

A counterexample to a conjecture of Somekawa*

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Abstract

We construct an example of a torus T over a field K for which the Galois symbol $K(K; T, T)/nK(K; T, T) \rightarrow H^2(K, T[n] \otimes T[n])$ is not injective for some n . Here $K(K; T, T)$ is the Milnor K -group attached to T introduced by Somekawa.

Introduction Let K be a field, m a positive integer and n an integer prime to the characteristic of K . Recently Rost and Voevodsky announced a proof of the bijectivity of the Galois symbol

$$(1) \quad K_m^M(K)/nK_m^M(K) \longrightarrow H^m(K, \mathbb{Z}/n\mathbb{Z}(m))$$

from Milnor K -groups to Galois cohomology (special cases have been obtained earlier by Merkurjev-Suslin, Rost and Voevodsky).

In [So], Somekawa has introduced certain *generalized Milnor K -groups* $K(K; A_1, \dots, A_m)$ attached to semi-abelian varieties A_1, \dots, A_m . If $A_1 = \dots = A_m = \mathbb{G}_m$ is the one-dimensional split torus they agree with the usual $K_m^M(K)$. If $m = 2$, $A_1 = \text{Jac}_X$ and $A_2 = \text{Jac}_Y$ are the Jacobians of smooth, projective and connected curves X and Y over K having a K -rational point, then $K(K; A_1, A_2)$ is the kernel of the Albanese map $\text{CH}_0(X \times Y)_{\text{deg}=0} \rightarrow \text{Alb}_{X \times Y}(K)$. In general, Somekawa has defined a Galois symbol

$$K(K; A_1, \dots, A_m)/nK(K; A_1, \dots, A_m) \longrightarrow H^m(K, A_1[n] \otimes \dots \otimes A_m[n])$$

and conjectured that it is always injective.

In this note we present a counterexample to the injectivity of the Galois symbol. Let us describe it briefly. Let L/K be a cyclic extension of degree n and σ a generator of the Galois group $\text{Gal}(L/K)$. Let T be the kernel of the norm map $\text{Res}_{L/K} \mathbb{G}_m \rightarrow \mathbb{G}_m$. We show that the norm $K(L; T, T) \rightarrow K(K; T, T)$ induces an isomorphism $K_2(L; T, T)/(1 - \sigma) \rightarrow K_2(K; T, T)$. On the other hand, the corresponding map of Galois cohomology groups

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$H^2(L, T[n] \otimes T[n]) / (1 - \sigma) \rightarrow H^2(K, T[n] \otimes T[n])$ is neither injective nor surjective (for a suitable choice of L/K). Note that, since T is split over L , the Galois symbol $K_2(L; T, T)/n \rightarrow H^2(L, T[n] \otimes T[n])$ is bijective. Consequently, $K_2(K; T, T)/n \rightarrow H^2(K, T[n] \otimes T[n])$ is in general not injective.

Algebraic groups as Mackey-functors Let K be a field. For a finite field extension L/K and commutative algebraic groups G over K and H over L we denote by G_L the base change of G to L and by $\text{Res}_{L/K} H$ the Weil restriction of H . The functor $G \mapsto G_L$ is left and right adjoint to $H \mapsto \text{Res}_{L/K} H$. In particular there are adjunction homomorphisms $\iota_{L/K} : G \rightarrow \text{Res}_{L/K} G_L$ and $N_{L/K} : \text{Res}_{L/K} G_L \rightarrow G$. When L/K is a Galois extension, the Galois group $\text{Gal}(L/K)$ acts canonically on $\text{Res}_{L/K} G_L$. The following simple result, whose proof will be left to the reader, will be used later.

Lemma 1 *Let L/K be a cyclic Galois extension of degree n , σ a generator of $\text{Gal}(L/K)$ and let G be a commutative algebraic group over K . Let G' be the kernel of $N_{L/K} : \text{Res}_{L/K} G_L \rightarrow G$ so that $G'_L \cong G_L^{n-1}$. Then the map*

$$\text{Res}_{L/K} (G_L)^{n-1} \cong \text{Res}_{L/K} G'_L \xrightarrow{N_{L/K}} G' \hookrightarrow \text{Res}_{L/K} G_L$$

is given on the i -th summand by $1 - \sigma^i$.

We denote by \mathcal{C}_K the category of finite reduced K -schemes. Thus each object of \mathcal{C}_K is isomorphic to $\text{Spec}(E_1 \times \dots \times E_r)$ where $E_1, \dots, E_r/K$ are finite field extensions. A commutative algebraic group G over K defines a *Mackey-functor*, i.e. a co- and contravariant functor $G : \mathcal{C}_K \rightarrow \text{Mod}_{\mathbb{Z}}$ satisfying (i), (ii) below. If $f : X \rightarrow Y$ is a morphism we denote by $f_* : G(X) \rightarrow G(Y)$ and $f^* : G(Y) \rightarrow G(X)$ the homomorphisms induced by co- and contravariant functoriality respectively.

(i) If $X = X_1 \coprod X_2 \in \text{Obj}(\mathcal{C}_K)$ then $G(X) = G(X_1) \oplus G(X_2)$.

(ii) If

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow g' & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

is a cartesian square in \mathcal{C}_K then $g^* f_* = (f')_*(g')^*$.

If $K \subseteq E_1 \subseteq E_2$ are two finite field extensions and $f : \text{Spec } E_2 \rightarrow \text{Spec } E_1$ the corresponding map in \mathcal{C}_K then f^* (resp. f_*) is given by $\iota_{E_2/E_1} : G(E_1) \rightarrow G(E_2)$ (resp. $N_{E_2/E_1} : G(E_2) \rightarrow G(E_1)$).

Local symbols We recall also the notion of a *local symbol* ([Se] and [So]) for G . Let $X \rightarrow \text{Spec } K$ is a proper non-singular algebraic curve (note that we do not assume that X is connected). Let $K(X)$ denote the ring of rational functions on X and $|X|$ the set of closed points of X . For $P \in |X|$ we denote by K_P the quotient field of the completion $\widehat{\mathcal{O}}_{X,P}$ of $\mathcal{O}_{X,P}$, by $v_P : K_P \rightarrow \mathbb{Z} \cup \{\infty\}$ the normalized valuation and by $K(P)$ the residue field of K_P . The local symbol at P is a homomorphism $\partial_P : (K_P)^* \otimes G(K_P) \rightarrow G(K(P))$. It is characterized by the following properties:

- (i) If $f \in (K_P)^*$ and $g \in G(\widehat{\mathcal{O}}_{X,P})$ then $\partial_P(f \otimes g) = v_P(f)g(P)$. Here $g(P)$ is the image of g under the canonical map $G(\widehat{\mathcal{O}}_{X,P}) \rightarrow G(K(P))$.
- (ii) For $f \in K(X)^*$ and $g \in G(K(X))$ we have $\sum_{P \in |X|} N_{K(P)/K}(\partial_P(f \otimes g)) = 0$.

Milnor K -groups attached to commutative algebraic groups Let G_1, \dots, G_m be commutative algebraic groups over K . In [So] Somekawa has introduced the Milnor K -group $K(K; G_1, \dots, G_m)$ (actually Somekawa considered only the case of semiabelian varieties though his construction works for arbitrary commutative algebraic groups). It is given as

$$K(K; G_1, \dots, G_m) = \left(\bigoplus_X G_1(X) \otimes \dots \otimes G_m(X) \right) / R$$

where X runs through all objects of \mathcal{C}_K and where the subgroup R is generated by the following elements:

(R1) If $f : X \rightarrow Y$ is a morphism in \mathcal{C}_K and if $x_{i_0} \in G_{i_0}(Y)$ for some i_0 and $x_i \in G_i(X)$ for $i \neq i_0$, then

$$x_1 \otimes \dots \otimes f_*(x_{i_0}) \otimes \dots \otimes x_m - f^*(x_1) \otimes \dots \otimes x_{i_0} \otimes \dots \otimes f^*(x_m) \in R.$$

(R2) Let $X \rightarrow \text{Spec } K$ be a proper non-singular curve, $f \in K(X)^*$ and $g_i \in G_i(K(X))$. Assume that for each $P \in |X|$ there exists $i(P)$ such that $g_i \in G_i(\widehat{\mathcal{O}}_{X,P})$ for all $i \neq i(P)$. Then

$$\sum_{P \in |X|} g_1(P) \otimes \dots \otimes \partial_P(f \otimes g_{i(P)}) \otimes \dots \otimes g_m(P) \in R.$$

For $X \in \mathcal{C}_K$ and $x_i \in G_i(X)$ for $i = 1, \dots, m$ we write $\{x_1, \dots, x_m\}_{X/K}$ for the image of $x_1 \otimes \dots \otimes x_m$ in $K(K; G_1, \dots, G_m)$ (elements of this form will be referred to as *symbols*).

A sequence of algebraic groups $G' \rightarrow G \rightarrow G''$ over K will be called *Zariski exact* if $G'(E) \rightarrow G(E) \rightarrow G''(E)$ is exact for every extension E/K . The proof of the following result is straightforward; hence will be omitted.

Lemma 2 *Let m be a positive integer and let $i \in \{1, \dots, m\}$. Let G_1, \dots, G_m be commutative algebraic groups over K and let $G'_i \rightarrow G_i \rightarrow G''_i \rightarrow 1$ be a Zariski exact sequence of commutative algebraic groups over K . Then the sequence*

$$K(K; G_1, \dots, G'_i, \dots) \rightarrow K(K; G_1, \dots, G_i, \dots) \rightarrow K(K; G_1, \dots, G''_i, \dots) \rightarrow 0$$

is exact as well.

The norm map Let G_1, \dots, G_m be commutative algebraic groups over K and let L/K be a finite extension. Set $K(L; G_1, \dots, G_m) := K(L; (G_1)_L, \dots, (G_m)_L)$. Then we have the norm map [So]

$$(2) \quad N_{L/K} : K(L; G_1, \dots, G_m) \longrightarrow K(K; G_1, \dots, G_m)$$

defined on symbols by $N_{L/K}(\{x_1, \dots, x_m\}_{X/L}) = \{x_1, \dots, x_m\}_{X/K}$ for any $X \in \mathcal{C}_L$ and $x_i \in G_i(X)$ ($i = 1, \dots, m$). We give another interpretation of (2) below when L/K is separable. It is based on the following result.

Lemma 3 *Let L/K be a finite separable extension and let i, m be positive integers with $i \leq m$. Let $G_1, \dots, G_{i-1}, G_{i+1}, \dots, G_m$ be commutative algebraic groups over K and let G_i be a commutative algebraic group over L . Then, we have an isomorphism*

$$K(K; G_1, \dots, \text{Res}_{L/K} G_i, \dots, G_m) \cong K(L; (G_1)_L, \dots, G_i, \dots, (G_m)_L).$$

Proof. To simplify the notation we assume that $i = m$. We denote by $\pi^{-1} : \mathcal{C}_K \rightarrow \mathcal{C}_L$ and $\pi : \mathcal{C}_L \rightarrow \mathcal{C}_K$ the functors

$$\pi^{-1}(X \rightarrow \text{Spec } K) := (X \otimes_K L \rightarrow \text{Spec } L),$$

$$\pi(Y \rightarrow \text{Spec } L) := (Y \rightarrow \text{Spec } L \rightarrow \text{Spec } K).$$

π is left adjoint to π^{-1} . For $X \in \mathcal{C}_K$ and $Y \in \mathcal{C}_L$ let

$$p_X : X \otimes_K L \longrightarrow X, \quad \iota_Y : Y \longrightarrow Y \otimes_K L.$$

be the adjunction morphisms. We define homomorphisms

$$\phi : K(K; G_1, \dots, G_{m-1}, \text{Res}_{L/K} G_m) \longrightarrow K(K; (G_1)_L, \dots, (G_{m-1})_L, G_m),$$

$$\psi : K(L; (G_1)_L, \dots, (G_{m-1})_L, G_m) \longrightarrow K(K; G_1, \dots, G_{m-1}, \text{Res}_{L/K} G_m).$$

as follows. For $X \in \mathcal{C}_K$, $x_1 \in G_1(X), \dots, x_{m-1} \in G_{m-1}(X)$ and $x_m \in G_m(X \otimes_K L)$ we put

$$\phi(\{x_1, \dots, x_m\}_{X/K}) = \{p^*(x_1), \dots, p^*(x_{m-1}), x_m\}_{(X \otimes_K L)/L}.$$

Conversely, for $Y \in \mathcal{C}_L$ and $y_1 \in G_1(Y), \dots, y_m \in G_m(Y)$ let

$$\psi(\{y_1, \dots, y_{m-1}, y_m\}_{Y/L} = \{y_1, \dots, y_{m-1}, \iota_*(y_m)\}_{Y/K}.$$

One can easily verify that these maps are well-defined and mutually inverse to each other. \square

Let G_1, \dots, G_m be commutative algebraic groups over K and let L/K be a finite separable extension. Take any $i \in \{1, \dots, m\}$. The map $N_{L/K} : \text{Res}_{L/K}(G_i)_L \rightarrow G_i$ induces a map $K(K; G_1, \dots, \text{Res}_{L/K}(G_i)_L, \dots, G_m) \rightarrow K(K; G_1, \dots, G_m)$, and the composition of it with the isomorphism ψ above coincides with the norm map (2). When L/K is a Galois extension, the action of $\text{Gal}(L/K)$ on $\text{Res}_{L/K}(G_i)_L$ induces its action on

$$K(L; G_1, \dots, G_m) \cong K(K; G_1, \dots, \text{Res}_{L/K}(G_i)_L, \dots, G_m)$$

and we have $N_{L/K} \circ \sigma = N_{L/K}$ for all $\sigma \in \text{Gal}(L/K)$. This action does not depend on the choice of i .

Lemma 4 *Let L/K be a cyclic Galois extension and let $\sigma \in \text{Gal}(L/K)$ be a generator. Suppose that for two different $i \in \{1, \dots, m\}$ the sequence*

$$(3) \quad \text{Res}_{L/K}(G_i)_K \xrightarrow{N_{L/K}} G_i \rightarrow 1$$

is Zariski exact. Then the sequence of abelian groups

$$K(L; G_1, \dots, G_m) \xrightarrow{1-\sigma} K(L; G_1, \dots, G_m) \xrightarrow{N_{L/K}} K(K; G_1, \dots, G_m) \rightarrow 0$$

is exact.

Proof. Suppose that (3) is exact for $i = m-1, m$. Let $G'_m := \text{Ker}(N_{L/K} : \text{Res}_{L/K}(G_m)_L \rightarrow G_m)$. By Lemmas 2 and 3 there are exact sequences

$$(4) \quad K(K; G_1, \dots, G'_m) \rightarrow K(L; G_1, \dots, G_m) \xrightarrow{N_{L/K}} K(K; G_1, \dots, G_m) \rightarrow 0$$

$$(5) \quad K(L; G_1, \dots, G_{m-1}, G'_m) \xrightarrow{N_{L/K}} K(K; G_1, \dots, G_{m-1}, G'_m) \rightarrow 0.$$

Since $(G'_m)_L \cong (G_m)_L^{n-1}$ ($n := [L : K]$) we can replace the first group of (5) by $K(L; G_1, \dots, G_m)^{n-1}$. By Lemma 1 the composite

$$K(L; G_1, \dots, G_m)^{n-1} \rightarrow K(K; G_1, \dots, G_{m-1}, G'_m) \rightarrow K(L; G_1, \dots, G_m)$$

is given on the i -th summand by $1 - \sigma^i$. The assertion follows. \square

Galois symbol Let G_1, \dots, G_m be connected commutative algebraic groups over K , and let n be an integer prime to the characteristic of K . For any finite extension L/K , we have a homomorphism [So]

$$(6) \quad h_L : K(L; G_1, \dots, G_m)/n \longrightarrow H^m(L, G_1[n] \otimes \dots \otimes G_m[n])$$

called the *Galois symbol*. This is characterized by the following properties.

- (i) If $x_i \in G_i(L)$ for $i = 1, \dots, m$, then $h_L(\{x_1, \dots, x_m\}_{L/L}) = (x_1) \cup \dots \cup (x_m)$. Here we write by (x_i) for the image of x_i in $H^1(L, G_i[n])$ by the connecting homomorphism associated to the exact sequence $1 \rightarrow G_i[n] \rightarrow G_i \xrightarrow{n} G_i \rightarrow 1$.
- (ii) If $M/L/K$ is a tower of finite extensions and if M/L is separable (resp. purely inseparable), then the diagram

$$\begin{array}{ccc} K(M; G_1, \dots, G_m)/n & \xrightarrow{h_M} & H^m(M, G_1[n] \otimes \dots \otimes G_m[n]) \\ \downarrow N_{M/L} & & \downarrow \\ K(L; G_1, \dots, G_m)/n & \xrightarrow{h_L} & H^m(L, G_1[n] \otimes \dots \otimes G_m[n]) \end{array}$$

is commutative, where the right vertical map is the corestriction (resp. the multiplication by $[M : L]$ under the identification $H^m(M, G_1[n] \otimes \dots \otimes G_m[n]) \cong H^m(L, G_1[n] \otimes \dots \otimes G_m[n])$).

Property (i) implies in particular that (6) coincides with the usual Galois symbol (1) in the case $G_1 = \dots = G_m = \mathbb{G}_m$. In [So] Remark 1.7, Somekawa conjectured that the Galois symbol associated to semiabelian varieties should be injective.

Galois cohomology of cyclic extensions Let L/K be a cyclic Galois extension of degree n and let σ be a generator of $G := \text{Gal}(L/K)$. For a discrete G_K -module M , tensoring the short exact sequence of G -modules

$$(7) \quad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[G] \xrightarrow{1-\sigma} \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

with M yields a distinguished triangle

$$(8) \quad M[1] \xrightarrow{\alpha} C^\bullet(M) \xrightarrow{\beta} M \xrightarrow{\gamma} M[2]$$

in the derived category $D(G_K)$. Here we denote by $C^\bullet(M)$ the complex

$$\text{Res}_{L/K} M \xrightarrow{1-\sigma} \text{Res}_{L/K} M$$

concentrated in degree -1 and 0 . The spectral sequence

$$E_1^{p,q} = H^q(K, C^p(M)) \implies E^{p+q} = H^{p+q}(K, C^\cdot(M))$$

induces short exact sequences

$$(9) \quad 0 \rightarrow H^q(L, M)_G \rightarrow H^q(K, C^\cdot(M)) \rightarrow H^{q+1}(L, M)^G \rightarrow 0.$$

It is easy to see that the composite

$$H^{q+1}(K, M) \xrightarrow{\alpha} H^q(K, C^\cdot(M)) \rightarrow H^{q+1}(L, M)^G$$

is the restriction and

$$H^q(L, M)_G \rightarrow H^q(K, C^\cdot(M)) \xrightarrow{\beta} H^q(K, M)$$

is induced by the corestriction. In particular we have $\gamma(H^q(K, M)) \subseteq \text{Ker}(\text{res} : H^{q+2}(K, M) \rightarrow H^{q+2}(L, M))$ hence

$$(10) \quad n\gamma(H^q(K, M)) = 0.$$

For an integer m prime to $\text{char } K$ and $r \in \mathbb{N}$ we write $\mathbb{Z}/m\mathbb{Z}(r) := \mu_m^{\otimes r}$ and

$$H^3(L/K, \mathbb{Z}/m\mathbb{Z}(2)) := \text{Ker}(H^3(K, \mathbb{Z}/m\mathbb{Z}(2)) \xrightarrow{\text{res}} H^3(L, \mathbb{Z}/m\mathbb{Z}(2))).$$

By restricting $\alpha : H^3(K, \mathbb{Z}/m\mathbb{Z}(2)) \rightarrow H^2(K, C^\cdot(\mathbb{Z}/m\mathbb{Z}(2)))$ to the subgroup $H^3(L/K, \mathbb{Z}/m\mathbb{Z}(2))$ and composing it with the inverse of the first map in (9) we obtain a map

$$(11) \quad H^3(L/K, \mathbb{Z}/m\mathbb{Z}(2)) \rightarrow \text{Ker}(H^2(L, \mathbb{Z}/m\mathbb{Z}(2))_G \xrightarrow{\text{cor}} H^2(K, \mathbb{Z}/m\mathbb{Z}(2))).$$

Lemma 5 *Assume that n is prime to $\text{char } K$ and $\mu_{n^2}(\overline{K}) \subset K$. Then the homomorphism (11) is injective for $m = n$.*

Proof. It is enough to show that $\gamma : H^1(K, \mathbb{Z}/n\mathbb{Z}(2)) \rightarrow H^3(K, \mathbb{Z}/n\mathbb{Z}(2))$ is zero. Consider the commutative diagram

$$\begin{array}{ccc} H^1(K, \mathbb{Z}/n\mathbb{Z}(2)) & \longrightarrow & H^1(K, \mathbb{Z}/n^2\mathbb{Z}(2)) \\ \downarrow \gamma & & \downarrow \gamma \\ H^3(K, \mathbb{Z}/n\mathbb{Z}(2)) & \longrightarrow & H^3(K, \mathbb{Z}/n^2\mathbb{Z}(2)) \end{array}$$

induced by the canonical injection $\mathbb{Z}/n\mathbb{Z}(2) \rightarrow \mathbb{Z}/n^2\mathbb{Z}(2)$. The assumption $\mu_{n^2}(\overline{K}) \subset K$ implies that the upper horizontal map can be identified with

$$K^*/(K^*)^n \longrightarrow K^*/(K^*)^{n^2}, x(K^*)^n \mapsto x^n(K^*)^{n^2}.$$

In particular the image is contained in $nH^1(K, \mathbb{Z}/n^2\mathbb{Z}(2))$. By (10) it is mapped under γ to $n\gamma(H^1(K, \mathbb{Z}/n^2\mathbb{Z}(2))) = 0$. On the other hand it is a simple consequence of the Merkurjev-Suslin theorem [MS] that the lower horizontal map is injective. Hence $\gamma(H^1(K, \mathbb{Z}/n\mathbb{Z}(2))) = 0$. \square

Example for the non-injectivity of the Galois symbol Let L/K be as in the last section and let $T := \text{Ker}(N_{L/K} : \text{Res}_{L/K} \mathbb{G}_m \rightarrow \mathbb{G}_m)$. We make the following assumptions

$$(12) \quad n \text{ is prime to char } K \text{ and } \mu_{n^2}(\overline{K}) \subset K,$$

$$(13) \quad H^3(L/K, \mathbb{Z}/n\mathbb{Z}(2)) \neq 0.$$

Proposition 6 *The Galois symbol $K(K; T, T)/n \rightarrow H^2(K, T[n] \otimes T[n])$ is not injective.*

Proof. Let σ be a generator of $G := \text{Gal}(L/K)$. The exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \text{Res}_{L/K} \mathbb{G}_m \xrightarrow{1-\sigma} \text{Res}_{L/K} \mathbb{G}_m \xrightarrow{N_{L/K}} \mathbb{G}_m \longrightarrow 1$$

yields two short exact sequences

$$(14) \quad 1 \longrightarrow \mathbb{G}_m \longrightarrow \text{Res}_{L/K} \mathbb{G}_m \longrightarrow T \longrightarrow 1,$$

$$(15) \quad 1 \longrightarrow T \longrightarrow \text{Res}_{L/K} \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 1.$$

Correspondingly, (7) induces two short exact sequences

$$(16) \quad 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[G] \rightarrow X \rightarrow 0, \quad 0 \rightarrow X \rightarrow \mathbb{Z}[G] \rightarrow \mathbb{Z} \rightarrow 0$$

where X denotes the cocharacter group of T . Note that the sequence (14) is Zariski exact by Hilbert 90. Since the map $\text{Res}_{L/K} \mathbb{G}_m \rightarrow T$ factors through $\text{Res}_{L/K} \mathbb{G}_m \rightarrow \text{Res}_{L/K} T \rightarrow T$ the sequence $\text{Res}_{L/K} T \rightarrow T \rightarrow 1$ is Zariski exact as well. By Lemma 4 the upper horizontal map in the diagram

$$\begin{array}{ccc} (K(L; T, T)/n)_G & \xrightarrow{N_{L/K}} & K(K; T, T)/n \\ \downarrow & & \downarrow \\ H^2(L, T[n] \otimes T[n])_G & \xrightarrow{\text{cor}} & H^2(K, T[n] \otimes T[n]) \end{array}$$

is an isomorphism. The vertical maps are Galois symbols. Since T_L is a split torus the left vertical map is an isomorphism by the Merkurjev-Suslin theorem [MS]. Thus to finish the proof it remains to show that the lower vertical arrow is not injective. Note that $T[n] \cong \mathbb{Z}/n\mathbb{Z}(1) \otimes X$. Hence the assertion follows from Lemma 5 and Lemma 7 below.

Lemma 7 *There exists homomorphisms of G -modules $e : \mathbb{Z} \rightarrow X \otimes_{\mathbb{Z}} X$ and $f : X \otimes_{\mathbb{Z}} X \rightarrow \mathbb{Z}$ such that $f \circ e : \mathbb{Z} \rightarrow \mathbb{Z}$ is multiplication by $n - 1$.*

Proof. For a G -module M we write M^\vee for the G -module $\text{Hom}(M, \mathbb{Z})$. Let $(,) : \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G] \longrightarrow \mathbb{Z}$ be the symmetric pairing given by

$$(17) \quad (g, g') = \begin{cases} 1 & \text{if } g = g', \\ 0 & \text{if } g \neq g'. \end{cases}$$

It yields an isomorphism $\mathbb{Z}[G] \rightarrow \mathbb{Z}[G]^\vee$. For a submodule $M \subseteq \mathbb{Z}[G]$ let

$$M^\perp = \{x \in \mathbb{Z}[G] \mid (x, m) = 0 \ \forall m \in M\}.$$

Then we have $X^\perp = \mathbb{Z}S$ and $(\mathbb{Z}S)^\perp = X$ where $S = \sum_{i=0}^{n-1} \sigma^i$. Thus (17) yields an isomorphism $X \cong (\mathbb{Z}[G]/\mathbb{Z}S)^\vee$. By (16) we have $\mathbb{Z}[G]/\mathbb{Z}S \cong X$, hence

$$X \otimes_{\mathbb{Z}} X \cong X \otimes_{\mathbb{Z}} X^\vee \cong \text{Hom}(X, X)$$

Thus it suffices to prove the assertion for $\text{Hom}(X, X)$. Obviously, for the two maps $e : \mathbb{Z} \rightarrow \text{Hom}(X, X), m \mapsto m \text{id}_X$ and $f : \text{Hom}(X, X) \rightarrow \mathbb{Z}, \tau \mapsto \text{Tr}(\tau)$ we have $f \circ e = \text{rank}(X) = n - 1$. \square

Remark It is easy to construct examples where the assumptions (12) and (13) above are satisfied. For instance if K is a 2-local field satisfying property (12) and L/K is any cyclic extension of degree n then property (13) holds by [Ka].

References

- [Ka] K. Kato, *A generalization of local class field theory by using K -groups I*, J. Fac. Sci. Univ. Tokyo **26** (1979), 303–376.
- [MS] A.S. Merkurjev, A.A. Suslin, *K -cohomology of Severi-Brauer varieties and the norm residue homomorphism*. Math. USSR Izvestiya **21** (1983), 307–340.
- [Se] J.-P. Serre, *Algebraic groups and class fields*. Graduate Texts in Mathematics **117**. Springer Verlag 1988.
- [So] M. Somekawa, *On Milnor K -groups attached to semi-abelian varieties*. K -theory **4** (1990), 105–119.

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