

THE TOWER OF K -THEORY OF TRUNCATED POLYNOMIAL ALGEBRAS

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Introduction

Suppose that A is a regular noetherian ring that is also an \mathbb{F}_p -algebra. Then it was proved by the author and Madsen [5, 7] that there is a long-exact sequence

$$\cdots \longrightarrow \bigoplus_{i \geq 0} \mathbb{W}_{i+1} \Omega_A^{q-2i} \xrightarrow{V_m} \bigoplus_{i \geq 0} \mathbb{W}_{m(i+1)} \Omega_A^{q-2i} \xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^m), (x)) \longrightarrow \cdots$$

which expresses the K -groups of the truncated polynomial algebra $A[x]/(x^m)$ relative to the ideal (x) in terms of the groups $\mathbb{W}_r \Omega_A^q$ of big de Rham-Witt forms; see also [4]. In this paper, we consider the map of relative K -groups

$$f_* : K_{q+1}(A[x]/(x^m), (x)) \rightarrow K_{q+1}(A[x]/(x^n), (x))$$

induced by the canonical projection

$$f : A[x]/(x^m) \rightarrow A[x]/(x^n).$$

To state the result, we recall from [7] that the big de Rham-Witt groups $\mathbb{W}_r \Omega_A^q$ are modules over the ring $\mathbb{W}(A)$ of big Witt vectors in A . In particular, the big de Rham-Witt groups $\mathbb{W}_r \Omega_A^q$ are modules over $\mathbb{W}(\mathbb{F}_p)$. The ring $\mathbb{W}(\mathbb{F}_p)$ is canonically isomorphic to the product indexed by the set I_p of positive integers that are not divisible by p of copies of the ring \mathbb{Z}_p of p -adic integers. A unit α of the total quotient ring of $\mathbb{W}(\mathbb{F}_p)$ determines a divisor $\text{div}(\alpha)$ on $\mathbb{W}(\mathbb{F}_p)$ and, conversely, the unit α is determined, up to multiplication by a unit of $\mathbb{W}(\mathbb{F}_p)$, by the divisor $\text{div}(\alpha)$. The ring $\mathbb{W}_r(\mathbb{F}_p)$ of big Witt vectors of length r in \mathbb{F}_p determines a divisor on $\mathbb{W}(\mathbb{F}_p)$ that we denote by $\text{div}(\mathbb{W}_r(\mathbb{F}_p))$.

THEOREM A. *Let A be a regular noetherian ring and an \mathbb{F}_p -algebra. Then the canonical projection $f : A[x]/(x^m) \rightarrow A[x]/(x^n)$ induces a map of long-exact sequences*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \bigoplus_{i \geq 0} \mathbb{W}_{i+1} \Omega_A^{q-2i} & \xrightarrow{V_m} & \bigoplus_{i \geq 0} \mathbb{W}_{m(i+1)} \Omega_A^{q-2i} & \xrightarrow{\varepsilon} & K_{q+1}(A[x]/(x^m), (x)) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \bigoplus_{i \geq 0} \mathbb{W}_{i+1} \Omega_A^{q-2i} & \xrightarrow{V_n} & \bigoplus_{i \geq 0} \mathbb{W}_{n(i+1)} \Omega_A^{q-2i} & \xrightarrow{\varepsilon} & K_{q+1}(A[x]/(x^n), (x)) \longrightarrow \cdots \end{array}$$

where the right-hand vertical map is the map of relative K -groups induced by the canonical projection, where the middle vertical map takes the i th summand of the domain to the i th summand of the target by the composition

$$\mathbb{W}_{m(i+1)} \Omega_A^{q-2i} \xrightarrow{\text{res}} \mathbb{W}_{n(i+1)} \Omega_A^{q-2i} \xrightarrow{m_\alpha} \mathbb{W}_{n(i+1)} \Omega_A^{q-2i}$$

of the restriction map and the multiplication by an element $\alpha = \alpha_p(m, n, i)$ of $\mathbb{W}(\mathbb{F}_p)$ that is determined, up to a unit, by the effective divisor

$$\operatorname{div}(\alpha) = \sum_{0 \leq h < i} (\operatorname{div}(\mathbb{W}_{m(h+1)}(\mathbb{F}_p)) - \operatorname{div}(\mathbb{W}_{n(h+1)}(\mathbb{F}_p))),$$

and where the left-hand vertical map is zero.

The divisor $\operatorname{div}(\alpha_p(m, n, i))$ depends not only on the integers m , n , and i , but also on the prime p . The lengths as \mathbb{Z}_p -module of the domain and target of the map

$$f_*: K_{2i+1}(\mathbb{F}_p[x]/(x^m), (x)) \rightarrow K_{2i+1}(\mathbb{F}_p[x]/(x^n), (x))$$

are equal to $(m-1)(i+1)$ and $(n-1)(i+1)$, respectively, and hence, do not depend on the prime p . By contrast, the lengths as a \mathbb{Z}_p -module of the kernel $K_{2i+1}(\mathbb{F}_p[x]/(x^m), (x^n))$ and cokernel $K_{2i}(\mathbb{F}_p[x]/(x^m), (x^n))$ are arithmetic functions of the prime p . We note that, if

$$\operatorname{div}(\alpha_p(m, n, i)) \geq \operatorname{div}(\mathbb{W}_{n(i+1)}(\mathbb{F}_p))$$

then the multiplication by α map

$$m_\alpha: \mathbb{W}_{n(i+1)}\Omega_A^{q-2i} \rightarrow \mathbb{W}_{n(i+1)}\Omega_A^{q-2i}$$

is the zero map. The following result records when this is the case.

THEOREM B. *The element $\alpha_p(m, n, i) \in \mathbb{W}(\mathbb{F}_p)$ satisfies the following:*

(i) *For every pair of integers $m > n > 1$ and for every prime number p , there exists an integer $i_0 = i_0(m, n, p)$ such that, for all integers $i \geq i_0$,*

$$\operatorname{div}(\alpha_p(m, n, i)) \geq \operatorname{div}(\mathbb{W}_{n(i+1)}(\mathbb{F}_p)).$$

(ii) *For every pair of integers $m > n > 1$, the integer $i_0(m, n, p)$ tends to infinity as the prime number p tends to infinity.*

(iii) *For every integer $n > 1$ and every prime number p , there exists an integer $m_0 = m_0(n, p)$ such that, for all $m \geq m_0(n, p)$ and for all positive integers i ,*

$$\operatorname{div}(\alpha_p(m, n, i)) \geq \operatorname{div}(\mathbb{W}_{n(i+1)}(\mathbb{F}_p)).$$

The proof of Thm. A is based on the result [5, Prop. 4.2.3] which expresses the K -theory of $A[x]/(x^m)$ relative to the ideal (x) in terms of the $RO(\mathbb{T})$ -graded equivariant homotopy groups of the topological Hochschild \mathbb{T} -spectrum $T(A)$. Here and throughout $\mathbb{T} = S(\mathbb{C})$ is the multiplicative group of complex numbers of modulus 1. More precisely, for every \mathbb{F}_p -algebra A , there is a long-exact sequence

$$\cdots \longrightarrow \lim_R \operatorname{TR}_{q-\lambda_d}^{r/m}(A) \xrightarrow{V_m} \lim_R \operatorname{TR}_{q-\lambda_d}^r(A) \xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^m), (x)) \longrightarrow \cdots,$$

and the corresponding sequence for the groups with $\mathbb{Z}/p^s\mathbb{Z}$ -coefficients is valid, for every ring A . We briefly explain the terms in this sequence. Let λ be a finite dimensional orthogonal \mathbb{T} -representation, let S^λ be the one-point compactification, and let $C_r \subset \mathbb{T}$ be the subgroup of order r . Then one defines

$$\operatorname{TR}_{q-\lambda}^r(A) = [S^q \wedge (\mathbb{T}/C_r)_+, T(A) \wedge S^\lambda]_{\mathbb{T}}$$

to be the abelian group of maps in the \mathbb{T} -stable homotopy category between the indicated \mathbb{T} -spectra. Suppose that $r = st$. Then there are maps

$$\begin{aligned} R_s: \operatorname{TR}_{q-\lambda}^r(A) &\rightarrow \operatorname{TR}_{q-\lambda'}^t(A) && \text{(restriction)} \\ F_s: \operatorname{TR}_{q-\lambda}^r(A) &\rightarrow \operatorname{TR}_{q-\lambda}^t(A) && \text{(Frobenius)} \\ V_s: \operatorname{TR}_{q-\lambda}^t(A) &\rightarrow \operatorname{TR}_{q-\lambda}^r(A) && \text{(Verschiebung)} \end{aligned}$$

where $\lambda' = \rho_s^*(\lambda^{C_s})$ is the \mathbb{T}/C_s -representation λ^{C_s} considered as a \mathbb{T} -representation via the root isomorphism $\rho_s: \mathbb{T} \rightarrow \mathbb{T}/C_s$ defined by $\rho_s(z) = z^{1/s}C_s$. In the case at hand,

$$d = d(m, r) = \left\lfloor \frac{r-1}{m} \right\rfloor$$

is the largest integer less than or equal to $(r-1)/m$, and λ_d is the orthogonal \mathbb{T} -representation

$$\lambda_d = \mathbb{C}(1) \oplus \mathbb{C}(2) \oplus \cdots \oplus \mathbb{C}(d),$$

where $\mathbb{C}(i) = \mathbb{C}$ with \mathbb{T} acting from the left by $z \cdot w = z^i w$. We prove the following general result which does not require the ring A to be regular or noetherian.

THEOREM C. *Let A be an \mathbb{F}_p -algebra, and let $f: A[x]/(x^m) \rightarrow A[x]/(x^n)$ be the canonical projection. Then there is a map of long-exact sequences*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \lim_R \mathrm{TR}_{q-\lambda_d}^{r/m}(A) & \xrightarrow{V_m} & \lim_R \mathrm{TR}_{q-\lambda_d}^r(A) & \xrightarrow{\varepsilon} & K_{q+1}(A[x]/(x^m), (x)) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \lim_R \mathrm{TR}_{q-\lambda_e}^{r/n}(A) & \xrightarrow{V_n} & \lim_R \mathrm{TR}_{q-\lambda_e}^r(A) & \xrightarrow{\varepsilon} & K_{q+1}(A[x]/(x^n), (x)) \longrightarrow \cdots, \end{array}$$

where $d = d(m, r) = \lfloor (r-1)/m \rfloor$ and $e = e(n, r) = \lfloor (r-1)/n \rfloor$, where the right-hand vertical map is the map of relative K -groups induced by the canonical projection, where the middle vertical map is the map of limits that is given by the maps

$$\iota(m, n, r)_q: \mathrm{TR}_{q-\lambda_d}^r(A) \rightarrow \mathrm{TR}_{q-\lambda_e}^r(A)$$

induced by the canonical inclusions $\iota(m, n, r): S^{\lambda_d} \rightarrow S^{\lambda_e}$, and where the left-hand vertical map is zero. The corresponding statement for the groups with $\mathbb{Z}/p^s\mathbb{Z}$ -coefficients is valid, for every ring A .

We remark that the derived limits corresponding to the limits that appear in the statement of Thm. C vanish.

The paper is organized as follows. In Sect. 1 we explain the p -typical decomposition of the terms in the long-exact sequences of Thms. A and C and restate the two theorems in this form. In Sect. 2 we prove Thm. C. Sect. 3 is concerned with a long-exact sequence related to the inclusion of λ_{a-1} in λ_a . In Sect. 4 we evaluate the map $\iota(m, n, r)_q: \mathrm{TR}_{q-\lambda_d}^r(A) \rightarrow \mathrm{TR}_{q-\lambda_e}^r(A)$ and prove Thm. A. Finally, in Sect. 5, we investigate the divisor $\mathrm{div}(\alpha_p(m, n, i))$ and prove Thm. B and some consequences for the tower of K -theory of truncated polynomial algebras.

1. p -typical decompositions

The long-exact sequences that appear in Thms. A and C of the introduction have canonical p -typical product decompositions. We here recall the p -typical decomposition and state the equivalent Thms. 1.7 and 1.3.

It is proved in [5, Prop. 4.2.5] that the groups $\mathrm{TR}_{q-\lambda}^r(A)$, which appear in the statement of Thm. C, decompose as a product of the p -typical groups

$$\mathrm{TR}_{q-\lambda}^u(A; p) = \mathrm{TR}_{q-\lambda}^{p^{u-1}}(A) = [S^q \wedge (\mathbb{T}/C_{p^{u-1}})_+, T(A) \wedge S^\lambda]_{\mathbb{T}}.$$

To state the result, we write $r = p^{u-1}r'$ where r' is not divisible by p . Let j be a divisor of r' , and let $\lambda' = \lambda'(j) = \rho_j^*(\lambda^{C_j})$. We let

$$\gamma_j: \mathrm{TR}_{q-\lambda}^r(A) \rightarrow \mathrm{TR}_{q-\lambda'}^u(A; p)$$

be the composite map

$$\mathrm{TR}_{q-\lambda}^r(A) \xrightarrow{F_j} \mathrm{TR}_{q-\lambda}^{r/j}(A) \xrightarrow{R_{r'/j}} \mathrm{TR}_{q-\lambda'}^{p^{u-1}}(A) = \mathrm{TR}_{q-\lambda'}^u(A; p)$$

and define

$$\gamma: \mathrm{TR}_{q-\lambda}^r(A) \rightarrow \prod_j \mathrm{TR}_{q-\lambda'}^u(A; p)$$

be the product of the maps γ_j as j ranges over the divisors of r' . The map γ is an isomorphism, for every $\mathbb{Z}_{(p)}$ -algebra A , by [5, Prop. 4.2.3]. The maps R_s , F_s , and V_s are similarly expressed as products of their p -typical analogs

$$\begin{aligned} R &= R_p: \mathrm{TR}_{q-\lambda}^u(A; p) \rightarrow \mathrm{TR}_{q-\lambda'}^{u-1}(A; p) && \text{(restriction).} \\ F &= F_p: \mathrm{TR}_{q-\lambda}^u(A; p) \rightarrow \mathrm{TR}_{q-\lambda}^{u-1}(A; p) && \text{(Frobenius)} \\ V &= V_p: \mathrm{TR}_{q-\lambda}^{u-1}(A; p) \rightarrow \mathrm{TR}_{q-\lambda}^u(A; p) && \text{(Verschiebung)} \end{aligned}$$

Suppose that $r = st$ and write $s = p^v s'$ and $t = p^{u-v-1} t'$ with s' and t' not divisible by p . Then there are commutative diagrams

$$\begin{array}{ccc} \mathrm{TR}_{q-\lambda}^r(A) & \xrightarrow{\gamma} & \prod_j \mathrm{TR}_{q-\lambda'}^u(A; p) \\ \downarrow R_s & & \downarrow R_s^\gamma \\ \mathrm{TR}_{q-\lambda'}^t(A) & \xrightarrow{\gamma} & \prod_j \mathrm{TR}_{q-\lambda'}^{u-v}(A; p) \end{array} \qquad \begin{array}{ccc} \mathrm{TR}_{q-\lambda}^r(A) & \xrightarrow{\gamma} & \prod_j \mathrm{TR}_{q-\lambda'}^u(A; p) \\ \begin{array}{c} \uparrow V_s \\ \downarrow F_s \end{array} & & \begin{array}{c} \uparrow V_s^\gamma \\ \downarrow F_s^\gamma \end{array} \\ \mathrm{TR}_{q-\lambda}^t(A) & \xrightarrow{\gamma} & \prod_j \mathrm{TR}_{q-\lambda'}^{u-v}(A; p) \end{array}$$

where the maps R_s^γ , F_s^γ , and V_s^γ are defined as follows: The map R_s^γ takes the factor indexed by a divisor j of t' to the factor indexed by the same divisor j of t' by the map R^v and annihilates the factors indexed by divisors j of r' that do not divide t' . The map F_s^γ takes the factor indexed by a divisor j of r' that is divisible by s' to the factor indexed by the divisor j/s' of t' by the map F^v and annihilates the remaining factors. Finally, the map V_s^γ takes the factor indexed by the divisor j of t' to the factor indexed by the divisor $s'j$ of r' by the map $s'V^v$.

It is now straightforward to check that the p -typical decomposition of the top long-exact sequence in the diagram in the statement of Thm. C of the introduction takes the form

$$\begin{aligned} \cdots &\rightarrow \prod_{j \in m' I_p} \lim_R \mathrm{TR}_{q-\lambda_d}^{u-v}(A; p) \xrightarrow{m' V^v} \prod_{j \in I_p} \lim_R \mathrm{TR}_{q-\lambda_d}^u(A; p) \\ &\xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^m), (x)) \xrightarrow{\partial} \prod_{j \in m' I_p} \lim_R \mathrm{TR}_{q-1-\lambda_d}^{u-v}(A; p) \rightarrow \cdots \end{aligned} \tag{1.1}$$

where we have written $m = p^v m'$ with m' not divisible by p , and where the integer d , which depends on m , j , and p , is defined by

$$d = d_p(m, u, j) = \left\lfloor \frac{p^{u-1} j - 1}{m} \right\rfloor.$$

In the top line, the right-hand product ranges over the set I_p of positive integers not divisible by p , and the left-hand product ranges over the subset $m' I_p \subset I_p$. Similarly, the bottom long-exact sequence in the diagram in the statement of Thm. C takes the form

$$\begin{aligned} \cdots &\rightarrow \prod_{j \in n' I_p} \lim_R \mathrm{TR}_{q-\lambda_e}^{u-w}(A; p) \xrightarrow{n' V^w} \prod_{j \in I_p} \lim_R \mathrm{TR}_{q-\lambda_e}^u(A; p) \\ &\xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^n), (x)) \xrightarrow{\partial} \prod_{j \in n' I_p} \lim_R \mathrm{TR}_{q-1-\lambda_e}^{u-w}(A; p) \rightarrow \cdots \end{aligned} \tag{1.2}$$

where $n = p^w n'$ with n' not divisible by p , and where $e = d_p(n, u, j)$. It follows that Thm. C of the introduction is equivalent to the following statement:

THEOREM 1.3. *Let A be an \mathbb{F}_p -algebra, and let $f: A[x]/(x^m) \rightarrow A[x]/(x^n)$ be the canonical projection. Then there is a map of long-exact sequence from the sequence (1.1) to the sequence (1.2) that is given, on the lower left-hand terms, by the map*

$$f_*: K_*(A[x]/(x^m), (x)) \rightarrow K_*(A[x]/(x^n), (x))$$

induced by the canonical projection, on the upper right-hand terms, by the map that takes the j th factor of the domain to the j th factor of the target by the map

$$\iota_p(m, n, j)_q: \lim_R \mathrm{TR}_{q-\lambda_d}^u(A; p) \rightarrow \lim_R \mathrm{TR}_{q-\lambda_e}^u(A; p)$$

induced from the canonical inclusions $\iota_p(m, n, j, u): S^{\lambda_d} \rightarrow S^{\lambda_e}$, and, on the upper left-hand terms, by the zero map. The corresponding statement for the groups with $\mathbb{Z}/p^s\mathbb{Z}$ -coefficients is valid for any ring A .

The limits systems that appear in the statement of Thm. 1.3 stabilize. In particular, the corresponding derived limits vanish. More precisely, we have the following result.

LEMMA 1.4. *In the commutative diagram*

$$\begin{array}{ccc} \lim_R \mathrm{TR}_{q-\lambda_d}^u(A; p) & \xrightarrow{\iota_p(m, n, j)_q} & \lim_R \mathrm{TR}_{q-\lambda_e}^u(A; p) \\ \downarrow R & & \downarrow R \\ \mathrm{TR}_{q-\lambda_d}^u(A; p) & \xrightarrow{\iota_p(m, n, j, u)_q} & \mathrm{TR}_{q-\lambda_e}^u(k; p), \end{array}$$

the left-hand vertical map is an isomorphism, if $q < 2d$, and right-hand vertical map is an isomorphism, if $q < 2e$.

Proof. We prove that the left-hand vertical map is an isomorphism, for $q < 2d$. Let r be a positive integer, let $d = d_p(m, r, j)$, and let $d' = d_p(m, r-1, j)$. It follows from [6, Thm. 2.2] that there is a long-exact sequence

$$\cdots \rightarrow \mathbb{H}_q(C_{p^{r-1}}, T(A) \wedge S^{\lambda_d}) \xrightarrow{N} \mathrm{TR}_{q-\lambda_d}^r(A; p) \xrightarrow{R} \mathrm{TR}_{q-\lambda_{d'}}^{r-1}(A; p) \rightarrow \cdots$$

The Borel homology group that appear on the left-hand side is the abutment of a first quadrant homology type spectral sequence

$$E_{s,t}^2 = H_s(C_{p^{r-1}}, \mathrm{TR}_{t-\lambda_d}^1(A; p)) \Rightarrow \mathbb{H}_{s+t}(C_{p^{r-1}}, T(A) \wedge S^{\lambda_d}),$$

and the groups $\mathrm{TR}_{q-\lambda_d}^1(A; p)$ are zero, for $q < 2d$. Hence, we may conclude that the map R in the long-exact sequence above is an isomorphism, for $q < 2d$. \square

We next recall from [7, Cor. 1.2.6] that the big de Rham-Witt groups $\mathbb{W}_r\Omega_A^q$ decompose as a product of the more familiar p -typical de Rham-Witt groups $W_s\Omega_A^q$ of Bloch-Deligne-Illusie [10]. More generally, there is a big de Rham-Witt group $\mathbb{W}_S\Omega_A^q$ associated with every subset $S \subset \mathbb{N}$ of the set of positive integers that is stable under division. The group $\mathbb{W}_\emptyset\Omega_A^q$ is zero, the group $\mathbb{W}_r\Omega_A^q$ is the big de Rham-Witt group associated with the set of positive integers less than or equal to r , and the p -typical de Rham-Witt group $W_s\Omega_A^q$ is the big de Rham-Witt group

$$W_s\Omega_A^q = \mathbb{W}_{\{1, p, \dots, p^{s-1}\}}\Omega_A^q.$$

For every pair of subset $T \subset S \subset \mathbb{N}$ that are stable under division, there is a restriction map

$$\mathrm{res}: \mathbb{W}_S\Omega_A^q \rightarrow \mathbb{W}_T\Omega_A^q.$$

We mention that, if $S \subset \mathbb{N}$ is the union of a family of subsets $S_\alpha \subset \mathbb{N}$ each of which is stable under division, then the restriction maps induce an isomorphism

$$\mathbb{W}_S \Omega_A^q \xrightarrow{\sim} \lim_{\text{res}} \mathbb{W}_{S_\alpha} \Omega_A^q.$$

Let $S \subset \mathbb{N}$ be a subset stable under division, let s be any positive integer, and let S/s be the set of positive integers t such that $st \in S$. Then there are maps

$$\begin{aligned} F_s &: \mathbb{W}_S \Omega_A^q \rightarrow \mathbb{W}_{S/s} \Omega_A^q && \text{(Frobenius)} \\ V_s &: \mathbb{W}_{S/s} \Omega_A^q \rightarrow \mathbb{W}_S \Omega_A^q && \text{(Verschiebung)}. \end{aligned}$$

Now, let $S \subset \mathbb{N}$ be a subset stable under division, let $j \in S$ be an integer that is not divisible by p , and let $u = u_p(S, j)$ be the largest positive integer such that $p^{u-1}j \in S$. We let

$$\eta_j: \mathbb{W}_S \Omega_A^q \rightarrow W_u \Omega_A^q$$

be the composite map

$$\mathbb{W}_S \Omega_A^q \xrightarrow{F_j} \mathbb{W}_{S/j} \Omega_A^q \xrightarrow{\text{res}} \mathbb{W}_{\{1, p, \dots, p^{u-1}\}} \Omega_A^q = W_u \Omega_A^q$$

and define

$$\eta: \mathbb{W}_S \Omega_A^q \rightarrow \prod_j W_u \Omega_A^q$$

to be the map given by the maps η_j as j ranges over all integers $j \in S$ that are not divisible by p . The map η is an isomorphism, for every $\mathbb{Z}_{(p)}$ -algebra, by [7, Cor. 1.2.6]. Again, the maps res , F_s , and V_s are expressed as products of their p -typical analogs

$$\begin{aligned} R &= \text{res}: W_s \Omega_A^q \rightarrow W_{s-1} \Omega_A^q && \text{(restriction).} \\ F &= F_p: W_s \Omega_A^q \rightarrow W_{s-1} \Omega_A^q && \text{(Frobenius)} \\ V &= V_p: W_{s-1} \Omega_A^q \rightarrow W_s \Omega_A^q && \text{(Verschiebung)} \end{aligned}$$

Let $T \subset S \subset \mathbb{N}$ be a pair of subsets stable under division. Then there is a commutative diagram

$$\begin{array}{ccc} \mathbb{W}_S \Omega_A^q & \xrightarrow{\eta} & \prod_j W_u \Omega_A^q \\ \downarrow \text{res} & & \downarrow \text{res}^\eta \\ \mathbb{W}_T \Omega_A^q & \xrightarrow{\eta} & \prod_j W_{u'} \Omega_A^q, \end{array}$$

where $u = u_p(S, j)$ and $u' = u_p(T, j)$, and where the map res^η takes the factor indexed by an integer $j \in T$ that is not divisible by p to the factor indexed by the same integer j by the map $R^{u-u'}$ and annihilates the remaining factors. Similarly, let $S \subset \mathbb{N}$ be a subset stable under division, and let s be a positive integer. We write $s = p^v s'$ with s' not divisible by p . Then there are commutative diagrams

$$\begin{array}{ccc} \mathbb{W}_S \Omega_A^q & \xrightarrow{\eta} & \prod_j W_u \Omega_A^q \\ V_s \uparrow \downarrow F_s & & V_s^\eta \uparrow \downarrow F_s^\eta \\ \mathbb{W}_{S/s} \Omega_A^q & \xrightarrow{\eta} & \prod_j W_{u-v} \Omega_A^q, \end{array}$$

where $u = u_p(S, j)$, and where the maps F_s^η and V_s^η are defined as follows: The map F_s^η takes the factor indexed by an integer $j \in S$ that is not divisible by p , is divisible by s' , and satisfies $p^v j \in S$ to the factor indexed by the integer $j/s' \in S/s$ by the map F^v and annihilates the remaining factors. The map V_s^η takes the factor indexed by $j \in S/s$ not divisible by p to the factor indexed by $s'j \in S$ by the map $s'V^v$.

Suppose that A is an \mathbb{F}_p -algebra and a regular noetherian ring. It is straightforward to verify that the p -typical decomposition of the top long-exact sequence in the diagram in the statement of Thm. A of the introduction takes the form

$$\begin{aligned} \cdots \rightarrow \bigoplus_{i \geq 0} \bigoplus_{j \in m'I_p} W_{s-v} \Omega_A^{q-2i} \xrightarrow{m'V^v} \bigoplus_{i \geq 0} \bigoplus_{j \in I_p} W_s \Omega_A^{q-2i} \\ \xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^m), (x)) \xrightarrow{\partial} \bigoplus_{i \geq 0} \bigoplus_{j \in m'I_p} W_{s-v} \Omega_A^{q-1-2i} \rightarrow \cdots \end{aligned} \quad (1.5)$$

where $s = s_p(m, i, j)$ is the unique integer that satisfies

$$p^{s-1}j \leq m(i+1) < p^s j,$$

if $1 \leq j \leq m(i+1)$, and 0, otherwise. Similarly, the bottom long-exact sequence in the diagram in Thm. A takes the form

$$\begin{aligned} \cdots \rightarrow \bigoplus_{i \geq 0} \bigoplus_{j \in n'I_p} W_{t-w} \Omega_A^{q-2i} \xrightarrow{n'V^w} \bigoplus_{i \geq 0} \bigoplus_{j \in I_p} W_t \Omega_A^{q-2i} \\ \xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^n), (x)) \xrightarrow{\partial} \bigoplus_{i \geq 0} \bigoplus_{j \in n'I_p} W_{t-w} \Omega_A^{q-1-2i} \rightarrow \cdots \end{aligned} \quad (1.6)$$

where $t = s_p(n, i, j)$.

Finally, we explain the p -typical decomposition of the divisor

$$\operatorname{div}(\alpha_p(m, n, i)) = \sum_{0 \leq h < i} (\operatorname{div}(\mathbb{W}_{m(h+1)}(\mathbb{F}_p)) - \operatorname{div}(\mathbb{W}_{n(h+1)}(\mathbb{F}_p)))$$

from Thm. A. We first recall the general theory of divisors from [2, §21]. Let X be a scheme. The sheaf of meromorphic functions \mathcal{M}_X is defined to be the sheaf associated with the presheaf that to an affine open subset $U \subset X$ associates the total quotient ring of the ring $\Gamma(U, \mathcal{O}_X)$. If $X = \operatorname{Spec} R$, and if $S \subset R$ is the subset of non-zero-divisors, then the canonical map

$$S^{-1}R \rightarrow \Gamma(X, \mathcal{M}_X)$$

is injective, but not always surjective. The sheaf \mathcal{M}_X is a sheaf of rings and contains \mathcal{O}_X as a sub-sheaf of rings, and the group of divisors $\operatorname{Div}(X)$ is defined to be the group of global sections of the quotient sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$ of the associated sheafs of units. The exact sequence

$$0 \rightarrow \mathcal{O}_X^* \rightarrow \mathcal{M}_X^* \rightarrow \mathcal{M}_X^*/\mathcal{O}_X^* \rightarrow 0$$

induces a long-exact sequence of sheaf cohomology groups which begins

$$0 \rightarrow \Gamma(X, \mathcal{O}_X^*) \rightarrow \Gamma(X, \mathcal{M}_X^*) \xrightarrow{\operatorname{div}} \operatorname{Div}(X) \xrightarrow{\partial} \operatorname{Pic}(X) \rightarrow \cdots$$

A divisor D on X determines and is determined by a invertible sub- \mathcal{O}_X -module $\mathcal{I} \subset \mathcal{M}_X$. Indeed, one shows that the sub- \mathcal{O}_X -module $\mathcal{I} \subset \mathcal{M}_X$ is invertible if and only if there exists a covering $\{U_\alpha\}$ of X and sections $f_\alpha \in \Gamma(U_\alpha, \mathcal{M}_X^*)$ such that $\mathcal{I}|_{U_\alpha} = \mathcal{O}_{U_\alpha} \cdot f_\alpha$. In particular, a closed subscheme $Z \subset X$ with the property that the quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_X$ that defines Z is an invertible \mathcal{O}_X -module defines a divisor $\operatorname{div}(Z) \in \operatorname{Div}(X)$.

We now consider $X = \operatorname{Spec}(\mathbb{W}(\mathbb{F}_p))$. The isomorphism

$$\eta: \mathbb{W}(\mathbb{F}_p) \xrightarrow{\sim} \prod_{j \in I_p} W(\mathbb{F}_p)$$

shows that an element $a \in \mathbb{W}(\mathbb{F}_p)$ is a non-zero-divisor if and only if $\eta_j(a) \in W(\mathbb{F}_p)$ is a non-zero-divisor, for all $j \in I_p$, and hence, we have an induced isomorphism

$$\eta: S^{-1}\mathbb{W}(\mathbb{F}_p) \xrightarrow{\sim} \prod_{j \in I_p} S^{-1}W(\mathbb{F}_p),$$

where $\mathbb{S} \subset \mathbb{W}(\mathbb{F}_p)$ and $S \subset W(\mathbb{F}_p)$ are the subsets of non-zero-divisors. It follows that we have the composite isomorphism

$$\text{ord}: (\mathbb{S}^{-1}\mathbb{W}(\mathbb{F}_p))^*/\mathbb{W}(\mathbb{F}_p)^* \rightarrow \prod_{j \in I_p} S^{-1}W(\mathbb{F}_p)^*/W(\mathbb{F}_p)^* \rightarrow \prod_{j \in I_p} \mathbb{Z}$$

given by the map η and the p -adic valuation. Now an element $\alpha \in (\mathbb{S}^{-1}\mathbb{W}(\mathbb{F}_p))^*$ satisfies

$$\text{div}(\alpha) = \text{div}(\mathbb{W}_{m(i+1)}(\mathbb{F}_p)),$$

if and only if $\text{ord}_j(\alpha) = s_p(m, i, j)$, for all $j \in I_p$, and since the integers $s_p(m, i, j)$ are non-negative, we have $\alpha \in \mathbb{S} \subset \mathbb{W}(\mathbb{F}_p)$. The following statement is equivalent to Thm. A of the introduction. The proof is given in Sect. 4 below.

THEOREM 1.7. *Let A be a regular noetherian ring and an \mathbb{F}_p -algebra. Then the canonical projection $f: A[x]/(x^m) \rightarrow A[x]/(x^n)$ induces a map of long-exact sequences from the sequence (1.5) to the sequence (1.6) that is given, on the lower left-hand terms, by the map*

$$f_*: K_*(A[x]/(x^m), (x)) \rightarrow K_*(A[x]/(x^n), (x))$$

induced by the canonical projection, on the upper right-hand terms, by the map that takes the (i, j) th summand of the domain to the (i, j) th summand of the target by the composite map

$$W_s\Omega_A^{q-2i} \xrightarrow{R^{s-t}} W_t\Omega_A^{q-2i} \xrightarrow{m_\alpha} W_t\Omega_A^{q-2i},$$

where the right-hand map is multiplication by a p -adic integer $\alpha = \alpha_p(m, n, i, j)$ whose p -adic valuation is given by the sum

$$v_p(\alpha_p(m, n, i, j)) = \sum_{0 \leq h < i} (s_p(m, h, j) - s_p(n, h, j)),$$

and, on the upper left-hand terms, by the zero map. Here $s_p(m, i, j)$ is the unique integer s that satisfies $p^{s-1}j \leq m(i+1) < p^s j$, if $1 \leq j \leq m(i+1)$, and 0, otherwise.

REMARK 1.8. (i) The integer $s_p(m, h, j) - s_p(n, h, j)$ is equal to the number of positive integers r such that $n(h+1) < p^{r-1}j \leq m(h+1)$.

(ii) We recall from [10, Prop. I.3.4] that the map $m_\alpha: W_t\Omega_A^{q-2i} \rightarrow W_t\Omega_A^{q-2i}$ given by the multiplication by a p -adic integer α of p -adic valuation v factors canonically

$$W_t\Omega_A^{q-2i} \xrightarrow{R^v} W_{t-v}\Omega_A^{q-2i} \xrightarrow{m_\alpha} W_t\Omega_A^{q-2i}$$

as the composite of the surjective restriction map R^v and an injective map m_α . There is no analog of this factorization for the big de Rham-Witt groups. Indeed, the quotient of $\mathbb{W}(\mathbb{F}_p)$ by the ideal generated by the element $\alpha_p(m, n, i) \in \mathbb{W}(\mathbb{F}_p)$ that appears in Thm. A is generally not of the form $\mathbb{W}_S(\mathbb{F}_p)$ for some subset $S \subset \mathbb{N}$ that is stable under division.

2. The cyclic bar-construction

In this section, we prove Thm. C of the introduction. We first recall the structure of the topological Hochschild \mathbb{T} -spectrum of the truncated polynomial algebra $A[x]/(x^m)$. We refer the reader to [4] for details on the topological Hochschild spectrum and the cyclic bar-construction.

It was proved in [6, Thm. 7.1], but see also [4, Prop. 4], that there is a canonical \mathcal{F} -equivalence of \mathbb{T} -spectra

$$\alpha: T(A) \wedge N^{\text{cy}}(\Pi_m) \xrightarrow{\sim} T(A[x]/(x^m))$$

from the smash product of the topological Hochschild \mathbb{T} -spectrum of A and the geometric realization $N^{\text{cy}}(\Pi_m)$ of the cyclic bar-construction $N^{\text{cy}}(\Pi_m)[-]$ of the pointed monoid

$$\Pi_m = \{0, 1, x, x^2, \dots, x^{m-1}\},$$

where 0 is the base point, and where $x^m = 0$, to the topological Hochschild \mathbb{T} -spectrum of $A[x]/(x^m)$. A map of \mathbb{T} -spectra is an \mathcal{F} -equivalence, if it induces an equivalence of C_r -fixed point spectra, for all finite subgroups $C_r \subset \mathbb{T}$, and hence, we have induced isomorphisms

$$[S^q \wedge (\mathbb{T}/C_r)_+, T(A) \wedge N^{\text{cy}}(\Pi_m)]_{\mathbb{T}} \xrightarrow{\alpha_*} [S^q \wedge (\mathbb{T}/C_r)_+, T(A[x]/(x^m))]_{\mathbb{T}} = \text{TR}_q^r(A[x]/(x^m)).$$

Moreover, the following diagram commutes

$$\begin{array}{ccc} T(A) \wedge N^{\text{cy}}(\Pi_m) & \xrightarrow{\alpha} & T(A[x]/(x^m)) \\ \downarrow \text{id} \wedge f'_* & & \downarrow f_* \\ T(A) \wedge N^{\text{cy}}(\Pi_n) & \xrightarrow{\alpha} & T(A[x]/(x^m)), \end{array}$$

where $f: A[x]/(x^m) \rightarrow A[x]/(x^n)$ and $f': \Pi_m \rightarrow \Pi_n$ are the canonical projections.

The cyclic bar-construction of Π_m decomposes as the wedge sum of pointed cyclic sets

$$N^{\text{cy}}(\Pi_m)[-] = \bigvee_{i \geq 0} N^{\text{cy}}(\Pi_m, i)[-],$$

where $N^{\text{cy}}(\Pi_m, i)[-] \subset N^{\text{cy}}(\Pi_m)[-]$ is the sub-pointed cyclic set whose k -simplices are the $(k+1)$ -tuples $(x^{i_0}, \dots, x^{i_k})$, where $i_0 + \dots + i_k = i$, and the base-point 0 . The geometric realization decomposes accordingly as a wedge sum of pointed \mathbb{T} -spaces

$$N^{\text{cy}}(\Pi_m) = \bigvee_{i \geq 0} N^{\text{cy}}(\Pi_m, i).$$

The homotopy type of the pointed \mathbb{T} -space $N^{\text{cy}}(\Pi_m, i)$ was determined [5, Thm. B]. The result is that $N^{\text{cy}}(\Pi_m, 0)$ is the discrete space $\{0, 1\}$ and that, if i is a positive integer, and if $d = d(m, i) = [(i-1)/m]$, then there is a canonical cofibration sequence of pointed \mathbb{T} -spaces

$$S^{\lambda_d} \wedge_{C_{i/m}} \mathbb{T}_+ \xrightarrow{\text{pr}} S^{\lambda_d} \wedge_{C_i} \mathbb{T}_+ \rightarrow N^{\text{cy}}(\Pi_m, i) \xrightarrow{\partial} S^{\lambda_d} \wedge_{C_{i/m}} \mathbb{T}_+[-1],$$

where the left-hand term is understood to be a point, if m does not divide i . More precisely, there is a canonical homotopy class of maps of pointed \mathbb{T} -spaces

$$\hat{\theta}_d: N^{\text{cy}}(\Pi_m, i) \rightarrow \text{cone}(S^{\lambda_d} \wedge_{C_{i/m}} \mathbb{T}_+ \xrightarrow{\text{pr}} S^{\lambda_d} \wedge_{C_i} \mathbb{T}_+)$$

from $N^{\text{cy}}(\Pi_m, i)$ to the mapping cone of the map pr and any map in this homotopy class is a weak equivalence of pointed \mathbb{T} -spaces. The canonical projection $f': \Pi_m \rightarrow \Pi_n$ induces a map of cofibration sequences which we identify in Prop. 2.2 below. But first, we recall from [5, §3] how the homotopy class of maps $\hat{\theta}_d$ is defined.

We view the standard simplex Δ^{i-1} as the convex hull of the set of group elements C_i inside the regular representation $\mathbb{R}[C_i]$. Let $\zeta_i \in C_i \subset \mathbb{T}$ be the generator $\zeta_i = \exp(2\pi\sqrt{-1}/i)$, and let $\Delta^{i-m} \subset \Delta^{i-1}$ be the convex hull of the group elements $1, \zeta_i, \dots, \zeta_i^{i-m}$. Then, by standard cyclic theory [12], there is a \mathbb{T} -equivariant homeomorphism

$$\varphi: (\Delta^{i-1}/C_i \cdot \Delta^{i-m}) \wedge_{C_i} \mathbb{T}_+ \xrightarrow{\sim} N^{\text{cy}}(\Pi_m, i).$$

The regular representation $\mathbb{R}[C_i]$ has the canonical direct sum decomposition

$$\mathbb{R}[C_i] = \begin{cases} \mathbb{R} \oplus \mathbb{C}(1) \oplus \dots \oplus \mathbb{C}(s), & \text{if } i = 2s + 1, \\ \mathbb{R} \oplus \mathbb{C}(1) \oplus \dots \oplus \mathbb{C}(s) \oplus \mathbb{R}_-, & \text{if } i = 2s + 2, \end{cases}$$

so if $d \leq s$, or equivalently, if $2d < i$, there is a canonical projection

$$\pi_d: \mathbb{R}[C_i] \rightarrow \mathbb{C}(1) \oplus \dots \oplus \mathbb{C}(d) = \lambda_d.$$

Suppose first that $md < i < m(d+1)$. We showed in [5, Thm. 3.1.2] that, in this case, the image of $C_i \cdot \Delta^{i-m} \subset \mathbb{R}[C_i]$ does not contain the origin $0 \in \lambda_d$. Hence, we may compose π_d with the radial projection away from a small ball around $0 \in \lambda_d$ to get a C_i -equivariant map

$$\theta_d: \Delta^{i-1}/C_i \cdot \Delta^{i-m} \rightarrow D(\lambda_d)/S(\lambda_d) = S^{\lambda_d}$$

whose homotopy class is well-defined. The composite map of pointed \mathbb{T} -spaces

$$\hat{\theta}_d: N^{\text{cy}}(\Pi_m, i) \xrightarrow{\varphi^{-1}} (\Delta^{i-1}/C_i \cdot \Delta^{i-m}) \wedge_{C_i} \mathbb{T}_+ \xrightarrow{\theta_d \wedge \text{id}} S^{\lambda_d} \wedge_{C_i} \mathbb{T}_+$$

is then the desired map $\hat{\theta}_d$. Its homotopy class is well-defined and [5, Prop. 3.2.6] shows that it is a weak equivalence.

Suppose next that $i = m(d+1)$. We let $\lambda'_d \subset \lambda_{d+1}$ be the image of the canonical inclusion $\iota: \lambda_d \rightarrow \lambda_{d+1}$ and let $\lambda_d^\perp \subset \lambda_{d+1}$ be the orthogonal complement of $\lambda'_d \subset \lambda_{d+1}$. The canonical projection from λ_{d+1} to $\mathbb{C}(d+1)$ restricts to an isomorphism of λ_d^\perp onto $\mathbb{C}(d+1)$, and we define $C'_m \subset \lambda_d^\perp$ to be the preimage by this isomorphism of $C_m \subset \mathbb{C}(d+1)$. It follows again from [5, Thm. 3.1.2] that the image $\pi_{d+1}(C_i \cdot \Delta^{i-m})$ does not contain the origin $0 \in \lambda_{d+1}$ and, in addition, [5, 3.3.5] shows that

$$\pi_{d+1}(C_i \cdot \Delta^{i-m}) \cap \lambda_d^\perp = C'_m.$$

We pick a small open ball $B \subset \lambda_{d+1} \setminus C'_m$ around a point in $S(\lambda_d^\perp)$ and define

$$U = (C_i \cdot B) \cap S(\lambda_{d+1}) \subset S(\lambda_{d+1}).$$

If the ball B is small enough, then the composition of the projection π_{d+1} and the radial projection away from a (different) small ball around $0 \in \lambda_{d+1}$ defines a C_i -equivariant map

$$\theta'_d: \Delta^{i-1}/C_i \cdot \Delta^{i-m} \rightarrow D(\lambda_{d+1})/(S(\lambda_{d+1}) \setminus U)$$

whose homotopy class is well-defined. Let $C(C_m) = \{0\} * C_m \subset D(\mathbb{C}(d+1))$ be the unreduced cone with base $C_m \subset \mathbb{C}(d+1)$ and with cone point $0 \in \mathbb{C}(d+1)$. Then the canonical homeomorphism of $D(\lambda_d) \times D(\mathbb{C}(d+1))$ onto $D(\lambda_{d+1})$ induces an inclusion into the target of the map θ'_d of the pointed C_i -space

$$\frac{D(\lambda_d) \times C(C_m)}{S(\lambda_d) \times C(C_m) \cup D(\lambda_d) \times C_m} = S^{\lambda_d} \wedge (S^0 * C_m).$$

One immediately verifies that this inclusion is a strong deformation retract of pointed C_i -spaces. Hence, the map θ'_d defines a homotopy class of maps of pointed C_i -spaces

$$\theta_d: \Delta^{i-1}/C_i \cdot \Delta^{i-m} \rightarrow S^{\lambda_d} \wedge (S^0 * C_m).$$

The composite map of pointed \mathbb{T} -spaces

$$\hat{\theta}_d: N^{\text{cy}}(\Pi_m, i) \xrightarrow{\varphi^{-1}} (\Delta^{i-1}/C_i \cdot \Delta^{i-m}) \wedge_{C_i} \mathbb{T}_+ \xrightarrow{\theta_d \wedge \text{id}} (S^{\lambda_d} \wedge (S^0 * C_m)) \wedge_{C_i} \mathbb{T}_+$$

is the desired map $\hat{\theta}_d$. Its homotopy class is well-defined and [5, Prop. 3.3.9] shows that it is a weak equivalence. In preparation for the proof of Prop. 2.2 below, we first prove the following result.

LEMMA 2.1. *Let m and n be positive integers with $m > n$.*

(i) *If $md < i < m(d+1)$ and $ne < i < n(e+1)$, then $d \leq e$ and the diagram*

$$\begin{array}{ccc} \Delta^{i-1}/C_i \cdot \Delta^{i-m} & \xrightarrow{\theta_d} & S^{\lambda_d} \\ \downarrow \text{pr} & & \downarrow \iota \\ \Delta^{i-1}/C_i \cdot \Delta^{i-n} & \xrightarrow{\theta_e} & S^{\lambda_e} \end{array}$$

commutes up to C_i -equivariant homotopy.

(ii) If $md < i < m(d+1)$ and $i = n(e+1)$, then $d \leq e$ and the diagram

$$\begin{array}{ccc} \Delta^{i-1}/C_i \cdot \Delta^{i-m} & \xrightarrow{\theta_d} & S^{\lambda_d} \\ \downarrow \text{pr} & & \downarrow \iota \\ \Delta^{i-1}/C_i \cdot \Delta^{i-n} & \xrightarrow{\theta_e} & S^{\lambda_e} \wedge (S^0 * C_n) \end{array}$$

commutes up to C_i -equivariant homotopy.

(iii) If $i = m(d+1)$ and $ne < i < n(e+1)$, then $d < e$ and the diagram

$$\begin{array}{ccc} \Delta^{i-1}/C_i \cdot \Delta^{i-m} & \xrightarrow{\theta_d} & S^{\lambda_d} \wedge (S^0 * C_m) \\ \downarrow \text{pr} & & \downarrow \iota \\ \Delta^{i-1}/C_i \cdot \Delta^{i-n} & \xrightarrow{\theta_e} & S^{\lambda_e} \end{array}$$

commutes up to C_i -equivariant homotopy.

(iv) If $i = m(d+1) = n(e+1)$, then $d < e$ and the diagram

$$\begin{array}{ccc} \Delta^{i-1}/C_i \cdot \Delta^{i-m} & \xrightarrow{\theta_d} & S^{\lambda_d} \wedge (S^0 * C_m) \\ \downarrow \text{pr} & & \downarrow \iota \\ \Delta^{i-1}/C_i \cdot \Delta^{i-n} & \xrightarrow{\theta_e} & S^{\lambda_e} \wedge (S^0 * C_n) \end{array}$$

commutes up to C_i -equivariant homotopy.

Proof. We first consider the case (i). Let $\lambda_e - \lambda_d \subset \lambda_e$ be the orthogonal complement of $\lambda_d \subset \lambda_e$. Then there is a canonical C_i -equivariant homeomorphism

$$D(\lambda_d) \times D(\lambda_e - \lambda_d) \xrightarrow{\sim} D(\lambda_e)$$

which restricts to a C_i -equivariant homeomorphism

$$S(\lambda_d) \times D(\lambda_e - \lambda_d) \cup D(\lambda_d) \times S(\lambda_e - \lambda_d) \xrightarrow{\sim} S(\lambda_e).$$

We note that the composition

$$\Delta^{i-1}/C_i \cdot \Delta^{i-m} \xrightarrow{\text{pr}} \Delta^{i-1}/C_i \cdot \Delta^{i-n} \xrightarrow{\theta_e} D(\lambda_e)/S(\lambda_e)$$

of the left-hand vertical map and the lower horizontal map in the diagram of part (i) of the statement factors through the map

$$\frac{D(\lambda_d) \times D(\lambda_e - \lambda_d)}{S(\lambda_d) \times D(\lambda_e - \lambda_d)} \rightarrow D(\lambda_e)/S(\lambda_e)$$

induced by the canonical homeomorphism above. But the latter map is homotopic to the composite map

$$\frac{D(\lambda_d) \times D(\lambda_e - \lambda_d)}{S(\lambda_d) \times D(\lambda_e - \lambda_d)} \xrightarrow{\text{pr}_1} D(\lambda_d)/S(\lambda_d) \xrightarrow{\iota} D(\lambda_e)/S(\lambda_e)$$

by the C_i -equivariant homotopy induced from the radial contraction

$$h: D(\lambda_e - \lambda_d) \times [0, 1] \rightarrow D(\lambda_e - \lambda_d)$$

defined by $h(z, t) = tz$. This homotopy, in turn, induces a C_i -equivariant homotopy from the composite $\iota \circ \theta_d$ to the composite $\theta_e \circ \text{pr}$ in the diagram in part (i) of the statement.

In the case (ii), we note similarly that the composition

$$\Delta^{i-1}/C_i \cdot \Delta^{i-m} \xrightarrow{\text{pr}} \Delta^{i-1}/C_i \cdot \Delta^{i-n} \xrightarrow{\theta_e} \frac{D(\lambda_e) \times C(C_n)}{S(\lambda_e) \times C(C_n) \cup D(\lambda_e) \times C_n}$$

of the left-hand vertical map and the lower horizontal map in the diagram in part (ii) of the statement factors through the canonical projection

$$\frac{D(\lambda_e) \times C(C_n)}{S(\lambda_e) \times C(C_n)} \rightarrow \frac{D(\lambda_e) \times C(C_n)}{S(\lambda_e) \times C(C_n) \cup D(\lambda_e) \times C_n}$$

Again, the latter map is homotopic to the composition

$$\frac{D(\lambda_e) \times C(C_n)}{S(\lambda_e) \times C(C_n)} \xrightarrow{\text{pr}_1} D(\lambda_e)/S(\lambda_e) \xrightarrow{\iota} \frac{D(\lambda_e) \times C(C_n)}{S(\lambda_e) \times C(C_n) \cup D(\lambda_e) \times C_n}$$

by the C_i -equivariant homotopy induced from the radial contraction

$$h: C(C_n) \times [0, 1] \rightarrow C(C_n)$$

defined by $h(z, t) = tz$. This homotopy induces the desired homotopy from the composite map $\iota \circ \theta_d$ to the composite map $\theta_e \circ \text{pr}$ in the diagram in part (ii) of the statement.

In the case (iii) of the statement, one proves as in the proof of part (i) of the statement that the composite map $\text{pr} \circ \theta_e$ is homotopic to the composition

$$\Delta^{i-1}/C_i \cdot \Delta^{i-m} \xrightarrow{\theta_{d+1}} D(\lambda_{d+1})/S(\lambda_{d+1}) \xrightarrow{\iota} D(\lambda_e)/S(\lambda_e).$$

But the map θ_{d+1} is homotopic to the composition

$$\Delta^{i-1}/C_i \cdot \Delta^{i-m} \xrightarrow{\theta_d} \frac{D(\lambda_d) \times C(C_m)}{S(\lambda_d) \times C(C_m) \cup D(\lambda_d) \times C_m} \xrightarrow{\iota} D(\lambda_{d+1})/S(\lambda_{d+1}).$$

Indeed, this is a direct consequence of the construction of the map λ_d . This proves the case (iii) of the statement. Finally, the proof of the case (iv) is analogous to the proof of the case (i). This completes the proof. \square

PROPOSITION 2.2. *Let $m > n \geq 1$ be integers and $f': \Pi_m \rightarrow \Pi_n$ the canonical projection. Let $i \geq 1$ be an integer, and set $d = \lfloor (i-1)/m \rfloor$ and $e = \lfloor (i-1)/n \rfloor$. Then there is a homotopy commutative diagram of pointed \mathbb{T} -spaces*

$$\begin{array}{ccccccc} S^{\lambda_d} \wedge_{C_{i/m}} \mathbb{T}_+ & \xrightarrow{\text{pr}} & S^{\lambda_d} \wedge_{C_i} \mathbb{T}_+ & \longrightarrow & N^{\text{cy}}(\Pi_m, i) & \xrightarrow{\partial} & S^{\lambda_d} \wedge_{C_{i/m}} \mathbb{T}_+[-1] \\ \downarrow * & & \downarrow \iota \wedge \text{id} & & \downarrow f'_* & & \downarrow * \\ S^{\lambda_e} \wedge_{C_{i/n}} \mathbb{T}_+ & \xrightarrow{\text{pr}} & S^{\lambda_e} \wedge_{C_i} \mathbb{T}_+ & \longrightarrow & N^{\text{cy}}(\Pi_n, i) & \xrightarrow{\partial} & S^{\lambda_e} \wedge_{C_{i/n}} \mathbb{T}_+[-1], \end{array}$$

where the rows are cofibration sequences, the map $\iota \wedge \text{id}$ is induced from the canonical inclusion of λ_d in λ_e , and the map $*$ is the null-map. The domain (resp. target) of the map $*$ is understood to be a point if m (resp. n) does not divide i .

Proof. As we recalled above, the statement that the rows in the diagram of the statement are cofibration sequences of pointed \mathbb{T} -spaces is equivalent to the statement that the maps of \mathbb{T} -spaces $\hat{\theta}_d$ are weak equivalences, and the latter statement is [5, Props. 3.2.6, 3.3.9]. The statement that the diagram commutes follows immediately from Lemma 2.1 upon applying the functor that to a pointed C_i -space X associates the pointed \mathbb{T} -space $X \wedge_{C_i} \mathbb{T}_+$ with the exception of the statement that the left-hand vertical map is null-homotopic. Only the case (iii)

needs proof. In this case, we have a homotopy commutative diagram of pointed C_i -spaces

$$\begin{array}{ccccccc} S^{\lambda_d} \wedge C_{m+} & \xrightarrow{\text{pr}_1} & S^{\lambda_d} & \longrightarrow & S^{\lambda_d} \wedge (S^0 * C_m) & \xrightarrow{\partial} & S^{\lambda_d} \wedge C_{m+}[-1] \\ \downarrow & & \parallel & & \downarrow & & \downarrow \\ S^{\lambda_d} \wedge S(\mathbb{C}(d+1))_+ & \longrightarrow & S^{\lambda_d} & \xrightarrow{\iota} & S^{\lambda_{d+1}} & \xrightarrow{\partial} & S^{\lambda_d} \wedge S(\mathbb{C}(d+1))_+[-1] \end{array}$$

which shows that the composite map $\iota \circ \text{pr}_1$ is null-homotopic. It follows that the composite map of pointed \mathbb{T} -spaces

$$S^{\lambda_d} \wedge_{C_{i/m}} \mathbb{T}_+ \xrightarrow{\text{pr}} S^{\lambda_d} \wedge_{C_i} \mathbb{T}_+ \xrightarrow{\iota \wedge \text{id}} S^{\lambda_{d+1}} \wedge_{C_i} \mathbb{T}_+$$

is null-homotopic as desired. \square

Proof of Thm. C. The proof that the horizontal sequences in the diagram in the statement are exact is given in [6, Prop. 4.2.3]. A direct, and perhaps more detailed, proof that the isomorphic sequences (1.1) and (1.2) are exact is given in [4, Prop. 11]. (The latter proof also shows that the derived limits corresponding to the limits that appear in the two sequences are zero.) It is immediately clear from either proof that Prop. 2.2 implies that the diagram in the statement commutes. \square

3. The map $\iota_*: \text{TR}_{q-\lambda_{a-1}}^u(A; p) \rightarrow \text{TR}_{q-\lambda_a}^u(A; p)$

In this section, we examine the map of the title. We first prove a general result for \mathbb{T} -spectra. A \mathbb{T} -spectrum is an object of the \mathbb{T} -stable homotopy category which, in turn, is the homotopy category of the model category of orthogonal \mathbb{T} -spectra [13]. We will follow the conventions of [8, §2] with respect to the tensor triangulated structure of the \mathbb{T} -stable category.

Let T be a \mathbb{T} -spectrum, and let $C_r \subset \mathbb{T}$ be the subgroup of order r . We define

$$\pi_q^{C_r}(T) = [S^q \wedge (\mathbb{T}/C_r)_+, T]_{\mathbb{T}}$$

to be the set of maps in the \mathbb{T} -stable homotopy category between the indicated \mathbb{T} -spectra. In particular,

$$\text{TR}_q^r(A) = \pi_q^{C_r}(T(A)).$$

There are canonical isomorphisms

$$\pi_q(T^{C_r}) = [S^q, T^{C_r}] \xrightarrow{\sim} [S^q \wedge (\mathbb{T}/C_r)_+, T^{C_r}]_{\mathbb{T}/C_r} \xrightarrow{\sim} [S^q \wedge (\mathbb{T}/C_r)_+, T]_{\mathbb{T}} = \pi_q^{C_r}(T),$$

where, in the middle term, T^{C_r} denotes the C_r -fixed point \mathbb{T}/C_r -spectrum, and where, on the left-hand side, T^{C_r} denotes the underlying non-equivariant spectrum of this \mathbb{T}/C_r -spectrum. We consider the cofibration sequence in the \mathbb{T} -stable homotopy category

$$\mathbb{T}_+ \xrightarrow{\pi} S^0 \xrightarrow{\iota} S^{\mathbb{C}(1)} \xrightarrow{\partial} S^1 \wedge \mathbb{T}_+ = \mathbb{T}_+[-1],$$

where π collapses \mathbb{T} onto the non-base point of S^0 , and where ι is the canonical inclusion. We identify the underlying pointed spaces of S^2 and $S^{\mathbb{C}(1)}$ by the isomorphism

$$\varphi: S^2 \rightarrow S^{\mathbb{C}(1)}$$

that takes the class of (a, b) in $S^2 = S^1 \wedge S^1$ to the class of $a + b\sqrt{-1}$ in $S^{\mathbb{C}(1)}$. The desuspension of the composition of φ and ∂ defines a map in the non-equivariant stable homotopy category

$$\sigma: S^1 \rightarrow \mathbb{T}_+$$

The cofibration sequence above induces a cofibration sequence in the non-equivariant stable homotopy category. The latter sequence splits. We thus obtain a direct sum diagram in the non-equivariant stable homotopy category

$$S^1 \begin{array}{c} \xrightarrow{\sigma} \\ \xleftarrow{\kappa} \end{array} \mathbb{T}_+ \begin{array}{c} \xrightarrow{\pi} \\ \xleftarrow{i} \end{array} S^0$$

where i is the section of π that takes the non-base point in S^0 to $1 \in \mathbb{T}$, and where the map κ is the corresponding retraction of the map σ . We define Connes' operator

$$d: \pi_q^{C_r}(T) \rightarrow \pi_{q+1}^{C_r}(T)$$

to be the map given by the composition

$$\pi_q(T^{C_r}) \xrightarrow{\ell_\sigma} \pi_{q+1}(\mathbb{T}_+ \wedge T^{C_r}) \xrightarrow{\rho_{r*}} \pi_{q+1}((\mathbb{T}/C_r)_+ \wedge T^{C_r}) \xrightarrow{\mu_*} \pi_{q+1}(T^{C_r}),$$

where the left-hand map is left multiplication by σ , where the middle map is induced by the isomorphism $\rho_r: \mathbb{T} \rightarrow \mathbb{T}/C_r$ defined by the r th root, and where the right-hand map is induced from the left action by \mathbb{T} on the underlying non-equivariant spectrum of T . It anti-commutes with the suspension isomorphism in the sense that the following diagram anti-commutes:

$$\begin{array}{ccc} \pi_q^{C_r}(T) & \xrightarrow{d} & \pi_{q+1}^{C_r}(T) \\ \downarrow \text{susp} \quad (-1) & & \downarrow \text{susp} \\ \pi_{q+1}^{C_r}(T) & \xrightarrow{d} & \pi_{q+2}^{C_r}(T). \end{array}$$

Moreover, $dd(x) = \eta \cdot d(x)$ and $F_r dV_r(x) = d(x) + (r-1)\eta \cdot x$, where $\eta \in \pi_1(S^0)$ is the Hopf class, and, if T is a ring \mathbb{T} -spectrum, d is a derivation for the multiplication; see [3, §1]. Finally, let $\gamma: X \wedge Y \rightarrow Y \wedge X$ be the canonical isomorphism.

LEMMA 3.1. *Let T be a \mathbb{T} -spectrum, and let r and j be relative prime positive integers. Then, for every integer q , there is a commutative diagram*

$$\begin{array}{ccc} \pi_{q-1}(T) \oplus \pi_q(T) & \xrightarrow{\text{susp} \oplus \text{susp}} & \pi_q(S^1 \wedge T) \oplus \pi_{q+1}(S^1 \wedge T) \\ \downarrow i_* \oplus i_* & & \uparrow \gamma_* \circ \kappa_* \oplus \gamma_* \circ \kappa_* \\ \pi_{q-1}(T \wedge (\mathbb{T}/C_j)_+) \oplus \pi_q(T \wedge (\mathbb{T}/C_j)_+) & & \pi_q(T \wedge (\mathbb{T}/C_j)_+) \oplus \pi_{q+1}(T \wedge (\mathbb{T}/C_j)_+) \\ \downarrow dV_r + V_r & & \uparrow (F_r, F_r d) \\ \pi_q^{C_r}(T \wedge (\mathbb{T}/C_j)_+) & \xlongequal{\quad} & \pi_q^{C_r}(T \wedge (\mathbb{T}/C_j)_+), \end{array}$$

where the top horizontal map is the suspension isomorphism. Moreover, the compositions of the left-hand vertical maps and the right-hand vertical maps are both isomorphisms.

Proof. The general formulas that we recalled before the statement show that

$$((\kappa_* \oplus \kappa_*) \circ (F_r, F_r d) \circ (dV_r + V_r) \circ (i_* \oplus i_*))(x, y) = ((\kappa_* \circ d \circ i_*)(x), (\kappa_* \circ d \circ i_*)(y)),$$

and one verifies that $\gamma_* \circ \kappa_* \circ d \circ i_*$ is equal to the suspension isomorphism. This shows that the diagram commutes. It was proved in [9, Prop. 3.4.1] that the composition of the left-hand vertical maps is an isomorphism, provided that $r = p^{v-1}$ is a power of a prime number p . The proof in the general case is completely similar. \square

Suppose now that r is a divisor in a . Then the isomorphism

$$\varphi: S^2 \rightarrow S^{\mathbb{C}(a)}$$

is an isomorphism of pointed C_r -spaces. We define

$$\hat{\varphi}: (\mathbb{T}/C_r)_+ \wedge S^2 \rightarrow (\mathbb{T}/C_r)_+ \wedge S^{\mathbb{C}(a)}$$

be the isomorphism of pointed \mathbb{T} -spaces that takes the class of (zC_r, x) to the class of $(zC_r, z\varphi(x))$. It follows that, if r is a divisor in a , we have an isomorphism

$$\varphi^\#: \mathrm{TR}_{q-\lambda_{a-1}}^r(A) \rightarrow \mathrm{TR}_{q+2-\lambda_a}^r(A)$$

defined by the composition

$$\begin{aligned} \mathrm{TR}_{q-\lambda_{a-1}}^r(A) &= [S^q \wedge (\mathbb{T}/C_r)_+, T(A) \wedge S^{\lambda_{a-1}}]_{\mathbb{T}} \\ &\xrightarrow{\mathrm{susp}} [S^{\mathbb{C}(a)} \wedge S^q \wedge (\mathbb{T}/C_r)_+, S^{\mathbb{C}(a)} \wedge T(A) \wedge S^{\lambda_{a-1}}]_{\mathbb{T}} \\ &\xrightarrow{\hat{\varphi}^*} [S^2 \wedge S^q \wedge (\mathbb{T}/C_r)_+, S^{\mathbb{C}(a)} \wedge T(A) \wedge S^{\lambda_{a-1}}]_{\mathbb{T}} \\ &\xrightarrow{\sim} [S^{q+2} \wedge (\mathbb{T}/C_r)_+, T(A) \wedge S^{\lambda_a}]_{\mathbb{T}} = \mathrm{TR}_{q+2-\lambda_a}^r(A), \end{aligned}$$

where the first map is the suspension isomorphism, and where the last map is the canonical isomorphism. We write $\varphi_{\#} = (\varphi^\#)^{-1}$.

PROPOSITION 3.2. *Let A be a $\mathbb{Z}_{(p)}$ -algebra, let $a, u \geq 1$ be integers, and let $v = v(a, u)$ be the minimum of u and $v_p(a) + 1$. Then there are natural long-exact sequences*

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ \mathrm{TR}_{q-1-\lambda_{a-1}}^v(A; p) \oplus \mathrm{TR}_{q-\lambda_{a-1}}^v(A; p) & & \mathrm{TR}_{q+1-\lambda_a}^u(A; p) \oplus \mathrm{TR}_{q-\lambda_{a-1}}^u(A; p) \\ \downarrow dV^{u-v} + V^{u-v} & & \downarrow \mathrm{pr}_2 \\ \mathrm{TR}_{q-\lambda_{a-1}}^u(A; p) & & \mathrm{TR}_{q-\lambda_{a-1}}^u(A; p) \\ \downarrow \iota_* & & \downarrow \iota_* = 0 \\ \mathrm{TR}_{q-\lambda_a}^u(A; p) & & \mathrm{TR}_{q-\lambda_a}^u(A; p) \\ \downarrow (\varphi_{\#} F^{u-v}, -\varphi_{\#} F^{u-v} d) & & \downarrow \mathrm{in}_1 \\ \mathrm{TR}_{q-2-\lambda_{a-1}}^v(A; p) \oplus \mathrm{TR}_{q-1-\lambda_{a-1}}^v(A; p) & & \mathrm{TR}_{q-\lambda_a}^u(A; p) \oplus \mathrm{TR}_{q-1-\lambda_{a-1}}^u(A; p) \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array}$$

where the left-hand sequence is valid, for $v < u$, and where the right-hand sequence is valid, for $v = u$.

Proof. We consider the cofibration sequence of pointed \mathbb{T} -spaces

$$S(\mathbb{C}(a))_+ \xrightarrow{\pi} S^0 \xrightarrow{\iota} S^{\mathbb{C}(a)} \xrightarrow{\delta} S^1 \wedge S(\mathbb{C}(a))_+$$

and the induced cofibration sequence in the \mathbb{T} -stable category

$$T \wedge S(\mathbb{C}(a))_+ \xrightarrow{\pi'} T \xrightarrow{\iota'} T \wedge S^{\mathbb{C}(a)} \xrightarrow{\delta'} S^1 \wedge T \wedge S(\mathbb{C}(a))_+$$

where $T = T(A) \wedge S^{\lambda_{a-1}}$. We also abbreviate $r = p^{u-1}$. The associated long-exact sequence of equivariant homotopy groups takes the form

$$\pi_q^{C_r}(T \wedge S(\mathbb{C}(a))_+) \xrightarrow{\pi'_*} \pi_q^{C_r}(T) \xrightarrow{\iota'_*} \pi_q^{C_r}(T \wedge S^{\mathbb{C}(a)}) \xrightarrow{\partial} \pi_{q-1}^{C_r}(T \wedge S(\mathbb{C}(a))_+),$$

where ∂ is the composition of ∂'_* and the inverse of the suspension isomorphism. We identify this sequence with the long-exact sequences of the statement. The terms $\pi_q^{C_r}(T)$ and $\pi_q^{C_r}(T \wedge S^{\mathbb{C}(a)})$ are equal to $\mathrm{TR}_{q-\lambda_{a-1}}^u(A; p)$ and $\mathrm{TR}_{q-\lambda_a}^u(A; p)$, respectively, and the map ι'_* is equal to the map ι_* . It remains to identify the remaining term and the two maps π'_* and ∂_* .

Suppose first that $u = v$ such that $\mathbb{C}(a)$ is C_r -trivial. Then the map $i: S^0 \rightarrow S(\mathbb{C}(a))_+$ that takes the non-base point in S^0 to $1 \in S(\mathbb{C}(a))_+$ is C_r -equivariant and defines a section of the map π'_* in the long-exact sequence of equivariant homotopy groups above. This completes the proof in the case $u = v$.

We next assume that $v < u$ and consider first the case $v = 1$ such that a and r are relatively prime. The map $i: S^0 \rightarrow S(\mathbb{C}(a))_+$ is a non-equivariant section of the projection π . The following diagram commutes

$$\begin{array}{ccc} \pi_q^{C_r}(T \wedge S(\mathbb{C}(a))_+) & \xrightarrow{\pi'_*} & \pi_q^{C_r}(T) \\ \uparrow dV_r + V_r & & \uparrow dV_r + V_r \\ \pi_{q-1}(T \wedge S(\mathbb{C}(a))_+) \oplus \pi_q(T \wedge S(\mathbb{C}(a))_+) & \xrightarrow{\pi'_* \oplus \pi'_*} & \pi_{q-1}(T) \oplus \pi_q(T) \\ \uparrow i'_* \oplus i'_* & & \\ \pi_{q-1}(T) \oplus \pi_q(T) & & \end{array}$$

and the composite map $(\pi'_* \oplus \pi'_*) \circ (i'_* \oplus i'_*)$ is the identity. Since $S(\mathbb{C}(a))$ is isomorphic to \mathbb{T}/C_a as a \mathbb{T} -space, Lemma 3.1 shows that the composition of the left-hand vertical maps is an isomorphism. We use this isomorphism to identify the left-hand term of the long-exact sequence of equivariant homotopy groups above. The composition of this isomorphism and the map π' is equal to the map

$$dV_r + V_r: \mathrm{TR}_{q-1-\lambda_{a-1}}^1(A; p) \oplus \mathrm{TR}_{q-\lambda_{a-1}}^1(A; p) \rightarrow \mathrm{TR}_{q-\lambda_{a-1}}^u(A; p)$$

as desired. Similarly, the following diagram commutes

$$\begin{array}{ccc} \pi_q^{C_r}(T \wedge S^{\mathbb{C}(a)}) & \xrightarrow{\partial_*} & \pi_{q-1}^{C_r}(T \wedge S(\mathbb{C}(a))_+) \\ \downarrow (F_r, -F_r, d) & & \downarrow (F_r, F_r, d) \\ \pi_q(T \wedge S^{\mathbb{C}(a)}) \oplus \pi_{q+1}(T \wedge S^{\mathbb{C}(a)}) & \xrightarrow{\partial_* \oplus \partial_*} & \pi_{q-1}(T \wedge S(\mathbb{C}(a))_+) \oplus \pi_q(T \wedge S(\mathbb{C}(a))_+) \\ & & \downarrow \mathrm{susp}^{-1} \circ \gamma_* \circ \kappa'_* \oplus \mathrm{susp}^{-1} \circ \gamma_* \circ \kappa'_* \\ & & \pi_{q-2}(T) \oplus \pi_{q-1}(T) \end{array}$$

and the composition of the lower horizontal map and the lower right-hand vertical map is equal to the map $\varphi_{\#} \oplus \varphi_{\#}$. Finally, Lemma 3.1 shows that the composition of the right-hand vertical maps is equal to the inverse of the map $dV_r i'_* \oplus V_r i'_*$. This completes the proof in the case where $1 = v < u$.

Finally, we suppose that $1 < v < u$ and abbreviate $r = p^{u-1}$, $t = p^{v-1}$, and $s = p^{u-v}$. We recall that the root isomorphism $\rho_t: \mathbb{T} \rightarrow \mathbb{T}/C_t$ induces an equivalence of categories ρ_t^* from

the \mathbb{T}/C_t -stable category to the \mathbb{T} -stable category. There is a commutative diagram

$$\begin{array}{ccccccc}
 \pi_q^{C_r}(T \wedge S(\mathbb{C}(a))_+) & \xrightarrow{\pi'_*} & \pi_q^{C_r}(T) & \xrightarrow{\iota'_*} & \pi_q^{C_r}(T \wedge S^{\mathbb{C}(a)}) & \xrightarrow{\partial} & \pi_{q-1}^{C_r}(T \wedge S(\mathbb{C}(a))_+) \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \pi_q^{C_s}(T' \wedge S(\mathbb{C}(j))_+) & \xrightarrow{\pi'_*} & \pi_q^{C_s}(T') & \xrightarrow{\iota'_*} & \pi_q^{C_s}(T' \wedge S^{\mathbb{C}(j)}) & \xrightarrow{\partial} & \pi_{q-1}^{C_s}(T' \wedge S(\mathbb{C}(j))_+)
 \end{array}$$

where $T = T(A) \wedge S^{\lambda_{a-1}}$ and $T' = \rho_t^*(T^{C_t})$, and where $j = a/t$. The vertical maps are the composition of the isomorphism $\rho_t^*: \pi_q^{C_r/C_t}(X^{C_t}) \rightarrow \pi_q^{C_s}(\rho_t^*(X^{C_t}))$ and the canonical isomorphism of $\pi_q^{C_r}(X)$ and $\pi_q^{C_r/C_t}(X^{C_t})$. The case of the lower sequence was treated above. This completes the proof. \square

REMARK 3.3. Let A be a regular noetherian ring that is also an \mathbb{F}_p -algebra. There is statement similar to Lemma 3.2 above for the p -typical de Rham-Witt complex of A . Suppose that $v < u$. Then there is a long-exact sequence

$$\cdots \xrightarrow{\beta} W_v \Omega_A^{q-1} \oplus W_v \Omega_A^q \xrightarrow{\alpha} W_u \Omega_A^q \xrightarrow{p^{u-v}} W_u \Omega_A^q \xrightarrow{\beta} W_v \Omega_A^q \oplus W_v \Omega_A^{q+1} \xrightarrow{\alpha} \cdots,$$

where $\alpha = dV^{u-v} + V^{u-v}$ and $\beta = (F^{u-v}, -F^{u-v}d)$. This sequence is a more precise version of the statement [11, III (3.3.3.3)]. We outline the steps in the proof. First, the basic case $A = \mathbb{F}_p$ is clear. Next, one uses [9, Thm. B] to show that the exactness of the sequence for A implies the exactness of the sequence for $A[x]$. Third, one uses [4, Prop. 6.2.3] of [10, Prop. I.1.14] to conclude that the sequence is exact whenever A is a smooth \mathbb{F}_p -algebra. Finally, one uses the theorem of Popescu [14] that every regular noetherian \mathbb{F}_p -algebra is a filtered colimit of smooth \mathbb{F}_p -algebra and the easy fact that the de Rham-Witt groups commute with colimits.

4. Proof of Thm. A

In this section, we prove Thm. 1.7, and hence, the equivalent Thm. A. We first evaluate the maps $\iota_p(m, n, j)_q$ in the basic case $A = \mathbb{F}_p$.

We recall that, if A is a commutative ring, the topological Hochschild \mathbb{T} -spectrum $T(A)$ is a commutative ring \mathbb{T} -spectrum [8, p. 14]. In particular, the group $\mathrm{TR}_0^u(A; p)$ is a commutative ring. We showed in [6, Thm. F] that there is a canonical isomorphism

$$\xi: W_u(A) \xrightarrow{\sim} \mathrm{TR}_0^u(A; p)$$

from the ring of p -typical Witt vectors of length u in A onto the ring $\mathrm{TR}_0^u(A; p)$. It also follows that the groups $\mathrm{TR}_{q-\lambda}^u(A; p)$ are modules over $\mathrm{TR}_0^u(A; p)$, and hence, over the ring $W(A)$.

We briefly recall the definition of the length of a module. Let R be a ring, and let M be an R -module. A chain of R -submodules $M_0 \subset M_1 \subset \cdots \subset M_s$ is said to have length s . The length $\mathrm{length}_R M$ is defined to be the maximal length of a chain of R -submodules of M . A $W(\mathbb{F}_p)$ -module M has a length s if and only if it is a finite torsion module. In this case, the order of M is p^s . The following result was proved in [6, Prop. 9.1]; see also [4, Thm. 3.1].

PROPOSITION 4.1. *Let p be a prime, and let a and u be positive integers. Then, for every non-negative integer i , the group $\mathrm{TR}_{2i-\lambda_a}^u(\mathbb{F}_p; p)$ is a cyclic $W(\mathbb{F}_p)$ -module of length*

$$\mathrm{length}_{W(\mathbb{F}_p)} \mathrm{TR}_{2i-\lambda_a}^u(\mathbb{F}_p; p) = \begin{cases} u, & \text{if } a \leq i, \\ u - s, & \text{if } [a/p^s] \leq i < [a/p^{s-1}] \text{ and } 1 \leq s < u, \\ 0, & \text{if } i < [a/p^{u-1}]. \end{cases}$$

In addition, the group $\mathrm{TR}_{q-\lambda_a}^u(\mathbb{F}_p; p)$ is zero, if q is a negative or odd integer.

We recall from the statement of Thm. 1.7 that the integer $s_p(m, i, j)$ is defined to be the larger of 0 and the unique integer that satisfies $p^{s-1}j \leq m(i+1) < p^s j$. We also recall from (1.1) that $d_p(m, u, j) = [(p^{u-1}j - 1)/m]$.

COROLLARY 4.2. *Let $m, u, u' \geq 1$ and $i \geq 0$ be integers, let $j \geq 1$ be an integer that is not divisible by p , and let $d = d_p(m, u, j)$ and $d' = d_p(m, u', j)$. Then the canonical projection*

$$\mathrm{pr}_{u'}: \lim_R \mathrm{TR}_{2i-\lambda_d}^u(\mathbb{F}_p; p) \rightarrow \mathrm{TR}_{2i-\lambda_{d'}}^{u'}(\mathbb{F}_p; p)$$

is an isomorphism, if $p^{u'}j > m(i+1)$, and the common group is a cyclic $W(\mathbb{F}_p)$ -module of length $s_p(m, i, j)$. The group $\lim_R \mathrm{TR}_{q-\lambda_d}^u(\mathbb{F}_p; p)$ is zero, if q is a negative or odd integer.

Proof. We proved in Lemma 1.4 that the map of the statement is an isomorphism, provided that $i < d_p(m, u' + 1, j) = [(p^{u'}j - 1)/m]$. This constraint on the integer u' is equivalent to the inequality $i + 1 \leq [(p^{u'}j - 1)/m]$, which is equivalent to $i + 1 \leq (p^{u'}j - 1)/m$, which, in turn, is equivalent to the stated inequality $m(i + 1) < p^{u'}j$. Starting from Prop. 4.1, a similar calculation shows that the length of the $W(\mathbb{F}_p)$ -module $\lim_R \mathrm{TR}_{2i-\lambda_d}^u(\mathbb{F}_p; p)$ is equal to $s_p(m, i, j)$ as stated. \square

LEMMA 4.3. *Let $m > n \geq 1$, $u \geq 1$, and $i \geq 0$ be integers. Let $1 \leq j \leq m(i+1)$ be an integer that is not divisible by p , and let $d = d_p(m, u, j)$ and $e = d_p(n, u, j)$. Then the map*

$$\iota_p(m, n, j, u)_{2i}: \mathrm{TR}_{2i-\lambda_d}^u(\mathbb{F}_p; p) \rightarrow \mathrm{TR}_{2i-\lambda_e}^u(\mathbb{F}_p; p)$$

takes a generator of the domain to the product of a generator of the target and an element $\alpha' = \alpha'_p(m, n, i, j, u)$ of $W(\mathbb{F}_p)$ with p -adic valuation

$$v_p(\alpha') = \sum_{d < a \leq e} \mathrm{length}_{W(\mathbb{F}_p)} \mathrm{TR}_{2i-\lambda_a}^{v(u, a)}(\mathbb{F}_p; p),$$

where $v(u, a) = \min\{u, v_p(a) + 1\}$.

Proof. The canonical inclusion of $\iota_p(m, n, j, u): \lambda_d \rightarrow \lambda_e$ is equal to the composition of the canonical inclusions $\iota: \lambda_{a-1} \rightarrow \lambda_a$ for $d < a \leq e$. Since the groups $\mathrm{TR}_{q-\lambda_a}^u(\mathbb{F}_p; p)$ are zero, for q odd, Lemma 3.2 above identifies the cokernel of the map

$$\iota_*: \mathrm{TR}_{2i-\lambda_{a-1}}^u(\mathbb{F}_p; p) \rightarrow \mathrm{TR}_{2i-\lambda_a}^u(\mathbb{F}_p; p)$$

with the $W(\mathbb{F}_p)$ -module $\mathrm{TR}_{2i-\lambda_a}^{v(u, a)}(\mathbb{F}_p; p)$. Equivalently, ι_* takes a generator of the domain to the product of a generator of the target and an element $\alpha_a \in W(\mathbb{F}_p)$ of p -adic valuation

$$v_p(\alpha_a) = \mathrm{length}_{W(\mathbb{F}_p)} \mathrm{TR}_{2i-\lambda_a}^{v(u, a)}(\mathbb{F}_p; p).$$

The lemma follows by an easy induction argument. \square

We proceed to manipulate the sum that appears in Lemma 4.3. To this end, we fix the positive integer u , and consider the bi-graded \mathbb{F}_p -vector space $E(u)$ defined by the associated graded for the p -adic filtration of the $W(\mathbb{F}_p)$ -modules that appear in Lemma 4.3,

$$E(u)_{i, a} = \bigoplus_{r \geq 0} \mathrm{gr}_p^r \mathrm{TR}_{2i-\lambda_a}^{v(u, a)}(\mathbb{F}_p; p).$$

The following results identifies the structure of this bi-graded \mathbb{F}_p -vector space.

LEMMA 4.4. *The bi-graded \mathbb{F}_p -vector space $E(u)$ is isomorphic to the bi-graded \mathbb{F}_p -vector space defined by*

$$A(u) = \bigoplus_{1 \leq r \leq u} S_{\mathbb{F}_p} \{x_r, \sigma_r\},$$

where $\deg x_r = (p^{r-1}, 1)$ and $\deg \sigma_r = (0, 1)$.

Proof. The statement is equivalent to the equality

$$\dim_{\mathbb{F}_p} A(u)_{a,i} = \text{length}_{W(\mathbb{F}_p)} \text{TR}_{2i-\lambda_a}^{v(u,a)}(\mathbb{F}_p; p),$$

which we prove by direct calculation. If $a \leq i$, then $A(u)_{a,i}$ has basis

$$x_1^a \sigma_1^{i-a}, x_2^{a/p} \sigma_2^{i-a/p}, \dots, x_{v(u,a)}^{a/p^{v(u,a)-1}} \sigma_{v(u,a)}^{i-a/p^{v(u,a)-1}},$$

and hence, $\dim_{\mathbb{F}_p} A(u)_{a,i} = v(u, a)$ as required. Similarly, if $a/p^s \leq i < a/p^{s-1}$ with $1 \leq s < v(u, a)$, then $A(u)_{a,i}$ has basis

$$x_{s+1}^{a/p^s} \sigma_{s+1}^{i-a/p^s}, x_{s+2}^{a/p^{s+1}} \sigma_{s+2}^{i-a/p^{s+1}}, \dots, x_{v(u,a)}^{a/p^{v(u,a)-1}} \sigma_{v(u,a)}^{i-a/p^{v(u,a)-1}},$$

and hence, $\dim_{\mathbb{F}_p} A(u)_{a,i} = v(u, a) - s$ as desired. Finally, if $i < a/p^{v(u,a)-1}$, then $A(u)_{a,i} = 0$. This completes the proof. \square

PROPOSITION 4.5. *Let $m > n \geq 1$ and $i \geq 0$ be integers, and let $1 \leq j \leq m(i+1)$ be an integer that is not divisible by p . Then the map*

$$\iota_p(m, n, j)_{2i}: \lim_R \text{TR}_{2i-\lambda_d}^u(\mathbb{F}_p; p) \rightarrow \lim_R \text{TR}_{2i-\lambda_e}^u(\mathbb{F}_p; p),$$

where $d = d_p(m, u, j)$ and $e = d_p(n, u, j)$, takes a generator of the domain to the product of a generator of the target and an element $\alpha = \alpha_p(m, n, i, j)$ of $W(\mathbb{F}_p)$ with p -adic valuation

$$v_p(\alpha) = \sum_{0 \leq h < i} (s_p(m, h, j) - s_p(n, h, j)),$$

where $s_p(m, h, j)$ is the larger of 0 and the unique integer s with $p^{s-1}j \leq h(m+1) < p^s j$.

Proof. By Cor. 4.2 above, we may instead consider the map

$$\iota_p(m, n, j, u)_{2i}: \text{TR}_{2i-\lambda_d}^u(\mathbb{F}_p; p) \rightarrow \text{TR}_{2i-\lambda_e}^u(\mathbb{F}_p; p),$$

for a fixed positive integer u with $p^u j > m(i+1)$. We showed in Lemma 4.3 above that this map takes a generator of the domain to the product of a generator of the target and an element $\alpha' = \alpha'_p(m, n, i, j, u)$ of $W(\mathbb{F}_p)$ whose p -adic valuation is given by the sum

$$v_p(\alpha') = \sum_{d < a \leq e} \text{length}_{W(\mathbb{F}_p)} \text{TR}_{2i-\lambda_a}^{v(u,a)}(\mathbb{F}_p; p)$$

where $v(u, a) = \min\{u, v_p(a) + 1\}$. We show that this sum is equal to the sum $v_p(\alpha)$ of the statement. It follows from Lemma 4.4 that

$$v_p(\alpha') = \dim_{\mathbb{F}_p} \left(\bigoplus_{d < a \leq e} A(u)_{a,i} \right).$$

The \mathbb{F}_p -vector space on the right-hand side has a basis given by the elements $x_r^k \sigma_r^l$, where $1 \leq r \leq u$, and where k and l are non-negative integers such that $k+l = i$ and $d < p^{r-1}k \leq e$.

Therefore, we have

$$\begin{aligned}
v_p(\alpha') &= \sum_{1 \leq r \leq u} \#\{0 \leq k \leq i \mid d < p^{r-1}k \leq e\} = \sum_{1 \leq r \leq u} \#\{1 \leq k \leq i \mid d < p^{r-1}k \leq e\} \\
&= \sum_{1 \leq k \leq i} \#\{1 \leq r \leq u \mid d < p^{r-1}k \leq e\} = \sum_{0 \leq h < i} \#\{1 \leq r \leq u \mid d < p^{r-1}(h+1) \leq e\} \\
&= \sum_{0 \leq h < i} (\#\{1 \leq r \leq u \mid d < p^{r-1}(h+1)\} - \#\{1 \leq r \leq u \mid e < p^{r-1}(h+1)\}),
\end{aligned}$$

where, we recall, $d = d_p(m, u, j)$ and $e = d_p(n, u, j)$. The inequality $d < p^{r-1}(h+1)$ is equivalent to the inequality $(p^{u-1}j - 1)/m < p^{r-1}(h+1)$ which, in turn, is equivalent to the inequality

$$p^{u-r}j < m(h+1).$$

Suppose that $s = s_p(m, h, j)$ satisfies $1 \leq s \leq u$. Then $p^{s-1}j \leq m(h+1) < p^s j$. Therefore, the inequality $p^{u-1}j < m(h+1)$ is equivalent to the inequality $u - r \leq s - 1$. Hence,

$$\#\{1 \leq r \leq u \mid d < p^{r-1}(h+1)\} = \#\{u - (s-1) \leq r \leq u\} = s = s_p(m, h, j).$$

Finally, if $m(h+1) < j$, we also find that

$$\#\{1 \leq r \leq u \mid d < p^{r-1}(h+1)\} = 0 = s_p(m, h, j).$$

The proof that $\#\{1 \leq r \leq u \mid e < p^{r-1}(h+1)\} = s_p(n, h, j)$ is similar. \square

Proof of Thm. A. We prove Thm. 1.7, and hence, Thm. A of the introduction.

Suppose that $A = \mathbb{F}_p$. We first identify the sequences (1.1) and (1.5) and the sequences (1.2) and (1.6). To this end, we recall that [6, Prop. 9.1] gives the following slightly stronger version of Prop. 4.1: Let a and u be positive integers, let i be a non-negative integer, and let

$$r = r_p(a, u) = \text{length}_{W(\mathbb{F}_p)} \text{TR}_{2i-\lambda_a}^u(\mathbb{F}_p; p).$$

Then we can choose a family of isomorphisms of $W(\mathbb{F}_p)$ -modules

$$\sigma_p(a, u, i): W_r(\mathbb{F}_p) \xrightarrow{\sim} \text{TR}_{2i-\lambda_a}^u(\mathbb{F}_p; p)$$

with the additional property that the following diagrams commute:

$$\begin{array}{ccc}
W_r(\mathbb{F}_p) & \xrightarrow{\sigma_p(a, u, i)} & \text{TR}_{2i-\lambda_a}^u(\mathbb{F}_p; p) \\
\uparrow \downarrow F & & \uparrow \downarrow F \\
W_{r-1}(\mathbb{F}_p) & \xrightarrow{\sigma_p(a, u-1, i)} & \text{TR}_{2i-\lambda_a}^{u-1}(\mathbb{F}_p; p).
\end{array}$$

Let $m \geq 1$ and $i \geq 0$ be integers, and let $j \geq 1$ be an integer not divisible by p . We know from Cor. 1.4 that, if $u' \geq 1$ is an integer such that $p^{u'}j > m(i+1)$, then the canonical projection

$$\text{pr}_{u'}: \lim_R \text{TR}_{2i-\lambda_d}^u(\mathbb{F}_p; p) \rightarrow \text{TR}_{2i-\lambda_{d'}}^{u'}(\mathbb{F}_p; p),$$

where $d = d_p(m, u, j)$ and $d' = d_p(m, u', j)$, is an isomorphism. Hence, the composition of the isomorphism $\sigma_p(d', u', i)$ and the inverse of the isomorphism $\text{pr}_{u'}$ defines an isomorphism

$$\tau_p(m, i, j, u'): W_s(\mathbb{F}_p) \xrightarrow{\sim} \lim_R \text{TR}_{2i-\lambda_d}^u(\mathbb{F}_p; p),$$

where $s = s_p(m, i, j)$. Then, if $p^{u'} > m(i+1)$, we define

$$\tau_p(m, i, u'): \bigoplus_{j \in I_p} W_s(\mathbb{F}_p) \rightarrow \prod_{j \in I_p} \lim_R \text{TR}_{2i-\lambda_d}^u(\mathbb{F}_p; p)$$

to be the isomorphism that takes the j th summand on the left-hand side to the j th factor on the right-hand side by the map $\tau_p(m, i, j, u')$. Similarly, we define

$$\tau'_p(m, i, u'): \bigoplus_{j \in m'I_p} W_{s-v}(\mathbb{F}_p) \rightarrow \prod_{j \in m'I_p} \lim_R \mathrm{TR}_{2i-\lambda_d}^{u-v}(\mathbb{F}_p; p),$$

where $m = p^v m'$ with m' not divisible by p , where $s = s_p(m, i, j)$, and where $d = d_p(m, u, j)$, to be the isomorphism that takes the j th summand on the left-hand side to the j th factor on the right-hand side by the isomorphism given by the composition of $\sigma_p(d', u' - v, i)$ and the inverse of the isomorphism

$$\mathrm{pr}_{u'}: \lim_R \mathrm{TR}_{2i-\lambda_d}^{u-v}(\mathbb{F}_p; p) \rightarrow \mathrm{TR}_{2i-\lambda_{d'}}^{u'-v}(\mathbb{F}_p; p).$$

Then, for every u' such that $p^{u'} > m(i+1)$, the diagram

$$\begin{array}{ccc} \bigoplus_{j \in m'I_p} W_{s-v}(\mathbb{F}_p) & \xrightarrow{m'V^v} & \bigoplus_{j \in I_p} W_s(\mathbb{F}_p) \\ \downarrow \tau'_p(m, i, u') & & \downarrow \tau_p(m, i, u') \\ \prod_{j \in m'I_p} \lim_R \mathrm{TR}_{2i-\lambda_d}^{u-v}(\mathbb{F}_p; p) & \xrightarrow{m'V^v} & \prod_{j \in I_p} \lim_R \mathrm{TR}_{2i-\lambda_d}^u(\mathbb{F}_p; p) \end{array}$$

commutes. This identifies the sequences (1.1) and (1.5). However, we do not know that the family of isomorphisms $\sigma_p(a, u, i)$ can be chosen with the additional property that, if both $p^{u'} > m(i+1)$ and $p^{u''} > m(i+1)$, the isomorphisms $\tau_p(m, i, u')$ and $\tau_p(m, i, u'')$ are equal. Therefore, we choose $u' = u'(m, i)$ to be the unique integer that satisfies

$$p^{u'-1} \leq m(i+1) < p^{u'}$$

and use the isomorphisms $\tau_p(m, i, u')$ and $\tau'_p(m, i, u')$ to identify (1.1) and (1.5). We use the same $u' = u'(m, i)$ and the isomorphisms $\tau_p(n, i, u')$ and $\tau'_p(n, i, u')$ to identify the sequences (1.2) and (1.6). In particular, the sequences (1.5) and (1.6) are exact as proved in [5, Thm. 4.2.10]. Finally, Prop. 4.5 shows there is a commutative diagram

$$\begin{array}{ccccc} W_s(\mathbb{F}_p) & \xrightarrow{R^{s-t}} & W_t(\mathbb{F}_p) & \xrightarrow{m_\alpha} & W_t(\mathbb{F}_p) \\ \downarrow \tau_p(m, i, j, u') & & & & \downarrow \tau_p(n, i, j, u') \\ \lim_R \mathrm{TR}_{2i-\lambda_d}^u(\mathbb{F}_p; p) & \xrightarrow{\iota_p(m, n, i, j)} & & & \lim_R \mathrm{TR}_{2i-\lambda_e}^u(\mathbb{F}_p; p), \end{array}$$

where the map m_α is given by multiplication by an element $\alpha = \alpha_p(m, n, i, j)$ of $W(\mathbb{F}_p)$ whose p -adic valuation is given by the sum

$$v_p(\alpha) = \sum_{0 \leq h < i} (s_p(m, h, j) - s_p(n, h, j)).$$

This completes the proof of Thm. 1.7 for $A = \mathbb{F}_p$.

Suppose next that A is any regular noetherian ring that is also an \mathbb{F}_p -algebra. The groups $\lim_R \mathrm{TR}_{q-\lambda_a}^u(A; p)$ were evaluated in [7], but see also [4, Thm. 16]. By the universal property of the de Rham-Witt complex, there is a canonical map of graded $W_u(\mathbb{F}_p)$ -algebras

$$\xi: W_u \Omega_A^* \rightarrow \mathrm{TR}_*^u(A; p)$$

that commutes with R , F , d , and V . Let $m \geq 1$ and $i \geq 0$ be integers, and let $u' = u'(m, i)$. Let j be an integer not divisible by p , and let $s = s_p(m, i, j)$. We consider the map

$$\tilde{\omega}_p(m, q, i, j, u'): \lim_R W_u \Omega_A^{q-2i} \otimes_{W(\mathbb{F}_p)} W_s(\mathbb{F}_p) \rightarrow \lim_R \mathrm{TR}_{q-\lambda_d}^u(A; p)$$

defined by the composition

$$\begin{aligned} \lim_R W_u \Omega_A^{q-2i} \otimes_{W(\mathbb{F}_p)} W_s(\mathbb{F}_p) &\rightarrow \lim_R \mathrm{TR}_{q-2i}^u(A; p) \otimes_{W(\mathbb{F}_p)} \lim_R \mathrm{TR}_{2i-\lambda_d}^u(\mathbb{F}_p; p) \\ &\rightarrow \lim_R \mathrm{TR}_{q-2i}^u(A; p) \otimes_{W(\mathbb{F}_p)} \lim_R \mathrm{TR}_{2i-\lambda_d}^u(A; p) \rightarrow \lim_R \mathrm{TR}_{q-\lambda_d}^u(A; p), \end{aligned}$$

where the first map is $\kappa \otimes \tau_p(m, i, j, u')$, where the second map is induced from the unit map $\mathbb{F}_p \rightarrow A$, and where the last map is induced by the $T(A)$ -module spectrum structure on $T(A) \wedge S^{\lambda_d}$. It follows from [4, Thm. 16] that the maps $\tilde{\omega}_p(m, q, i, j, u')$ factor through the canonical projections

$$\mathrm{pr}_s \otimes \mathrm{id}: \lim_R W_u \Omega_A^{q-2i} \otimes_{W(\mathbb{F}_p)} W_s(\mathbb{F}_p) \rightarrow W_s \Omega_A^{q-2i}$$

and that the induced maps $\omega_p(m, q, i, j, u')$ define an isomorphism

$$\omega_p(m, q, j) = \bigoplus_{i \geq 0} \omega_p(m, q, i, j, u'): \bigoplus_{i \geq 0} W_s \Omega_A^{q-2i} \xrightarrow{\sim} \lim_R \mathrm{TR}_{q-\lambda_d}^u(A; p),$$

where $u' = u'(m, i)$ and $s = s_p(m, i, j)$. We define

$$\omega_p(m, q): \bigoplus_{i \geq 0} \bigoplus_{j \in I_p} W_s \Omega_A^{q-2i} \rightarrow \prod_{j \in I_p} \lim_R \mathrm{TR}_{q-\lambda_d}^u(A; p)$$

to be the isomorphism that takes the j summand on the left-hand side to the j th factor on the right-hand side by the isomorphism $\omega_p(m, q, j)$. We define the isomorphism

$$\omega'_p(m, q): \bigoplus_{i \geq 0} \bigoplus_{j \in m' I_p} W_{s-v} \Omega_A^{q-2i} \rightarrow \prod_{j \in m' I_p} \lim_R \mathrm{TR}_{q-\lambda_d}^{u-v}(A; p)$$

in a completely similar manner substituting the isomorphisms $\tau'_p(m, i, j, u')$ for the isomorphisms $\tau_p(m, i, j, u')$ in the definition of $\omega_p(m, q)$. Then the diagram

$$\begin{array}{ccc} \bigoplus_{i \geq 0} \bigoplus_{j \in m' I_p} W_{s-v} \Omega_A^{q-2i} & \xrightarrow{m' V^v} & \bigoplus_{i \geq 0} \bigoplus_{j \in I_p} W_s \Omega_A^{q-2i} \\ \downarrow \omega'_p(m, q) & & \downarrow \omega_p(m, q) \\ \prod_{j \in m' I_p} \lim_R \mathrm{TR}_{q-\lambda_d}^{u-v}(A; p) & \xrightarrow{m' V^v} & \prod_{j \in I_p} \lim_R \mathrm{TR}_{q-\lambda_d}^u(A; p) \end{array}$$

commutes. Hence, the isomorphisms $\omega_p(m, q)$ and $\omega'_p(m, q)$ identify the sequences (1.1) and (1.5). In particular, the sequence (1.5) is exact. Similarly, the isomorphisms $\omega_p(n, q)$ and $\omega'_p(n, q)$ identify the sequences (1.2) and (1.6). Finally, we have a commutative diagram

$$\begin{array}{ccccc} W_s \Omega_A^{q-2i} & \xrightarrow{R^{s-t}} & W_t \Omega_A^{q-2i} & \xrightarrow{m_\alpha} & W_t \Omega_A^{q-2i} \\ \downarrow \omega_p(m, q, i, j, u') & & & & \downarrow \omega_p(n, q, i, j, u') \\ \lim_R \mathrm{TR}_{2i-\lambda_d}^u(A; p) & \xrightarrow{\iota_p(m, n, i, j)} & & & \lim_R \mathrm{TR}_{2i-\lambda_e}^u(A; p), \end{array}$$

where the top horizontal map is multiplication by the same element $\alpha = \alpha_p(m, n, i, j)$ of $W(\mathbb{F}_p)$ as in the basic case $A = \mathbb{F}_p$. This completes the proof of Thm. 1.7. \square

5. The divisor $\mathrm{div}(\alpha_p(m, n, i))$

In this section, we examine the divisor $\mathrm{div}(\alpha_p(m, n, i))$ on $\mathbb{W}(\mathbb{F}_p)$ and prove Thm. B of the introduction. In addition, we derive some consequences of the result for the tower of K -theory of truncated polynomial algebras.

LEMMA 5.1. Suppose that the integer $t = s_p(n, i, j)$ satisfies

$$\frac{m-n}{mn} \cdot \frac{p^t-1}{p-1} \cdot j \geq 2t.$$

Then $v_p(\alpha_p(m, n, i, j)) \geq s_p(n, i, j)$.

Proof. We recall that

$$v_p(\alpha_p(m, n, i, j)) = \sum_{0 \leq h < i} (s_p(m, h, j) - s_p(n, h, j)).$$

In this sum, every summand is non-negative, and the summand indexed by h is positive if and only if there exists an integer $r \geq 1$ with

$$n(h+1) < p^{r-1}j \leq m(h+1).$$

We estimate the number of indices $0 \leq h < i$ that satisfy this inequality, for some $r \geq 1$. For a given $r \geq 1$, the number of integers h that satisfy the inequality

$$n(h+1) < p^{r-1}j \leq m(h+1)$$

is at least

$$\frac{(m-n)p^{r-1}j}{mn} - 1.$$

If $t = s_p(n, i, j) = 0$, then the statement of the lemma is trivially true. So assume that $t > 0$. Then t is the unique integer that satisfies $p^{t-1}j \leq n(i+1) < p^tj$. Now, if $1 \leq r \leq t$, then

$$n(h+1) < p^{r-1}j \leq p^{t-1}j \leq h(i+1),$$

and hence, $h < i$. Therefore, we find

$$v_p(\alpha_p(m, n, i, j)) \geq \sum_{1 \leq r \leq t} \left(\frac{(m-n)p^{r-1}j}{mn} - 1 \right) = \frac{m-n}{mn} \cdot \frac{p^t-1}{p-1} \cdot j - t.$$

Hence, if the latter integer is greater than or equal to t , or equivalently, if

$$\frac{m-n}{mn} \cdot \frac{p^t-1}{p-1} \cdot j \geq 2t,$$

then $v_p(\alpha_p(m, n, i, j)) \geq s_p(n, i, j)$. \square

LEMMA 5.2. Suppose that $i \geq n/(m-n)$ and $v_p(\alpha_p(m, n, i-1, j)) \geq s_p(n, i-1, j)$. Then also $v_p(\alpha_p(m, n, i, j)) \geq s_p(n, i, j)$.

Proof. The assumption that $i \geq n/(m-n)$ implies that $mi \geq n(i+1)$ which, in turn, implies that $s_p(m, i-1, j) \geq s_p(n, i, j)$. Finally, $v_p(\alpha_p(m, n, i-1, j)) \geq s_p(n, i-1, j)$ implies that $v_p(\alpha_p(m, n, i, j)) \geq s_p(m, i-1, j)$. The statement follows. \square

Proof of Thm. B. We first show part (i) of the statement, which, by the discussion preceding Thm. 1.7, is equivalent to the statement that there exists an integer $i_0 = i_0(m, n, p)$ such that, for all integers $j \geq 1$ not divisible by p , $v_p(\alpha_p(m, n, i, j)) \geq s_p(n, i, j)$. Suppose first that $j \geq 2mn/(m-n)$. Then the inequality in the statement of Lemma 5.1 is satisfied, for all $t \geq 1$, and therefore, $v_p(\alpha_p(m, n, i, j)) \geq s_p(n, i, j)$, for all $i \geq 0$. Suppose next that $1 \leq j < 2mn/(m-n)$. The inequality of the statement of Lemma 5.1 is satisfied, if t is large enough. Since $t = s_p(n, i, j)$ tends to infinity as i tends to infinity, it follows that there

exists an integer $i_0(m, n, p, j)$ such that $v_p(\alpha_p(m, n, i, j)) \geq s_p(n, i, j)$, for all $i \geq i_0(m, n, p, j)$. This proves that part (i) of the statement holds with $i_0(m, n, p)$ equal to the maximum of the integers $i_0(m, n, p, j)$, where $1 \leq j < 2mn/(m-n)$.

We next prove part (ii) of the statement. Suppose that $p > m(i+1)$. Then, for every integer $0 \leq h \leq i$, and for every integer $1 \leq j \leq n$ not divisible by p , $s_p(m, i, j) = s_p(n, i, j) = 1$, and hence, $v_p(\alpha_p(m, n, i, j)) < s_p(n, i, j)$. Therefore, if $p > m(i+1)$, then $i_0(m, n, p) > i$. It follows that $i_0(m, n, p) \geq [(p-1)/m]$ which tends to infinity as p tends to infinity.

Finally, we prove part (iii) of the statement. We first note that, for fixed i and j , the integer $s_p(m, i, j)$ tends to infinity as m tends to infinity. Hence, there exists $m_0 = m_0(n, p, i, j)$ such that $v_p(\alpha_p(m, n, i, j)) \geq s_p(n, i, j)$, for $m \geq m_0(n, p, i, j)$. We assume that the integer $m_0(n, p, i, j)$ is chosen minimal with this property. An induction argument based on Lemma 5.2 then shows that $m_0(n, p, i, j) \leq m_0(n, p, i-1, j)$, for $i \geq n/(m-n)$. Moreover, since $s_p(n, i, j) = 0$, for $j > n(i+1)$, we have $m_0(n, p, i, j) = n$, for $j > n(i+1)$. This shows that the statement of part (iii) holds with $m_0(n, p)$ equal to the maximum of the finitely many integers $m_0(n, p, i, j)$, where $0 \leq i \leq n/(m-n)$ and $1 \leq j \leq n(i+1)$. \square

LEMMA 5.3. *Let A be an \mathbb{F}_p -algebra and suppose that A is generated by N elements as an algebra over the subring A^p of p th powers. Then, for every subset $S \subset \mathbb{N}$ stable under division, the big de Rham-Witt group $\mathbb{W}_S \Omega_A^q$ is zero, if $q > N + 1$.*

Proof. It suffices to show that, for all $u \geq 1$, the p -typical de Rham-Witt group $W_u \Omega_A^q$ is zero, if $q > N + 1$; compare Sect. 1 above. We first show that Ω_A^q is zero, if $q > N$. By assumption, there exists a surjective ring homomorphism $f: A^p[x_1, \dots, x_N] \rightarrow A$ from a polynomial algebra in finitely many variables over A^p . This induces map

$$f_*: \Omega_{A^p[x_1, \dots, x_N]}^q \rightarrow \Omega_A^q$$

is again surjective. Moreover, every element of the domain is a sum of elements of the form

$$\omega = \eta dx_{i_1} \dots dx_{i_s}$$

where $0 \leq s \leq N$, where $1 \leq i_1 < \dots < i_s \leq N$, and where $\eta \in \Omega_{A^p}^{q-s}$. Now, we claim that $f_*(\eta) = 0$ unless $s = q$. Indeed, the element $\eta \in \Omega_{A^p}^{q-s}$ can be written as a sum of elements of the form $b_0 db_1 \dots db_{q-s}$, where $b_0, \dots, b_{q-s} \in A^p$, and

$$f_*(b_0 db_1 \dots db_{q-s}) = f(b_0) df(b_1) \dots df(b_{q-s}).$$

But $f(b_i) = a_i^p$, for some $a_i \in A$, and hence, $df(b_i) = d(a_i^p) = pa_i^{p-1} da_i = 0$. It follows that the image of f_* is zero, if $q > N$. This shows that Ω_A^q is zero, if $q > N$, as stated.

Finally, we show by induction on $u \geq 1$ that $W_u \Omega_A^q$ is zero, if $q > N + 1$. The case $u = 1$ holds, since the canonical map $\Omega_A^q \rightarrow W_1 \Omega_A^q$ is an isomorphism. Finally, the induction step follows from the exact sequence

$$\Omega_A^q \oplus \Omega_A^{q-1} \xrightarrow{V^{u-1} + dV^{u-1}} W_u \Omega_A^q \xrightarrow{R} W_{u-1} \Omega_A^q \longrightarrow 0$$

which is proved in [8, Prop. 3.2.6]. \square

THEOREM 5.4. *Let A be a regular noetherian ring and an \mathbb{F}_p -algebra and assume that A is finitely generated as an algebra over the subring $A^p \subset A$ of p th powers. Let m and n be positive integers with $m > n + 1$. Then there exists an integer $q_0 > 1$ such that the map*

$$f_*: K_q(A[x]/(x^m), (x)) \rightarrow K_q(A[x]/(x^n), (x))$$

induced by the canonical projection is zero, for all $q \geq q_0$.

Proof. Since $m > n + 1$, we can choose $n < k < m$ and write f as the composition of the canonical projections $g: A[x]/(x^m) \rightarrow A[x]/(x^k)$ and $h: A[x]/(x^k) \rightarrow A[x]/(x^n)$. We consider the following maps of long-exact sequences from Thm. A.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \bigoplus_{i \geq 0} \mathbb{W}_{m(i+1)} \Omega_A^{q-2i} & \xrightarrow{\varepsilon} & K_{q+1}(A[x]/(x^m), (x)) & \xrightarrow{\partial} & \bigoplus_{i \geq 0} \mathbb{W}_{i+1} \Omega_A^{q-1-2i} \longrightarrow \cdots \\
& & \downarrow g'_* & & \downarrow g_* & & \downarrow 0 \\
\cdots & \longrightarrow & \bigoplus_{i \geq 0} \mathbb{W}_{k(i+1)} \Omega_A^{q-2i} & \xrightarrow{\varepsilon} & K_{q+1}(A[x]/(x^k), (x)) & \xrightarrow{\partial} & \bigoplus_{i \geq 0} \mathbb{W}_{i+1} \Omega_A^{q-1-2i} \longrightarrow \cdots \\
& & \downarrow h'_* & & \downarrow h_* & & \downarrow 0 \\
\cdots & \longrightarrow & \bigoplus_{i \geq 0} \mathbb{W}_{n(i+1)} \Omega_A^{q-2i} & \xrightarrow{\varepsilon} & K_{q+1}(A[x]/(x^n), (x)) & \xrightarrow{\partial} & \bigoplus_{i \geq 0} \mathbb{W}_{i+1} \Omega_A^{q-1-2i} \longrightarrow \cdots
\end{array}$$

It follows from Lemma 5.3 and Thm. B(i) that there exists an integer q_0 such that both the maps g'_* and h'_* are zero, for $q \geq q_0$. Finally, a diagram chase based on the diagram above shows that the composite map $f_* = h_* \circ g_*$ is zero, for $q \geq q_0$. \square

Suppose that A is a regular noetherian ring and an \mathbb{F}_p -algebra. We let

$$\varepsilon_i: \mathbb{W}_{n(i+1)} \Omega_A^{q-2i} \rightarrow K_{q+1}(A[x]/(x^n), (x))$$

be the i th summand of the map ε in the long-exact sequence

$$\cdots \longrightarrow \bigoplus_{i \geq 0} \mathbb{W}_{i+1} \Omega_A^{q-2i} \xrightarrow{V_n} \bigoplus_{i \geq 0} \mathbb{W}_{n(i+1)} \Omega_A^{q-2i} \xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^n), (x)) \longrightarrow \cdots$$

The image of the map ε_0 has the following interpretation.

THEOREM 5.5. *Let A be a regular noetherian ring that is also an \mathbb{F}_p -algebra, and let n be a positive integer. Then*

$$\begin{aligned}
& \bigcap_{m > n} \operatorname{im} (f_*: K_q(A[x]/(x^m), (x)) \rightarrow K_q(A[x]/(x^n), (x))) \\
& = \operatorname{im} (\varepsilon_0: \mathbb{W}_n \Omega_A^{q-1} \rightarrow K_q(A[x]/(x^n), (x)))
\end{aligned}$$

Proof. Let $g: A[x]/(x^m) \rightarrow A[x]/(x^k)$ and $h: A[x]/(x^k) \rightarrow A[x]/(x^n)$ be the canonical projections, with $m > k > n$. A diagram based on the diagram from the proof of Thm. 5.4 above shows that, for every integer $q \geq 0$, we have inclusions

$$\begin{aligned}
& \operatorname{im} \left(\bigoplus_{i \geq 0} \mathbb{W}_{m(i+1)} \Omega_A^{q-2i} \xrightarrow{h'_* g'_*} \bigoplus_{i \geq 0} \mathbb{W}_{n(i+1)} \Omega_A^{q-2i} \xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^n), (x)) \right) \\
& \subset \operatorname{im} \left(K_{q+1}(A[x]/(x^m), (x)) \xrightarrow{f_*} K_{q+1}(A[x]/(x^n), (x)) \right) \\
& \subset \operatorname{im} \left(\bigoplus_{i \geq 0} \mathbb{W}_{k(i+1)} \Omega_A^{q-2i} \xrightarrow{h'_*} \bigoplus_{i \geq 0} \mathbb{W}_{n(i+1)} \Omega_A^{q-2i} \xrightarrow{\varepsilon} K_{q+1}(A[x]/(x^n), (x)) \right)
\end{aligned}$$

The maps $h'_* g'_*$ and h'_* map the summand indexed by $i = 0$ to the summand indexed by $i = 0$ by the appropriate restriction map, which is surjective. Thm. B shows that, on summands indexed by $i > 1$, the maps $h'_* g'_*$ and h'_* are zero, if m is sufficiently large. The statement follows. \square

REMARK 5.6. Let k be a perfect field of positive characteristic p , and let n be a positive integer. It follows from Thms. A and B that, for $m \gg n$, the composite map

$$\mathbb{W}_{n(i+1)}(k) \xrightarrow{\varepsilon} K_{2i+1}(k[x]/(x^n), (x)) \xrightarrow{\delta} K_{2i}(k[x]/(x^m), (x^n))$$

induces an isomorphism

$$\Delta'_n : \mathbb{W}_{n(i+1)}(k)/V_n \mathbb{W}_{i+1}(k) \xrightarrow{\sim} K_{2i}(k[x]/(x^m), (x^n)).$$

We expect that, for $i = 1$, the map Δ'_n is equal to the map Δ_n of Stienstra [16].

REMARK 5.7. One generally expects that, for every scheme X , there exists an Atiyah-Hirzebruch type spectral sequence

$$E_{s,t}^2 = H^{t-s}(X; \mathbb{Z}(t)) \Rightarrow K_{s+t}(X)$$

from the motivic cohomology of X to the algebraic K -theory of X . Such a sequence has been constructed in the case where X is a smooth scheme over a field. However, in general, there presently does not exist a definition of motivic cohomology that could serve as the E^2 -term of a spectral sequence of this form. However, see [1, 15]. The long-exact sequence relating $K(A[x]/(x^n), (x))$ and the big de Rham-Witt groups and Thm. 5.5 suggests that, for $X = \text{Spec}(A[x]/(x^n))$, where A is a regular noetherian \mathbb{F}_p -algebra, the spectral sequence takes the form

$$E_{s,t}^1 = \mathbb{W}_s \Omega_A^{t-(s-1)} \oplus \mathbb{W}_{(s+1)n} \Omega_A^{t-s} \Rightarrow K_{s+t}(A[x]/(x^n), (x))$$

with the d^1 -differential induced by the Verschiebung operator

$$V_n : \mathbb{W}_s \Omega_A^{t-(s-1)} \rightarrow \mathbb{W}_{sn} \Omega_A^{t-(s-1)}$$

and with all higher differentials zero. The E^2 -term is of this hypothetical spectral sequence then is our candidate for the value of the motivic cohomology groups of $A[x]/(x^n)$ relative to the ideal (x) . We note that this value of the motivic cohomology groups is in agreement with the Beilinson-Soulé vanishing conjecture that $H^q(A[x]/(x^n), (x); \mathbb{Z}(t))$ is zero, for $q \leq 0$.

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