

Category of Noncommutative CW Complexes. II[†]

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Abstract

We introduce in this paper the notion of noncommutative Serre fibration (shortly, NCSF) and show that up to homotopy, every NCCW complex morphism is some noncommutative Serre fibration. We then deduce a six-term exact sequence for the periodic cyclic homology and for K-theory of an arbitrary noncommutative Serre fibration. We also show how to use this technique to compute K-theory and cyclic theory of some noncommutative quotients.

Introduction

It is well-known that for short exact sequences of closed two-sided ideals and C*-algebras of type

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \quad (0.1)$$

where A is some (closed) two-sided ideal in B and $C \cong B/A$ the quotient C*-algebra, the periodic cyclic homology admits six-term exact sequences of type

$$\begin{array}{ccccc} \mathrm{HP}_*(A) & \longrightarrow & \mathrm{HP}_*(B) & \longrightarrow & \mathrm{HP}_*(C) \\ \partial_{*+1} \uparrow & & & & \downarrow \partial_* \\ \mathrm{HP}_{*+1}(C) & \longleftarrow & \mathrm{HP}_{*+1}(B) & \longleftarrow & \mathrm{HP}_{*+1}(A) \end{array} \quad (0.2)$$

The same is true for K-theory

$$\begin{array}{ccccc} \mathrm{K}_*(A) & \longrightarrow & \mathrm{K}_*(B) & \longrightarrow & \mathrm{K}_*(C) \\ \partial_{*+1} \uparrow & & & & \downarrow \partial_* \\ \mathrm{K}_{*+1}(C) & \longleftarrow & \mathrm{K}_{*+1}(B) & \longleftarrow & \mathrm{K}_{*+1}(A) \end{array} \quad (0.3)$$

It is natural to ask whether the condition of A being an ideal is necessary for this exact sequence. We observe that this condition is indeed not necessarily to be satisfied. It is enough to consider the relative generalized homology in place of the generalized homology of the quotient algebra,

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what we don't have in general for a pair of algebra and subalgebra. We use the corresponding notion of noncommutative mapping cylinder, noncommutative mapping cone and suspension etc. introduced in [D1] in order to manage the situation. The ideas are therefore originated from algebraic topology.

For more special pairs, some thing like ordinary Serre fibrations with the *homotopy lifting property* (HLP), we have more properties. For this we arrive to a more general condition of a *noncommutative Serre fibration* (NCSF) as some homomorphism of algebras with the so called HLP (Homotopy Lifting Property). But we do restrict to consider only the so called *noncommutative CW-complexes*, [D1]. We prove that up to homotopy we can change any homomorphism between noncommutative CW-complexes in order to have a noncommutative Serre fibration. This main theorem let us then to apply it to study noncommutative decomposition series for operator algebras, started in [D2]

Let us describe the contents of the paper. In Section 1 we define some noncommutative object like noncommutative (NC) mapping cylinder, NC mapping cone, NC suspension, etc.... and the HLP (Homotopy Lifting Property) and NCCW complex. As the main result, we prove that in the category of NCCW complexes, every morphism is homotopic to a Serre fibration. We deduce in Section 2 the hexagon diagram for the periodic cyclic homology and K-theories of NC Serre fibration. In the ordinary algebraic topology one uses namely some computation of cohomology of spheres to compute homology of orthogonal and unitary groups. In our noncommutative theory, the cyclic homology of quantum orthogonal and quantum unitary groups are known from the representation theory we then use those to compute homology of quantum spheres as noncommutative quotients. In a next paper we prove also that for NC Serre fibrations there is also spectral sequences converging to HP and K theories with E_2 term like $E_{p,q}^2 = \text{HP}_p(B, A; \text{HP}_q(A)) \implies \text{HP}_{p+q}(B)$, and $E_{p,q}^2 = \text{HP}_p(B, A; K_q(A)) \implies K_{p+q}(B)$. In the works [D2][D3] we reduced the problem of studying the structure of an arbitrary GCR C^* -algebra (of type I) to some filtration by a towers of ideals with good enough quotients, what permit to use spectral sequence ideas into studying. The paper is the revised version of the preliminary version in arXiv:math.QA/0211048.

1 Noncommutative Serre Fibrations

We start with the notion of NCCW, introduced by S. Eilers, T. A. Loring and G. K. Pedersen [ELP] and G. Pedersen [P], see also [D1].

Definition 1.1 A *dimension 0 NCCW complex* is defined, following [P] as a finite sum of C^* algebras of finite linear dimension, i.e. a sum of finite dimensional matrix algebras,

$$A_0 = \bigoplus_k M_{n(k)}. \quad (1.1)$$

In dimension n , an *NCCW complex* is defined as a sequence $\{A_0, A_1, \dots, A_n\}$ of C^* -algebras A_k obtained each from the previous one by the pullback construction

$$\begin{array}{ccccccc} 0 & \longrightarrow & I_0^k F_k & \longrightarrow & A_k & \xrightarrow{\pi} & A_{k-1} & \longrightarrow & 0 \\ & & \parallel & & \downarrow \rho_k & & \downarrow \sigma_k & & \\ 0 & \longrightarrow & I_0^k F_k & \longrightarrow & I^k F_k & \xrightarrow{\partial} & \mathbf{S}^{k-1} F_k & \longrightarrow & 0, \end{array} \quad (1.2)$$

where F_k is some C^* -algebra of finite linear dimension, ∂ the restriction morphism, σ_k the connecting morphism, ρ_k the projection on the first coordinates and π the projection on the second coordinates in the presentation

$$A_k = \mathbf{I}^k F_k \bigoplus_{\mathbf{S}^{k-1} F_k} A_{k-1} \quad (1.3)$$

Definition 1.2 We say that the morphism $f : A \rightarrow B$ admits the so called *Homotopy Lifting Property (HLP)* if for every algebra C and every morphism $\varphi : A \rightarrow C$ such that there is some morphism $\tilde{\varphi} : B \rightarrow C$ satisfying $\varphi = \tilde{\varphi} \circ f$, and for every homotopy $\varphi_t : A \rightarrow C$, $\varphi_0 = \varphi$, there exists a homotopy $\tilde{\varphi}_t : B \rightarrow C$, $\tilde{\varphi}_0 = \tilde{\varphi}$, such that for every t , $\varphi_t = \tilde{\varphi}_t \circ f$, i.e. the following diagram is commutative

$$\begin{array}{ccc} & B & \\ & \uparrow & \searrow \tilde{\varphi}_t, \tilde{\varphi}_0 = \tilde{\varphi} \\ A & \xrightarrow{\varphi_t} & C \\ & \varphi_0 = \tilde{\varphi}_0 \circ f & \end{array} \quad (1.4)$$

Definition 1.3 A morphism of C^* -algebras $f : A \rightarrow B$ with HLP axiom is called a *noncommutative Serre fibration (NCSF)*.

Theorem 1.4 *In the category of NCCW complexes, for every morphism $f : A \rightarrow B$, there is some homotopies $A \sim A'$, $B \sim B'$ and a morphism $f' : A' \rightarrow B'$ which is a NC Serre fibration.*

Before prove this theorem we do recall [D1] the following notions of noncommutative cylinder and noncommutative mapping cone.

Definition 1.5 (NC cone) For C^* -algebras the *NC cone* of A is defined as the tensor product with $C_0((0, 1])$, i.e.

$$\text{Cone}(A) := C_0((0, 1]) \otimes A. \quad (1.5)$$

Definition 1.6 (NC suspension) For C^* -algebras the *NC suspension* of A is defined as the tensor product with $C_0((0, 1))$, i.e.

$$\mathbf{S}(A) := C_0((0, 1)) \otimes A. \quad (1.6)$$

Remark 1.7 *If A admits a NCCW complex structure, the same have the cone $\text{Cone}(A)$ of A and the suspension $\mathbf{S}(A)$ of A .*

Definition 1.8 (NC mapping cylinder) Consider a map $f : A \rightarrow B$ between C^* -algebras. The *NC mapping cylinder* $\text{Cyl}(f : A \rightarrow B)$ is defined by the pullback diagram [D1]

$$\begin{array}{ccc} \text{Cyl}(f) & \xrightarrow{pr_1} & C[0, 1] \otimes A \\ pr_2 \downarrow & & \downarrow f \circ ev(1) \\ B & \xrightarrow{id} & B \end{array} \quad (1.7)$$

where $\text{ev}(1)$ is the map of evaluation at the point $1 \in [0, 1]$. It can be also defined directly as follows. In the algebra $\mathbf{C}(\mathbf{I}) \otimes A \oplus B$ consider the closed two-sided ideal $\langle \{1\} \otimes a - f(a), \forall a \in A \rangle$, generated by elements of type $\{1\} \otimes a - f(a), \forall a \in A$. The quotient algebra

$$\text{Cyl}(f) = \text{Cyl}(f : A \rightarrow B) := (\mathbf{C}(\mathbf{I}) \otimes A \oplus B) / \langle \{1\} \otimes a - f(a), \forall a \in A \rangle \quad (1.8)$$

is called the *NC mapping cylinder* and denote it by $\text{Cyl}(f : A \rightarrow B)$.

Remark 1.9 *It is easy to show that A is included in $\text{Cyl}(f : A \rightarrow B)$ as $\mathbf{C}\{0\} \otimes A \subset \text{Cyl}(f : A \rightarrow B)$ and B is included in also $B \subset \text{Cyl}(f : A \rightarrow B)$.*

Definition 1.10 (NC mapping cone) The NC mapping cone $\text{Cone}(\varphi)$ is defined from the pull-back diagram

$$\begin{array}{ccc} \text{Cone}(\varphi) & \xrightarrow{\text{pr}_1} & \mathbf{C}_0(0, 1] \otimes A \\ \text{pr}_2 \downarrow & & \downarrow f \circ \text{ev}(1), \\ B & \xrightarrow{\text{id}} & B \end{array} \quad (1.9)$$

where $\text{ev}(1)$ is the map of evaluation at the point $1 \in [0, 1]$. It can be also directly defined as follows. In the algebra $\mathbf{C}((0, 1]) \otimes A \oplus B$ consider the closed two-sided ideal $\langle \{1\} \otimes a - f(a), \forall a \in A \rangle$, generated by elements of type $\{1\} \otimes a - f(a), \forall a \in A$. We define the *mapping cone* as the quotient algebra

$$\text{Cone}(f) = \text{Cone}(f : A \rightarrow B) := (\mathbf{C}_0((0, 1]) \otimes A \oplus B) / \langle \{1\} \otimes a - f(a), \forall a \in A \rangle. \quad (1.10)$$

Remark 1.11 *It is easy to show that B is included in $\text{Cone}(f : A \rightarrow B)$.*

PROOF OF THEOREM 1.4.

STEP 1. From the definition $\text{Cyl}(f : A \rightarrow B)$, we see that

Lemma 1.12 *$B' = \text{Cyl}(f : A \rightarrow B)$ is homotopic to B and $A \cong A \otimes \mathbf{C}(\{0\}) \hookrightarrow B'$.*

We can therefore from now on suppose that $A \hookrightarrow B$.

STEP 2. Let us denote $\Pi(A, B)$ the space of all piece-wise linear curve starting from A and ending in B . Every element in this space is a linear combination of elements from B , which is a NCCW complex and therefore

Lemma 1.13 *The algebra $\Pi(A, B)$ is also a NCCW complex in sense of Definition 1.1.*

STEP 3. The following lemma is clear.

Lemma 1.14 *$A' := \Pi(A, B)$ is homotopic to A .*

Indeed, by change of parametrization $\gamma(t) \mapsto \gamma(st), t \in \mathbf{I} = [0, 1]$, for $s \in \mathbf{I} = [0, 1]$ we can produce a homotopy between curves and their starting points.

STEP 4. Let us consider the morphism $A' = \Pi(A, B) \longrightarrow B$ which corresponds to each piece-wise linear path its end-point.

Lemma 1.15 *This map $A' \longrightarrow B$ satisfy the HLP axiom.*

Indeed if we have some piece-wise linear curve in $A' = \Pi(A, B)$, we have a family of piece-wise-linear curves in B , starting from A . The end-points give us the necessary homotopy from B to C . The theorem is proven. \square

2 Six-term exact sequences

Let us recall the definition of the periodic cyclic theory functor HP, see [C], [L] for more details.

Definition 2.1 Consider a C*-algebra with unity 1. We use the maximum norm for the tensor product of *-algebras. The cyclic complex $C_*(A)$ is defined as

$$C_n(A) := A \otimes \underbrace{\bar{A} \otimes \cdots \otimes \bar{A}}_{n \text{ times}}. \quad (2.1)$$

Define the Hochschild operators

$$C_0(A) := A \xleftarrow{b, b'} C_1(A) = A \otimes \bar{A} \xleftarrow{b, b'} \cdots \xleftarrow{b, b'} C_n(A) \xleftarrow{b, b'} \cdots, \quad (2.2)$$

where $\bar{A} := A/C$.

$$\mathbf{b}'(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n. \quad (2.3)$$

$$\mathbf{b}(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}. \quad (2.4)$$

It is easy to check that

$$\mathbf{b}^2 = 0, \quad (\mathbf{b}')^2 = 0. \quad (2.5)$$

One defines the cyclic operator $t : C_n(A) \rightarrow C_n(A)$ following the formula

$$\mathbf{t}(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := (-1)^{n+1} a_n \otimes a_0 \otimes \cdots \otimes a_{n-1} \quad (2.6)$$

It is easy to check that

$$\mathbf{t}^{n+1} = \mathbf{1}. \quad (2.7)$$

Define the operators

$$N = \mathbf{1} + \mathbf{t} + \cdots + \mathbf{t}^n. \quad (2.8)$$

From this cyclic complex we have the bi-complex $\mathcal{C}(A) = \{\mathcal{C}_{p,q}\}$

$$\begin{array}{cccccccc} \dots & & \vdots & & \vdots & & \vdots & & \vdots & & \dots \\ & & \downarrow b' & & \downarrow b & & \downarrow b' & & \downarrow b & & \\ \dots & \xrightarrow{\mathbf{t}-1} & C_3(A) & \xrightarrow{\mathbf{N}} & C_3(A) & \xrightarrow{\mathbf{t}-1} & C_3(A) & \xrightarrow{\mathbf{N}} & C_3(A) & \xrightarrow{\mathbf{t}-1} & \dots \\ & & \downarrow b' & & \downarrow b & & \downarrow b' & & \downarrow b & & \\ \dots & \xrightarrow{\mathbf{t}-1} & C_2(A) & \xrightarrow{\mathbf{N}} & C_2(A) & \xrightarrow{\mathbf{t}-1} & C_2(A) & \xrightarrow{\mathbf{N}} & C_2(A) & \xrightarrow{\mathbf{t}-1} & \dots \\ & & \downarrow b' & & \downarrow b & & \downarrow b' & & \downarrow b & & \\ \dots & \xrightarrow{\mathbf{t}-1} & C_1(A) & \xrightarrow{\mathbf{N}} & C_1(A) & \xrightarrow{\mathbf{t}-1} & C_1(A) & \xrightarrow{\mathbf{N}} & C_1(A) & \xrightarrow{\mathbf{t}-1} & \dots \\ & & \downarrow b' & & \downarrow b & & \downarrow b' & & \downarrow b & & \\ \dots & \xrightarrow{\mathbf{t}-1} & C_0(A) & \xrightarrow{\mathbf{N}} & C_0(A) & \xrightarrow{\mathbf{t}-1} & C_0(A) & \xrightarrow{\mathbf{N}} & C_0(A) & \xrightarrow{\mathbf{t}-1} & \dots \end{array} \quad (2.9)$$

The *total complex* is defined as

$$\text{Tot } \mathcal{C}(A)_i = \prod_{p+q=i \pmod{2}} \mathcal{C}_{p,q}, \quad i = 0, 1 \quad (2.10)$$

with differentials

$$B = \mathbf{d}_v + \mathbf{d}_h, \quad (2.11)$$

where \mathbf{d}_v and \mathbf{d}_h are the differential following the vertical or horizontal direction, correspondingly. The homology of the total complex is called the *periodic cyclic homology*

$$\text{HP}_i(A) := H_i(\text{Tot } \mathcal{C}(A)), \quad i = 0, 1. \quad (2.12)$$

It is well-known that periodic cyclic homology HP_i are a generalized homology functors, introduced by A. Connes [C], [L].

Theorem 2.2 *For every NC Serre fibration $f : A \longrightarrow B$, the corresponding hexagon in periodic cyclic homology holds*

$$\begin{array}{ccccc} \text{HP}_*(A) & \longrightarrow & \text{HP}_*(B) & \longrightarrow & \text{HP}_*(B, A) \\ \partial_{*+1} \uparrow & & & & \downarrow \partial_* \\ \text{HP}_{*+1}(B, A) & \longleftarrow & \text{HP}_{*+1}(B) & \longleftarrow & \text{HP}_{*+1}(A) \end{array} \quad (2.13)$$

Before prove the theorem we do introduce a noncommutative suspension.

Definition 2.3 *Noncommutative (shortly, NC) Suspension SA of A is by definition $A \otimes \mathbf{C}_0(0, 1)$.*

Following the construction of the NC mapping cone of a morphism $f : A \rightarrow B$, we have [D1]

Lemma 2.4 *There is an exact sequence of algebras*

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathbf{S}^2 A & \longrightarrow & \mathbf{S} \text{Cone}(f : A \rightarrow B) & \longrightarrow & \mathbf{S} \text{Cyl}(f : A \rightarrow B) & \longrightarrow \\ & & & & & & & \\ & \longrightarrow & \mathbf{S} A & \longrightarrow & \text{Cone}(f : A \rightarrow B) & \longrightarrow & \text{Cyl}(f : A \rightarrow B) & \longrightarrow A. \end{array} \quad (2.14)$$

The next lemma is a natural consequence from the well-known Künneth formula and Bott Periodicity of HP.

Lemma 2.5 *For the periodic cyclic homology HP, there is a natural isomorphism*

$$\text{HP}_*(SA) \cong \text{HP}_{*\pm 1}(A). \quad (2.15)$$

PROOF OF THEOREM 2.2.

□

Because in the category of NCCW-complexes, every morphism is homotopic to a NC Serre fibration, we have

Corollary 2.6 *In the category of NCCW-complexes each map $f : A \rightarrow B$ admits a six-term exact sequence*

$$\begin{array}{ccccc} \mathrm{HP}_*(A) & \longleftarrow & \mathrm{HP}_*(B) & \longleftarrow & \mathrm{HP}_*(B, A) \\ \partial_{*+1} \downarrow & & & & \uparrow \partial_* \\ \mathrm{HP}_{*+1}(B, A) & \longrightarrow & \mathrm{HP}_{*+1}(B) & \longrightarrow & \mathrm{HP}_{*+1}(A). \end{array} \quad (2.16)$$

Proof. It is well-known Bott Periodicity of type $\mathrm{HP}_*(\mathbf{S}^2 A) \cong \mathrm{HP}_*(A)$ and $\mathrm{HP}_*(\mathbf{S}^2 A) \cong \mathrm{HP}_*(A)$. From the long exact sequence (up-to homotopy)

$$\begin{array}{ccccccc} A & \longleftarrow & \mathrm{Cyl}(f : A \rightarrow B) & \longleftarrow & \mathrm{Cone}(i : A \rightarrow B) & \longleftarrow & \mathbf{S}A \\ \cdot & \longleftarrow & \mathbf{S} \mathrm{Cyl}(f : A \rightarrow B) & \longleftarrow & \mathbf{S} \mathrm{Cone}(i : A \rightarrow B) & \longleftarrow & \mathbf{S}^2 A \longrightarrow \dots \end{array} \quad (2.17)$$

we have the first connecting homomorphism

$$\partial_* : \mathrm{HP}_{*+1}(A) \longrightarrow \mathrm{HP}_*(B, A) \quad (2.18)$$

which gives us the first exact sequence

$$\begin{array}{ccccc} \mathrm{HP}_*(A) & \longleftarrow & \mathrm{HP}_*(B) & \longleftarrow & \mathrm{HP}_*(B, A) \\ & & & & \uparrow \partial_* \\ & & & & \mathrm{HP}_{*+1}(A). \end{array} \quad (2.19)$$

Then apply for the suspension SA and use the Bott 2-periodicity in HP-theory we have the second part

$$\begin{array}{ccccc} \mathrm{HP}_*(A) \cong \mathrm{HP}_{*+2}(A) & & & & \\ \partial_{*+1} \downarrow & & & & \\ \mathrm{HP}_{*+1}(B, A) & \longrightarrow & \mathrm{HP}_{*+1}(B) & \longrightarrow & \mathrm{HP}_{*+1}(A) \end{array} \quad (2.20)$$

of the commutative diagram. \square

For the K-theory we have the analogous results

Theorem 2.7 *For every NC Serre fibration $f : A \rightarrow B$, the corresponding hexagon in K-theory holds*

$$\begin{array}{ccccc} \mathrm{K}_*(A) & \longleftarrow & \mathrm{K}_*(B) & \longleftarrow & \mathrm{K}_*(B, A) \\ \partial_{*+1} \downarrow & & & & \uparrow \partial_* \\ \mathrm{K}_{*+1}(B, A) & \longrightarrow & \mathrm{K}_{*+1}(B) & \longrightarrow & \mathrm{K}_{*+1}(A), \end{array} \quad (2.21)$$

and we have also

Corollary 2.8 *In the category of NC CW-complexes each map $f : A \rightarrow B$ admits a six-term exact sequence*

$$\begin{array}{ccccc} \mathrm{K}_*(A) & \longleftarrow & \mathrm{K}_*(B) & \longleftarrow & \mathrm{K}_*(B, A) \\ \partial_{*+1} \downarrow & & & & \uparrow \partial_* \\ \mathrm{K}_{*+1}(B, A) & \longrightarrow & \mathrm{K}_{*+1}(B) & \longrightarrow & \mathrm{K}_{*+1}(A). \end{array} \quad (2.22)$$

3 Application

In the ordinary algebraic topology one used namely the computation of cohomology of spheres to compute homology of orthogonal and unitary groups. In our noncommutative theory, the cyclic homology of quantum orthogonal and quantum unitary groups are known from the representation theory we then use to compute homology of quantum spheres as noncommutative quotients.

3.1 Quotients of quantum orthogonal groups

Let us consider the quantum orthogonal groups $C_q^*(\mathrm{SO}(n))$. The natural inclusion $\mathrm{SO}(n-1) \hookrightarrow \mathrm{SO}(n)$ give us the homomorphism

$$C_q^*(\mathrm{SO}(n-1) \setminus \mathrm{SO}(n)) \hookrightarrow C_q^*(\mathrm{SO}(n)). \quad (3.1)$$

We have therefore up to homotopy a NC Serre fibration what for simplicity we denote again $(C_q^*(\mathrm{SU}(n)) : C_q^*(\mathrm{SU}(n-1))) :=$

$$\Pi(C_q^*(\mathrm{SO}(n-1) \setminus \mathrm{SO}(n)), \mathrm{Cyl}(\cdot, C_q^*(\mathrm{SO}(n))), C_q^*(\mathrm{SO}(n-1) \setminus \mathrm{SO}(n))) \quad (3.2)$$

and our associated exact sequence (up to homotopy of algebras) as

$$\begin{array}{ccccccc} C_q^*(\mathrm{SO}(n-1) \setminus \mathrm{SO}(n)) & \longleftarrow & C_q^*(\mathrm{SO}(n)) & \longleftarrow & \mathrm{Cone}(C_q^*(\mathrm{SO}(n-1) \setminus \mathrm{SO}(n)), C_q^*(\mathrm{SO}(n))) & \longleftarrow & \dots \\ \mathbf{S} C_q^*(\mathrm{SO}(n-1) \setminus \mathrm{SO}(n)) & \longleftarrow & \mathbf{S} C_q^*(\mathrm{SO}(n)) & \longleftarrow & \mathbf{S} \mathrm{Cone}(C_q^*(\mathrm{SO}(n-1) \setminus \mathrm{SO}(n)), C_q^*(\mathrm{SO}(n))) & \longleftarrow & \dots \\ & & & & \longleftarrow & \mathbf{S}^2 C_q^*(\mathrm{SO}(n-1) \setminus \mathrm{SO}(n)) & \longleftarrow \dots \end{array} \quad (3.3)$$

We can now apply the six-term exact sequence for Serre fibrations and have

$$\begin{array}{ccccc} \mathrm{HP}_*(C_q^*(\mathrm{SO}(n-1) \setminus \mathrm{SO}(n))) & \longleftarrow & \mathrm{HP}_*(C_q^*(\mathrm{SO}(n))) & \longleftarrow & \mathrm{HP}_*(C_q^*(\mathrm{SO}(n)) : C_q^*(\mathrm{SO}(n-1))) \\ \partial_{*+1} \downarrow & & & & \uparrow \partial_* \\ \mathrm{HP}_{*+1}(C_q^*(\mathrm{SO}(n)) : C_q^*(\mathrm{SO}(n-1))) & \longrightarrow & \mathrm{HP}_{*+1}(C_q^*(\mathrm{SO}(n))) & \longrightarrow & \mathrm{HP}_{*+1}(C_q^*(\mathrm{SO}(n-1) \setminus \mathrm{SO}(n))) \end{array} \quad (3.4)$$

In this six-term exact sequences the $\mathrm{HP}_*(C_q^*(\mathrm{SO}(n))) \cong H_{DR}^*(\mathbb{T})^W$ are well-known, where \mathbb{T} is a fixed maximal torus in $\mathrm{SO}(n)$ and $W = W(\mathbb{T}) = \mathcal{N}(\mathbb{T})/\mathbb{T}$ is the Weyl group of this maximal torus, see [DKT1]-[DKT2]. We can fix an immersion $\mathrm{SO}(n-1) \hookrightarrow \mathrm{SO}(n)$ so that the corresponding tori are included by immersions one-into-another.

3.2 Quotients of quantum unitary groups

By analogy, we consider the quantum unitary groups $C_q^*(\mathrm{SU}(n))$. The natural inclusion $\mathrm{SU}(n-1) \hookrightarrow \mathrm{SU}(n)$ give us the homomorphism

$$C_q^*(\mathrm{SU}(n-1) \setminus \mathrm{SU}(n)) \hookrightarrow C_q^*(\mathrm{SU}(n)). \quad (3.5)$$

We have therefore up to homotopy a NC Serre fibration what for simplicity we denote again $(C_q^*(\mathrm{SU}(n)) : C_q^*(\mathrm{SU}(n-1))) :=$

$$\Pi(C_q^*(\mathrm{SU}(n-1) \setminus \mathrm{SU}(n)), \mathrm{Cyl}(\cdot, C_q^*(\mathrm{SU}(n))), C_q^*(\mathrm{SU}(n-1) \setminus \mathrm{SU}(n))) \quad (3.6)$$

and our associated exact sequence (up to homotopy of algebras) as

$$\begin{array}{ccccccc}
C_q^*(SU(n-1) \setminus SU(n)) & \longleftarrow & C_q^*(SU(n)) & \longleftarrow & \text{Cone}(C_q^*(SU(n-1) \setminus SU(n)), C_q^*(SU(n))) & \longleftarrow & \\
\mathbf{S} C_q^*(SU(n-1) \setminus SU(n)) & \longleftarrow & \mathbf{S} C_q^*(SU(n)) & \longleftarrow & \mathbf{S} \text{Cone}(C_q^*(SU(n-1) \setminus SU(n)), C_q^*(SU(n))) & & \\
& & & & \longleftarrow & \mathbf{S}^2 C_q^*(SU(n-1) \setminus SU(n)) & \longleftarrow \dots
\end{array} \tag{3.7}$$

We can now also apply the six-term exact sequence for Serre fibrations and have

$$\begin{array}{ccccc}
\text{HP}_*(C_q^*(SU(n-1) \setminus SU(n))) & \longleftarrow & \text{HP}_*(C_q^*(SU(n))) & \longleftarrow & \text{HP}_*(C_q^*(SU(n)) : C_q^*(SU(n-1))) \\
\partial_{*+1} \downarrow & & & & \uparrow \partial_* \\
\text{HP}_{*+1}(C_q^*(SU(n)) : C_q^*(SU(n-1))) & \longrightarrow & \text{HP}_{*+1}(C_q^*(SU(n))) & \longrightarrow & \text{HP}_{*+1}(C_q^*(SU(n-1) \setminus SU(n)))
\end{array} \tag{3.8}$$

In this six-term exact sequences the $\text{HP}_*(C_q^*(SU(n))) \cong H_{DR}^*(\mathbb{T})^W$ are well-known, where \mathbb{T} is a fixed maximal torus in $SU(n)$ and $W = W(\mathbb{T}) = \mathcal{N}(\mathbb{T})/\mathbb{T}$ is the Weyl group of this maximal torus, see [DKT1]-[DKT2]. We can fix an immersion $SU(n-1) \hookrightarrow SU(n)$ so that the corresponding tori are included by immersions one-into-another.

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