  $Z$ , with smooth complement  $X$ . Choose a resolution of singularities. More precisely, consider in addition the following diagram, assumed to be cartesian:

$$\begin{array}{ccc} X & \hookrightarrow & \tilde{X} \xleftarrow{\tilde{i}} D \\ \parallel & & \pi \downarrow \quad \downarrow \pi \\ X & \hookrightarrow & X^* \xleftarrow{i} Z \end{array}$$

where  $\pi$  is proper (and birational),  $\tilde{X}$  is smooth (and proper), and  $D$  is a divisor with normal crossings, whose irreducible components  $D_m$  are smooth (and proper).

**Remark 2.1.** Note that  $\tilde{X}$ , as a smooth and proper surface, is projective: Zariski proved this result for algebraically closed base fields in [Z, p. 54], and [SGA1VIII, Cor. 7.7] allows to descend to arbitrary base fields.

Theorem 1.1 suggests how to construct the intersection motive; in particular, it should be a canonical direct complement of  $\bigoplus_m h^2(D_m)$  in  $h(\tilde{X})$ . Recall [S, Sect. 1.13] that the  $h^2(D_m)$  are canonically defined as quotient objects of the motives  $h(D_m)$ . Hence there is a canonical morphism

$$\tilde{i}^* : h(\tilde{X}) \longrightarrow \bigoplus_m h(D_m) \twoheadrightarrow \bigoplus_m h^2(D_m)$$

of Chow motives. Similarly [S, Sect. 1.11], there is a canonical morphism

$$\tilde{i}_* : \bigoplus_m h^0(D_m)(-1) \hookrightarrow \bigoplus_m h(D_m)(-1) \longrightarrow h(\tilde{X}) .$$

Here, the twist by  $(-1)$  denotes the tensor product with the Lefschetz motive  $\mathbb{L} = h^2(\mathbb{P}^1)$ . The following is a special case of [CtMi, Sect. 2.5].

**Theorem 2.2.** (i) *The composition  $\alpha := \tilde{i}^* \tilde{i}_*$  is an isomorphism of Chow motives.*

(ii) *The composition  $p := \tilde{i}_* \alpha^{-1} \tilde{i}^*$  is an idempotent on  $h(\tilde{X})$ . Hence so is the difference  $\text{id}_{\tilde{X}} - p$ .*

(iii) *The image  $\text{im } p$  is canonically isomorphic to  $\bigoplus_m h^2(D_m)$ .*

*Proof.* (ii) and (iii) are formal consequences of (i). The formula “ $\phi_* \phi^* = \text{deg } \phi$ ” for finite morphisms  $\phi$  [S, Sect. 1.10] shows that we may

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<sup>1</sup>This principle also explains why the problem of constructing the intersection motive of a proper curve  $\overline{C}$  is not very interesting: the intersection motive of  $\overline{C}$  is equal to the motive of the normalization  $C^*$  of  $\overline{C}$  (which is smooth and projective).

prove our claim after a finite extension of our ground field  $k$ . In particular, we may assume that all components  $D_m$  are geometrically irreducible, with field of constants equal to  $k$ . We then have canonical isomorphisms  $h^0(D_m) \cong h(\mathbf{Spec} k)$  and  $h^2(D_m) \cong \mathbb{L}$ . Denote by  $i_m$  the closed immersion of  $D_m$  into  $\tilde{X}$ . The map  $\alpha$  in question equals

$$\bigoplus_{m,n} i_m^* i_{n,*} : \bigoplus_n h^0(D_n)(-1) \longrightarrow \bigoplus_m h^2(D_m).$$

For each pair  $(m, n)$ , the composition  $i_m^* i_{n,*}$  is an endomorphism of  $\mathbb{L}$ . Now the degree map induces an isomorphism

$$\mathrm{End}(\mathbb{L}) = CH^0(\mathbf{Spec} k) \xrightarrow{\sim} \mathbb{Q}.$$

We leave it to the reader to show that under this isomorphism, the endomorphism  $i_m^* i_{n,*}$  is mapped to the intersection number  $D_n \cdot D_m$ . Our claim follows from the non-degeneracy of the intersection pairing on the components of  $D$  [Mm, p. 6]. **q.e.d.**

Following [CtMi, p. 158], we propose the following definition.

**Definition 2.3.** The *intersection motive* of  $\overline{X}$  is defined as

$$h_{!*}(\overline{X}) := (\tilde{X}, \mathrm{id}_{\tilde{X}} - p, 0) \in CHM(k).$$

Here, we follow the standard notation for Chow motives (see e.g. [S, Sect. 1.4]). Idempotents on Chow motives have an image; by definition, the image of the idempotent  $\mathrm{id}_{\tilde{X}} - p$  on the Chow motive  $(\tilde{X}, \mathrm{id}_{\tilde{X}}, 0) = h(\tilde{X})$  is  $(\tilde{X}, \mathrm{id}_{\tilde{X}} - p, 0) = h_{!*}(\overline{X})$ . Note that by definition, we have the equality  $h_{!*}(\overline{X}) = h_{!*}(X^*)$ .

Theorem 2.2 shows that there is a canonical decomposition

$$h(\tilde{X}) = h_{!*}(\overline{X}) \oplus \bigoplus_m h^2(D_m)$$

in  $CHM(k)$ . By Theorem 1.1 and Remark 1.4, the Betti, resp.  $\ell$ -adic realization of the intersection motive (for the base fields for which this realization exists) coincides with intersection cohomology of  $\overline{X}$  (and of  $X^*$ ).

**Proposition 2.4.** *As before, denote by  $X^*$  the normalization of  $\overline{X}$ . The definition of  $h_{!*}(\overline{X})$  is independent of the choices of the finite sub-scheme  $Z$  containing the singularities  $X^*$ , and of the desingularization  $\tilde{X}$  of  $X^*$ .*

This statement is going to be proved together with the functoriality properties of the intersection motive, whose formulation we prepare now. Consider a dominant morphism  $f : \overline{X} \rightarrow \overline{Y}$  of proper surfaces over  $k$ . By the universal property of the normalization  $Y^*$  of  $\overline{Y}$ , it induces a morphism, still denoted  $f$ , between  $X^*$  and  $Y^*$ . It is generically finite. Hence we can find a finite closed subscheme  $W$  of  $Y^*$  containing the singularities, and such that

the pre-image under  $f$  of  $Y := Y^* - W$  is dense, and smooth. The closed sub-scheme  $f^{-1}(W)$  of  $X$  contains the singularities of  $X^*$ . We thus can find a morphism  $F$  of desingularizations of  $X^*$  and  $Y^*$  of the type considered before:

$$\begin{array}{ccc} \tilde{X} & \xleftarrow{i_D} & D \\ F \downarrow & & \downarrow F \\ \tilde{Y} & \xleftarrow{i_C} & C \end{array}$$

This means that  $\tilde{X}$  and  $\tilde{Y}$  are smooth, and  $D$  and  $C$  are divisors with normal crossings, whose irreducible components  $D_m$  resp.  $C_n$  are smooth, and lying over finite closed sub-schemes of  $X^*$  and  $Y^*$ , respectively. Choose and fix such a diagram. Note that if the original morphism  $f : \bar{X} \rightarrow \bar{Y}$  is finite, then the diagram  $F$  can be chosen to be cartesian.

**Proposition 2.5.** (i) *The pull-back  $F^* : h(\tilde{Y}) \rightarrow h(\tilde{X})$  maps the sub-object  $h_{1*}(\bar{Y})$  of  $h(\tilde{Y})$  to the sub-object  $h_{1*}(\bar{X})$  of  $h(\tilde{X})$ .*

(ii) *The push-forward  $F_* : h(\tilde{X}) \rightarrow h(\tilde{Y})$  maps the quotient  $h_{1*}(\bar{X})$  of  $h(\tilde{X})$  to the quotient  $h_{1*}(\bar{Y})$  of  $h(\tilde{Y})$ .*

(iii) *The composition  $F_* F^* : h_{1*}(\bar{Y}) \rightarrow h_{1*}(\bar{Y})$  equals multiplication with the degree of  $f$ .*

(iv) *If  $f$  is finite, and if the morphism  $F$  is chosen to be cartesian, then both  $F^*$  and  $F_*$  respect the decompositions*

$$h(\tilde{Y}) = h_{1*}(\bar{Y}) \oplus \bigoplus_n h^2(C_n)$$

and

$$h(\tilde{X}) = h_{1*}(\bar{X}) \oplus \bigoplus_m h^2(D_m)$$

of  $h(\tilde{Y})$  and of  $h(\tilde{X})$ , respectively.

*Proof.* By definition, there are (split) exact sequences

$$0 \longrightarrow h_{1*}(\bar{X}) \longrightarrow h(\tilde{X}) \xrightarrow{i_D^*} \bigoplus_m h^2(D_m) \longrightarrow 0$$

and

$$0 \longrightarrow \bigoplus_m h^0(D_m)(-1) \xrightarrow{i_{D,*}} h(\tilde{X}) \longrightarrow h_{1*}(\bar{X}) \longrightarrow 0 ;$$

similarly for  $\tilde{Y}$  and  $C$ . Obviously, the first sequence is contravariant, and the second is covariant. This proves parts (i) and (ii). Part (iii) follows from this, and from the corresponding formula for  $F_* F^*$  on the motive of  $\tilde{Y}$  [S, Sect. 1.10]; note that the degree of  $F$  equals the one of  $f$ . If  $F$  is cartesian, then the above sequences are both co- and contravariant thanks to the base

change formulae  $F_*i_D^* = i_C^*F_*$  and  $F^*i_{C,*} = i_{D,*}F^*$ . This proves part (iv). **q.e.d.**

*Proof of Proposition 2.4.* First, let us show that for a fixed choice of  $Z$ , the definition of  $h_{l_*}(\overline{X})$  is independent of the choice of the desingularization  $\widetilde{X}$  of  $X^*$ . Using that the system of such desingularizations is filtering, we reduce ourselves to the situation considered in Proposition 2.5, with  $f = \text{id}$ . We thus have a cartesian diagram

$$\begin{array}{ccc} \widetilde{X} & \xleftarrow{i_D} & D \\ F \downarrow & & \downarrow F \\ \widetilde{X}' & \xleftarrow{i_C} & C \end{array}$$

Let us denote by  $h_{l_*}(\overline{X})$  and  $h'_{l_*}(\overline{X})$  the two intersection motives formed with respect to  $\widetilde{X}$  and  $\widetilde{X}'$ , respectively. We want to show that  $F^* : h'_{l_*}(\overline{X}) \rightarrow h_{l_*}(\overline{X})$  is an isomorphism. The scheme  $\widetilde{X}'$  is normal, and the morphism  $F$  is proper. By the valuative criterion of properness, the locus of points of  $\widetilde{X}'$  where  $F^{-1}$  is not defined is of dimension zero. Let  $P$  be a point in this locus. If the fibre over  $P$  were finite, then  $F$  would be quasi-finite near  $P$ . Since it is proper, it would be finite. But since both its source and target are normal, it would be an isomorphism near  $P$ , contrary to our assumption. This shows that the fibre over  $P$  is of dimension one. Since the fibre is connected [EGA3, Cor. (4.3.12)], it is pure of dimension one, i.e., it is a divisor. By the universal property of the blow-up,  $\widetilde{X}$  dominates the blow-up of  $\widetilde{X}'$  in the points  $P_1, \dots, P_r$  where  $F$  is not an isomorphism. This blow-up lies between  $\widetilde{X}$  and  $\widetilde{X}'$ , and satisfies the same conditions on desingularizations. Repeating this argument and using the fact that  $\widetilde{X}$  is Noetherian, one sees that this process stops at some point;  $F$  is therefore the composition of blow-ups in points. By induction, we may assume that  $F$  equals the blow-up of  $\widetilde{X}'$  in one point  $P$ . The exceptional divisor  $E := F^{-1}(P)$  is a projective bundle (of rank one) over  $P$ . It is also one of the irreducible components  $D_m$  of  $D$ ; in fact, the morphism  $F$  induces a bijection between the components of  $D$  other than  $E$  and the components  $C_n$  of  $C$ . Denote by  $i_E$  the closed immersion of  $E$  into  $\widetilde{X}$ . By Manin's computation of the motive of a blow-up [S, Thm. 2.8], the sequence

$$0 \longrightarrow h(\widetilde{X}') \xrightarrow{F^*} h(\widetilde{X}) \xrightarrow{i_E^*} h^2(E) \longrightarrow 0$$

is (split) exact. But obviously, so is

$$0 \longrightarrow \bigoplus_n h^2(C_n) \xrightarrow{F^*} \bigoplus_m h^2(D_m) \xrightarrow{i_E^*} h^2(E) \longrightarrow 0.$$

Hence  $F^*$  maps the kernel  $h'_{l_*}(\overline{X})$  of  $i_C^*$  isomorphically to the kernel  $h_{l_*}(\overline{X})$  of  $i_D^*$ .

In the same way, one shows that enlarging  $Z$  by adding non-singular points of  $X^*$  does not change the value of  $h_{1*}(\overline{X})$ . **q.e.d.**

Recall the definition of the *dual* of a Chow motive [S, Sect. 1.15]. For example, for any desingularization  $\tilde{X}$  of  $X^*$ , the dual of  $(\tilde{X}, \text{id}_{\tilde{X}}, 0) = h(\tilde{X})$  is given by  $(\tilde{X}, \text{id}_{\tilde{X}}, 2) = h(\tilde{X})(2)$ .

**Proposition 2.6.** *The dual of the intersection motive  $h_{1*}(\overline{X})$  is canonically isomorphic to  $h_{1*}(\overline{X})(2)$ .*

*Proof.* By definition, the dual of  $(\tilde{X}, \text{id}_{\tilde{X}} - p, 0)$  equals  $(\tilde{X}, {}^t(\text{id}_{\tilde{X}} - p), 2)$ , where  ${}^t$  denotes the transposition of cycles in  $\tilde{X} \times \tilde{X}$ . But  $p$  is symmetric: in fact,  ${}^t(\tilde{i}^*) = \tilde{i}_*$ , and  ${}^t(\tilde{i}_*) = \tilde{i}^*$ .

One checks as in the proof of Proposition 2.4 that this identification of  $h_{1*}(\overline{X})^*$  with  $h_{1*}(\overline{X})(2)$  does not depend on the choice of  $\tilde{X}$ . **q.e.d.**

### 3 The Künneth filtration of the intersection motive

We continue to consider the situation of Section 2. Thus,  $\overline{X}$  is a proper surface over the base field  $k$  with normalization  $X^*$ , and we fix

$$X \hookrightarrow X^* \xleftarrow{i} Z$$

where  $i$  is a closed immersion of a finite sub-scheme  $Z$ , with smooth complement  $X$ . In addition, we consider the following cartesian diagram:

$$\begin{array}{ccc} X & \hookrightarrow & \tilde{X} \xleftarrow{i} D \\ \parallel & & \pi \downarrow \quad \downarrow \pi \\ X & \hookrightarrow & X^* \xleftarrow{i} Z \end{array}$$

where  $\pi$  is proper,  $\tilde{X}$  is smooth and proper (hence projective), and  $D$  is a divisor with normal crossings, whose irreducible components  $D_m$  are smooth. The aim of this section is to recall Murre's construction of *Künneth decompositions* of the motive of  $\tilde{X}$  [Mr1], following Scholl's presentation [S, Chap. 4], and to study the resulting filtration on the intersection motive.

Thus, fix (i) a hyperplane section  $C \subset \tilde{X}$  that is a smooth curve (observe that  $C$  might only be defined over a finite extension  $k'$  of  $k$ ). As explained in [S, Sect. 4.3], the embedding of  $C$  into  $\tilde{X}$  induces an isogeny  $P \rightarrow J$  from the Picard variety to the Albanese variety of  $\tilde{X}$ . This isogeny is actually independent of the choice of the smooth curve  $C$  representing the fixed very ample class in  $CH^1(\tilde{X})$  (and a non-zero multiple of the isogeny is defined

over  $k$ ). Fix (ii) an isogeny  $\beta : J \rightarrow P$  such that the composition of the two isogenies equals multiplication by  $n > 0$ . Finally, fix (iii) a 0-cycle  $T$  of degree one on  $C$ . Then by [S, Thm. 3.9],  $\beta$  corresponds to a symmetric cycle class

$$\tilde{\beta} \in CH^1(\tilde{X} \times \tilde{X})$$

satisfying the condition  $p_{\tilde{X},*}(\tilde{\beta} \cdot [\tilde{X} \times T]) = 0 \in CH^1(\tilde{X})$ , where  $p_{\tilde{X}}$  is the first projection from the product  $\tilde{X} \times \tilde{X}$  to  $\tilde{X}$ .

One then defines [S, Sect. 4.3] projectors  $\pi_0 := [T \times \tilde{X}]$  and  $\pi_4 := {}^t\pi_0 = [\tilde{X} \times T]$ , as well as  $p_1 := \frac{1}{n}\tilde{\beta} \cdot [C \times \tilde{X}]$  and  $p_3 := {}^t p_1$ . All orthogonality relations are satisfied, including  $p_3 p_1 = 0$ , except that  $p_1 p_3$  is not necessarily equal to zero. This is why a modification is necessary: one puts  $\pi_1 := p_1 - \frac{1}{2} p_1 p_3$  and  $\pi_3 := {}^t \pi_1 = p_3 - \frac{1}{2} p_1 p_3$ .<sup>2</sup> This, together with  $\pi_2 := \text{id}_{\tilde{X}} - \pi_0 - \pi_1 - \pi_3 - \pi_4$ , gives a full auto-dual set of orthogonal projectors. We thus get a Künneth decomposition of  $h(\tilde{X})$  (first over  $k'$ , then by pushing down, over  $k$ ):

$$h(\tilde{X}) = {}'h^0(\tilde{X}) \oplus {}'h^1(\tilde{X}) \oplus {}'h^2(\tilde{X}) \oplus {}'h^3(\tilde{X}) \oplus {}'h^4(\tilde{X}),$$

with

$${}'h^n(\tilde{X}) := (\tilde{X}, \pi_n, 0) \subset (\tilde{X}, \text{id}_{\tilde{X}}, 0) = h(\tilde{X}), \quad 0 \leq n \leq 4.$$

**Definition 3.1.** (a) The *Künneth filtration* of  $h(\tilde{X})$  is the ascending filtration of  $h(\tilde{X})$  by sub-motives induced by a Künneth decomposition of  $h(\tilde{X})$ :

$$0 \subset h^0(\tilde{X}) \subset h^{\leq 1}(\tilde{X}) \subset h^{\leq 2}(\tilde{X}) \subset h^{\leq 3}(\tilde{X}) \subset h^{\leq 4}(\tilde{X}) = h(\tilde{X}),$$

where we set  $h^{\leq r}(\tilde{X}) := \bigoplus_{n=0}^r {}'h^n(\tilde{X})$ ,  $r \leq 4$ .

(b) The  $n$ -th *Künneth component* of  $h(\tilde{X})$ ,  $0 \leq n \leq 4$ , is the sub-quotient of  $h(\tilde{X})$  defined by

$$h^n(\tilde{X}) := h^{\leq n}(\tilde{X}) / h^{\leq n-1}(\tilde{X}).$$

**Remark 3.2.** The sub-objects  $h^{\leq n}(\tilde{X})$  are direct factors of  $h(\tilde{X})$ , hence the sub-quotients  $h^n(\tilde{X})$  exist. Similarly, one may define the quotients

$$h^{\geq r}(\tilde{X}) := h(\tilde{X}) / h^{\leq r-1}(\tilde{X})$$

of  $h(\tilde{X})$ .

Note that a number of choices is involved in the construction of the projectors  $\pi_0, \dots, \pi_4$ : mainly, a very ample line bundle  $\mathcal{L}$  on  $\tilde{X}$ , and a 0-cycle on a smooth curve in the divisor class corresponding to  $\mathcal{L}$ . The following is the content of [KMrP, Thm. 14.3.10 i):

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<sup>2</sup>This differs from Murre's original solution [Mr1, Rem. 6.5], where one takes  $p_1 - p_1 p_3$  and  $p_3$  instead of  $\pi_1$  and  $\pi_3$ .

**Proposition 3.3.** *The Künneth filtration of  $h(\tilde{X})$  is independent of the choices made in the construction of the Künneth decomposition.*

**Remark 3.4.** (a) In particular, the Künneth components  $h^n(\tilde{X})$  are canonically defined sub-quotients of  $h(\tilde{X})$ .

(b) *A posteriori*, one may define the notion of Künneth decomposition of  $h(\tilde{X})$  as being a decomposition splitting the Künneth filtration. Such decompositions include the ones obtained by Murre’s construction, but there could be others.

Our aim (see Theorem 3.8) is to deduce from the Künneth filtration of  $h(\tilde{X})$  a filtration of the intersection motive  $h_{l*}(\overline{X}) \subset h(\tilde{X})$ :

$$0 \subset h_{l*}^0(\overline{X}) \subset h_{l*}^{\leq 1}(\overline{X}) \subset h_{l*}^{\leq 2}(\overline{X}) \subset h_{l*}^{\leq 3}(\overline{X}) \subset h_{l*}^{\leq 4}(\overline{X}) = h_{l*}(\overline{X}) .$$

The idea is of course to take the “induced” filtration. But since we are working in a category which is only pseudo-Abelian, we need to proceed with some care. Recall the quotient  $\bigoplus_m h^2(D_m)$  and the sub-object  $\bigoplus_m h^0(D_m)$  of  $\bigoplus_m h(D_m)$ .

**Proposition 3.5.** *The Künneth filtration of  $h(\tilde{X})$  satisfies the following conditions.*

- (1) *Duality  $h(\tilde{X})^\vee \xrightarrow{\sim} h(\tilde{X})(2)$  induces isomorphisms*  

$$h^{\leq r}(\tilde{X})^\vee \xrightarrow{\sim} h^{\geq 4-r}(\tilde{X})(2) .$$

- (2) *The composition of morphisms*

$$h^{\leq 1}(\tilde{X}) \hookrightarrow h(\tilde{X}) \xrightarrow{\tilde{i}^*} \bigoplus_m h(D_m) \longrightarrow \bigoplus_m h^2(D_m)$$

*equals zero.*

*Proof.* The Künneth filtration satisfies (1) since the decompositions obtained by Murre’s construction are auto-dual:  $'h^n(\tilde{X})^\vee \cong 'h^{4-n}(\tilde{X})(2)$  under the duality  $h(\tilde{X})^\vee \cong h(\tilde{X})(2)$ .

By [J, Prop. 5.8], condition (2) is a consequence of Murre’s Conjecture B [Mr2, Sect. 1.4] on the triviality of the action of the  $\ell$ -th Künneth projector on  $CH^j(Y)$ , for  $\ell > 2j$ . Here,  $Y$  equals the product of  $\tilde{X}$  and  $D_m$ ,  $j = 2$ , and  $\ell = 5, 6$ . Note that for products of a surface and a curve, the conjecture is known to hold (see [Mr3, Lemma 8.3.2] for the case  $j = 2$ ).

But since the argument proving (2) is rather explicit, we may just as well give it for the convenience of the reader. We need to compute the composition of correspondences

$$h(\tilde{X}) \xrightarrow{\pi_n} h(\tilde{X}) \xrightarrow{\tilde{i}^*} \bigoplus_m h(D_m) \xrightarrow{pr} \bigoplus_m h^2(D_m) ,$$

for  $n = 0, 1$ . The composition is zero if and only if it is zero after base change to a finite field extension. Hence we may assume that all  $D_m$  are geometrically irreducible, with field of constants  $k$ . Then the  $h^2(D_m)$  equal  $\mathbb{L}$ , and the composition  $pr \circ \tilde{i}^*$  corresponds to the cycle class

$$([D_m])_m \in \bigoplus_m CH^1(\tilde{X})$$

on  $\coprod_m \tilde{X} \times \mathbf{Spec} k$ . By definition of the composition of correspondences, we then find

$$p_{\tilde{X},*} \circ \tilde{i}^* \circ \pi = (p_{\tilde{X},*}(\pi \cdot [\tilde{X} \times D_m]))_m \in \bigoplus_m CH^1(\tilde{X}),$$

for any  $\pi \in CH^2(\tilde{X} \times \tilde{X})$ . Here as before,  $p_{\tilde{X}}$  is the first projection from the product  $\tilde{X} \times \tilde{X}$  to  $\tilde{X}$ . Let us fix  $m$ . We need to show that for  $n = 0, 1$ , the cycle class

$$p_{\tilde{X},*}(\pi_n \cdot [\tilde{X} \times D_m]) \in CH^1(\tilde{X})$$

is zero. For  $n = 0$ , this is easy: the intersection

$$\pi_0 \cdot [\tilde{X} \times D_m] = [T \times \tilde{X}] \cdot [\tilde{X} \times D_m] = [T \times D_m]$$

has one-dimensional fibres under  $p_{\tilde{X}}$ . Therefore, its push-forward under  $p_{\tilde{X}}$  is zero.

For  $n = 1$ , observe first that by definition of  $\pi_1$ , and by associativity of composition of correspondences, it suffices to show that

$$p_{\tilde{X},*}(p_1 \cdot [\tilde{X} \times D_m]) = 0.$$

By definition, the intersection  $p_1 \cdot [\tilde{X} \times D_m]$  is a non-zero multiple of

$$\tilde{\beta} \cdot [C \times \tilde{X}] \cdot [\tilde{X} \times D_m].$$

By the projection formula, the image under  $p_{\tilde{X},*}$  of this cycle equals the image under the push-forward  $CH^0(C) \rightarrow CH^1(\tilde{X})$  of

$$p_{1,*}(\tilde{\beta}_C \cdot [C \times D_m]),$$

where  $\tilde{\beta}_C$  denotes the pull-back of  $\tilde{\beta}$  to  $C \times \tilde{X}$ , and  $p_1$  the projection from  $C \times \tilde{X}$  to  $C$ . Denote by  $p_2$  the projection from this product to  $\tilde{X}$ . Now symmetry of  $\tilde{\beta}$  and the condition  $p_{\tilde{X},*}(\tilde{\beta} \cdot [\tilde{X} \times T]) = 0$  imply that

$$p_{2,*}(\tilde{\beta}_C \times [T \times \tilde{X}]) = 0 \in CH^1(\tilde{X}).$$

It follows that

$$p_{2,*}(\tilde{\beta}_C \times [T \times D_m]) = 0 \in CH^1(D_m).$$

In particular, the degree  $a$  of this 0-cycle is zero. But since  $T$  is of degree one, we have

$$p_{1,*}(\tilde{\beta}_C \cdot [C \times D_m]) = a[C] \in CH^0(C).$$

**q.e.d.**

Given that duality  $h(D_m)^\vee \xrightarrow{\sim} h(D_m)(1)$  induces an isomorphism

$$h^0(D_m)^\vee \xrightarrow{\sim} h^2(D_m)(1),$$

it is easy to see that the morphism  $\tilde{i}_*$  dual to the one from condition (2)

$$\bigoplus_m h^0(D_m) \hookrightarrow \bigoplus_m h(D_m) \xrightarrow{\tilde{i}_*} h(\tilde{X})(1) \longrightarrow h^{\geq 3}(\tilde{X})(1)$$

is zero, i.e., the map  $\tilde{i}_* : \bigoplus_m h^0(D_m) \rightarrow h(\tilde{X})(1)$  factors through the sub-motive  $h^{\leq 2}(\tilde{X})(1)$ . On the other hand, by condition (2), the inverse image  $\tilde{i}^* : h(\tilde{X}) \rightarrow \bigoplus_m h^2(D_m)$  factors through the quotient motive  $h^{\geq 2}(\tilde{X})$ . It follows that the composition

$$\alpha = \tilde{i}^* \tilde{i}_* : \bigoplus_m h^0(D_m)(-1) \longrightarrow \bigoplus_m h^2(D_m)$$

considered in Section 2 factors naturally through  $h^2(\tilde{X})$ . By Theorem 2.2 (i), the morphism  $\alpha$  is an isomorphism.

**Definition 3.6.** Define the motive  $h_{!*}^2(\overline{X})$  as the kernel of

$$\tilde{i}_* \alpha^{-1} \tilde{i}^* : h^2(\tilde{X}) \longrightarrow h^2(\tilde{X}).$$

Note that  $\tilde{i}_* \alpha^{-1} \tilde{i}^*$  is an idempotent on  $h^2(\tilde{X})$ ; it therefore admits a kernel. Its image is of course canonically isomorphic (via  $\tilde{i}^*$ ) to  $\bigoplus_m h^2(D_m)$ . Dually, the image of the projector  $\text{id}_{h^2(\tilde{X})} - \tilde{i}_* \alpha^{-1} \tilde{i}^*$  is  $h_{!*}^2(\overline{X})$ . Its kernel is canonically isomorphic (via  $\tilde{i}_*$ ) to  $\bigoplus_m h^0(D_m)(-1)$ .

**Remark 3.7.** In [KMrP, Sect. 14.2.2], the *transcendental part*  $t^2(\tilde{X})$  of the motive of the surface  $\tilde{X}$  is defined, as a complement in  $h^2(\tilde{X})$  of the algebraic, i.e., “Néron–Severi”-part  $h^2(\tilde{X})_{\text{alg}}$ . It follows that under the projection from  $h^2(\tilde{X})$ , the transcendental part  $t^2(\tilde{X})$  maps monomorphically to  $h_{!*}^2(\overline{X})$ .

By condition (2) from Proposition 3.5, the projector  $p = \tilde{i}_* \alpha^{-1} \tilde{i}^*$  on  $h(\tilde{X})$  used to define  $h_{!*}(\overline{X})$  gives rise to compatible factorizations

$$p^{\geq r} := \tilde{i}_* \alpha^{-1} \tilde{i}^* : h^{\geq r}(\tilde{X}) \longrightarrow h^{\geq r}(\tilde{X}), \quad r \leq 2$$

and

$$p^{\leq r} := \tilde{i}_* \alpha^{-1} \tilde{i}^* : h^{\leq r}(\tilde{X}) \longrightarrow h^{\leq r}(\tilde{X}), \quad r \geq 2,$$

all of which are again idempotent. Consequently, we get (split) exact sequences of motives

$$\begin{aligned} 0 &\longrightarrow h^{\leq 1}(\tilde{X}) \longrightarrow \ker(p^{\leq 2}) \longrightarrow h_{\dagger*}^2(\bar{X}) \longrightarrow 0, \\ 0 &\longrightarrow \ker(p^{\leq 2}) \longrightarrow \ker(p^{\leq 3}) \longrightarrow h^3(\tilde{X}) \longrightarrow 0 \end{aligned}$$

etc.

**Theorem 3.8.** (i) *The Künneth filtration of  $h(\tilde{X})$*

$$0 \subset h^0(\tilde{X}) \subset h^{\leq 1}(\tilde{X}) \subset h^{\leq 2}(\tilde{X}) \subset h^{\leq 3}(\tilde{X}) \subset h^{\leq 4}(\tilde{X}) = h(\tilde{X})$$

*induces a filtration of the intersection motive  $h_{\dagger*}(\bar{X})$*

$$0 \subset h_{\dagger*}^0(\bar{X}) \subset h_{\dagger*}^{\leq 1}(\bar{X}) \subset h_{\dagger*}^{\leq 2}(\bar{X}) \subset h_{\dagger*}^{\leq 3}(\bar{X}) \subset h_{\dagger*}^{\leq 4}(\bar{X}) = h_{\dagger*}(\bar{X}).$$

*It is uniquely defined by the following property: both the canonical projection from  $h(\tilde{X})$  to  $h_{\dagger*}(\bar{X})$  and the canonical inclusion of  $h_{\dagger*}(\bar{X})$  into  $h(\tilde{X})$  are morphisms of filtered motives. The filtration is split in the sense that all  $h_{\dagger*}^{\leq r}(\bar{X})$  admit direct complements in  $h_{\dagger*}(\bar{X})$ . In particular, the quotients*

$$h_{\dagger*}^{\geq r}(\bar{X}) := h_{\dagger*}(\bar{X})/h_{\dagger*}^{\leq r-1}(\bar{X})$$

*of  $h_{\dagger*}(\bar{X})$  exist.*

(ii) *The filtration of  $h_{\dagger*}(\bar{X})$  is independent of the choice of desingularization  $\tilde{X}$ .*

(iii) *Duality  $h_{\dagger*}(\bar{X})^\vee \xrightarrow{\sim} h_{\dagger*}(\bar{X})(2)$  (Proposition 2.6) induces isomorphisms*

$$h_{\dagger*}^{\leq r}(\bar{X})^\vee \xrightarrow{\sim} h_{\dagger*}^{\geq 4-r}(\bar{X})(2).$$

*Proof.* Define

$$h_{\dagger*}^{\leq r}(\bar{X}) := h^{\leq r}(\tilde{X}) \quad \text{for } r \leq 1$$

and

$$h_{\dagger*}^{\leq r}(\bar{X}) := \ker(p^{\leq r}) \quad \text{for } r \geq 2.$$

Claim (i) is a consequence of the compatibility of the idempotents  $p^{\leq r}$ , (ii) is a consequence of Proposition 2.5 (iv), and (iii) follows from symmetry of  $p$ . **q.e.d.**

**Definition 3.9.** (a) *The filtration*

$$0 \subset h_{\dagger*}^0(\bar{X}) \subset h_{\dagger*}^{\leq 1}(\bar{X}) \subset h_{\dagger*}^{\leq 2}(\bar{X}) \subset h_{\dagger*}^{\leq 3}(\bar{X}) \subset h_{\dagger*}^{\leq 4}(\bar{X}) = h_{\dagger*}(\bar{X}).$$

*from Theorem 3.8 is called the Künneth filtration of  $h_{\dagger*}(\bar{X})$ .*

(b) *The  $n$ -th Künneth component of  $h_{\dagger*}(\bar{X})$ ,  $0 \leq n \leq 4$ , is the sub-quotient of  $h_{\dagger*}(\bar{X})$  defined by*

$$h_{\dagger*}^n(\bar{X}) := h_{\dagger*}^{\leq n}(\bar{X})/h_{\dagger*}^{\leq n-1}(\bar{X}).$$

For future reference, let us note the following immediate consequence of our construction.

**Proposition 3.10.** *Let  $n$  be an integer unequal to two. Then there is a canonical isomorphism of motives*

$$h_{l_*}^n(\overline{X}) \xrightarrow{\sim} h^n(\tilde{X}) .$$

**Remark 3.11.** One may define the notion of Künneth decomposition of the intersection motive as being a decomposition splitting the Künneth filtration. Adding the complement  $\oplus_m h^2(D_m)$  of  $h_{l_*}(\overline{X})$  in  $h(\tilde{X})$ , one gets a Künneth decomposition of  $h(\tilde{X})$  (in the abstract sense of Remark 3.4 (b)). With these choices, both the canonical projection from  $h(\tilde{X})$  to  $h_{l_*}(\overline{X})$  and the canonical inclusion of  $h_{l_*}(\overline{X})$  into  $h(\tilde{X})$  are morphisms of graded motives. It is not clear to me whether such Künneth decompositions of  $h(\tilde{X})$  can be obtained using Murre's construction recalled earlier, when  $D$  has more than one component. The problem is the relation

$$p_{\tilde{X},*}(p_3 \cdot [\tilde{X} \times D_m]) = 0$$

(we use the same notation as in the proof of Proposition 3.5). The cycle class in question is a non-zero multiple of

$$p_{\tilde{X},*}(\tilde{\beta} \cdot [\tilde{X} \times C \cdot D_m]) .$$

For any fixed  $m$ , the Künneth decomposition of  $h(\tilde{X})$  can be *chosen* such that this cycle class vanishes: take  $T$  to be equal to  $\frac{1}{d}[C \cdot D_m]$ , where  $d$  is the degree of  $C \cdot D_m$ .

## 4 Hard Lefschetz for the intersection motive

We continue to consider a proper surface  $\overline{X}$  over the base field  $k$ . Let us consider the Künneth filtration

$$0 \subset h_{l_*}^0(\overline{X}) \subset h_{l_*}^{\leq 1}(\overline{X}) \subset h_{l_*}^{\leq 2}(\overline{X}) \subset h_{l_*}^{\leq 3}(\overline{X}) \subset h_{l_*}^{\leq 4}(\overline{X}) = h(\overline{X})_{l_*}$$

of the intersection motive. The aim of this section is to prove the following.

**Theorem 4.1.** *Let  $\mathcal{L}$  be a line bundle on  $\overline{X}$ .*

(i) *There is a morphism of motives*

$$c_{\mathcal{L}} : h_{l_*}(\overline{X})(-1) \longrightarrow h_{l_*}(\overline{X}) ,$$

*which is uniquely characterized by the following two properties:*

- (1) *If  $\overline{X}$  is smooth, then  $c_{\mathcal{L}}$  equals the cup-product with the first Chern class of  $\mathcal{L}$  on  $h(\overline{X})(-1) = h_{l_*}(\overline{X})(-1)$  [S, Sect. 2.1].*

(2) The morphism  $c_{\mathcal{L}}$  is contravariantly functorial with respect to dominant morphisms  $g : \overline{Y} \rightarrow \overline{X}$  of proper surfaces over  $k$ : the diagram

$$\begin{array}{ccc} h_{1*}(\overline{Y})(-1) & \xrightarrow{c_{g^*\mathcal{L}}} & h_{1*}(\overline{Y}) \\ g^*(-1)\uparrow & & \uparrow g^* \\ h_{1*}(\overline{X})(-1) & \xrightarrow{c_{\mathcal{L}}} & h_{1*}(\overline{X}) \end{array}$$

(see Proposition 2.5 (i)) commutes.

(ii) If  $\mathcal{L}'$  is a second line bundle on  $\overline{X}$ , then

$$c_{\mathcal{L} \otimes \mathcal{L}'} = c_{\mathcal{L}} + c_{\mathcal{L}'} .$$

In other words, the map

$$\mathrm{Pic}(\overline{X}) \longrightarrow \mathrm{Hom}(h_{1*}(\overline{X})(-1), h_{1*}(\overline{X})) , \mathcal{L} \longmapsto c_{\mathcal{L}}$$

is a morphism of groups.

(iii) The morphism  $c_{\mathcal{L}}$  is filtered in the following sense: it induces morphisms

$$c_{\mathcal{L}} : h_{1*}^{\leq n-2}(\overline{X})(-1) \longrightarrow h_{1*}^{\leq n}(\overline{X})$$

and hence, morphisms

$$c_{\mathcal{L}} : h_{1*}^{n-2}(\overline{X})(-1) \longrightarrow h_{1*}^n(\overline{X})$$

for all  $n \in \mathbb{Z}$ .

(iv) If ( $\overline{X}$  is projective and)  $\mathcal{L}$  or  $\mathcal{L}^{-1}$  is ample, then

$$c_{\mathcal{L}}^2 = c_{\mathcal{L}} \circ c_{\mathcal{L}} : h_{1*}^0(\overline{X})(-2) \longrightarrow h_{1*}^4(\overline{X})$$

and

$$c_{\mathcal{L}} : h_{1*}^1(\overline{X})(-1) \longrightarrow h_{1*}^3(\overline{X})$$

are isomorphisms.

Part (iv) of this result should be seen as the motivic analogue of the Hard Lefschetz Theorem for intersection cohomology [BBD, Thm. 6.2.10].

In order to prepare the proof of Theorem 4.1, let us recall the ingredients of the proof when  $\overline{X}$  is smooth (in which case Theorem 4.1 is of course known). The morphism  $c_{\mathcal{L}}$  then equals the cup-product with the first Chern class, which can be described as follows. Push-forward via the diagonal  $\Delta$  of a class in  $CH^1(\overline{X})$  gives a (symmetric) class in  $CH^3(\overline{X} \times_k \overline{X})$ . In the category  $CHM(k)$ , this vector space equals the group of morphisms from  $h(\overline{X})(-1)$  to  $h(\overline{X})$ . We define  $c_{\mathcal{L}} := \Delta_*([\mathcal{L}])$ . From this description, properties (i) (2) (for smooth  $\overline{Y}$ ) and (ii) are immediate. Recall that  $\overline{X}$ , as a smooth and proper surface, is projective. Since the group  $\mathrm{Pic}(\overline{X})$  is generated by the classes of very ample line bundles, in order to prove (iii) and (iv), we may (by (ii)) assume that  $\mathcal{L}$  is very ample. In addition, we may prove the claims after base

change to a finite extension of  $k$ , and hence assume that  $\overline{X}$  is geometrically connected, and that  $\mathcal{L}$  is represented by a smooth curve  $C$  embedded into  $\overline{X}$  via the closed immersion  $i_C$ . The morphism  $c_{\mathcal{L}}$  then equals the composition of

$$i_C^*(-1) : h(\overline{X})(-1) \longrightarrow h(C)(-1)$$

and of

$$i_{C,*} : h(C)(-1) \longrightarrow h(\overline{X}) .$$

By auto-duality of the Künneth filtrations for  $C$  and for  $\overline{X}$ , it suffices for (iii) to show that  $i_C^* : h(\overline{X}) \rightarrow h(C)$  is a morphism of filtered motives. But this follows from [Mr3, Lemma 8.3.2] and [J, Prop. 5.8]. As for (iv), observe that identifying  $h^0(\tilde{X})(-2)$  and  $h^4(\tilde{X})$  with  $\mathbb{Q}(-2)$  allows to relate the morphism  $c_{\mathcal{L}}^2 : h^0(\tilde{X})(-2) \rightarrow h^4(\tilde{X})$  to the self-intersection number  $C \cdot C$ , which is strictly positive since  $\mathcal{L}$  is very ample. We may quote [S, Thm. 4.4 (ii)] for  $c_{\mathcal{L}} : h^1(\tilde{X})(-1) \rightarrow h^3(\tilde{X})$ .

Given the contravariance property of the intersection motive (Proposition 2.5 (i)), it is now clear what remains to be done in order to prove Theorem 4.1 in the generality we stated it. First note that in our statement, we may replace  $\overline{X}$  by its normalization  $X^*$ . Indeed,  $h_{l*}(\overline{X}) = h_{l*}(X^*)$ , and the morphism  $X^* \rightarrow \overline{X}$  being finite, the pull-back of an ample line bundle on  $\overline{X}$  is ample on  $X^*$ . Next, fix a cartesian diagram

$$\begin{array}{ccccc} X & \hookrightarrow & \tilde{X} & \xleftarrow{i} & D \\ \parallel & & \pi \downarrow & & \downarrow \pi \\ X & \hookrightarrow & X^* & \longleftarrow & Z \end{array}$$

which is a desingularization of  $X^*$ . Thus,  $\pi$  is proper,  $\tilde{X}$  is smooth and proper (hence projective),  $Z$  is finite, and  $D$  a divisor with normal crossings, whose irreducible components  $D_m$  are smooth. We need to show that for any line bundle  $\mathcal{L}$  on  $X^*$ , the composition

$$h_{l*}(\overline{X})(-1) \hookrightarrow h(\tilde{X})(-1) \xrightarrow{c_{\pi^*\mathcal{L}}} h(\tilde{X})$$

lands in  $h_{l*}(\overline{X}) \subset h(\tilde{X})$  — this will then be our definition of  $c_{\mathcal{L}}$  — and that we have the Hard Lefschetz Theorem 4.1 (iv). In fact, we shall prove a more general result.

**Variant 4.2.** *Let  $\tilde{\mathcal{L}}$  be a line bundle on  $\tilde{X}$ , whose restrictions to all  $D_m$  are trivial (for example, the pull-back of a line bundle on  $X^*$ ).*

(i) *The restriction of the morphism of motives*

$$c_{\tilde{\mathcal{L}}} : h(\tilde{X})(-1) \longrightarrow h(\tilde{X})$$

to the sub-motive  $h_{!*}(\overline{X})(-1)$  induces a morphism  $h_{!*}(\overline{X})(-1) \rightarrow h_{!*}(\overline{X})$ . In other words, there is a commutative diagram

$$\begin{array}{ccc} h(\tilde{X})(-1) & \xrightarrow{c_{\tilde{\mathcal{L}}}} & h(\tilde{X}) \\ \pi^*(-1) \uparrow & & \uparrow \pi^* \\ h_{!*}(\overline{X})(-1) & \xrightarrow{c_{\tilde{\mathcal{L}}}} & h_{!*}(\overline{X}) \end{array}$$

(ii) If  $\tilde{\mathcal{L}}'$  is a second line bundle on  $\tilde{X}$  with trivial restrictions to all  $D_m$ , then

$$c_{\tilde{\mathcal{L}} \otimes \tilde{\mathcal{L}}'} = c_{\tilde{\mathcal{L}}} + c_{\tilde{\mathcal{L}}'} .$$

(iii) The morphism  $c_{\tilde{\mathcal{L}}}$  is filtered: it induces morphisms

$$c_{\tilde{\mathcal{L}}} : h_{!*}^{\leq n-2}(\overline{X})(-1) \longrightarrow h_{!*}^{\leq n}(\overline{X})$$

for all  $n \in \mathbb{Z}$ .

(iv) Assume that  $\tilde{\mathcal{L}}$  is the line bundle associated to a divisor  $C$  on  $\tilde{X}$  such that  $C - \sum_m a_m D_m$  or  $-C - \sum_m a_m D_m$  is ample for a suitable choice of integers  $a_m \geq 0$  (for example,  $\tilde{\mathcal{L}} = \pi^* \mathcal{L}$  for an ample line bundle  $\mathcal{L}$  on  $X^*$ ). Then

$$c_{\tilde{\mathcal{L}}}^2 : h_{!*}^0(\overline{X})(-2) \longrightarrow h_{!*}^4(\overline{X})$$

and

$$c_{\tilde{\mathcal{L}}} : h_{!*}^1(\overline{X})(-1) \longrightarrow h_{!*}^3(\overline{X})$$

are isomorphisms.

*Proof.* In order to prove (i), we have to check that the composition

$$h_{!*}(\overline{X})(-1) \xrightarrow{\pi^*(-1)} h(\tilde{X})(-1) \xrightarrow{c_{\tilde{\mathcal{L}}}} h(\tilde{X}) \xrightarrow{\tilde{i}_* \alpha^{-1} \tilde{i}^*} h(\tilde{X})$$

is zero. Since the formation of Chern classes is compatible with pull-backs, the composition  $\tilde{i}^* c_{\tilde{\mathcal{L}}}$  equals

$$h(\tilde{X})(-1) \xrightarrow{\oplus_m i_m^*} \bigoplus_m h(D_m)(-1) \xrightarrow{\oplus_m c_{i_m^* \tilde{\mathcal{L}}}} \bigoplus_m h(D_m) \longrightarrow \bigoplus_m h^2(D_m) ,$$

where  $i_m$  denotes the immersion of  $D_m$  into  $\tilde{X}$ . But by assumption, the morphisms  $c_{i_m^* \tilde{\mathcal{L}}} : h(D_m)(-1) \rightarrow h(D_m)$  are all zero.

Claims (ii) and (iii) hold since they hold for  $c_{\tilde{\mathcal{L}}} : h(\tilde{X})(-1) \rightarrow h(\tilde{X})$ .

As for (iv), observe that according to Proposition 3.10,

$$h_{!*}^n(\overline{X}) \cong h^n(\tilde{X}) , \quad n \neq 2 .$$

Thus, we have to prove that

$$c_{\tilde{\mathcal{L}}}^2 : h^0(\tilde{X})(-2) \longrightarrow h^4(\tilde{X})$$

and

$$c_{\tilde{\mathcal{L}}} : h^1(\tilde{X})(-1) \longrightarrow h^3(\tilde{X})$$

are isomorphisms. As before, the claim for  $c_{\tilde{\mathcal{L}}}^2$  is essentially equivalent to showing that the self-intersection number  $C \cdot \tilde{C}$  is non-zero. Since the restriction of  $\tilde{\mathcal{L}}$  to any of the  $D_m$  is trivial, we have the formula

$$C \cdot C = (\pm C - \sum_m a_m D_m) \cdot (\pm C - \sum_m a_m D_m) - (\sum_m a_m D_m) \cdot (\sum_m a_m D_m).$$

The intersection matrix  $(D_n \cdot D_m)_{n,m}$  is negative definite [Mm, p. 6], hence the matrix  $((a_n D_n) \cdot (a_m D_m))_{n,m}$  is negative semi-definite. It follows that the term  $(\sum_m a_m D_m) \cdot (\sum_m a_m D_m)$  is non-positive. Hence

$$C \cdot C \geq (\pm C - \sum_m a_m D_m) \cdot (\pm C - \sum_m a_m D_m).$$

But one of the divisors  $C - \sum_m a_m D_m$ ,  $-C - \sum_m a_m D_m$  is ample. Therefore, its self-intersection number is strictly positive.

In order to prove the claim for  $c_{\tilde{\mathcal{L}}} : h^1(\tilde{X})(-1) \rightarrow h^3(\tilde{X})$ , observe first that by (ii), we may assume  $C - \sum_m a_m D_m$  to be very ample. By passing to a finite extension of  $k$ , we find a smooth curve  $H$  embedded into  $\tilde{X}$  via the closed immersion  $i_H$ , and such that there is an equivalence of divisors

$$C - \sum_m a_m D_m \sim H.$$

In particular,  $H$  is very ample, and

$$c_{\tilde{\mathcal{L}}} = i_{H,*} i_H^* + \sum_m a_m i_{m,*} i_m^* : h^1(\tilde{X})(-1) \longrightarrow h^3(\tilde{X}).$$

Hard Lefschetz 4.1 (iv) tells us that  $i_{H,*} i_H^*$  is an isomorphism. In order to see that the same still holds after adding the “error term”  $\sum_m a_m i_{m,*} i_m^*$ , we need to recall more details of the proof.

In fact, as follows from [S, Prop. 4.5], the full sub-category of motives isomorphic to  $h^1(Y)$ , for smooth projective varieties  $Y$  over  $k$ , is equivalent to the category of Abelian varieties over  $k$  up to isogeny. More precisely, this equivalence is such that  $h^1(Y)$  corresponds to the Picard variety  $P_Y$ , and that the motive  $h^{2d_Y-1}(d_Y - 1)$  (for  $Y$  of pure dimension  $d_Y$ ) corresponds to the Albanese variety  $A_Y$ . Furthermore, for a morphism  $f : Y_1 \rightarrow Y_2$ , the pull-back of motives  $f^* : h^1(Y_2) \rightarrow h^1(Y_1)$  corresponds to  $f^* : P_{Y_2} \rightarrow P_{Y_1}$ , while the push-forward  $f_* : h^{2d_{Y_1}-1}(d_{Y_1} - 1) \rightarrow h^{2d_{Y_2}-1}(d_{Y_2} - 1)$  (for  $Y_i$  of pure dimension  $d_{Y_i}$ ,  $i = 1, 2$ ) corresponds to  $f_* : A_{Y_1} \rightarrow A_{Y_2}$ . Proving that  $c_{\tilde{\mathcal{L}}}$  is an isomorphism of motives is thus equivalent to proving the following statement: the composition of

$$I^* : P_{\tilde{X}} \longrightarrow P_H \times_k \prod_m (P_{D_m})^{a_m}$$

with its dual

$$I_* : A_H \times_k \prod_m (A_{D_m})^{a_m} \longrightarrow A_{\tilde{X}}$$

is an isogeny from the Picard variety of  $\tilde{X}$  to the Albanese variety of  $\tilde{X}$ . Here,  $I$  denotes the morphism from the disjoint union of  $H$  and  $a_m$  copies of  $D_m$ , for all  $m$ , to  $\tilde{X}$ . Also, we have identified the Picard and the Albanese varieties of the curves  $H$  and  $D_m$  to the respective Jacobians, using the fact that these are canonically principally polarized.

The decisive ingredient of the proof is [We, Cor. 1 of Thm. 7], which states that since  $H$  is very ample, the kernel of  $i_H^* : P_{\tilde{X}} \rightarrow P_H$  is finite. The same is thus true for  $I^*$ . Now observe that a polarization on an Abelian variety (such as  $P_H \times_k \prod_m (P_{D_m})^{a_m}$ ) induces a polarization on any sub-Abelian variety. The composition  $I_* I^*$  is therefore an isogeny. **q.e.d.**

## 5 The motive of the exceptional divisor

At this point, we need to enlarge the category of motives we are working in since we wish to consider motives of genuinely singular varieties. Let us first set up the notation, which follows that of [V1]. From now on, our base field  $k$  is assumed to be perfect. We write  $Sch/k$  for the category of schemes which are separated and of finite type over  $k$ , and  $Sm/k$  for the full sub-category of objects of  $Sch/k$  which are smooth over  $k$ . Recall the definition of the category  $SmCor(k)$  [V1, p. 190]: its objects are those of  $Sm/k$ . Morphisms from  $Y$  to  $X$  are given by the group  $c(Y, X)$  of *finite correspondences* from  $Y$  to  $X$ . The category  $Shv_{Nis}(SmCor(k))$  of *Nisnevich sheaves with transfers* [V1, Def. 3.1.1] is the category of those contravariant additive functors from  $SmCor(k)$  to Abelian groups, whose restriction to  $Sm/k$  is a sheaf for the Nisnevich topology. Inside the derived category  $D^-(Shv_{Nis}(SmCor(k)))$  of complexes bounded from above, one defines the full triangulated sub-category  $DM_-^{eff}(k)$  of *effective motivic complexes* over  $k$  [V1, p. 205, Prop. 3.1.13] as the one consisting of objects whose cohomology sheaves are *homotopy invariant* [V1, Def. 3.1.10]. The inclusion of  $DM_-^{eff}(k)$  into  $D^-(Shv_{Nis}(SmCor(k)))$  admits a left adjoint  $\mathbf{RC}$ , which is induced from the functor

$$\underline{C}_* : Shv_{Nis}(SmCor(k)) \longrightarrow C^-(Shv_{Nis}(SmCor(k)))$$

which maps  $F$  to the simple complex associated to the *singular simplicial complex* [V1, p. 207, Prop. 3.2.3]. One defines a functor  $L$  from  $Sch/k$  to  $Shv_{Nis}(SmCor(k))$ : it associates to  $X$  the Nisnevich sheaf with transfers  $c(\bullet, X)$ ; note that the above definition of  $c(Y, X)$  still makes sense when  $X \in Sch/k$  is not necessarily smooth. One defines the *motive*  $M(X)$  as  $\mathbf{RC}(L(X))$ . We shall use the same symbol for  $M(X) \in DM_-^{eff}(k)$  and for

its canonical representative  $\underline{C}_*(L(X))$  in  $C^-(Shv_{Nis}(SmCor(k)))$ . There is a second functor  $L^c$ , which associates to  $X \in Sch/k$  the Nisnevich sheaf of quasi-finite correspondences [V1, p. 223, 224]. One defines the *motive with compact support*  $M^c(X)$  of  $X \in Sch/k$  as  $\mathbf{RC}(L^c(X))$ . It coincides with  $M(X)$  if  $X$  is proper.

A second, more geometric approach to motives is the one developed in [V1, Sect. 2.1].<sup>3</sup> There, the triangulated category  $DM_{gm}^{eff}(k)$  of *effective geometrical motives* over  $k$  is defined. There is a canonical full triangulated embedding of  $DM_{gm}^{eff}(k)$  into  $DM_-^{eff}(k)$  [V1, Thm. 3.2.6], which maps the geometrical motive of  $X \in Sm/k$  [V1, Def. 2.1.1] to  $M(X)$ . Using this embedding, we consider  $M(X)$  as an object of  $DM_{gm}^{eff}(k)$ . The *Tate motive*  $\mathbb{Z}(1)$  in  $DM_{gm}^{eff}(k)$  is defined as the *reduced motive* of  $\mathbb{P}_k^1$  [V1, p. 192], shifted by  $-2$ . There is a canonical direct sum decomposition

$$M(\mathbb{P}_k^1) = \mathbb{Z}(0) \oplus \mathbb{Z}(1)[2] .$$

The category  $DM_{gm}(k)$  of *geometrical motives* over  $k$  is obtained from the category  $DM_{gm}^{eff}(k)$  by inverting  $\mathbb{Z}(1)$ . All categories  $DM_{gm}^{eff}(k)$ ,  $DM_{gm}(k)$ ,  $D^-(Shv_{Nis}(SmCor(k)))$ , and  $DM_-^{eff}(k)$  are tensor triangulated, and admit unit objects, which we denote by the same symbol  $\mathbb{Z}(0)$  [V1, Prop. 2.1.3, Cor. 2.1.5, p. 206, Thm. 3.2.6]. For  $M \in DM_{gm}(k)$  and  $n \in \mathbb{Z}$ , write  $M(n)$  for the tensor product  $M \otimes \mathbb{Z}(n)$ . According to [V3], the functor  $DM_{gm}^{eff}(k) \rightarrow DM_{gm}(k)$  is a full triangulated embedding (see [V1, Thm. 4.3.1] for a proof when  $k$  admits resolution of singularities).

Let us denote by  $DM_{gm}^{eff}(k)_{\mathbb{Q}}$  and  $DM_{gm}(k)_{\mathbb{Q}}$  the tensor product with  $\mathbb{Q}$  of the categories  $DM_{gm}^{eff}(k)$  and  $DM_{gm}(k)$ , respectively. The relation to Chow motives is given by the following result due to Voevodsky.

**Theorem 5.1.** (i) *There is a natural contravariant  $\mathbb{Q}$ -linear tensor functor*

$$R : CHM(k) \longrightarrow DM_{gm}(k)_{\mathbb{Q}} .$$

*R is fully faithful.*

(ii) *For any smooth projective variety  $S$  over  $k$ , the functor  $R$  maps the Chow motive  $h(S)$  to the motive  $M(S) \in DM_{gm}^{eff}(k)_{\mathbb{Q}} \subset DM_{gm}(k)_{\mathbb{Q}}$ .*

(iii) *The functor  $R$  maps the Lefschetz motive  $\mathbb{L} = \mathbb{Q}(-1)$  to the motive  $\mathbb{Z}(1)[2]$ , compatibly with the decompositions*

$$h(\mathbb{P}_k^1) = \mathbb{Q}(0) \oplus \mathbb{Q}(-1)$$

---

<sup>3</sup>In fact, this is the approach which will be almost exclusively used in this and the next section. The approach using Nisnevich sheaves described above occurs only when we verify that the motives of certain singular varieties do belong to the category  $DM_{gm}^{eff}(k)$ .

in  $CHM(k)$  and

$$M(\mathbb{P}_k^1) = \mathbb{Z}(0) \oplus \mathbb{Z}(1)[2]$$

in  $DM_{gm}^{eff}(k)$ .

*Proof.* The essential point of the proof is to show equality of morphisms:

$$\mathrm{Hom}_{CHM(k)}(h(Y)(-q), h(X)) = \mathrm{Hom}_{DM_{gm}(k)_{\mathbb{Q}}}(M(X), M(Y)(q)[2q])$$

for smooth projective varieties  $X$  and  $Y$  over  $k$  and  $q \geq 0$ . Duality in  $DM_{gm}(k)_{\mathbb{Q}}$  [A, Thm. 18.4.1.1] ([V1, Thm. 4.3.7] if  $k$  admits resolution of singularities) allows us to reduce to the case  $Y = \mathbf{Spec} k$ , in which case the claim follows from [V2, Cor. 2]. **q.e.d.**

**Example 5.2.** Fix a proper surface  $\bar{X}$  over  $k$ . Recall the Künneth filtration of the intersection motive

$$0 \subset h_{!*}^0(\bar{X}) \subset h_{!*}^{\leq 1}(\bar{X}) \subset h_{!*}^{\leq 2}(\bar{X}) \subset h_{!*}^{\leq 3}(\bar{X}) \subset h_{!*}^{\leq 4}(\bar{X}) = h_{!*}(\bar{X}),$$

the quotients

$$h_{!*}^{\geq r}(\bar{X}) := h_{!*}(\bar{X})/h_{!*}^{\leq r-1}(\bar{X}),$$

and the Künneth components

$$h_{!*}^n(\bar{X}) = h_{!*}^{\leq n}(\bar{X})/h_{!*}^{\leq n-1}(\bar{X})$$

(Definition 3.9). Let us write  $M^{!*}(\bar{X}) := R(h_{!*}(\bar{X}))$ ,

$$M_{\geq r}^{!*}(\bar{X}) := R(h_{!*}^{\geq r}(\bar{X})),$$

$$M_{\leq n}^{!*}(\bar{X}) := R(h_{!*}^{\leq n}(\bar{X})),$$

$$M_n^{!*}(\bar{X}) := R(h_{!*}^n(\bar{X})).$$

We thus have exact triangles

$$M_{\geq r+1}^{!*}(\bar{X}) \longrightarrow M^{!*}(\bar{X}) \longrightarrow M_{\leq r}^{!*}(\bar{X}) \xrightarrow{\delta} M_{\geq r+1}^{!*}(\bar{X})[1],$$

$$M_n^{!*}(\bar{X}) \longrightarrow M_{\leq n}^{!*}(\bar{X}) \longrightarrow M_{\leq n-1}^{!*}(\bar{X}) \xrightarrow{\delta} M_n^{!*}(\bar{X})[1]$$

in  $DM_{gm}^{eff}(k)_{\mathbb{Q}}$ , which are all split in the sense that the boundaries  $\delta$  are zero.

For the rest of this section, fix a (not necessarily proper) surface  $\bar{X}$  over  $k$ , and a cartesian diagram

$$\begin{array}{ccccc} X & \hookrightarrow & \tilde{X} & \longleftarrow & D \\ \parallel & & \pi \downarrow & & \downarrow \pi \\ X & \hookrightarrow & X^* & \longleftarrow & Z \end{array}$$

which is a desingularization of the normalization  $X^*$ . Thus,  $\pi$  is proper,  $\tilde{X}$  is smooth,  $Z$  is finite, and  $D$  a divisor with normal crossings, whose irreducible components  $D_m$  are smooth projective curves. The exact triangle associated to the closed covering of  $D$  by the  $D_m$  [V1, Prop. 4.1.3] shows that  $M(D)$  belongs to the category  $DM_{gm}^{eff}(k)$ .

**Definition 5.3.** Define Chow motives  $h^0(D)$  and  $h^2(D)$  as follows.

- (a)  $h^0(D) := h(S)$ , where  $S$  equals the spectrum of the ring of global sections of the structure sheaf of  $D$ .
- (b)  $h^2(D) := \bigoplus_m h^2(D_m)$ .

Let us write  $M_0(D) := R(h^0(D))$  and  $M_2(D) := R(h^2(D))$ . The morphism  $D \rightarrow S$  and the inclusions  $i_m$  of the components  $D_m$  into  $D$  induce morphisms  $M(D) \rightarrow M_0(D)$  and  $M_2(D) \rightarrow M(D)$  in  $DM_{gm}^{eff}(k)_{\mathbb{Q}}$ .

**Lemma 5.4.** *The morphism  $M(D) \rightarrow M_0(D)$  is a split epimorphism, and  $M_2(D) \rightarrow M(D)$  is a split monomorphism. The composition of the two morphisms  $M_2(D) \rightarrow M(D) \rightarrow M_0(D)$  is trivial.*

*Proof.* The composition

$$\bigoplus_m R(h^0(D_m)) \longrightarrow \bigoplus_m R(h^2(D_m)) = \bigoplus_m M(D_m) \longrightarrow M(D) \longrightarrow M_0(D)$$

is a split epimorphism, hence so is  $M(D) \rightarrow M_0(D)$ . The composition

$$M_2(D) \longrightarrow M(D) \longrightarrow M(\tilde{X})$$

is a split monomorphism (Theorem 2.2 (i)), hence so is  $M_2(D) \rightarrow M(D)$ . The last claim is obvious. **q.e.d.**

It follows that the objects

$$M_{\geq 1}(D) := \ker(M(D) \rightarrow M_0(D)) ,$$

$$M_{\leq 1}(D) := M(D)/M_2(D) ,$$

and

$$M_1(D) := \ker(M_{\leq 1}(D) \rightarrow M_0(D)) = M_{\geq 1}(D)/M_2(D)$$

exist. They give rise to what we might call the Künneth filtration of  $M(D)$ :

$$M(D) =: M_{\leq 2}(D) \twoheadrightarrow M_{\leq 1}(D) \twoheadrightarrow M_0(D) ,$$

$$M_2(D) \hookrightarrow M_{\geq 1}(D) \hookrightarrow M_{\geq 0}(D) := M(D) .$$

Note that there are split exact triangles

$$M_2(D) \longrightarrow M(D) \longrightarrow M_{\leq 1}(D) \xrightarrow{\delta=0} M_2(D)[1] ,$$

$$M_1(D) \longrightarrow M_{\leq 1}(D) \longrightarrow M_0(D) \xrightarrow{\delta=0} M_1(D)[1]$$

in  $DM_{gm}^{eff}(k)_{\mathbb{Q}}$ .

**Remark 5.5.** Unlike  $M_0(D)$  and  $M_2(D)$ , the sub-quotient  $M_1(D)$  should not in general be expected to come from a Chow motive. Indeed, as we shall see, the “kernel” of

$$\bigoplus_{n < m} M(D_n \cap D_m)[1] \longrightarrow \bigoplus_m M_0(D_m)[1]$$

contributes to  $M_1(D)$ .

## 6 An extension of motives

We continue to study the situation

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{j}} & \tilde{X} & \xleftarrow{\tilde{i}} & D \\ \parallel & & \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{j} & X^* & \xleftarrow{\quad} & Z \end{array}$$

fixed in Section 5, but assume in addition that the surface  $\overline{X}$  is proper. The morphism  $\tilde{i}_* : M(D) \rightarrow M(\tilde{X})$  will be at the base of the construction of extensions of two Chow motives in  $DM_{gm}^{eff}(k)_{\mathbb{Q}}$  (Theorem 6.6). One of the motives is  $M_0^*(\overline{X})(1)$ , and the other one is  $M_1(D)[-1]$ . Let us start with a number of elementary observations.

**Lemma 6.1.** *The composition*

$$M(D) \xrightarrow{\tilde{i}_*} M(\tilde{X}) \twoheadrightarrow M^{!*}(\overline{X})$$

*factors uniquely through a morphism  $\tilde{i}_* : M_{\leq 1}(D) \rightarrow M^{!*}(\overline{X})$ .*

*Proof.* We identify  $M^{!*}(\overline{X})$  with the categorical quotient of  $M(\tilde{X})$  by  $M_2(D)$ . The composition in question thus vanishes on  $M_2(D)$ . It therefore factors uniquely over the categorical quotient  $M_{\leq 1}(D)$  of  $M(D)$  by  $M_2(D)$ .

**q.e.d.**

**Remark 6.2.** If  $k$  admits resolution of singularities, then we have *localization* for the motive with compact support [V1, Prop. 4.1.5]. In our situation, this means that there is a canonical exact triangle

$$M(D) \xrightarrow{\tilde{i}_*} M(\tilde{X}) \xrightarrow{\tilde{j}^*} M^c(X) \longrightarrow M(D)[1].$$

From this, one deduces easily that  $\tilde{i}_* : M_{\leq 1}(D) \rightarrow M^{!*}(\overline{X})$  sits in an exact triangle

$$M_{\leq 1}(D) \xrightarrow{\tilde{i}_*} M^{!*}(\overline{X}) \xrightarrow{j^*} M^c(X) \longrightarrow M_{\leq 1}(D)[1].$$

In particular, the cone of  $\tilde{i}_*$  is independent of the choice of  $\tilde{X}$ , as is its target  $M^{!*}(\bar{X})$ .

Consider the sub-object  $M_1(D)$  of  $M_{\leq 1}(D)$ , and the quotient  $M_0^{!*}(\bar{X})$  of  $M^{!*}(\bar{X})$ .

**Lemma 6.3.** *The composition*

$$M_1(D) \hookrightarrow M_{\leq 1}(D) \xrightarrow{\tilde{i}_*} M^{!*}(\bar{X}) \twoheadrightarrow M_0^{!*}(\bar{X})$$

is trivial.

*Proof.* The motive  $M_0^{!*}(\bar{X})$  equals  $M_0(\tilde{X}) := R(h^0(\tilde{X}))$  (Proposition 3.10), hence the composition

$$M_{\leq 1}(D) \xrightarrow{\tilde{i}_*} M^{!*}(\bar{X}) \twoheadrightarrow M_0^{!*}(\bar{X})$$

equals the composition

$$M_{\leq 1}(D) \twoheadrightarrow M_0(D) \xrightarrow{\tilde{i}_*} M_0(\tilde{X}).$$

It is therefore trivial on  $M_1(D)$ .

**q.e.d.**

In other words, the morphism  $\tilde{i}_* : M_{\leq 1}(D) \rightarrow M^{!*}(\bar{X})$  respects the Künneth filtrations. It induces a morphism  $\tilde{i}_* : M_1(D) \rightarrow M_{\geq 1}^{!*}(\bar{X})$ . Consider the quotient  $M_1^{!*}(\bar{X})$  of  $M_{\geq 1}^{!*}(\bar{X})$ .

**Proposition 6.4.** *Assume that all geometric irreducible components of  $D$  are of genus zero.*

(i) *The object  $M_1(D)[-1]$  of  $DM_{gm}^{eff}(k)_{\mathbb{Q}}$  is an Artin motive, i.e., it is isomorphic to the motive of some zero-dimensional variety over  $k$ . More precisely, there is a canonical exact sequence of Artin motives*

$$0 \longrightarrow M_1(D)[-1] \longrightarrow \bigoplus_{n < m} M(D_n \cap D_m) \longrightarrow \bigoplus_m M_0(D_m).$$

(ii) *The composition*

$$M_1(D) \xrightarrow{\tilde{i}_*} M_{\geq 1}^{!*}(\bar{X}) \twoheadrightarrow M_1^{!*}(\bar{X})$$

is trivial.

*Proof.* (i) Consider the closed covering of  $D$  by the  $D_m$ . It induces an exact triangle

$$\bigoplus_{n < m} M(D_n \cap D_m) \longrightarrow \bigoplus_m M(D_m) \longrightarrow M(D) \longrightarrow \bigoplus_{n < m} M(D_n \cap D_m)[1]$$

[V1, Prop. 4.1.3]. Given the definition of  $M_2$ , we get an exact triangle

$$\bigoplus_{n < m} M(D_n \cap D_m) \longrightarrow \bigoplus_m M_{\leq 1}(D_m) \longrightarrow M_{\leq 1}(D) \longrightarrow \bigoplus_{n < m} M(D_n \cap D_m)[1].$$

But the  $M_1(D_m)$  are zero by assumption. Hence the exact triangle takes the form

$$\bigoplus_{n < m} M(D_n \cap D_m) \longrightarrow \bigoplus_m M_0(D_m) \longrightarrow M_{\leq 1}(D) \longrightarrow \bigoplus_{n < m} M(D_n \cap D_m)[1] ;$$

it thus belongs to the full triangulated sub-category  $d_{\leq 0} DM_{gm}^{eff}(k)_{\mathbb{Q}}$  generated by motives of dimension 0. This triangulated sub-category is canonically equivalent to the bounded derived category of the Abelian semi-simple category of Artin motives (with  $\mathbb{Q}$ -coefficients) over  $k$  [V1, Prop. 3.4.1 and Remark 2 following it]. The sequence

$$\bigoplus_{n < m} M(D_n \cap D_m) \longrightarrow \bigoplus_m M_0(D_m) \longrightarrow M_0(D) \longrightarrow 0$$

of Artin motives is exact. From this and the above exact triangle, we see that  $M_1(D)[-1]$  is an Artin motive, which fits into an exact sequence

$$0 \longrightarrow M_1(D)[-1] \longrightarrow \bigoplus_{n < m} M(D_n \cap D_m) \longrightarrow \bigoplus_m M_0(D_m) .$$

(ii) The motive  $M_1^{!*}(\overline{X})$  equals  $M_1(\tilde{X})$  (Proposition 3.10). We shall show triviality of

$$\mathrm{Hom}_{DM_{gm}^{eff}(k)_{\mathbb{Q}}}(M(Y)[1], M_1(\tilde{X}))$$

for any zero-dimensional variety  $Y$  over  $k$ . Hard Lefschetz

$$M_1(\tilde{X}) \cong M_3(\tilde{X})(-1)[-2]$$

and duality in  $DM_{gm}(k)_{\mathbb{Q}}$  imply that this group is isomorphic to

$$\mathrm{Hom}_{DM_{gm}(k)_{\mathbb{Q}}}(M_1(\tilde{X}) \otimes M(Y)(-1)[-1], \mathbb{Z}(0)) ,$$

which equals the direct factor  $\mathrm{Hom}_{DM_{gm}^{eff}(k)_{\mathbb{Q}}}(M_1(\tilde{X}) \otimes M(Y), \mathbb{Z}(1)[1])$  of

$$\mathrm{Hom}_{DM_{gm}^{eff}(k)_{\mathbb{Q}}}(M(\tilde{X} \times_k Y), \mathbb{Z}(1)[1]) .$$

According to [V1, Cor. 3.4.3], for any smooth variety  $W$  over  $k$ , the group  $\mathrm{Hom}_{DM_{gm}^{eff}(k)_{\mathbb{Q}}}(M(W), \mathbb{Z}(1)[1])$  equals the group of global sections  $\Gamma(W, \mathbb{G}_m)$ , tensored with  $\mathbb{Q}$ . The inclusion of  $\mathrm{Hom}_{DM_{gm}^{eff}(k)_{\mathbb{Q}}}(M_0(\tilde{X}) \otimes M(Y), \mathbb{Z}(1)[1])$  into  $\mathrm{Hom}_{DM_{gm}^{eff}(k)_{\mathbb{Q}}}(M(\tilde{X} \times_k Y), \mathbb{Z}(1)[1])$  is therefore an isomorphism ( $\tilde{X} \times_k Y$  is proper). **q.e.d.**

Under the assumption of Proposition 6.4, we thus get a canonical morphism

$$M_1(D) \xrightarrow{\tilde{i}_*} M_{\geq 2}^{!*}(\overline{X}) \longrightarrow M_2^{!*}(\overline{X}) ,$$

which we may interpret as a one-extension  $\mathbb{E}$  in  $DM_{gm}^{eff}(k)_{\mathbb{Q}}$  of the Artin motive  $M_1(D)[-1]$  by  $M_2^{!*}(\overline{X})[-2]$ .

**Remark 6.5.** (a) Remark 6.2 shows where to look for a natural candidate for the cone of  $\mathbb{E} : M_1(D) \rightarrow M_2^{!*}(\overline{X})$ : it should be a canonical sub-quotient of the motive with compact support  $M^c(X)$ .

(b) Note that the object  $M_1(D)$  is trivial if  $X^*$  is smooth.

(c) Without the assumption of Proposition 5.4, we still get morphisms

$$M_1(D) \longrightarrow M_2^{!*}(\overline{X}) ,$$

by composing  $\tilde{i}_* : M_1(D) \rightarrow M_{\geq 1}^{!*}(\overline{X})$  with projections  $p_2$  from  $M_{\geq 1}^{!*}(\overline{X})$  to its direct factor  $M_2^{!*}(\overline{X})$ . In special cases, the dependence on the choice of the projection  $p_2$  may be controlled.

Using the morphisms  $c_{\mathcal{L}}$  studied in Section 4, we can relate  $M_2^{!*}(\overline{X})[-2]$  to a tensor product of an Artin motive and the Tate twist  $\mathbb{Z}(1)$ .

**Theorem 6.6.** *Assume that all geometric irreducible components  $D_m$  of  $D$  are of genus zero. There is a canonical morphism of vector spaces*

$$\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{Ext}_{DM_{gm}^{eff}(k)_{\mathbb{Q}}}^1(M_1(D)[-1], M_0^{!*}(\overline{X})(1)) .$$

Here,  $\mathrm{Ext}_{DM_{gm}^{eff}(k)_{\mathbb{Q}}}^1(\bullet, *)$  denotes  $\mathrm{Hom}_{DM_{gm}^{eff}(k)_{\mathbb{Q}}}(\bullet, * [1])$ .

*Proof.* Denote by  $\mathrm{Pic}(\tilde{X})'$  the group of line bundles on  $\tilde{X}$ , whose restrictions to all  $D_m$  are trivial. First, define a morphism

$$\mathrm{Pic}(\tilde{X})' \longrightarrow \mathrm{Ext}_{DM_{gm}^{eff}(k)_{\mathbb{Q}}}^1(M_1(D)[-1], M_0^{!*}(\overline{X})(1)) .$$

by mapping the class of  $\mathcal{L} \in \mathrm{Pic}(\tilde{X})'$  to the image of

$$\mathbb{E} \in \mathrm{Ext}_{DM_{gm}^{eff}(k)_{\mathbb{Q}}}^1(M_1(D)[-1], M_2^{!*}(\overline{X})[-2])$$

under  $R(c_{\mathcal{L}}) : M_2^{!*}(\overline{X})[-2] \rightarrow M_0^{!*}(\overline{X})(1)$  (Variant 4.2 (iii)). Next, consider the restriction

$$\tilde{j}^* : \mathrm{Pic}(\tilde{X})' \longrightarrow \mathrm{Pic}(X) .$$

We shall see that the non-degeneracy of the intersection pairing on the  $D_m$  [Mm, p. 6] implies that the map  $\tilde{j}^* \otimes \mathbb{Q}$  is an isomorphism. Indeed, any element in the kernel of  $\tilde{j}^* : \mathrm{Pic}(\tilde{X}) \rightarrow \mathrm{Pic}(X)$  is represented by a linear combination  $\sum_m a_m D_m$  of the  $D_m$ . If the class of  $\sum_m a_m D_m$  belongs to  $\mathrm{Pic}(\tilde{X})'$ , then its intersection numbers with all  $D_m$  must be zero. Thus the vector  $(a_m)_m$  is in the kernel of the intersection matrix, hence it is zero. For the surjectivity of

$$\tilde{j}^* \otimes \mathbb{Q} : \mathrm{Pic}(\tilde{X})' \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} ,$$

observe that  $\tilde{j}^* : \text{Pic}(\tilde{X}) \rightarrow \text{Pic}(X)$  is surjective. The non-degeneracy of the intersection matrix shows that any divisor  $C$  on  $\tilde{X}$  can be modified by a rational linear combination of the  $D_m$  such that the difference  $C'$  has trivial intersection numbers with all the  $D_m$ . Since these are supposed to be of genus zero, the restriction of  $C'$  to all  $D_m$  is principal. **q.e.d.**

Given a sub-scheme  $Z_\infty$  of the finite scheme  $Z$ , we may consider the pre-image  $D_\infty \subset D$  of  $Z_0$  under  $\pi$ , and define  $M_1(D_\infty)$  as before. It is a canonical direct factor of  $M_1(D)$ .

**Corollary 6.7.** *Assume that all geometric irreducible components  $D_m$  of  $D$  are of genus zero. There is a canonical morphism of vector spaces*

$$\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \text{Ext}_{DM_{gm}^{eff}(k)_{\mathbb{Q}}}^1(M_1(D_\infty)[-1], M_0^{1*}(\bar{X})(1)) .$$

According to Remark 6.5 (b), our construction yields the zero morphism if  $X^*$  is smooth. Thus, it can be interesting only if the normalization of our surface  $\bar{X}$  presents singularities.

## 7 Motivic interpretation of a construction of A. Caspar

In this final section, our base field will be equal to  $\mathbb{Q}$ . Let us first recall the geometric setting studied in [Cs]. Let  $F$  be a real quadratic extension of  $\mathbb{Q}$  with discriminant  $d$ . Assume that the class number in the narrow sense of  $F$  equals one. Let  $X'$  be the *Hilbert modular surface* of full level associated to  $F$  [vdG, Sect. X.4]. Denote by  $X^*$  its *Baily-Borel compactification*, and by  $X$  the smooth part of  $X'$ . All these surfaces are normal and geometrically connected. The complement of  $X^* - X'$  consists of one  $\mathbb{Q}$ -rational point, denoted  $\infty$  (the *cusps* of  $X^*$ ). The finite sub-scheme  $Z := (X^* - X)_{\text{red}}$  includes the cusp, but also the singularities of  $X'$ . There is a canonical desingularization

$$\begin{array}{ccccc} X & \hookrightarrow & \tilde{X} & \longleftarrow & D \\ \parallel & & \pi \downarrow & & \downarrow \pi \\ X & \hookrightarrow & X^* & \longleftarrow & Z \end{array}$$

$\tilde{X}$  is a smooth, and  $D$  a divisor with normal crossings, whose irreducible components are smooth. Furthermore, all geometric irreducible components of  $D$  are of genus zero. The irreducible components of the pre-image  $D_\infty \subset D$  of  $\infty$  under  $\pi$  are isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ , and form a polygon: for the complex surface underlying  $\tilde{X}$ , this is due to Hirzebruch [vdG, Chap. II]; that the construction descends to  $\mathbb{Q}$  follows from [R, Sect. 5].

**Lemma 7.1.** *The Artin motive  $M_1(D_\infty)[-1]$  is canonically isomorphic to  $\mathbb{Z}(0)$ .*

*Proof.* As in Proposition 6.4,  $M_1(D_\infty)[-1]$  equals the kernel of

$$\bigoplus_{n < m} M(D_n \cap D_m) \longrightarrow \bigoplus_m M_0(D_m),$$

where  $D_m$  are the components of  $D_\infty$ . Since  $D_\infty$  is a polygon, all  $M_0(D_m)$  are equal to  $\mathbb{Z}(0)$ . So are the  $M(D_n \cap D_m)$  for consecutive indices  $n, m$ , and the morphism  $M(D_{m-1} \cap D_m) \longrightarrow M_0(D_{m-1}) \oplus M_0(D_m)$  equals the difference  $\mathbb{Z}(0) \rightarrow \mathbb{Z}(0)^2$ . Hence  $M_1(D_\infty)[-1] = \mathbb{Z}(0)$ . **q.e.d.**

The variety  $\tilde{X}$  being geometrically connected, we get

$$M_0^{!*}(\overline{X}) = M_0(\tilde{X}) = \mathbb{Z}(0).$$

Corollary 6.7 thus yields the following.

**Theorem 7.2.** *Let  $k$  be an extension of  $\mathbb{Q}$ . Denote by  $X_k$  the base change of  $X$  to  $k$ .*

(i) *There is a canonical morphism of vector spaces*

$$cl_{\text{KCE}} : \text{Pic}(X_k) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \text{Ext}_{DM_{gm}^{eff}(k)_{\mathbb{Q}}}^1(\mathbb{Z}(0), \mathbb{Z}(1)) = k^* \otimes_{\mathbb{Z}} \mathbb{Q}.$$

(ii) *The morphism  $cl_{\text{KCE}}$  is compatible with field extensions in the following sense: if  $k_1 \hookrightarrow k_2$  is a morphism of extensions of  $\mathbb{Q}$ , then the diagram*

$$\begin{array}{ccc} \text{Pic}(X_{k_2}) & \xrightarrow{cl_{\text{KCE}}} & k_2^* \otimes_{\mathbb{Z}} \mathbb{Q} \\ \otimes_{k_1}^{k_2} \uparrow & & \uparrow \\ \text{Pic}(X_{k_1}) & \xrightarrow{cl_{\text{KCE}}} & k_1^* \otimes_{\mathbb{Z}} \mathbb{Q} \end{array}$$

*commutes. In particular,  $cl_{\text{KCE}}$  commutes with the action of the automorphism group of  $k$ .*

*Proof.* The only point to be verified is the equality

$$\text{Ext}_{DM_{gm}^{eff}(k)_{\mathbb{Q}}}^1(\mathbb{Z}(0), \mathbb{Z}(1)) = k^* \otimes_{\mathbb{Z}} \mathbb{Q}.$$

But this is the content of [V1, Cor. 3.4.3].

**q.e.d.**

As suggested by the terminology of [Cs], the image of a line bundle  $\mathcal{L}$  in  $\text{Pic}(X_k) \otimes_{\mathbb{Z}} \mathbb{Q}$  under  $cl_{\text{KCE}}$  will be called the *Kummer–Chern–Eisenstein extension* associated to  $\mathcal{L}$ .

Now consider the case  $k = F$ . Denote by  $\chi$  the primitive Dirichlet character modulo  $d$ , and by  $\mathbb{Z}(\chi)$  the Artin motive over  $\mathbb{Q}$  associated to  $\chi$ , i.e., the vector space  $\mathbb{Q}$  on which the absolute Galois group of  $\mathbb{Q}$  acts via  $\chi$ . Compatibility of  $cl_{\text{KCE}}$  with the action of the Galois group implies the following.

**Corollary 7.3.** *Denote by  $\text{Pic}(X)^{\chi=-1} \subset \text{Pic}(X_F)$  the subgroup of line bundles in  $\text{Pic}(X_F)$ , on which the non-trivial automorphism of  $F$  acts by  $[\mathcal{L}] \mapsto [\mathcal{L}^{-1}]$ . Denote by  $(F^*)^{\chi=-1} \subset F^*$  the kernel of the norm. Then the restriction of  $cl_{\text{KCE}}$  to  $\text{Pic}(X)^{\chi=-1} \otimes_{\mathbb{Z}} \mathbb{Q}$  induces a morphism of vector spaces*

$$cl_{\text{KCE}} : \text{Pic}(X)^{\chi=-1} \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \text{Ext}_{DM_{gm}^{eff}(\mathbb{Q})_{\mathbb{Q}}}^1(\mathbb{Z}(\chi), \mathbb{Z}(1)) = (F^*)^{\chi=-1} \otimes_{\mathbb{Z}} \mathbb{Q} .$$

Let  $\sigma_1, \sigma_2$  be the (real) embeddings of  $F$  into  $\mathbb{C}$ . We consider the two line bundles  $\mathcal{L}_i$  on  $X_F$ ,  $i = 1, 2$ , characterized by their factors of automorphy  $(c\tau_i + d)^2$  over  $\mathbb{C}$ . They are conjugate under the action of the Galois group of  $F$  over  $\mathbb{Q}$  [vdG, Prop. XI.(2.7)]. The tensor product  $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$  belongs to  $\text{Pic}(X)^{\chi=-1} \otimes_{\mathbb{Z}} \mathbb{Q}$ . We propose ourselves to identify its image under the map  $cl_{\text{KCE}}$  of Corollary 7.3.

In order to do so, consider the  $\ell$ -adic realization, for a fixed prime number  $\ell$  [DG, Sect. 1.5]. It is a triangulated covariant functor

$$r_{\ell} : DM_{gm}^{eff}(\mathbb{Q})_{\mathbb{Q}} \longrightarrow D^{-}(\text{Shv}_{et}(\mathbf{Spec} \mathbb{Q}), \mathbb{Q}_{\ell})$$

to the “derived category” of constructible  $\mathbb{Q}_{\ell}$ -sheaves on  $\mathbf{Spec} \mathbb{Q}$  [E].

**Proposition 7.4.** *Denote by  $\mathbb{Q}_{\ell}(\chi)$  the Artin motive  $\mathbb{Z}(\chi)$ , tensored with  $\mathbb{Q}_{\ell}$ , and by  $\mathbb{Q}_{\ell}(1)$  the  $\ell$ -adic Tate twist. Let  $(F^*)^{\wedge}$  be the  $\ell$ -adic completion of  $F^*$ . There is a canonical isomorphism*

$$\text{Ext}_{(\text{Shv}_{et}(\mathbf{Spec} \mathbb{Q}), \mathbb{Q}_{\ell})}^1(\mathbb{Q}_{\ell}(\chi), \mathbb{Q}_{\ell}(1)) \xrightarrow{\sim} ((F^*)^{\wedge})^{\chi=-1} \otimes_{\mathbb{Z}} \mathbb{Q} ,$$

fitting into a commutative diagram

$$\begin{array}{ccc} \text{Ext}_{DM_{gm}^{eff}(\mathbb{Q})_{\mathbb{Q}}}^1(\mathbb{Z}(\chi), \mathbb{Z}(1)) & \xrightarrow{\sim} & (F^*)^{\chi=-1} \otimes_{\mathbb{Z}} \mathbb{Q} \\ r_{\ell} \downarrow & & \downarrow \\ \text{Ext}_{(\text{Shv}_{et}(\mathbf{Spec} \mathbb{Q}), \mathbb{Q}_{\ell})}^1(\mathbb{Q}_{\ell}(\chi), \mathbb{Q}_{\ell}(1)) & \xrightarrow{\sim} & ((F^*)^{\wedge})^{\chi=-1} \otimes_{\mathbb{Z}} \mathbb{Q} \end{array}$$

The following appears plausible.

**Assumption 7.5.** The functor  $r_{\ell}$  maps the localization triangle

$$M(D) \longrightarrow M(\tilde{X}) \longrightarrow M^c(X) \longrightarrow M(D)[1]$$

from [V1, Prop. 4.1.5] to the dual of the localization triangle for constructible  $\mathbb{Q}_{\ell}$ -sheaves

$$R\Gamma_c(X, \mathbb{Q}_{\ell}) \longrightarrow R\Gamma(\tilde{X}, \mathbb{Q}_{\ell}) \longrightarrow R\Gamma(D, \mathbb{Q}_{\ell}) \longrightarrow R\Gamma_c(X, \mathbb{Q}_{\ell})[1] .$$

Under this assumption, our construction shows (see Remark 6.5 (a)) that the image of  $cl_{\text{KCE}}(\mathcal{L}_1 \otimes \mathcal{L}_2^{-1})$  under  $r_{\ell}$  is the extension of Galois modules constructed in [Cs, Section 1.2]. One of the main results of [loc. cit.] describes this extension.

**Theorem 7.6.** *Assume that  $d$  is a prime congruent to 1 modulo 4. Denote by  $\varepsilon \in \mathcal{O}_F^*$  the generator of the totally positive units, and by  $\zeta_F$  the Dedekind zeta function of  $F$ . Under Assumption 7.5, the composition of  $r_\ell$  and the isomorphism from Proposition 7.4 maps*

$$cl_{KCE}(\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}) \in \text{Ext}_{DM_{gm}^{eff}(\mathbb{Q})_{\mathbb{Q}}}^1(\mathbb{Z}(\chi), \mathbb{Z}(1))$$

to  $\varepsilon^{\pm \frac{1}{2}\zeta_F(-1)^{-1}} \in ((F^*)^\wedge)^{\chi=-1} \otimes_{\mathbb{Z}} \mathbb{Q}$ .

*Proof.* This is [Cs, Thm. 2.5]. The ambiguity concerning the sign in the exponent comes from the fact that we have made no distinguished choice of an embedding of  $F$  into  $\mathbb{R}$  (hence the rôles of  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are symmetric).

**q.e.d.**

This result allows us to identify  $cl_{KCE}(\mathcal{L}_1 \otimes \mathcal{L}_2^{-1})$  itself.

**Corollary 7.7.** *Under Assumption 7.5, and with the conditions and notations of Theorem 7.6,*

$$cl_{KCE}(\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}) = \varepsilon^{\pm \frac{1}{2}\zeta_F(-1)^{-1}} \in (F^*)^{\chi=-1} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

*Proof.* This follows from Theorem 7.6, and from injectivity of

$$(F^*)^{\chi=-1} \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow ((F^*)^\wedge)^{\chi=-1} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

**q.e.d.**

The question posed in [Cs, Sect. 1.4] remains open. The “classical” geometric construction of the one-extension in

$$\text{Ext}_{DM_{gm}^{eff}(k)_{\mathbb{Q}}}^1(\mathbb{Z}(0), \mathbb{Z}(1))$$

mapping to  $1 \neq x \in k^* \subset k^* \otimes_{\mathbb{Z}} \mathbb{Q}$  under the identification of [V1, Cor. 4.3.4] is via a motivic interpretation of the first cohomology group of  $\mathbb{G}_m$  relative to the two points 1 and  $x$ . Is there a geometric link between this construction (for  $x = \varepsilon^{\pm \frac{1}{2}\zeta_F(-1)^{-1}}$ ) and  $cl_{KCE}(\mathcal{L}_1 \otimes \mathcal{L}_2^{-1})$ ?

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