

Category of Noncommutative CW complexes. III*

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Abstract

We prove in this paper a noncommutative version of Leray Theorem and then Leray-Serre Spectral Theorem for noncommutative Serre fibrations: for NC Serre fibration there are converging spectral sequences with E^2 terms as $E_{p,q}^2 = \text{HP}_p(A; \text{HP}_q(B, A)) \implies \text{HP}_{p+q}(B)$ and $E_{p,q}^2 = \text{HP}_p(A; \text{K}_q(B, A)) \implies \text{K}_{p+q}(B)$.

Key Words: NC Serre fibration, noncommutative CW-complexes, Leray-Serre Spectral Theorem

Introduction

The ideas of using spectral sequences to operator algebras was started in [D1]. In that work the author constructed for an arbitrary algebras some CCR- or CT- composition series and tried to define the structure of the algebras through Busby invariants of extensions. This reduced to the ideas of using spectral sequences. However, the technique of spectral sequence meets some difficulties, for example convergence problem. In this paper we consider the NCCW structure in place of the composition series in order to manage the spectral sequences in computing HP or K groups for some NCCW complexes.

It is well-known that for every short exact sequence of ideal and algebras

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \quad (0.1)$$

where A is some ideal in B and $C \cong B/A$ is the quotient algebra, there is a natural spectral sequence converging to the cyclic periodic homology with E_2 -term

$$E_{p,q}^2 = \text{HP}_p(A; \text{HP}_q(B, A)) \implies \text{HP}_n(B), \text{ with } n = p + q \quad (0.2)$$

and a spectral sequence converging to the K-theory with E_2 term

$$E_{p,q}^2 = \text{HP}_p(A; \text{K}_q(B, A)) \implies \text{K}_n(B), \text{ with } n = p + q. \quad (0.3)$$

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It is natural to ask whether the condition of A being an ideal is necessary for this exact sequence. We observe [D3] that this condition is indeed not necessarily to be satisfied. From this we arrive to a more general condition of a noncommutative Serre fibration as some homomorphism of algebras with the so called HLP (*Homotopy Lifting Problem*). But we do restrict to consider only the so called noncommutative CW-complexes. We proved that up to homotopy we can change any homomorphism between noncommutative CW-complexes and the CW-complexes themselves to have a noncommutative Serre fibration. We can then deduce the spectral sequences for the cyclic periodic homology and K-theory of an arbitrary NC Serre fibration and of an arbitrary morphism of NC CW-complexes. It seems to the author that recently the notion of NC CW-complex is introduced, see e.g. [C] and here we work with spectral sequences for general maps between NC CW-complexes.

Let us describe the contents of the paper. In Section 1 we construct for noncommutative (NC) CW-complexes the Leray spectral sequences. Next, in the section 2, we prove that for NC Serre fibration there are spectral sequences with E^2 terms as

$$E_{p,q}^2 = \text{HP}_p(A; \text{HP}_q(B, A)) \implies \text{HP}_{p+q}(B) \quad (0.4)$$

and

$$E_{p,q}^2 = \text{HP}_p(A; \text{K}_q(B, A)) \implies \text{K}_{p+q}(B). \quad (0.5)$$

1 Leray Spectral Sequence Theorem for NCCW

We start with the notion of NCCW, introduced by S. Eilers, T. A. Loring and G. K. Pedersen [ELP] and G. Pedersen [P]. For reader's convenience, we repeat some definitions from [D3].

Definition 1.1 A *dimension 0 NCCW complex* is defined, following [P] as a finite sum of C^* algebras of finite linear dimension, i.e. a sum of finite dimensional matrix algebras,

$$A_0 = \bigoplus_k \mathbf{M}_{n(k)}. \quad (1.1)$$

In dimension n , an *NCCW complex* is defined as a sequence $\{A_0, A_1, \dots, A_n\}$ of C^* -algebras A_k obtained each from the previous one by the pullback construction

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{I}_0^k F_k & \longrightarrow & A_k & \xrightarrow{\pi} & A_{k-1} & \longrightarrow & 0 \\ & & \parallel & & \downarrow \rho_k & & \downarrow \sigma_k & & \\ 0 & \longrightarrow & \mathbf{I}_0^k F_k & \longrightarrow & \mathbf{I}^k F_k & \xrightarrow{\partial} & \mathbf{S}^{k-1} F_k & \longrightarrow & 0, \end{array} \quad (1.2)$$

where F_k is some C^* -algebra of finite linear dimension, ∂ the restriction morphism, σ_k the connecting morphism, ρ_k the projection on the first coordinates and π the projection on the second coordinates in the presentation

$$A_k = \mathbf{I}^k F_k \bigoplus_{\mathbf{S}^{k-1} F_k} A_{k-1} \quad (1.3)$$

Definition 1.2 We say that the morphism $f : A \longrightarrow B$ admits the so called *Homotopy Lifting Property (HLP)* if for every algebra C and every morphism $\varphi : A \longrightarrow C$ such that there is some morphism $\tilde{\varphi} : B \longrightarrow C$ satisfying $\varphi = \tilde{\varphi} \circ f$, and for every homotopy $\varphi_t : A \longrightarrow C$, $\varphi_0 = \varphi$, there exists a homotopy $\tilde{\varphi}_t : B \longrightarrow C$, $\tilde{\varphi}_0 = \tilde{\varphi}$, such that for every t , $\varphi_t = \tilde{\varphi}_t \circ f$, i.e. the following diagram is commutative

$$\begin{array}{ccc}
 & B & \\
 & \uparrow f & \searrow \tilde{\varphi}_t, \tilde{\varphi}_0 = \tilde{\varphi} \\
 A & \xrightarrow{\varphi_t} & C \\
 & \varphi_0 = \tilde{\varphi}_0 \circ f &
 \end{array} \tag{1.4}$$

Definition 1.3 A morphism of C^* -algebras $f : A \longrightarrow B$ with HLP axiom is called a *noncommutative Serre fibration (NCSF)*.

Theorem 1.4 *In the category of NCCW complexes, for every morphism $f : A \longrightarrow B$, there is some homotopies $A \sim A'$, $B \sim B'$ and a morphism $f' : A' \longrightarrow B'$ which is a NC Serre fibration.*

Before prove this theorem we do recall [D1] the following notions of noncommutative cylinder and noncommutative mapping cone.

Definition 1.5 (NC cone) For C^* -algebras the *NC cone* of A is defined as the tensor product with $C_0((0, 1])$, i.e.

$$\text{Cone}(A) := C_0((0, 1]) \otimes A. \tag{1.5}$$

Definition 1.6 (NC suspension) For C^* -algebras the *NC suspension* of A is defined as the tensor product with $C_0((0, 1))$, i.e.

$$\mathbf{S}(A) := C_0((0, 1)) \otimes A. \tag{1.6}$$

Remark 1.7 *If A admits a NCCW complex structure, the same have the cone $\text{Cone}(A)$ of A and the suspension $\mathbf{S}(A)$ of A .*

Definition 1.8 (NC mapping cylinder) Consider a map $f : A \rightarrow B$ between C^* -algebras. The *NC mapping cylinder* $\text{Cyl}(f : A \rightarrow B)$ is defined by the pullback diagram [D1]

$$\begin{array}{ccc}
 \text{Cyl}(f) & \xrightarrow{pr_1} & C[0, 1] \otimes A \\
 pr_2 \downarrow & & \downarrow f \circ \text{ev}(1) \\
 B & \xrightarrow{id} & B
 \end{array} \tag{1.7}$$

where $ev(1)$ is the map of evaluation at the point $1 \in [0, 1]$. It can be also defined directly as follows. In the algebra $\mathbf{C}(\mathbf{I}) \otimes A \oplus B$ consider the closed two-sided ideal $\langle \{1\} \otimes a - f(a), \forall a \in A \rangle$, generated by elements of type $\{1\} \otimes a - f(a), \forall a \in A$. The quotient algebra

$$\text{Cyl}(f) = \text{Cyl}(f : A \rightarrow B) := (\mathbf{C}(\mathbf{I}) \otimes A \oplus B) / \langle \{1\} \otimes a - f(a), \forall a \in A \rangle \quad (1.8)$$

is called the *NC mapping cylinder* and denote it by $\text{Cyl}(f : A \rightarrow B)$.

Remark 1.9 *It is easy to show that A is included in $\text{Cyl}(f : A \rightarrow B)$ as $\mathbf{C}\{0\} \otimes A \subset \text{Cyl}(f : A \rightarrow B)$ and B is included in also $B \subset \text{Cyl}(f : A \rightarrow B)$.*

Definition 1.10 (NC mapping cone) The NC mapping cone $\text{Cone}(\varphi)$ is defined from the pullback diagram

$$\begin{array}{ccc} \text{Cone}(\varphi) & \xrightarrow{pr_1} & \mathbf{C}_0(0, 1] \otimes A \\ pr_2 \downarrow & & \downarrow f \circ ev(1), \\ B & \xrightarrow{id} & B \end{array} \quad (1.9)$$

where $ev(1)$ is the map of evaluation at the point $1 \in [0, 1]$. It can be also directly defined as follows. In the algebra $\mathbf{C}((0, 1]) \otimes A \oplus B$ consider the closed two-sided ideal $\langle \{1\} \otimes a - f(a), \forall a \in A \rangle$, generated by elements of type $\{1\} \otimes a - f(a), \forall a \in A$. We define the *mapping cone* as the quotient algebra

$$\text{Cone}(f) = \text{Cone}(f : A \rightarrow B) := (\mathbf{C}_0((0, 1]) \otimes A \oplus B) / \langle \{1\} \otimes a - f(a), \forall a \in A \rangle. \quad (1.10)$$

Remark 1.11 *It is easy to show that B is included in $\text{Cone}(f : A \rightarrow B)$.*

Theorem 1.12 (Noncommutative Leray Theorem for HP-groups) *Let*

$$\{0\} = A_{-1} \subseteq A_0 \subseteq \cdots \subseteq A_k = A \quad (1.11)$$

be a NCCW complex structure of A . Then for every $r \geq 0$, and all p, q , there are groups $E_{p,q}^r$ such that $E_{p,q}^r = 0$ for all $p < 0$ or $p > k$, and homomorphisms

$$d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r \quad (1.12)$$

such that $d_{p-r,q+r-1}^r \circ d_{p,q}^r = 0$ and that

1.

$$E_{p,q}^{r+1} = \ker d_{p,q}^r / \text{Im } d_{p+r,q-r+1}^r. \quad (1.13)$$

2.

$$E_{p,q}^0 = C_q(A_p, A_{p-1}). \quad (1.14)$$

3. $d_{p,q}^0$ is coincided with the boundary operator

$$\partial : C_{p+q}(A_p, A_{p-1}) \rightarrow C_{p+q-1}(A_p, A_{p-1}) \quad (1.15)$$

4.

$$E_{p,q}^1 = \text{HP}_{p+q}(A_p, A_{p-1}). \quad (1.16)$$

5. $d_{p,q}^1$ is coincided with the homomorphism

$$\partial_* : \text{HP}_{p+q}(A_p, A_{p-1}) \rightarrow \text{HP}_{p+q-1}(A_{p-1}, A_{p-2}), \quad (1.17)$$

associated with the triple (A_p, A_{p-1}, A_{p-2}) .

6.

$$E_{p,q}^\infty = \frac{\text{Im}(\text{HP}_{p+q}(A_p) \rightarrow \text{HP}_{p+q}(A))}{\text{Im}(\text{HP}_{p+q}(A_{p-1}) \rightarrow \text{HP}_{p+q}(A))} = \frac{{}_{(p)}\text{HP}_{p+q}(A)}{{}_{(p-1)}\text{HP}_{p+q}(A)}. \quad (1.18)$$

PROOF. The proof is quite similar to the same one in the classical algebraic topology. There is no need of special modification. \square

Theorem 1.13 (Noncommutative Leray Theorem for \mathbf{K} -groups) *Let*

$$\{0\} = A_{-1} \subseteq A_0 \subseteq \cdots \subseteq A_k = A \quad (1.19)$$

be NCCW complex structure of A . Then for every $r \geq 0$, and all p, q , there are groups $E_{p,q}^r$ such that $E_{p,q}^r = 0$ for all $p < 0$ or $p > k$, and homomorphisms

$$d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1} \quad (1.20)$$

such that $d_{p-r,q+r-1}^r \circ d_{p,q}^r = 0$ and that

1.

$$E_{p,q}^{r+1} = \ker d_{p,q}^r / \text{Im } d_{p+r,q-r+1}^r. \quad (1.21)$$

2.

$$E_{p,q}^0 = C(A_p, A_{p-1}). \quad (1.22)$$

3. $d_{p,q}^0$ is coincided with the boundary operator

$$\partial : C_{p+q}(A_p, A_{p-1}) \rightarrow C_{p+q-1}(A_p, A_{p-1}). \quad (1.23)$$

4.

$$E_{p,q}^1 = \mathbf{K}_{p+q}(A_p, A_{p-1}) \quad (1.24)$$

5. $d_{p,q}^1$ is coincided with the homomorphism

$$\partial_* : K_{p+q}(A_p, A_{p-1}) \rightarrow K_{p+q-1}(A_{p-1}, A_{p-2}), \quad (1.25)$$

associated with the triple (A_p, A_{p-1}, A_{p-2}) .

6.

$$E_{p,q}^\infty = \frac{\text{Im}(K_{p+q}(A_p) \rightarrow K_{p+q}(A))}{\text{Im}(K_{p+q}(A_{p-1}) \rightarrow K_{p+q}(A))} = \frac{{}_{(p)}K_{p+q}(A)}{{}_{(p-1)}K_{p+q}(A)}. \quad (1.26)$$

PROOF. The proof is quite similar to the same one in the classical algebraic topology. There is no need of special modification. \square

2 Spectral Sequences for Noncommutative Serre Fibrations

Definition 2.1 Let $A \hookrightarrow B$ be a Serre fibration. The system of all homotopy equivalences

$$i : (B : A) := \Pi(A, \text{Cone}(B, A)) \rightarrow B \quad (2.1)$$

is called the **system of local coefficients**. We say that a system of local coefficients is **simple** iff the K -theory and cyclic theory of $(B : A)$ is independent of the choice of i .

From now on we consider only NC Serre fibrations with simple system of local coefficients.

Lemma 2.2 For the Serre fibration $f : A \hookrightarrow B$ such that the spectrum $\hat{B} \setminus \hat{A}$ has a cellular decomposition

$$\hat{B} \setminus \hat{A} = \bigcup_{\alpha} (\mathbf{D}_{\alpha}^p, \mathbf{S}_{\alpha}^{p-1}), \quad \mathbf{D}_{\alpha}^p = \mathbf{D}^p, \mathbf{S}_{\alpha}^{p-1} = \mathbf{S}^{p-1} \quad (2.2)$$

i.e.

$$B = A \bigoplus_{\mathbf{S}^{p-1}F_p} \mathbf{I}^p F_p \quad (2.3)$$

for some finite set F_p , we have

$$A \otimes \mathbf{C}([0, 1]) \oplus B/(A \times \{0\} \oplus f(A) \oplus A \otimes \mathbf{C}(\{1\})) \cong A \otimes \mathcal{K} \otimes \mathbf{C}(\hat{B} \setminus \hat{A}) \quad (2.4)$$

Theorem 2.3 For an arbitrary NC Serre fibration $A \hookrightarrow B$ with simple system of local coefficients, we have

1.

$$E_{p,q}^1 = \mathcal{C}_p(A; \text{HP}_q(B, A)), \quad (2.5)$$

2.

$$d_{p,q}^1 = \partial : \mathcal{C}_p(A; \text{HP}_q(B, A)) \rightarrow \mathcal{C}_{p-1}(A; \text{HP}_q(B, A)), \quad (2.6)$$

3.

$$E_{p,q}^2 = \text{HP}_p(A; \text{HP}_q(B, A)). \quad (2.7)$$

PROOF. In the category of NCCW complexes, we have the fact that the graded de Rham cohomology of a cell is isomorphic to the periodic cyclic homology of the corresponding NC cell. And therefore

$$\begin{aligned} E_{p,q}^1 &= \frac{\ker\{\text{HP}_q(A_p, A_{p-1} \rightarrow \text{HP}_q(A_{p-1}, A_{p-2}))\}}{\ker\{\text{HP}_q(A_p, A_{p-1} \rightarrow \text{HP}_q(A_p, A_{p-1}))\}} = \\ &= \text{HP}_{p+q}(A_p, A_{p-1}) \cong \text{HP}\left(\bigvee_{\text{p-dimensional cells}} C(\mathbf{S}^p) \otimes \mathcal{K} \otimes A\right) \cong \\ &\cong \bigvee_{\text{p-dimensional cells}} \text{HP}_{p+q}(C(\mathbf{D}^p) \otimes \mathcal{K} \otimes B, C(\mathbf{S}^{p-1}) \otimes \mathcal{K} \otimes A) \cong \\ &\cong \bigvee_{\text{p-dimensional cells}} \text{HP}_q(\mathbf{C}) \otimes \text{HP}_q(A) \cong \mathcal{C}_q(B, A; \text{HP}_q(\mathbf{C})). \end{aligned} \quad (2.8)$$

It is easy to see that

$$\tilde{d}_{p,q}^1 = \delta : \mathcal{C}_p(A, \tilde{\text{HP}}_q(B, A)) \rightarrow \mathcal{C}_{p+1}(A; \tilde{\text{HP}}_q(B, A)) \quad (2.9)$$

and therefore

$$\tilde{E}_{p,q}^2 = \text{HP}(A; \tilde{\text{HP}}_q(B, A)). \quad (2.10)$$

□

Corollary 2.4 *In the category of NCCW complexes for an arbitrary map $A \rightarrow B$, we have*

1.

$$E_{p,q}^1 = \mathcal{C}_p(A, \text{HP}_q(B, A)), \quad (2.11)$$

2.

$$d_{p,q}^1 = \partial : \mathcal{C}_p(A; \text{HP}_q(B, A)) \rightarrow \mathcal{C}_{p-1}(A, \text{HP}_q(B, A)), \quad (2.12)$$

3.

$$E_{p,q}^2 = \text{HP}_p(A; \text{HP}_q(B, A)). \quad (2.13)$$

By analogy we have also the same results for K-groups

Theorem 2.5 *For an arbitrary NC Serre fibration $A \hookrightarrow B$ with simple system of local coefficients, we have*

1.

$$E_{p,q}^1 = \mathcal{C}_p(A, K_q(B, A)), \quad (2.14)$$

2.

$$d_{p,q}^1 = \partial : \mathcal{C}_p(A, K_q(B, A)) \rightarrow \mathcal{C}_{p-1}(A, K_q(B, A)), \quad (2.15)$$

3.

$$E_{p,q}^2 = \text{HP}_p(A; K_q(B, A)). \quad (2.16)$$

PROOF. In the category of NCCW complexes, we have the fact that the graded de Rham cohomology of a cell is isomorphic to the periodic cyclic homology of the corresponding NC cell. And therefore

$$\begin{aligned} E_{p,q}^1 &= \frac{\ker\{K(A_p, A_{p-1} \rightarrow K(A_{p-1}, A_{p-2}))\}}{\ker\{K(A_p, A_{p-1} \rightarrow K(A_p, A_{p-1}))\}} = \\ &= K_{p+q}(A_p, A_{p-1}) \cong K\left(\bigvee_{\text{p-dimensional cells}} C(\mathbf{D}^p) \otimes \mathcal{K} \otimes A\right) \cong \\ &\cong \bigvee_{\text{p-dimensional cells}} K_{p+q}(C(\mathbf{D}^p) \otimes \mathcal{K} \otimes A, \mathbf{S}^{p-1} \otimes \mathcal{K} \otimes A) \cong \\ &\cong \bigvee_{\text{p-dimensional cells}} K_q(A) \cong \mathcal{C}_q(A; K_q(B, A)). \end{aligned} \quad (2.17)$$

It is easy to see that

$$\tilde{d}_{p,q}^1 = \delta : \mathcal{C}_p(A; \tilde{K}_q(B, A)) \rightarrow \mathcal{C}_{p+1}(A; \tilde{K}_q(B, A)) \quad (2.18)$$

and therefore

$$\tilde{E}_{p,q}^2 = \text{HP}_p(A; \tilde{K}_q(B, A)). \quad (2.19)$$

□

Corollary 2.6 *In the category of NCCW complexes for an arbitrary morphism $A \rightarrow B$, we have*

1.

$$E_{p,q}^1 = \mathcal{C}_p(A; K_q(B, A)), \quad (2.20)$$

2.

$$d_{p,q}^1 = \partial : \mathcal{C}_p(A; K_q(B, A)) \rightarrow \mathcal{C}_{p-1}(A; K_q(B, A)), \quad (2.21)$$

3.

$$E_{p,q}^2 = \text{HP}_p(A; K_q(B, A)). \quad (2.22)$$

The following remark was suggested by C. Schochet

Remark 2.7 These theorem are true because both K-theory and cyclic theory have very strong excision properties (analogous in spaces to the property that given a space X and a closed subspace Y with $f : Y \rightarrow X$ the inclusion, that the natural map $\text{Cone}(f) \rightarrow X/Y$ induces an isomorphism on K-theory and cyclic theory (but this fails for many other theories.))

Remark 2.8 In a subsequent paper, we shall prove that for the induced noncommutative Serre fibration with simple system of local coefficients there is also noncommutative analog of the Eilenberg-Moore spectral sequences, which are collapsing at E^2 terms if the system of local coefficients are simple.

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