Simplicial radditive functors

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September 2007

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1 Introduction

Let C be a category with finite coproducts $\amalg$ and an initial object $\emptyset$. A contravariant functor $F : C \to Sets$ is called radditive if $F(\emptyset) = pt$ and for any X, Y in C the natural map $F(X \amalg Y) \to F(X) \times F(Y)$ is a bijection. Some examples of categories which are equivalent to the categories of radditive functors are given in 3.1-3.4.

Categories of simplicial radditive functors provide a context for doing homotopy theory which is sufficiently general to accommodate most interesting examples and at the same time is richer than the context of general model categories.

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[2] This is a new version of the paper which appeared in September 2000 under the title "Δ-closed classes".

[3] Work on this paper was supported by NSF grant 0403367.

In the first section we work with classes of morphisms between simplicial objects of a general category with finite coproducts. We prove several results about classes which are Δ-closed i.e. such that for a morphism $f : B_{**} \to B'_{**}$ of bi-simplicial objects whose rows $f_{i*} : B_{i*} \to B'_{i*}$ are in E one has $\Delta(f) \in E$.

Then we introduce the category of radditive functors and show that it has all limits and colimits. The category of simplicial radditive functors $\Delta^{op}Rad(C)$ on any C carries a closed model structure which we call the projective c.m.s. The associated homotopy category $H(C)$ is an invariant of C. If C is additive then $Rad(C)$ is equivalent to the abelian category of contravariant additive functors from C to Ab and $H(C)$ is equivalent to the "non-negative" part of its derived category.

Next we consider the question of how to generate classes of morphisms which produce good localizations of $H(C)$. There are two standard methods in the additive case - to close a generating class with respect to some standard operations or to use the the concepts of local objects and local equivalences. Our notion of the $\bar{\Delta}$-closed class provides a non-additive analog of the first method.

Categories for which the class of projective equivalences in $\Delta^{op}Rad(C)$ is closed under finite coproducts are called *grainy*. For grainy categories we are able to completely analyze the relationship between $\bar{\Delta}$-closures and local closures which leads to several strong functoriality theorems.

I am very grateful to Charles Weibel who made a great number of useful suggestions both for the original version of the paper and for the new one.

2 Elementary properties of Δ-closed classes

2.1 Δ-closed classes

Let C be a category and $\Delta^{op}C$ the category of simplicial objects over C. Following [10] define a unit homotopy from a morphism $f : A \to B$ to a morphism $g : A \to B$ in $\Delta^{op}C$ as a collection of morphisms $h^n_i : A_n \to B_n$ where $n \geq 0$ and $i = -1, \ldots, n$ satisfying the following conditions:

1. $h^n_{-1} = f_n$, $h^n_n = g_n$ where f_n and g_n are the components of f and g
2. $\partial_i h_j = h_{j-1}\partial_i$ if $i \leq j$, $\partial_i h_j = h_j \partial_i$ if $i > j$
3. $s_i h_j = h_{j+1} s_i$ if $i \leq j$, $s_i h_j = h_j s_i$ if $i > j$.

If C has coproducts (resp. finite coproducts), K is a set (resp. a finite set) and X an object of C we let $\amalg_K X$ denote the coproduct of K copies of X. Similarly for a simplicial set (resp. finite simplicial set) K and an object X of $\Delta^{op}C$ we let $X \boxtimes K$ denote the simplicial object with terms $X_n \boxtimes K_n$.

Example 2.1 If C is the category of sets then $X \boxtimes K = X \times K$. If C is the category of pointed sets then $X \boxtimes K = X \wedge (K_+)$.

One verifies easily (see [10, Prop. 2.1]) that a homotopy from f to g in the sense of the definition given above is the same as a morphism $h : A \boxtimes \Delta^1 \to B$ such that $h \circ (Id \boxtimes \partial_0) = f$ and $h \circ (Id \boxtimes \partial_1) = g$.

Two morphisms are called homotopic if they can be connected by a chain of unit homotopies (going in either direction). A morphism $f : A \to B$ in $\Delta^{op}C$ is called a homotopy equivalence if there exist a morphism $g : B \to A$ such that the compositions gf and fg are homotopic to the corresponding identity morphisms.

Definition 2.2 *Let $\mathcal{C}$ be a category. A class E of morphisms in $\Delta^{op}\mathcal{C}$ is called Δ-closed if the following conditions hold.*

1. *All homotopy equivalences are in E.*
2. *If f and g are morphisms such that the composition gf is defined and two out of three morphisms f, g, gf are in E then the third is in E.*
3. *If $f : B \to B'$ is a morphism of bisimplicial objects over C such that the rows or columns of f are in E then the diagonal morphism $\Delta(f)$ is in E*

A functor $F : \Delta^{op}C \to \Delta^{op}D$ is called a term-wise functor if there is a functor $C \to D$ which defines F in the standard way. Any term-wise functor takes homotopic morphisms to homotopic morphisms and if we define F on bisimplicial objects setting $(F(X))_{ij} = F(X_{ij})$ we have $F \circ \Delta = \Delta \circ F$. This implies the following result.

Lemma 2.3 *Let C, C' be categories and $F : \Delta^{op}C \to \Delta^{op}C'$ a term-wise functor. Then for any class of morphisms E in $\Delta^{op}C$ one has*

$$F(cl_\Delta(E)) \subset cl_\Delta(F(E))$$

If C has finite coproducts we say that a class is $(\Delta, \amalg_{<\infty})$-closed if it is Δ-closed and closed under finite coproducts. We denote the smallest Δ-closed class containing a class E by $cl_\Delta(E)$ and use a similar notation for the $(\Delta, \amalg_{<\infty})$-closed classes.

Proposition 2.4 *Let C be a category with finite coproducts and an initial object and E a class of morphisms in $\Delta^{op}C$. Then one has*

$$cl_{\Delta,\amalg_{<\infty}}(E) = cl_{\Delta}(E \amalg Id_C).$$

Where $E \amalg Id_C$ is the class of morphisms of the form $f \amalg Id_X$ for $f \in E$ and $X \in C$.

Proof: Since identities belong to any Δ-closed class the right hand side is contained in the left hand side. On the other hand E is contained in the right hand side and the right hand side is Δ-closed. It remains to show that the right hand side is closed under finite coproducts. For a pair of morphisms $f : X \to X'$, $g : Y \to Y'$ we have

$$f \amalg g = (Id_{X'} \amalg g) \circ (f \amalg Id_Y)$$

and therefore it is sufficient to check that for a morphism f in $cl_{\Delta}(E \amalg Id)$ and an object X in $\Delta^{op}C$ one has $f \amalg Id_X \in cl_{\Delta}(E \amalg Id)$. Consider the class H of morphisms f in $cl_{\Delta}(E \amalg Id)$ for which this holds for any X in $\Delta^{op}C$. For a simplicial object X with terms $X_i \in C$ we can write $f \amalg Id_X$ as the diagonal of a morphism of bisimplicial objects whose rows are of the form $f \amalg Id_{X_i}$. This implies that $E \amalg Id_X \subset H$. On the other hand it is easy to see that H is Δ-closed. Thus $H = cl_{\Delta}(E \amalg Id)$.

If C, C' have finite coproducts and $F : \Delta^{op}C \to \Delta^{op}C'$ is a term-wise functor which commutes with finite coproducts then we can extend the statement of Lemma 2.3 to $cl_{\Delta,\amalg_{<\infty}}(E)$. If F does not commute with coproducts we get the following analog of Lemma 2.3.

Proposition 2.5 *Let C, C' be categories with an initial object and finite coproducts and $F : C \to C'$ a functor. Then for any class of morphisms E in $\Delta^{op}C$ one has*

$$F(cl_{\Delta,\amalg_{<\infty}}(E)) \subset cl_{\Delta}(F(E \amalg Id_C))$$

where $E \amalg Id_C$ is the class of morphisms of the form $f \amalg Id_X$ for $f \in E$ and $X \in C$.

Proof: It follows immediately from Lemma 2.3 and Proposition 2.4.

Definition 2.6 *A morphism $e : A \to X$ in a category C is called a coprojection if there exists a morphism $f : Y \to X$ such that $f \amalg e$ is an isomorphism. A morphism $f : A \to X$ in $\Delta^{op}C$ is called a term-wise coprojection if for any $i \geq 0$ the morphism $f_i : A_i \to X_i$ is a coprojection.*

For any morphism $f : B \to A$ and any object C the square

$$\begin{tikzcd} B \arrow[r, "e_B"] \arrow[d] & B \amalg C \arrow[d] \\ A \arrow[r, "e_A"] & A \amalg C \end{tikzcd} \tag{1}$$

is a push-out square. This shows that in a category with finite coproducts there exist push-outs for all pairs of morphisms (e, f) such that e is a coprojection. Therefore the same is true for pairs of morphisms (e, f) in $\Delta^{op}C$ such that e is a term-wise coprojection.

Definition 2.7 *A square is called an elementary exact square if it is isomorphic to the push-out square for a pair of morphisms (e, f) where e is a term-wise coprojection.*

For any commutative square Q of the form

$$\begin{tikzcd} B \arrow[r, "e_B"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ A \arrow[r, "e_A"] & X \end{tikzcd} \tag{2}$$

denote by K_Q the object defined by the elementary exact square

$$\begin{tikzcd} B \amalg B \arrow[r] \arrow[d] & B \boxtimes \Delta^1 \arrow[d] \\ A \amalg Y \arrow[r] & K_Q \end{tikzcd} \tag{3}$$

and by $p_Q : K_Q \to X$ the obvious morphism. For a morphism $f : X \to X'$ the object K_{Q_f} defined by the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & X' \arrow[d] \\ X \arrow[r, "f"] & X' \end{tikzcd} \tag{4}$$

is called the cylinder of f and denoted by $cyl(f)$.

Lemma 2.8 *The morphisms $X' \to cyl(f)$ and $cyl(f) \to X'$ are mutually inverse homotopy equivalences.*

Proof: The object $cyl(f)$ is defined by the elementary exact square

$$\begin{tikzcd} X \arrow[r, "Id\boxtimes\partial_0"] \arrow[d] & X \boxtimes \Delta^1 \arrow[d] \\ X' \arrow[r] & cyl(f) \end{tikzcd} \tag{5}$$

The composition $X' \to cyl(f) \to X'$ is the identity. The homotopy from the identity of $cyl(f)$ to the composition $cyl(f) \to X' \to cyl(f)$ is given by the morphism $cyl(f)\boxtimes\Delta^1 \to cyl(f)$ which equals to the projection $X'\boxtimes\Delta^1 \to X'$ on $X'\boxtimes\Delta^1$ and to the morphism $Id\boxtimes h$ on

$$(X'\boxtimes\Delta^1)\boxtimes\Delta^1 = X'\boxtimes(\Delta^1 \times \Delta^1)$$

where $h : \Delta^1 \times \Delta^1 \to \Delta^1$ is the usual homotopy from the identity to the composition $\Delta^1 \to \Delta^0 \xrightarrow{\partial_0} \Delta^1$.

Lemma 2.9 *For an elementary exact square Q of the form (2) the morphism $p_Q : K_Q \to X$ belongs to $cl_\Delta(\emptyset)$.*

Proof: The object K_Q is the diagonal of the bisimplicial object whose rows are K_{Q_i} where Q_i is the square formed by the i-th terms of A, B and Y and p_Q is the diagonal of the morphism whose terms are $p_{Q_i} : K_{Q_i} \to X_i$. Therefore, it is sufficient to prove the statement of the lemma for a square in C of the form (1). Since $K_{Q_1 \amalg Q_2} = K_{Q_1} \amalg K_{Q_2}$ and a square of the form (1) is a coproduct of a square of the form (4) and a transpose of such a square our result follows from Lemma 2.8.

Lemma 2.10 *Let $f = (f_A, f_B, f_Y, f_X) : Q \to Q'$ be a morphism of commutative squares of the form (2) then one has*

$$(K(f) : K_Q \to K_{Q'}) \in cl_{\Delta,\amalg_{<\infty}}(\{f_A, f_B, f_Y\})$$

Proof: The object K_Q is isomorphic to the diagonal of a bisimplicial object whose rows are of the form $A \amalg Y \amalg (\amalg_n B)$ and this isomorphism is natural with respect to morphisms of squares.

Lemma 2.11 *Let C be a category with finite coproducts. Then for any elementary exact square in $\Delta^{op}C$ of the form (2) one has*

$$e_A \in cl_{\Delta,\amalg_{<\infty}}(\{e_B\})$$
$$v \in cl_{\Delta,\amalg_{<\infty}}(\{u\})$$

Proof: To prove the first inclusion consider the morphism of squares of the form

$$\left(\begin{tikzcd} B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & A \end{tikzcd}\right) \longrightarrow \left(\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d] \\ A \arrow[r] & X \end{tikzcd}\right) \tag{6}$$

The morphism $e_A : A \to X$ can be represented as the composition

$$A \to K_{Q_0} \to K_Q \to X$$

The first and the third arrows are in $cl_{\Delta,\amalg_{<\infty}}(\emptyset)$ by Lemma 2.9. The second one is in $cl_{\Delta,\amalg_{<\infty}}(e_B)$ by Lemma 2.10. We conclude that $e_A \in cl_{\Delta,\amalg_{<\infty}}(e_B)$.

To prove the second inclusion one applies the same reasoning to the morphism of squares

$$\left(\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}\right) \longrightarrow \left(\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d] \\ A \arrow[r] & X \end{tikzcd}\right)$$

Lemma 2.12 *For any commutative square Q of the form (2) the morphism $A \to K_Q$ belongs to $cl_{\Delta,\amalg_{<\infty}}(\{e_B\})$.*

Proof: Follows from Lemma 2.8 and Lemma 2.10 in the same way as in the proof of Lemma 2.11 if one considers the morphism of commutative squares

$$\left(\begin{array}{ccc} B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}\right) \longrightarrow \left(\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}\right)$$

2.2 $\bar{\Delta}$-closed classes

Definition 2.13 *A class of morphisms E in a category C is said to be closed under filtering colimits if for any pair of filtering systems $(X_i)_{i\in I}$, $(Y_i)_{i\in I}$ such that $X = colim_i X_i$ and $Y = colim_i Y_i$ exist and any morphism of systems $(f_i) : (X_i) \to (Y_i)$ such that $f_i \in E$ one has $f = colim_i f_i \in E$.*

Lemma 2.14 *Let E be a class closed under filtering colimits. Then it is closed under "filtering compositions" i.e. for a filtering system $(X_i)_{i\in I}$ such that $X = colim_i X_i$ exists and all morphisms $X_i \to X_j$ are in E, the morphisms $X_i \to X$ are in E.*

Proof: Observe first that by replacing I with the category i/I of morphisms $i \to j$ and our original system with the system $(i \to j) \mapsto X_j$ we do not change the colimit. Now we can apply Definition 2.13 to the obvious morphism from the constant system $(i \to j) \mapsto X_i$ to the system $(i \to j) \mapsto X_j$.

Definition 2.15 *Let C be a category. A class of morphisms E in $\Delta^{op}C$ is called $\bar{\Delta}$-closed if it is $(\Delta, \amalg_{<\infty})$-closed and closed under filtering colimits.*

Proposition 2.16 *Let C, C' be categories with all (small) coproducts and $F : C \to C'$ a functor which commutes with filtering colimits*[4]*. Then for any class of morphisms E in $\Delta^{op}C$ one has*

$$F(cl_{\bar{\Delta}}(E)) \subset cl_{\bar{\Delta}}(F(E \amalg Id_C))$$

where $E \amalg Id_C$ is the class of morphisms of the form $f \amalg Id_X$ for $f \in E$ and $X \in C$.

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[4] The ones which exist in C.

Proof: Let H be a class of morphisms f such that for any X in $\Delta^{op}C$ one has

$$F(f \amalg Id_X) \in cl_{\bar{\Delta}} F(E \amalg Id_C))$$

One verifies immediately that H is Δ-closed. Since F commutes with filtering colimits it is closed under filtering colimits. Finally since for $f : X \to X'$ and $g : Y \to Y'$ one has $f \amalg g = (f \amalg Id'_Y) \circ (Id_X \amalg g)$ it is closed under finite coproducts. Therefore it is $\bar{\Delta}$-closed. An obvious application of the diagonal shows that it contains E.

Corollary 2.17 *Let C, C' and $F : C \to C'$ be as in Proposition 2.16. Then one has*

$$F(cl_{\bar{\Delta}}(\emptyset)) \subset cl_{\bar{\Delta}}(\emptyset).$$

Remark 2.18 If there is a class of objects A in C such that A is closed under finite coproducts and any object of C is a filtering colimit of objects from A then in the conclusion of Proposition 2.16 one can replace $E \amalg Id_C$ with $E \amalg Id_A$.

Lemma 2.19 *The class of weak equivalences in $\Delta^{op}Sets$ is $\bar{\Delta}$-closed.*

Proof: The fact that the class of weak equivalences satisfies the first two conditions of Definition 2.2 is easy to see. The fact that it satisfies the third one is proved in [6, Lemma 5.3.1 p.129]. The fact that the class of weak equivalences is closed under filtering colimits is well known and follows easily from the definition of weak equivalences.

Proposition 2.20 *Let C be a category with small coproducts, $f : K \to K'$ a weak equivalence of simplicial sets and X an object of C. Then one has*

$$(Id_X \boxtimes f : X \boxtimes K \to X \boxtimes K') \in cl_{\bar{\Delta}}(\emptyset)$$

Proof: Let $Ex = colim_n Ex_n$ be the Kan completion functor. Since f is a weak equivalence the map $Ex(f)$ is a homotopy equivalence and therefore the same holds for the morphism $Id_X \boxtimes Ex(f)$. It remains to show that the map

$$X \boxtimes K \to X \boxtimes Ex(K)$$

belongs to $cl_{\bar{\Delta}}(\emptyset)$. Since any $\bar{\Delta}$-closed class is closed under countable compositions it is sufficient to show that the maps

$$X \boxtimes Ex_n(K) \to X \boxtimes Ex_{n+1}(K)$$

belong to $cl_{\bar{\Delta}}(\emptyset)$. By definition of Ex_n (see e.g. [2]) we have a square of the form

$$\begin{tikzcd}
\amalg_{\Lambda^{n,k}\to Ex_n(K)} X\boxtimes\Lambda^{n,k} \arrow[r] \arrow[d] & X\boxtimes Ex_n(K) \arrow[d] \\
\amalg_{\Lambda^{n,k}\to Ex_n(K)} X\boxtimes\Delta^{n} \arrow[r] & X\boxtimes Ex_{n+1}(K)
\end{tikzcd}$$

which is clearly elementary exact. It remains to notice that the left vertical arrow is a homotopy equivalence and thus the right vertical arrow is in $cl_{\Delta,\amalg_{\le\infty}}(\emptyset)$ by Lemma 2.11.

Proposition 2.20 and Lemma 2.19 imply the following.

Corollary 2.21 *The class $cl_{\bar{\Delta}}(\emptyset)$ in $\Delta^{op}Sets$ coincides with the class of weak equivalences.*

Let I be a small category and C be a category with all small coproducts. Let further $\Phi : I \to C$ be a functor. Define a simplicial object $hocolim(\Phi)$ in C setting

$$hocolim(\Phi)_n = \coprod_{X_0\to\ldots\to X_n} \Phi(X_0)$$

where the coproduct is taken over all sequences of morphisms in I of length $n+1$. One defines the face and degeneracy morphisms in the obvious way (it is the same construction as in [1] only now the target category is not *Sets* but C). Note that

$$hocolim(\Phi)_0 = \coprod_{X\in I} \Phi(X)$$

and in particular for any X in I we have a morphism $\Phi(X) \to hocolim(\Phi)$.

Lemma 2.22 *If I has a final object p then the morphism $\Phi(p) \to hocolim(\Phi)$ is a homotopy equivalence.*

Proof: Note that since p is final we have a canonical morphism $X \to p$ for each X in I. By the universal property of the coproduct it gives us a morphism $hocolim(\Phi)_0 \to \Phi(p)$ and one verifies easily that it defines a morphism $\pi : hocolim(\Phi) \to \Phi(p)$. Thus we have two morphisms going in the opposite directions and it is not difficult to explicitly construct the homotopy required for them to be mutually inverse homotopy equivalences. One can also prove the lemma in the following more abstract way.

Let I_0 be the set of objects of I considered as a discreet category. Let $i : I_0 \to I$ be the obvious functor. Composition with i gives a functor

$$i_* : Funct(I, C) \to Funct(I_0, C)$$

If C has coproducts it has a left adjoint i^* which takes a functor Φ to the functor

$$Y \mapsto \coprod_{X \to Y} \Phi(X)$$

where the coproduct is over all arrows in I ending in Y. This pair of adjoint functors defines in the usual way a functor

$$L : Funct(I, C) \to \Delta^{op} Funct(I, C)$$

such that $L(\Phi)_n = (i^* i_*)^n(\Phi)$ and a morphism $\eta : L(\Phi) \to \Phi$. By [1] we know that for any Φ the morphism $i_*(\eta) : i_* L(\Phi) \to i_* \Phi$ is a homotopy equivalence. Let

$$ev_p : Funct(I_0, C) \to C$$

be the functor of the form $\Psi \mapsto \Psi(p)$. One observes easily that the morphism

$$\pi : hocolim(\Phi) \to \Phi(p)$$

is $ev_p(i_*(\eta))$ and in particular it is a homotopy equivalence. Since $\Phi(p) \to hocolim(\Phi)$ is a one-sided inverse to π it is a homotopy equivalence as well.

Proposition 2.23 *Let I be a small filtering category, C a category with coproducts and $\Phi : I \to \Delta^{op} C$ a functor. Then the canonical morphism $c : \Delta hocolim(\Phi) \to \operatorname{colim} \Phi$ is in $cl_{\bar{\Delta}}(\emptyset)$.*

Proof: Since $cl_{\bar{\Delta}}(\emptyset)$ is closed under the diagonals we may assume that Φ is a functor with values in C. For X in I let I/X be the category of objects of I over X and let Φ_X be the composition of Φ with the obvious functor $I/X \to X$. Let $c_X : hocolim(\Phi_X) \to \Phi(X)$ be the obvious morphism. The objects $hocolim(\Phi_X)$ are functorial in X and one has

$$hocolim(\Phi) = \operatorname*{colim}_{X \in I} hocolim(\Phi_X)$$

such that $c = \operatorname{colim}_{X \in I} c_X$ (it is easy to see when I is a partially ordered set and the general case is similar). Since X is the final object of I/X the morphisms c_X are homotopy equivalences by Lemma 2.22. In particular they are in $cl_{\bar{\Delta}}(\emptyset)$. We conclude that c is in $cl_{\bar{\Delta}}(\emptyset)$.

Proposition 2.24 *Let C be a category with all coproducts. Then a class E is $\bar{\Delta}$-closed iff it is Δ-closed, closed under all coproducts and contains $cl_{\bar{\Delta}}(\emptyset)$.*

Proof: The "only if" part is obvious. To prove the "iff" part it is sufficient to check that a class E satisfying the conditions of the proposition is closed under filtering colimits. It follows immediately from Proposition 2.23 and the explicit form of the *hocolim* construction.

3 Homotopy theory of simplicial radditive functors

3.1 Radditive functors

Let C be a category with finite coproducts and an initial object 0. Denote by $Rad(C)$ the full subcategory of the category of contravariant functors from C to sets which consists of functors F such that $F(0) = pt$ and for any finite family of objects X_i, $i \in I$ the map

$$F(\amalg_{i\in I} X_i) \to \prod_{i\in I} F(X_i)$$

is bijective. The objects of $Rad(C)$ will be called radditive functors.

Example 3.1 Let A be a commutative ring and C the category of finitely generated free commutative algebras over A. Then $Rad(C)$ is equivalent to the category of all commutative algebras over A. Similarly, the category of radditive functors on the category of finitely generated free groups is equivalent to the category of all groups.

Example 3.2 If C is an additive category then $Rad(C)$ is equivalent to the category of additive contravariant functors from C to abelian groups. See Lemma 6.1.

Example 3.3 Recall that a presheaf on a small category is a contravariant functor from this category to the category of sets. Let C be a small category and $C^{\amalg}$ the full subcategory of the category of presheaves on C which consists of finite coproducts of representable presheaves. Then $Rad(C^{\amalg})$ is equivalent to the category of presheaves on C.

Example 3.4 For an object X of a category C let X_+ be the pointed presheaf on C obtained from the presheaf represented by X by the addition of a disjoint base point. Let $C_+^{\amalg<\infty}$ be the full subcategory of the category of pointed presheaves on C which consists of coproducts of objects of the form X_+. Then $Rad(C_+^{\amalg<\infty})$ is the category of pointed presheaves on C.

This example has a generalization which we will state as a lemma.

Lemma 3.5 *Let C be a category with finite coproducts and a final object pt and C_+ the full subcategory of pointed objects in C which consists of objects of the form $X_+ = X \amalg pt$. Then $Rad(C_+)$ is equivalent to the category of pointed objects in $Rad(C)$.*

Proof: Note that $pt = \emptyset_+$ is the initial object in C_+. Therefore for any $F \in Rad(C_+)$ one has $F(pt) = pt$. Let $Rad(C)_\bullet$ be the category of pointed objects in $Rad(C)$. Define a functor

$$\phi : Rad(C_+) \to Rad(C)_\bullet$$

by the rule $\phi(F) : X \to F(X_+)$ with the distinguished point in $F(X_+)$ being the image of $F(pt) \to F(X_+)$.

Define a functor

$$\psi : Rad(C)_\bullet \to Rad(C_+)$$

by the rule $\phi(G) : X_+ \to G(X)$. For a morphism $f : X_+ \to Y_+$ we define $\psi(G)(f)$ as the composition

$$G(Y) \to G(Y) \times G(pt) \cong G(Y_+) \to G(X_+) \cong G(X) \times G(pt) \to G(X).$$

Where the first morphism is the product of the identity with the distinguished point in $G(pt)$. One verifies easily that ϕ and ψ are mutually inverse equivalences.

Any representable functor is radditive by definition of coproducts. Therefore we have a full embedding $C \to Rad(C)$ which sends an object to the corresponding representable functor. The following lemma is straightforward.

Lemma 3.6 *The functor $C \to Rad(C)$ commutes with finite coproducts.*

Lemma 3.7 *The category $Rad(C)$ has limits.*

Proof: The limit of a diagram $F : I \to Rad(C)$ is the same as its limit in the category of contravariant functors with values in sets i.e.

$$(limF(i))(U) = lim(F(i)(U)).$$

Lemma 3.8 *Let $F : I \to Rad(C)$ be a filtering system of radditive functors. Then the colimit $colimF(i)$ of F in the category of functors is radditive and gives a colimit of F in the category of radditive functors.*

Proof: The second statement follows immediately from the first. The first one follows from the fact that filtering colimits of sets commute with finite products.

Recall that a coequalizer diagram $X \rightrightarrows Y$ is called *reflexive* if both morphisms have a common right inverse (section) $Y \to X$.

Lemma 3.9 *Let* $X \rightrightarrows Y$ *be a reflexive coequalizer diagram of radditive functors. Then the coequalizer of this diagram in the category of presheaves is a radditve functor and a coequalizer of* $X \rightrightarrows Y$ *in the category of radditive functors.*

Proof: The first statement follows from the fact that finite products of sets commute with coequalizers of reflexive pairs. The second follows from the first.

Lemma 3.10 *Let* X_α *be a collection of objects of* C*. Denote by the same symbols the radditive functors represented by* X_α*. Define* $\cup X_\alpha$ *as the functor given by*

$$U \mapsto colim_{I\subset A} Hom(U, \amalg_{i\in I} X_i)$$

where I *run through finite subsets of* A*. Then* $\cup_\alpha X_\alpha$ *is radditive and it is a coproduct of* X_α *in the category of radditive functors.*

For a set S and a radditive functor X we let $X\boxtimes S$ denote the coproduct of S copies of X.

Proposition 3.11 *The inclusion functor*

$$Rad(C) \to Funct(C^{op}, Sets)$$

has a left adjoint r*.*

Proof: Let F be a functor which is not necessarily radditive. Consider the coequalizer diagram in $Rad(C)$ of the form

$$\cup_{(p:U\to V)\in C} U\boxtimes F(V) \rightrightarrows \cup_{W\in C} W\boxtimes F(W) \tag{7}$$

where one arrow maps the summand U corresponding to $(p : U \to V, f \in F(V))$ to the summand corresponding to $(U, F(p)(f))$ by the identity and the other one maps it to the summand corresponding to (V, f) by p. Let $r(F)$ be the coequalizer of these two maps in the category of functors. Since (7) is reflexive via $W \mapsto Id_W$ this functor is radditive by Lemma 3.9 and one verifies easily that for any radditive G one has

$$Hom(F, G) = Hom(r(F), G)$$

Remark 3.12 As the following example shows the functor r is not, in general, left exact i.e. it does not commute with finite limits. Therefore, radditive functors can not be thought of as sheaves with respect to some topology on C.

Example 3.13 Let C be the category of finitely generated free abelian groups such that $Rad(C)$ is equivalent to the category of all abelian groups. Consider the functor F defined by the push-out square

$$\begin{tikzcd}
\mathbf{Z} \amalg \mathbf{Z} \arrow[r] \arrow[d] & \mathbf{Z} \times \mathbf{Z} \arrow[d, "p"] \\
0 \arrow[r] & F
\end{tikzcd}$$

where $\mathbf{Z}$ is the functor represented by $\mathbf{Z}$ and $\amalg$ and $\times$ are in the category of all functors from C to $Sets$. Let $i : \mathbf{Z} \to F$ be the composition of the diagonal $\mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$ with p. One verifies easily that it is a monomorphism. On the other hand $r(F) = 0$ and therefore $r(i)$ is not a monomorphism.

Lemma 3.14 *The category $Rad(C)$ has all small colimits.*

Proof: For a diagram $F : I \to Rad(C)$ of radditive functors one gets $colimF$ applying the functor r of Proposition 3.11 to the colimit of F in the category of functors.

As a corollary of Lemmas 3.8 and 3.9 and of the proof of Proposition 3.11 we get the following characterization of radditive functors.

Lemma 3.15 *A contravariant functor from C to $Sets$ is radditive iff it is the coequalizer of a reflexive pair whose terms are filtering colimits of representable functors.*

Note that this lemma allows one to define the notion of a radditive functor on a category C without the assumption that C has finite coproducts. However in this case it is unclear why $Rad(C)$ has limits or colimits.

Let C, C' be two categories with finite coproducts and $F : C \to C'$ be a functor. Let F^* be the usual inverse image functor on presheaves of sets on C and C'.

Proposition 3.16 *For any radditive functor X the functor $F^*(X)$ is radditive.*

Proof: It follows from Lemma 3.15 since F^* takes representable functors to representable functors and commutes with colimits.

We let F^{rad} denote the functor $Rad(C) \to Rad(C')$ defined by F^* according Proposition 3.16.

Lemma 3.17 *For any F the functor F^{rad} commutes with filtering colimits and reflexive coequalizers.*

Proof: It follows from Lemma 3.8 and Lemma 3.9.

Lemma 3.18 *If F preserves finite coproducts then F^{rad} is left adjoint to the obvious functor $F_* : Rad(C') \to Rad(C)$ defined by F.*

Proof: Follows immediately from the definitions.

We will distinguish two subcategories in $Rad(C)$. The category $C^{\#}$ which consists of filtering colimits of systems of representable functors and the category $[C]$ which consists of coproducts of families of representable functors.

One can easily see that for any $F : C \to D$ the functor F^{rad} takes $C^{\#}$ to $D^{\#}$.

Example 3.19 Let C be the category of pointed finite simplicial sets, D the category of free finite simplicial sets and $F : C \to D$ the forgetful functor. Then one can easily see that F^{rad} does not take $[C]$ to $[D]$ since an infinite wedge of finite simplicial sets can not be represented as an infinite coproduct of finite simplicial sets.

3.2 Projective closed model structure

Let $\Delta^{op}Rad(C)$ be the category of simplicial objects in $Rad(C)$ or, equivalently, the category of radditive functors with values in simplicial sets. For a simplicial set K and $X \in \Delta^{op}Rad(C)$ let $Hom_{\boxtimes}(K, X)$ be the simplicial radditive functor which takes $U \in C$ to $S(K, X(U))$ where $S(-,-)$ is the simplicial function object in the category of simplicial sets. The following lemma is straightforward.

Lemma 3.20 *The category $\Delta^{op}Rad(C)$ together with the simplicial function space $S(-,-)$ and functors $X \mapsto X \boxtimes K$, $X \mapsto Hom_{\boxtimes}(K, X)$ is a simplicial category in the sense of [4, p.81].*

A morphism of radditive simplicial functors $f : X \to Y$ is called a projective equivalence if, for any U in C, the map of simplicial sets $X(U) \to Y(U)$ defined by f is a weak equivalence.

Example 3.21

1. The equivalence of Example 3.2 identifies projective equivalences with quasi-siomorphisms of the coresponding normalized complexes. See Lemma 6.4.

2. The equivalence
$$Rad(C^{\amalg}) \to PreShv(C)$$
of Example 3.3 identifies projective equivalences of simplicial radditive functors on $C^{\amalg}$ with section-wise equivalences of simplicial presheaves on C.

3. The equivalence
$$Rad(C_{+}) \to Rad(C)_{\bullet}$$
of Lemma 3.5 identifies projective equivalences of simplicial radditive functors on C_{+} with projective equivalences of pointed simplicial radditive functors on C.

Example 3.22 The radditivization functor r need not take projective equivalences of simplicial presheaves to projective equivalences of radditive functors. Indeed, Example 3.13 provides us with an additive category and a monomorphism $f : F \to G$ of presheaves such that $r(f)$ is not a monomorphism. Consider $g : cone(f) \to \pi_0(cone(f))$. Since f is a mono, g is a projective equivalence.

Consider $r(g) = (cone(r(f)) \to \pi_0(cone(r(f))))$. By Lemma 6.3 we have for any $X \in C$
$$cone(r(f))(X) = cone_{Ab}(r(F) \to r(G))$$
where the cone is in the category of simplicial abelian groups. Since $r(f)$ is not a mono it has a non-trivial π_1 and therefore $r(g)$ is not a projective equivalence.

Define the homotopy category $H(C)$ of C as the localization of the category of simplicial radditive functors with respect to the class of projective equivalences.

Lemma 3.23 *The class of projective equivalences of simplicial radditive functors is Δ-closed.*

Proof: Follows from Lemma 2.3 applied to functors $U \mapsto X(U)$ and Lemma 2.19.

In view of Lemma 3.8 we also have the following result.

Lemma 3.24 *The class of projective equivalences of simplicial radditive functors is closed under filtering colimits.*

Proposition 3.25 *The class of projective equivalences contains the class $cl_{\bar{\Delta}}(\emptyset)$.*

Proof: Applying Corollary 2.17 to the functor of sections over an object $U \in C$ we conclude that for any element $f : X \to Y$ in $cl_{\bar{\Delta}}(\emptyset)$ and any U in C the map of simplicial sets $f_U : X(U) \to Y(U)$ belongs to $cl_{\bar{\Delta}}(\emptyset)$. Since the class of weak equivalences of simplicial sets is $\bar{\Delta}$-closed we conclude that it is a weak equivalence.

Remark 3.26 The class of projective equivalences of simplicial radditive functors is not, in general, closed with respect to finite coproducts. In particular it need not be $\bar{\Delta}$-closed. Consider the situation of Example 3.1. Recall that the coproducts in the category of commutative algebras are tensor products. Given a weak equivalence of simplicial algebras $f : X \to Y$ and a (simplicial) algebra Z the morphism $f \boxtimes Id_Z$ need not be a weak equivalence unless Z is flat or X and Y are flat.

Following [6] we say that, for a given class of morphisms E, a morphism is an E-cell if it is a filtering composition of morphisms which are push-outs of elements of E. We further say that a morphism is E-injective if it has the right lifting property with respect to morphisms in E.

Let I be the set of morphisms of the form

$$U \boxtimes \partial\Delta^n \to U \boxtimes \Delta^n$$

where U runs through all objects in C and $n \geq 0$. Let J be the set of morphisms of the form

$$U \boxtimes \Lambda^{n,k} \to U \boxtimes \Delta^n$$

where U runs through all objects in C, $n \geq 0$ and $k = 0, \ldots, n$.

Let us say that a morphism $f : X \to Y$ is a projective fibration if for any U in C the map of simplicial sets $X(U) \to Y(U)$ defined by f is a Kan fibration. One can easily see that a morphism is a projective fibration if and only if it is J-injective and a trivial projective fibration (i.e. a projective fibration and a projective equivalence) if and only if it is I-injective.

Finally, let us say that a morphism is a projective cofibration if it has the left lifting property with respect to trivial projective fibrations.

Theorem 3.27 *The classes of projective equivalences, projective fibrations and projective cofibrations form a finitely generated closed model structure on the category of simplicial radditive functors.*

Proof: In view of (see [6, Theorem 2.1.19]) and since $Rad(C)$ has all small limits and colimits we need to verify the following conditions:

1. The domains and codomains of elements of I and J are small.
2. Any J-cell is a cofibration and a projective equivalence.
3. A morphism is I-injective iff it is a projective fibration (i.e. J-injective) and a projective equivalence.

The first statement is clear. In fact domains and codomains of elements of I and J are compact i.e. small relative to the countable cardinal. A standard argument shows that elements of J are I-cells and therefore J-cells are I-cells and in particular cofibrations. Let us show that any J-cell is a projective equivalence. The class of projective equivalences is closed under filtering compositions because of the corresponding properties of morphisms of simplicial sets and Lemma 3.8. Therefore it is sufficient to show that any push-out of an element of J is a projective equivalence. Consider a push-out square of the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\
X \arrow[r] & X'
\end{tikzcd}$$

where f is an element of J. Then f is a term-wise coprojection, this square is elementary exact and by Lemma 2.11 we conclude that f' belongs to $cl_{\Delta,\amalg_{<\infty}}(\{f\})$. Since f is a homotopy equivalence we conclude that f' is in $cl_{\Delta,\amalg_{<\infty}}(\emptyset)$ and therefore it is a projective equivalence by Proposition 3.25.

The fact that any I-injective morphisms are exactly projective fibrations which are projective equivalences is clear.

For a class of morphisms E, a *sequential* E-cell is a countable composition of push-outs of coproducts of elements of E. One verifies easily that any sequential E-cell is an E-cell.

Proposition 3.28 *A morphism $f : X \to Y$ in $\Delta^{op}Rad(C)$ is a projective cofibration (resp. a trivial projective cofibration) if and only if there exist morphisms $s : Y \to E$ and $p : E \to Y$ such that $ps = Id$ and sf is a sequential I-cell (resp. sequential J-cell).*

Proof: The fact that we can find s and p such that $ps = Id$ and sp is an I-cell (resp. J-cell) is proved in [6, Corollary 2.1.15]. Since domains and codomains of elements of I and J are compact we can actually choose s and p such that sp is a sequential I- or J-cell.

Corollary 3.29 *Let $f : X \to Y$ be a projective cofibration. Then there exist morphisms $s : Y \to E$ and $p : E \to Y$ such that $ps = Id$ and each term of sf is of the form $X \to X \amalg X'$ where X' is a coproduct of representable radditive functors.*

Proposition 3.30 *Let $f : K \to L$ be a weak equivalence of simplicial sets and $j : B \to A$ a term-wise coprojection. Then the morphism*

$$A \boxtimes K \to A \boxtimes K \amalg_{B \boxtimes K} B \boxtimes L$$

is a projective equivalence.

Proof: Consider the push-out square

$$\begin{tikzcd}
B \boxtimes K \arrow[r] \arrow[d] & B \boxtimes L \arrow[d] \\
A \boxtimes K \arrow[r] & X
\end{tikzcd}$$

Since $B \to A$ is a term-wise coprojection the same holds for $B \boxtimes K \to A \boxtimes K$ and therefore by Lemma 2.11 we conclude that the morphism $A \boxtimes K \to X$ belongs to $cl_{\Delta, \amalg_{<\infty}}(\{Id_B \boxtimes f\})$. By Proposition 2.20 we conclude that this morphism is contained in $cl_{\bar{\Delta}}(\emptyset)$ and therefore is a projective equivalence by Proposition 3.25.

Corollary 3.31 *For any object X of $\Delta^{op}Rad(C)$ and any weak equivalence of simplicial sets $K \to L$ the morphism $X \boxtimes K \to X \boxtimes L$ is a projective equivalence.*

Proposition 3.32 *Let $j : B \to A$ be a projective cofibration and $i : K \to L$ a cofibration of simplicial sets. Then the morphism*

$$h(i,j) : H(i,j) = B \boxtimes L \amalg_{B \boxtimes K} A \boxtimes K \to A \boxtimes L$$

is a cofibration. If i of j is a projective equivalence then so is $h(i,j)$.

Proof: Let us show first that $h(i,j)$ is a cofibration. Proposition 3.28 implies that it is sufficient to show that the class of morphisms j, such that $h(i,j)$ is a cofibration for all i, contains elements of I, i.e. morphisms of the form $U \boxtimes \partial\Delta^n \to U \boxtimes \Delta^n$, and is closed under retracts, coproducts, push-outs and sequential compositions. For j of the form $U \boxtimes \partial\Delta^n \to U \boxtimes \Delta^n$ the morphism $h(i,j)$ is of the form $U \boxtimes K' \to U \boxtimes L'$ where $K' \to L'$ is a monomorphism of simplicial sets. Any such morphism is a cofibration because it has the right lifting property for trivial projective fibrations. The fact that our class is closed under all required operations is also easy to check.

Assume that i is a projective equivalence. In view of Proposition 3.28 it is a retract of a term-wise coprojection and we may assume that it is a term-wise coprojection. Then Lemma 3.30 implies easily that $h(i,j)$ is a projective equivalence.

Assume that j is a projective equivalence. Then j is a projective trivial cofibration and therefore, by Proposition 3.28, a retract of a J-cell. For an element of J and any i the morphism $h(i,j)$ is a clearly a projective equivalence. Checking again that the class of trivial cofibrations j such that $h(i,j)$ is a trivial cofibration is closed under retracts, push-outs, coproducts and sequential compositions we conclude that $h(j,i)$ is a projective equivalence if j is.

Corollary 3.33 *For any cofibrant X and any monomorphism of simplicial sets $K \to L$ the map $X \boxtimes K \to X \boxtimes L$ is a projective cofibration. In particular for a cofibrant X and a simplicial set K the object $X \boxtimes K$ is cofibrant.*

Corollary 3.34 *The projective closed model structure is simplicial i.e. it satisfies the axiom SM7 (see e.g. [4, p.89]).*

Proof: We have to show that for a projective cofibration $j : A \to B$ and a projective fibration $q : X \to Y$ the morphism

$$S(B,X) \to S(A,X) \times_{S(A,Y)} S(B,Y)$$

is a Kan fibration which is a weak equivalence if j or q is a weak equivalence. This follows by adjunction from Proposition 3.32.

Lemma 3.35 *Let $f : X \to X'$ be a projective equivalence between cofibrant objects and K be a simplicial set. Then $f \boxtimes Id_K$ is a projective equivalence.*

Proof: The morphism $f \boxtimes Id_K$ is the diagonal of a morphism of bisimplicial objects whose terms are coproducts of copies of f. A coproduct of copies of f is a filtering colimit of finite coproducts of copies of f. It remains to check that a finite coproduct of projective equivalences of cofibrant objects is a projective equivalence. This holds in any model category. See e.g. [6, Lemma 5.2.6, p.126].

The functorial behavior of the projective c.m.s with respect to functors which respect finite coproducts is given in the following lemma. More advanced aspects of functoriality will be discussed after we introduce the standard simplicial resolution construction in the next section.

Lemma 3.36 *Let $F : C \to C'$ be a functor which commutes with finite coproducts. Then one has:*

1. *The functor F_* respects projective equivalences and projective fibrations*
2. *The functor F^{rad} respects projective cofibrations and trivial projective cofibrations.*

Proof: The first statement follows from the definitions. The second follows from Lemma 3.18 and the characterization of cofibrations and trivial cofibrations in terms of the lifting properties.

Example 3.37 Even for $F : C \to C'$ which commutes with finite coproducts the functor F^{rad} need not respect all projective equivalences. Consider for example the functor F^{rad} defined by the functor $C^{\amalg} \to C$. Using the equivalence between $Rad(C^{\amalg})$ and presheaves on C one can see that F^{rad} in this case is the radditivization functor of Proposition 3.11. The fact that it need not respect projective equivalences is demonstrated in Example 3.22.

Let Δ_{Mon} be the subcategory of monomorphisms in the standard simplicial category Δ. A contravariant functor from Δ_{Mon} to a category is a "simplicial object with no degeneracies". Let π_* be the obvious forgetful functor from $\Delta^{op}C$ to $\Delta^{op}_{Mon}C$. General argument shows that if C has colimits then π_* has a left adjoint π^*. In fact, since any morphism in Δ has a canonical decomposition into an epimorphism followed by a monomorphism, one needs only finite coproducts to define π^*. For a functor $Z = (Z_i)$ from Δ_{Mon} to C the simplicial object $\pi^*(Z)$ has terms of the form

$$\pi^*(Z)_i = \amalg_{[i]\to[j]} Z_j$$

where $[i] \to [j]$ runs through epimorphisms from $[i]$ to $[j]$ in Δ (see [12, Ex.8.1.5]).

Definition 3.38 *Let C be a category with finite coproducts. An object X in $\Delta^{op}Rad(C)$ is called degeneracy free if it belongs to the image of the functor π^*.*

If $Z = (Z_n)$ is an object of Δ^{op}_{Mon} and $X = \pi^*(Z)$ the corresponding degeneracy free simplicial object we say that X is based on the sequence (Z_n). Since $Rad(C)$ has colimits, for any simplicial object X we can define the skeletons $sk_n(X)$ in the usual way such that $X = colim sk_n(X)$ and for each n one has a push-out square of the form

$$\begin{tikzcd} L_n(X)\boxtimes\Delta^n \amalg_{L_n(X)\boxtimes\partial\Delta^n} X_n\boxtimes\partial\Delta^n \arrow[r] \arrow[d] & sk_{n-1}(X) \arrow[d] \\ X_n\boxtimes\Delta^n \arrow[r] & sk_n(X) \end{tikzcd} \tag{8}$$

where $L_n(X) = (sk_{n-1}(X))_n$ is the n-th latching object of X.

Lemma 3.39 *Let X be a degeneracy free object based on (Z_n). Then for any $n \geq 0$ there is a push-out square of the form*

$$\begin{tikzcd} Z_n\boxtimes\partial\Delta^n \arrow[r] \arrow[d] & sk_{n-1}(X) \arrow[d] \\ Z_n\boxtimes\Delta^n \arrow[r] & sk_n(X) \end{tikzcd}$$

Proof: See [4, Cor. 1.14, p.358].

Proposition 3.40 *Let B be a bisimplicial object such that the simplicial object formed by the rows of B is degeneracy free based on a sequence Z_n. Then there are elementary exact squares*

$$\begin{tikzcd} Z_n\boxtimes\partial\Delta^n \arrow[r] \arrow[d] & \Delta(sk_{n-1}(B)) \arrow[d] \\ Z_n\boxtimes\Delta^n \arrow[r] & \Delta(sk_n(B)) \end{tikzcd}$$

and

$$\Delta(B) = colim_n \Delta(sk_n(B)).$$

Proof: It follows easily from Lemma 3.39 since Δ commutes with colimits.

Corollary 3.41 *Let B be a bisimplicial radditive functor such that the simplicial object in $\Delta^{op}Rad(C)$ formed by the rows of B is degeneracy free on cofibrant objects Z_n. Then $\Delta(B)$ is cofibrant.*

Proof: It follows immediately from Proposition 3.40 and Corollary 3.33.

Lemma 3.42 *Let X be an object of $\Delta^{op}Rad(C)$ which is a degeneracy free object based on a sequence (Z_n) where Z_n are in $[C]$. Then one has:*

1. *the terms of X are in $[C]$*
2. *X is cofibrant in the projective model structure.*

Proof: It follows from Lemma 3.39. For the second statement one uses the fact that for $Z \in [C]$ the morphism $Z\boxtimes\partial\Delta^n \to Z\boxtimes\Delta^n$ is a cofibration which is a particular case of Proposition 3.32.

3.3 Standard simplicial resolution of a functor

Let C be a category with finite coproducts. Recall that we let $[C]$ denote the full subcategory in $Rad(C)$ which consists of coproducts of representable functors. In this section we construct a functor

$$Lres : Rad(C) \to \Delta^{op}[C]$$

which sends a radditive functor to its simplicial "resolution" by coproducts of representable functors. We start with a more general construction. Let R be a category with coproducts, C a small subcategory in R and $[C]$ the full subcategory of R which consists of objects isomorphic to coproducts of objects in C. Define a functor

$$G_C : R \to \Delta^{op}[C]$$

as follows. For F in R the object of n-simplexes of $G(F)$ is

$$G(F)_n = \amalg_{X_0\to\ldots\to X_n} X_0 \boxtimes Hom(X_n, F) \tag{9}$$

where $X_0 \to \ldots \to X_i$ run through all sequences of morphisms of length i in C. The face morphism $\partial_i : G_n(F) \to G_{n-1}(F)$ maps the summand X_0 indexed by $(X_0 \xrightarrow{g_0} \ldots \xrightarrow{g_{n-1}} X_n \xrightarrow{f} F)$ to:

X_1 indexed by $(X_1 \xrightarrow{g_1} \ldots \xrightarrow{g_{n-1}} X_n \xrightarrow{f} F)$ if $i = 0$

X_0 indexed by $(X_1 \xrightarrow{g_1} \ldots \to \widehat{X}_i \to \ldots \xrightarrow{g_{n-1}} X_n \xrightarrow{f} F)$, where the hat indicates that the corresponding object is omitted and the incoming and outgoing morphisms are composed, if $i = 1, \ldots, n-1$

X_0 indexed by $(X_1 \xrightarrow{g_1} \ldots \xrightarrow{g_{n-2}} X_{n-1} \xrightarrow{f\circ g_{n-1}} F)$ if $i = n$

The degeneracy morphism $s_i : G_n(F) \to G_{n+1}(F)$ for $i = 0, \ldots, n$ maps the summand X_0 indexed by $(X_0 \xrightarrow{g_0} \ldots \xrightarrow{g_{n-1}} X_n \xrightarrow{f} F)$ to the summand X_0 indexed by $(X_0 \xrightarrow{g_0} \ldots \to X_i \xrightarrow{Id} X_i \to \ldots \xrightarrow{g_{n-1}} X_n \xrightarrow{f} F)$.

Remark 3.43 Our construction is a particular case of the simplicial replacement of a diagram considered in [1, XII.5]. The object $G(F)$ can be identified with the simplicial replacement of the diagram $C/F \to [C]$. In particular for any simplicial object X in D the simplicial set $S(G(F), X)$ can be identified with the homotopy limit of the diagram of simplicial sets $U \mapsto Hom(U, X)$ indexed by $(C/F)^{op}$.

Lemma 3.44 *For any Y in C the morphism $G(Y) \to Y$ is a homotopy equivalence.*

Remark 3.45 We do not know whether or not this morphism is a homotopy equivalence for Y in $[C]$.

Proof: The terms of the simplicial object $G(Y)$ are given by

$$G_n(Y) = \amalg_{X_0\to\ldots\to X_n\to Y} X_0. \tag{10}$$

The embedding of Y to $G_n(Y)$ on the summand indexed by $Y \overset{Id}{\to} \ldots \overset{Id}{\to} Y$ commutes with faces and degeneracies. This gives us a morphism $Y \to G(Y)$ such that the composition $Y \to G(Y) \to Y$ is the identity. Let us show that the composition in the other direction is homotopic to identity of $G(Y)$. We will use the description of simplicial homotopies given at the beginning of Section 2.1. According to it a simplicial homotopy between two maps $f, g : A \to B$ is given by a collection of morphisms

$$h_i^n : A_n \to B_n$$

where $i = -1, \ldots, n$ which satisfy certain relations. For $i = -1$ we set h_i^n on the summand X_0 indexed by $(X_0 \to \ldots \to X_n \to Y)$ to be the obvious map to the summand Y indexed by the sequence of identities. For $i \geq 0$ we set h_i^n on the summand X_0 indexed by $(X_0 \to \ldots \to X_n \to Y)$ to be the identity map to the summand X_0 indexed by $(X_0 \to \ldots \to X_i \to Y \to \ldots \to Y)$ where the map $X_i \to Y$ is the composition of the sequence $X_i \to X_{i+1} \to \ldots \to Y$ and the morphisms between Y's are identities. One verifies by an explicit check that this collection is a homotopy from the composition $G(Y) \to Y \to G(Y)$ to the identity.

Let us consider now the case when C is a category with finite coproducts and $R = Rad(C)$. We get a functor

$$Rad(C) \to \Delta^{op}[C]$$

which we denote by $Lres$ and call the standard simplicial resolution functor. The following result follows immediately from our construction and Lemma 3.8.

Lemma 3.46 *The functor Lres commutes with filtering colimits.*

Remark 3.47 The functor $Lres$ does not commute with finite coproducts. The example of C being the category of finitely generated free abelian groups and $Rad(C)$ being all abelian groups shows that in general it is not possible to find a functor from $Rad(C)$ to $\Delta^{op}[C]$ which commutes with finite coproducts and satisfies the conclusion of Lemma 3.52.

Recall that we let $C^{\#}$ denote the full subcategory of objects in $Rad(C)$ which are filtering colimits of representable functors. Combining Lemma 3.44 with Lemma 3.46 we get the following result.

Lemma 3.48 *For any $X \in C^{\#}$ the morphism $Lres(X) \to X$ is in $cl_{\bar{\Delta}}(\emptyset)$.*

Lemma 3.49 *The functor Lres takes projective equivalences to elements of $cl_{\bar{\Delta}}(\emptyset)$.*

Proof: If $f : X \to X'$ is a projective equivalence then the columns of the map of bisimplicial objects $Lres(X) \to Lres(X')$ are of the form $\amalg U \boxtimes f$ where $U \in ob(C)$ and f are weak equivalences of simplicial sets. By Proposition 2.20 they belong to $cl_{\bar{\Delta}}(\emptyset)$.

Lemma 3.50 *Let $f : X \to Y$ be a morphism of simplicial objects over $C^{\#}$ which is a projective equivalence as a morphism in $\Delta^{op}Rad(C)$. Then f belongs to $cl_{\bar{\Delta}}(\emptyset)$.*

Proof: Consider the square

$$\begin{tikzcd}
Lres(X) \arrow[r] \arrow[d] & Lres(Y) \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

The vertical arrows are in $cl_{\bar{\Delta}}(\emptyset)$ by Lemma 3.48. The upper horizontal arrow is in $cl_{\bar{\Delta}}(\emptyset)$ by Lemma 3.49. Therefore the lower horizontal arrow is in $cl_{\bar{\Delta}}(\emptyset)$.

Combining Proposition 3.25 with Lemma 3.50 we get the following result.

Proposition 3.51 *The class of morphisms in $\Delta^{op}C^{\#}$ which are projective equivalences as morphisms of simplicial radditive functors coincides with $cl_{\bar{\Delta}}(\emptyset)$. In particular it is $\bar{\Delta}$-closed.*

We will write $Lres(X)$ instead of $\Delta(Lres(X))$ for object X in $\Delta^{op}Rad(C)$.

Lemma 3.52 *For any radditive functor F the morphism $Lres(F) \to F$ is a projective equivalence.*

Proof: Let $0 \to Cof(F) \to F$ be the standard factorization of the morphism $0 \to F$ such that the first arrow is a sequential I-cell and the second one has the right lifting property for elements of I i.e. is a trivial projective fibration. Consider the commutative square

$$\begin{tikzcd}
Lres(Cof(F)) \arrow[r] \arrow[d] & Cof(F) \arrow[d] \\
Lres(F) \arrow[r] & F
\end{tikzcd}$$

The upper horizontal morphism is the diagonal of a morphism which is row-wise of the form $Lres(T_i) \to T_i$ where T_i are the terms of $Cof(F)$. Since elements of I are term-wise coprojections each term of $Cof(F)$ is a filtering colimit of objects of C. Therefore Lemma 3.44 implies that each morphism $Lres(T_i) \to T_i$ is a filtering colimit of homotopy equivalences and in particular a projective equivalence. The left vertical arrow is the diagonal of a morphism which is column-wise of the form $\amalg U \boxtimes Cof(F)(V) \to \amalg U \boxtimes F(V)$. Since $Cof(F)(V) \to F(V)$ is a weak equivalence of simplicial sets this morphism belongs to $cl_{\bar{\Delta}}(\emptyset)$ by Proposition 2.20. Proposition 3.51 implies that it is a projective equivalence. We conclude that three out of four arrows in our square are projective equivalences and therefore the fourth is a projective equivalence.

For a functor $F : C \to D$ denote by $iso(F)$ the class of morphisms f in C such that $f(f)$ is an isomorphism.

Proposition 3.53 *Let C be a small category with finite coproducts. Then the functor $\Phi : \Delta^{op}[C] \to H(C)$ is a strict localization*[5] *and $iso(\Phi) = cl_{\bar{\Delta}}(\emptyset)$.*

Proof: It is sufficient to show that Φ is a localization. For the fact that any morphism in $H(C)$ is isomorphic to the image of a morphism from $\Delta^{op}[C]$ see the proof of Lemma 5.4. The fact that $iso(\Phi) = cl_{\bar{\Delta}}(\emptyset)$ follows from Lemma 3.50. Let $H'(C)$ be the localization of $\Delta^{op}[C]$ with respect to $cl_{\bar{\Delta}}(\emptyset)$. To prove the proposition we will construct two functors

$$[inc] : H'(C) \to H(C)$$

and

$$[Lres] : H(C) \to H'(C)$$

together with isomorphisms $[inc][Lres] \to Id_{H(C)}$ and $[Lres][inc] \to Id_{H'(C)}$. Let inc be the inclusion of $\Delta^{op}[C]$ to $\Delta^{op}Rad(C)$. Proposition 3.51 implies that inc takes elements of $cl_{\bar{\Delta}}(\emptyset)$ to projective equivalences and thus defines a functor $[inc]$ between the localized categories. We denote by $[Lres]$ the functor $H(C) \to H'(C)$ defined by $\Delta Lres$ according to Lemma 3.49. For any simplicial radditive functor X we have an obvious morphism $inc(Lres(X)) \to X$ and for any X in $\Delta^{op}[C]$ an obvious morphism $Lres(inc(X)) \to X$. These morphisms define natural transformations $[inc][Lres] \to Id_{H(C)}$ and $[Lres][inc] \to Id_{H'(C)}$. Lemmas 3.52 and 3.48 imply that they are isomorphisms.

The standard simplicial resolution functor can be used to define "derived" functors associated with arbitrary functors between categories with finite coproducts. Let $F : C \to C'$ be a general functor. Set

$$\mathbf{L}F^{rad} = F^{rad} \circ Lres.$$

Lemma 3.54 *One has:*

1. *The functor $\mathbf{L}F^{rad}$ takes projective equivalences to projective equivalences.*
2. *For any $X \in \Delta^{op}C^{\#}$ the morphism $\mathbf{L}F^{rad}(X) \to F^{rad}(X)$ is a projective equivalence.*

Proof: If $X \to Y$ is a projective equivalence then $Lres(f) \in cl_{\bar{\Delta}}(\emptyset)$ by Lemmas 3.50 and 3.52. Therefore $F^{rad}(Lres(f)) \in cl_{\bar{\Delta}}(\emptyset)$ by Corolllary 2.17 and it is a projective equivalence by Proposition 3.25. If $X \in \Delta^{op}C^{\#}$ then $Lres(X) \to X$ is in $cl_{\bar{\Delta}}(\emptyset)$ by Lemma 3.50 and

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[5] A functor is called a strict localization if it is a localization and any morphism in the target category is isomorphic to the image of a morphism in the source category.

therefore $\mathbf{L}F^{rad}(X) \to F^{rad}(X)$ is a projective equivalence again by Corollary 2.17 and Proposition 3.25.

For a composable pair of functors $C \xrightarrow{F} D \xrightarrow{G} E$ one has a natural transformation

$$\mathbf{L}G^{rad} \circ \mathbf{L}F^{rad} \to \mathbf{L}(G \circ F)^{rad}.$$

Lemma 3.55 *For any $X \in \Delta^{op}Rad(C)$ the morphism*

$$\mathbf{L}G^{rad}(\mathbf{L}F^{rad}(X)) \to \mathbf{L}(G \circ F)^{rad}(X)$$

is a projective equivalence.

Proof: We have to show that for $X \in \Delta^{op}[C]$ the morphism

$$G^{rad} LresF^{rad}(X) \to G^{rad}F^{rad}(X)$$

is a projective equivalence. The diagonal argument reduces the problem to the case $X \in [C]$. Then $F^{rad}(X)$ is a filtering colimit of representable functors and since both G^{rad} and $Lres$ commute with filtering colimits our result follows from Lemma 3.54(2) and Lemma 3.24.

Lemma 3.56 *Let $F : C \to C'$ be a functor which commutes with finite coproducts. Then $\mathbf{L}F^*$ and F_* define a pair of adjoint functors between $H(C)$ and $H(C')$.*

Proof: Define the adjunctions as follows. For $\mathbf{L}F^{rad}F_* \to Id$ we take

$$F^{rad}LresF_* \to F^{rad}F_* \xrightarrow{a} Id$$

and for $Id \to F_*\mathbf{L}F^{rad}$ we take

$$Id \to Lres \xrightarrow{b} F_*F^{rad}Lres.$$

where the first arrow is the inverse in $H(C)$ of the morphism $Lres \to Id$. We need to verify that these morphisms are indeed adjunctions i.e. that the compositions

$$F_* \to F_*\mathbf{L}F^{rad}F_* \to F_*$$

$$\mathbf{L}F^{rad} \to \mathbf{L}F^{rad}F_*\mathbf{L}F^{rad} \to \mathbf{L}F^{rad}$$

are identities in $H(C)$ and $H(C')$ respectively. For the first composition it follows from the diagram

$$\begin{tikzcd}
LresF_* \arrow[r] \arrow[d] & F_*F^{rad}LresF_* \arrow[r] \arrow[d] & F_* \arrow[d] \\
F_* \arrow[r] & F_*F^{rad}F_* \arrow[r] & F_*
\end{tikzcd}$$

and the fact that (a, b) form an adjunction between F_* and F^{rad}. For the second composition consider the diagram

$$\begin{tikzcd}
F^{rad}LresId & F^{rad}LresLres \arrow[l] \arrow[r] \arrow[d] & F^{rad}LresF_*F^{rad}Lres \arrow[r] \arrow[d] & F^{rad}Lres \arrow[d] \\
 & F^{rad}IdLres \arrow[r] & F^{rad}F_*F^{rad}Lres \arrow[r] & F^{rad}Lres
\end{tikzcd}$$

The lower composition is identity since (a, b) is an adjunction. To check that the upper one is the identity in $H(C')$ it remains to verify that the morphisms

$$F^{rad}LresLres \to F^{rad}IdLres$$

$$F^{rad}LresLres \to F^{rad}LresId$$

coincide in $H(C')$. Since all the functors involved respect projective equivalences it is sufficient to check it for $X \in \Delta^{op}[C]$. For such an X it follows from the commutative square

$$\begin{tikzcd}
LresLres(X) \arrow[r] \arrow[d] & LresId(X) \arrow[d] \\
IdLres(X) \arrow[r] & X
\end{tikzcd}$$

3.4 Grainy categories

Definition 3.57 *A category C with finite coproducts is called grainy if the class of projective equivalences in $\Delta^{op}Rad(C)$ is closed under finite coproducts.*

Example 3.58

1. An additive category is grainy. See Lemma 6.2.
2. For any D the category $D^{\amalg}$ is grainy. It follows from the discussion of Example 2 and the fact that the class of projective equivalences of presheaves is closed under finite coproducts.
3. For any D the category $(D^{\amalg})_+$ is grainy. It follows from the discussion of Examples 3, 2 and the fact that the coproduct on pointed presheaves preserves projective equivalences.
4. Let C_A be the category of free finitely generated commutative algebras over a commutative ring A (see Example 3.1) such that $Rad(C_A)$ is the category of all commutative algebras over A. The category C_A is grainy if A is a field but not if A is a more general ring.

Lemma 3.59 *Let C be a grainy category. Then the class of projective equivalences in $\Delta^{op}Rad(C)$ is $\bar{\Delta}$-closed.*

Proof: By definition this class is closed under finite coproducts. By Lemma 3.24 it is closed under filtering colimits and by Lemma 3.23 it is Δ-closed.

Lemma 3.60 *Let C be a grainy category, $f : X \to X'$ a projective equivalence in $\Delta^{op}Rad(C)$ and K a simplicial set. Then $f \boxtimes Id_K : X \boxtimes Id_K \to X' \boxtimes Id_K$ is a projective equivalence.*

Proof: The morphism $f \boxtimes Id_K$ is the diagonal of a morphism of bisimplicial objects whose rows are coproducts of copies of f. The statement of the lemma follows now from Lemma 3.59.

Lemma 3.61 *Let C be a grainy and E a class of morphisms in $\Delta^{op}C^{\#}$. Then one has*

$$Lres(cl_{\bar{\Delta}}(E \cup W_{proj})) \subset cl_{\bar{\Delta}}(E)$$

where on the left hand side E is considered as a class of morphisms in $\Delta^{op}Rad(C)$ and on the right hand side as a class of morphisms in $\Delta^{op}C^{\#}$.

Proof: Denote the class $Lres^{-1}(cl_{\bar{\Delta}}(E))$ by H. We need to show that it is $\bar{\Delta}$-closed and contains E. For $f : X \to Y$ in E we have $Lres(f) \in cl_{\bar{\Delta}}(E)$ by Lemma 3.48. By Lemma 3.46 it is closed under filtering colimits. The fact that it is Δ-closed is straightforward. It remains to check that for $f : X \to X'$ and $g : Y \to Y'$ such that $Lres(f), Lres(g) \in cl_{\bar{\Delta}}(E)$ one has $Lres(f \amalg g) \in cl_{\bar{\Delta}}(E)$. Consider the diagram

$$\begin{tikzcd}
Lres(X) \amalg Lres(Y) \arrow[r] \arrow[d] & Lres(X') \amalg Lres(Y') \arrow[d] \\
Lres(X \amalg Y) \arrow[r] & Lres(X' \amalg Y')
\end{tikzcd}$$

It is upper horizontal arrow is in $cl_{\bar{\Delta}}(E)$ by our assumption. Since C is grainy, the vertical arrows are projective equivalences between objects of $\Delta^{op}C^{\#}$. This implies that they are in $cl_{\bar{\Delta}}(\emptyset)$ and, therefore, the lower horizontal arrow is in $cl_{\bar{\Delta}}(E)$.

Let π_* be the forgetful functor from $\Delta^{op}C$ to $\Delta^{op}_{Mon}C$ considered at the end of Section 3.2. We call the functor $\pi^*\pi_*$ the wrapping functor and denote it by Wr. For any simplicial object X the object $Wr(X)$ is degeneracy free and its terms are given by the formula

$$Wr(X)_i = \amalg_{[i]\to[j]} X_j \tag{11}$$

where $[i] \to [j]$ runs through epimorphisms from $[i]$ to $[j]$ in Δ and X_j are the terms of X. If X belongs to $\Delta^{op}[C]$ then $Wr(X)$ is cofibrant by Lemma 3.42. For any X, the adjunction defines a natural morphism $Wr(X) \to X$.

Theorem 3.62 *Let C be a grainy category. Then for any $X \in \Delta^{op}Rad(C)$ the morphism $p_X : Wr(X) \to X$ is a projective equivalence.*

Proof: Consider $Lres(X)$ as a bi-simplicial object whose rows are of the form $Lres(X_m)$. Then its n-th column is of the form

$$\amalg_{U_0 \to \ldots \to U_n} U_0 \boxtimes X(U_n)$$

where the coproduct is in $Rad(C)$ over all sequences of morphisms $U_0 \to \ldots \to U_n$ in C. Let Wr^v be the vertical wrapping functor i.e. the wrapping functor applied column by column. Consider the following square:

$$\begin{tikzcd} \Delta Wr^v(Lres(X)) \arrow[r, "a"] \arrow[d, "b"'] & \Delta Lres(X) \arrow[d, "c"'] \\ Wr(X) \arrow[r, "p"] & X \end{tikzcd}$$

Let us show that the morphisms a, b, c are projective equivalences and therefore p is a projective equivalence by the 2-out-of-3 property. We have $c \in W_{proj}$ by Lemma 3.52. The morphism b is the diagonal of the morphism whose i-th row is of the form

$$\amalg_{[i] \to [j]} Lres(X_j) \to \amalg_{[i] \to [j]} X_j.$$

where the coproduct is over all the order-preserving surjections $[i] \to [j]$. These morphisms are projective equivalences by Lemma 3.52 and by the fact that for a grainy C the class of projective equivalences is closed under coproducts. Therefore b is a projective equivalence since the class of projective equivalences is Δ-closed. The morphism a is the diagonal of the morphism whose columns are of the form

$$\amalg_{U_0 \to \ldots \to U_n} U_0 \boxtimes Wr(X(U_n)) \to \amalg_{U_0 \to \ldots \to U_n} U_0 \boxtimes X(U_n) \tag{12}$$

where $Wr(X(U_n))$ is the wrapping functor on the simplicial *set* $X(U_n)$. Since for a simplicial set S, the morphism $Wr(S) \to S$ is a weak equivalence we conclude by Corollary 3.31 that the morphisms (12) are coproducts of projective equivalences and therefore projective equivalences. Theorem is proved.

Lemma 3.63 *Let C be a grainy category. Then the projective closed model structure on $\Delta^{op}Rad(C)$ is left proper i.e. for any push-out square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "g'"] \\ B \arrow[r] & Y \end{tikzcd}$$

where f is a cofibration and g is a projective equivalence, g' is a projective equivalence.

Proof: By Corollary 3.29 we may represent f as a composition $f = psf$ where $ps = Id$ and the terms of $c = fs$ are of the form $X_n \to X_n \amalg X'_n$ where $X'_n \in [C]$. The push-out g' is a retract of g'' where g'' is the push-out of g by c. Therefore it is enough to show that g'' is a projective equivalence. The morphism g'' is the diagonal of a morphism whose rows are of the form $g \amalg Id_{X'_n}$ which finishes the proof by Lemma 3.59.

4 E-local equivalences

4.1 Functors $Ex_{E,N}$

Lemma 4.1 *Let C be a small category with finite coproducts and A a set of objects of $Rad(C)$. Then for a sufficiently large ordinal R, any family $(X_r)_{r \in R}$ and any $Y \in A$ one has*

$$Hom(Y, X) = colim_r Hom(Y, X_r).$$

Proof: Since filtering colimits of radditive functors are computed section-by-section we may embed $Rad(C)$ into the category of presheaves on C. In this context the result is easy to deduce.

Proposition 4.2 *Let C be a bounded category with small coproducts and let E and N be two sets of morphisms in $\Delta^{op}Rad(C)$ such that the elements of N are term-wise coprojections. Then there exist a functor*

$$Ex_{E,N} : \Delta^{op}Rad(C) \to \Delta^{op}Rad(C)$$

and a natural transformation $Id \to Ex_{E,N}$ with the following properties:

1. *for any X in $\Delta^{op}Rad(C)$ the morphism $X \to Ex_{E,N}(X)$ belongs to $cl_{\bar{\Delta}}(E)$*
2. *for any $f : X \to X'$ from E and any Y the map*

$$S(X', Ex_{E,N}(Y)) \to S(X, Ex_{E,N}(Y))$$

   is a weak equivalence

3. *for any $f : X \to X'$ from N and any Y the map*

$$S(X', Ex_{E,N}(Y)) \to S(X, Ex_{E,N}(Y))$$

   is a Kan fibration.

Proof: For a morphism $f : X \to X'$ denote by $i_f : X \to cyl(f)$ the composition

$$X \overset{Id\boxtimes\partial_1}{\to} X\boxtimes\Delta^1 \to cyl(f).$$

Note that i_f is a term-wise coprojection. Let R be an ordinal satisfying the condition of Lemma 4.1 with respect to the set of sources and targets of morphisms from E and N. We are going to define functors

$$Ex_r : \Delta^{op}C \to \Delta^{op}C$$

for $r \in R$ and natural transformations $Ex_r \to Ex_s$ for $r < s$ and then set

$$Ex_{E,N} = colim_r Ex_r$$

To avoid the complications related to large ordinals we will assume that R is the countable ordinal i.e. $R = \mathbf{N}$.

Set $Ex_0(X) = X$ and define $Ex_{n+1}(X)$ for even n by the push-out square

$$\begin{tikzcd}
\amalg_{(f:Y\to Y')\in E, i\geq 0, g\in A(f,i)} \; (cyl(f)\boxtimes\partial\Delta^i \amalg_{Y\boxtimes\partial\Delta^i} Y\boxtimes\Delta^i) \arrow[r] \arrow[d] & Ex_n(X) \arrow[d] \\
\amalg_{(f:Y\to Y')\in E, i\geq 0, g\in A(f,i)} \; cyl(f)\boxtimes\Delta^i \arrow[r] & Ex_{n+1}(X)
\end{tikzcd} \tag{13}$$

where

$$A(f,i) = Hom(cyl(f)\boxtimes\partial\Delta^i \amalg_{Y\boxtimes\partial\Delta^i} Y\boxtimes\Delta^i, Ex_n(X))$$

and for odd n by the push out square

$$\begin{tikzcd}
\amalg_{(f:Y\to Y')\in N, i\geq 0, 0\geq k\geq i, g\in B(f,i)} \; (Y'\boxtimes\Lambda^{i,k} \amalg_{Y\boxtimes\Lambda^{i,k}} Y\boxtimes\Delta^i) \arrow[r] \arrow[d] & Ex_n(X) \arrow[d] \\
\amalg_{(f:Y\to Y')\in E, i\geq 0, 0\geq k\geq i, g\in B(f,i)} \; Y'\boxtimes\Delta^i \arrow[r] & Ex_{n+1}(X)
\end{tikzcd} \tag{14}$$

where

$$B(f,i) = Hom(Y'\boxtimes\Lambda^{i,k} \amalg_{Y\boxtimes\Lambda^{i,k}} Y\boxtimes\Delta^i, Ex_n(X))$$

Our construction is well defined, i.e. the required push-outs exist, because the left vertical arrows in squares of both types are term-wise coprojections (for squares (14) this follows from our assumption that elements of N are term-wise coprojections). Define $Ex_{E,N}$ as $colim_{n\geq 0} Ex_n$. By Lemma 2.14, $cl_{\bar\Delta}(E)$ is closed under filtering compositions. Therefore, to prove the first condition of the lemma it is sufficient to check that the morphisms $Ex_n \to Ex_{n+1}$ belong to $cl_{\bar\Delta}(E)$. Consider first the squares (13). In view of Lemma 2.11 it is sufficient to show that the morphism

$$cyl(f)\boxtimes\partial\Delta^i \amalg_{Y\boxtimes\partial\Delta^i} Y\boxtimes\Delta^i \to cyl(f)\boxtimes\Delta^i$$

belongs to $cl_{\bar\Delta}(f)$. This morphism fits into the commutative diagram

$$\begin{tikzcd}
Y\boxtimes\partial\Delta^i \arrow[r] \arrow[d] & cyl(f)\boxtimes\partial\Delta^i \arrow[d] & \\
Y\boxtimes\Delta^i \arrow[r] & cyl(f)\boxtimes\partial\Delta^i \amalg_{Y\boxtimes\partial\Delta^i} Y\boxtimes\Delta^i \arrow[r] & cyl(f)\boxtimes\Delta^i
\end{tikzcd} \tag{15}$$

The definition of the morphism $Y \to cyl(f)$ implies easily that it belongs to $cl_{\bar{\Delta}}(f)$. Thus for any simplicial set K the morphism $Y \boxtimes K \to cyl(f) \boxtimes K$ belongs to $cl_{\bar{\Delta}}(f)$. In particular the upper horizontal arrow in (15) and the composition of the two lower horizontal ones belong to $cl_{\bar{\Delta}}(f)$. By Lemma 2.11 we conclude that the first lower horizontal arrow is in $cl_{\bar{\Delta}}(f)$ and therefore the second one is in $cl_{\bar{\Delta}}(f)$ by the two out of three property.

The morphisms

$$Y' \boxtimes \Lambda^{i,k} \amalg_{Y \boxtimes \Lambda^{i,k}} Y \boxtimes \Delta^i \to Y' \boxtimes \Delta^i$$

used in squares (14) fit into commutative diagrams

$$\begin{tikzcd}
Y \boxtimes \Lambda^{i,k} \arrow[r] \arrow[d] & Y \boxtimes \Delta^i \arrow[d] & \\
Y' \boxtimes \Lambda^{i,k} \arrow[r] & Y' \boxtimes \Lambda^{i,k} \amalg_{Y \boxtimes \Lambda^{i,k}} Y \boxtimes \Delta^i \arrow[r] & Y' \boxtimes \Delta^i
\end{tikzcd} \tag{16}$$

and one argues in a similar way using the fact that the upper horizontal arrow and the composition of the lower horizontal ones are in $cl_{\bar{\Delta}}(\emptyset)$ by Proposition 2.20 since the inclusions $\Lambda^{i,k} \to \Delta^i$ are weak equivalences of simplicial sets.

To prove the second condition observe first that for any $f : X \to X'$ the morphism $cyl(f) \to X'$ is a homotopy equivalence by Lemma 2.8 and therefore it is sufficient to show that for f in E the map of simplicial sets

$$S(cyl(f), Ex_{E,N}(Y)) \to S(X, Ex_{E,N}(Y))$$

defined by $i_f : X \to cyl(f)$ is a weak equivalence. By our assumption that Lemma 4.1 holds for the countable ordinal this map is isomorphic to the map

$$colim_n S(cyl(f), Ex_n(Y)) \to colim_n S(X, Ex_n(Y))$$

Our choice of the squares (13) implies that it has the right lifting property with respect to all embeddings $\partial\Delta^i \to \Delta^i$. Any such map is a trivial fibration and in particular a weak equivalence.

The third condition follows from the fact that, because of our choice of squares (14), for any $f : X \to X'$ in N the map

$$colim_n S(X', Ex_n(Y)) \to colim_n S(X, Ex_n(Y))$$

has the right lifting property with respect to the embeddings $\Lambda^{i,k} \to \Delta^i$.

The construction of $Ex_{E,N}$ immediately implies the following result.

Lemma 4.3 *Assume that sources and targets of elements of E and N are in $\Delta^{op}C^{\#}$. Then for any $X \in \Delta^{op}C^{\#}$ one has*

$$Ex_{E,N}(X) \in \Delta^{op}C^{\#}.$$

Remark 4.4 If the ordinal R used in the proof of Proposition 4.2 is countable than the obvious analog of Lemma 4.3 holds for $[C]$. In general I do not know whether this is the case since it is not clear that colimits of general filtering systems of coprojections exist in $[C]$.

Proposition 4.5 *Let E, N be two sets of morphisms between cofibrant objects such that the elements of E are term-wise coprojections and cofibrations. Then for any X the map $X \to Ex_{E,N}(X)$ is a cofibration.*

Proof: A filtered composition of cofibrations is a cofibration. A push-out of a cocofibration is a cocofibration. A coproduct of cofibrations is a cofibration. Therefore it is sufficient to verify that the morphisms which form the left hand side vertical arrows of (13) and (14) are cofibrations. In both cases the morphisms are of the type considered in Lemma 3.32. In the first case the corresponding morphisms j are of the form $i_f : Y \to cyl(f)$ for $f : Y \to Y'$ in E and in the second case they are elements of N. Therefore the statement for squares (14) follows directly from Lemma 3.32 and our assumption on N. For the squares of the first type it follows from Lemma 3.32 and the following result.

Lemma 4.6 *Let $f : Y \to Y'$ be a morphism between two cofibrant objects. Then the morphism $i_f : Y \to cyl(f)$ is a cofibration.*

Proof: The morphism i_f may be defined as the composition of the two lower horizontal arrows of the diagram

$$\begin{tikzcd}
 & Y \amalg Y \arrow[r] \arrow[d, "Id \amalg f"'] & Y \boxtimes \Delta^1 \arrow[d] \\
Y \arrow[r] & Y \amalg Y' \arrow[r] & cyl(f)
\end{tikzcd}$$

The upper arrow is a cofibration by Corollary 3.33 since $Y \amalg Y = Y \partial \Delta^1$. Therefore the second lower arrow is a cofibration. The first lower arrow is a cofibration because Y' is cofibrant.

4.2 Basic constructions

Definition 4.7 *A class of morphisms E in $\Delta^{op}Rad(C)$ is called admissible if the domains and codomains of its members are cofibrant.*

Definition 4.8 *Let E be an admissible class of morphisms in $\Delta^{op}Rad(C)$. An object Y of this category is called E-local if it is projectively fibrant and for any element $f : X \to X'$ in E and any simplicial set K the map*

$$Hom_{H(C)}(X' \boxtimes K, Y) \to Hom_{H(C)}(X \boxtimes K, Y) \tag{17}$$

defined by f, is bijective.

Lemma 4.9 *Let E be an admissible class of morphisms in $\Delta^{op}Rad(C)$. An object Y of this category is E-local iff it is projectively fibrant and for any element $f : X \to X'$ of E the map of simplicial sets*

$$S(\tilde{X}', Y) \to S(\tilde{X}, Y)$$

defined by f, is a weak equivalence.

Proof: Note first that $S(\tilde{X}', Y)$ and $S(\tilde{X}, Y)$ are Kan simplicial sets. A map between two such sets is a weak equivalence iff for any K it induces the bijection on homotopy classes of maps from K. These homotopy classes of maps are identified with the sides of (17).

Definition 4.10 *Let E be an admissible class of morphisms in $\Delta^{op}Rad(C)$. A morphism $f : X \to X'$ is called a (left) E-local equivalence if for any E-local Y the map*

$$Hom_{H(C)}(X', Y) \to Hom_{H(C)}(X, Y)$$

defined by f is bijective.

Lemma 4.11 *Let E be an admissible class of morphisms in $\Delta^{op}Rad(C)$. A morphism $f : X \to X'$ is an E-local equivalence if for any E-local Y and any cofibrant replacement $\tilde{f} : \tilde{X} \to \tilde{X}'$ of f the map of simplicial sets*

$$S(\tilde{X}', Y) \to S(\tilde{X}, Y)$$

defined by $\tilde{f}$ is a weak equivalence.

Proof: We may clearly assume that X and X' are cofibrant. For a simplicial set K let $Hom_{\boxtimes}(K, -)$ be the functor right adjoint to $K \boxtimes (-)$ which was defined at the beginning of Section 3.2. Then for any E-local Y the object $Hom_{\boxtimes}(K, Y)$ is again E-local and our result follows by the same reasoning as in the proof of Lemma 4.9.

Remark 4.12 Note that the class of $X \boxtimes K$ in $H(C)$ for a cofibrant X depends only on the class of X in $H(C)$. This shows that the classes of E-local objects and E-local equivalences depend only on the the image of E in $H(C)$.

We denote the class of (left) E-local equivalences by $cl_l(E)$. The goal of this and the next few sections is to analize the relations between $cl_l(E)$ and $cl_{\bar{\Delta}}(E)$. The following lemmas summarize some obvious properties of E-local objects and E-local equivalences.

Lemma 4.13 *An E-local equivalence whose domain and codomain are E-local is a projective equivalence.*

Proof: Let $f : X \to Y$ be an E-local equivalence between E-local objects. The definition of E-local equivalences show that the composition with f defines bijections between morphisms between X and Y in $H(C)$. Therefore it is an iso in $H(C)$ i.e. a projective equivalence.

Set

$$H(C, E) = \Delta^{op} Rad(C)[cl_l(E)^{-1}] = H(C)[cl_l(E)^{-1}].$$

Lemma 4.14 *The class $cl_l(E)$ is saturated i.e. it coincides with the class of morphisms which become isomorphisms in $H(C, E)$.*

Proof: It follows immediately from the universal properties of localization and the definition of $cl_l(E)$.

Lemma 4.15 *Let E be a admissible set of morphisms in $\Delta^{op} Rad(C)$. Let N be the set of morphisms of the form $0 \to U$ for $U \in C$ and $Ex_{E,N}$ the corresponding functor of Proposition 4.2. Then for any X the object $Ex_{E,N}(X)$ is E-local.*

Proof: Follows immediately from Lemma 4.9 and Proposition 4.2.

Proposition 4.16 *For any admissible E the class of E-local equivalences is closed under filtering colimits.*

Proof: Let $(f_i)_{i \in I} : (X_i)_{i \in I} \to (Y_i)_{i \in I}$ be a morphism of filtering systems such that the morphisms $X_i \to Y_i$ are E-local equivalences. We need to show that the morphism

$$colim_I X_i \to colim_I Y_i$$

is an E-local equivalence. Applying the cofibrant replacement functor and using the fact that the class of projective equivalences is closed under filetring colimits we may assume that the object X_i and Y_i are cofibrant.

Since homotopy limits of fibrant simplicial sets respect weak equivalences the morphism

$$hocolim_I X_i \to hocolim_I Y_i$$

is an E-local equivalence. On the other hand the morphisms from *hocolim* to *colim* are in $cl_{\bar{\Delta}}(\emptyset)$ by Proposition 2.23 and therefore they are projective equivalences by Proposition 3.25.

Corollary 4.17 *Let E be an admissible class and $(X_i)_{i\in I}$ a filtering system such that the morphisms $X_i \to X_j$ are E-local equivalences. Then the morphisms*

$$X_j \to hocolim_{i\in I} X_i$$

are E-local equivalences.

Proof: See the proof of Lemma 2.14.

Lemma 4.18 *Consider a push-out square Q of the form*

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

where $B \to A$ is a cofibration and an E-local equivalence and Y is cofibrant. Then $Y \to X$ is a cofibration and an E-local equivalence.

Proof: By Lemma 4.11 it is enough to check that for an E-local Z the map $S(X, Z) \to S(Y, Z)$ is a weak equivalence. This follows from the fact that the pull-back of a trivial fibration of simplicial sets is a trivial fibration applied to the square $S(Q, Z)$.

Lemma 4.19 *Let E be admissible, X be a cofibrant object and Ex the functor of Lemma 4.15. Then $X \to Ex(X)$ is an E-local equivalence.*

Proof: In view of Lemmas 4.18 and 4.17 it is sufficient to verify that the objects which appear on the right of squares (13) and (14) are cofibrant and that the corresponding morphisms are cofibrations and E-local equivalences. This follows from the proof of Proposition 4.5 and Lemma 4.18.

Proposition 4.20 *Let E be admissible. Then for any X in $\Delta^{op}Rad(C)$ there exists a diagram*

$$X \leftarrow Cof(X) \to Ex(Cof(X))$$

where morphisms are E-local equivalences and $Ex(Cof(X))$ is E-local.

Proof: It is enough to take $Cof(X)$ to be the cofibrant replacement of X and Ex to be the functor of Lemmas 4.15 and 4.19.

Proposition 4.21 *For a c-admissible class E the functor $H(C) \to H(C,E)$ has a right adjoint which identifies $H(C,E)$ with the full subcategory of E-local objects in $H(C)$.*

Proof: It follows immediately from Proposition 4.20 and the following general lemma.

Lemma 4.22 *Let C be a category and E a class of morphisms in C such that the localization $C[E^{-1}]$ exists. Let Z be an object such that for any $f : X \to Y$ in E the map $Hom_C(Y,Z) \to Hom_C(X,Z)$ defined by f is a bijection. Then for any $U \in C$ the map*

$$Hom_C(U,Z) \to Hom_{C[E^{-1}]}(U,Z)$$

is a bijection.

Proof: See [].

Corollary 4.23 *For any admissible E the functor*

$$\Delta^{op}Rad(C) \to H(C,E)$$

is a strict localization i.e. any morphism in the target category is isomorphic to the image of a morphism in the source category.

Proof: It is the composition of two functors the first one being a strict localization by Proposition 3.53 and the second one being a strict localization in view of Proposition 4.21.

Lemma 4.24 *Let $E \subset E'$ be two admissible classes in $\Delta^{op}Rad(C)$. Then the functor*

$$H(C,E) \to H(C,E')$$

is the localization with respect to the class of morphisms $f : X \to Y$ such that for any E'-local Z in $\Delta^{op}Rad(C)$ the map

$$Hom_{H(C,E)}(Y,Z) \to Hom_{H(C,E)}(X,Z)$$

is a bijection.

Proof: Corollary 4.23 implies easily that our functor is a localization with respect to the image of $cl_l(E')$ in $H(C,E')$. Since the class of E'-local objects contains the class of E-local objects for any E'-local Z and any X one has

$$Hom_{H(C)}(X,Z) = Hom_{H(C,E)}(X,Z)$$

which implies that the image of $cl_l(E')$ in $H(C,E)$ coincides with the class described in the lemma.

Proposition 4.25 *For an admissible E one has*

$$cl_l(E) \subset cl_{\bar{\Delta}}(E \cup W_{proj})$$

where W_{proj} is the class of projective equivalences.

Proof: Let $f : X \to Y$ be in $cl_l(E)$. Taking a cofibrant replacement of f and using the 2-out-of-3 property for cl_{Δ} we may assume that X and Y are cofibrant. Consider now the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Ex(X) \arrow[d] \\ Y \arrow[r] & Ex(Y) \end{tikzcd}$$

The horizontal arrows are in $cl_l(E)$ by Lemma 4.19. Hence the right vertical one is in this class and by Lemmas 4.15 and 4.13 it is a projective equivalence. On the other hand by Proposition 4.2(1) the horizontal arrows are in $cl_{\bar{\Delta}}(E \cup W_{proj})$. We conclude that the left vertical arrow is in $cl_{\bar{\Delta}}(E \cup W_{proj})$.

For a class of morphisms E in $\Delta^{op}Rad(C)$ let $E \cap C^{\#}$ (resp. $E \cap [C]$) denote the subset of elements of E whose domains and codomains are in $\Delta^{op}C^{\#}$ (resp. in $\Delta^{op}[C]$).

Proposition 4.26 *Let E be an admissible set of morphisms in $\Delta^{op}[C]$. Then one has*

$$cl_l(E) \cap [C] \subset cl_{\bar{\Delta}}(E)$$

where the $\bar{\Delta}$-closure on the right is in $\Delta^{op}[C]$.

Proof: Let $f : X \to Y$ be an element of $cl_l(E) \cap [C]$. As in the proof of Proposition 4.25 consider a cofibrant replacement $Cof(f) : Cof(X) \to Cof(Y)$ of f where we use the standard cofibrant replacement functor Cof. This functor takes any object of $\Delta^{op}Rad(C)$ to an object of $\Delta^{op}[C]$. By Lemma 3.50 we conclude that the morphisms $Cof(X) \to X$ and $Cof(Y) \to Y$ are in $cl_{\bar{\Delta}}(\emptyset)$. Therefore it is enought to show that $Cof(f)$ is in $cl_{\bar{\Delta}}(E)$. Consider the diagram

$$\begin{tikzcd} Cof(X) \arrow[r] \arrow[d] & Ex(Cof(X)) \arrow[d] \\ Cof(Y) \arrow[r] & Ex(Cof(Y)) \end{tikzcd}$$

where Ex is the functor of Lemma 4.15. The horizontal arrows are in $cl_l(E)$ by Lemma 4.19. Therefore the right hand side vertical arrow is in $cl_l(E)$. Since the objects in question are E-local by Lemma 4.15 the right vertical arrow is a projective equivalence. The construction of Ex together with our assumption that the domains and codomains of elements of E are in $[C]$ imply that $Ex(Cof(X))$ and $Ex(Cof(Y))$ are in $\Delta^{op}[C]$. Applying again Lemma 3.50 we conclude that the right vertical arrow is in $cl_{\bar{\Delta}}(\emptyset)$.

On the other hand the horizontal arrows are in $cl_{\bar{\Delta}}(E)$ by Proposition 4.2(1) and therefore the left hand side vertical arrow is in $cl_{\bar{\Delta}}(E)$.

Denote by $H(C, E)$ the localization of the category $\Delta^{op}Rad(C)$ with respect to the class of E-local equivalences and let $\Phi : \Delta^{op}[C] \to H(C, E)$ be the obvious functor.

Proposition 4.27 *Let E be an admissible class of morphisms in $\Delta^{op}Rad(C)$. Then the functor Φ is a localization and $iso(\Phi) = cl_l(E) \cap [C]$.*

Proof: The functor Φ is the composition of two functors

$$\Delta^{op}[C] \to H(C) \to H(C, E)$$

where the first one is a strict localization by Proposition 3.53 and the second one is a localization by definition. Therefore it is a localization. The fact that $iso(\Phi) = cl_l(E) \cap [C]$ follows immediately from Lemma 4.14.

Remark 4.28 Both Proposition 4.26 and Proposition 4.27 remain valid if we replace $[C]$ with $C^{\#}$.

4.3 E-local equivalences in grainy categories

Lemma 4.29 *Let C be a grainy category. Then for any class E the class $cl_l(E)$ is closed under coproducts.*

Proof: Follows from the fact that in a grainy category a coproduct of projective equivalences is a projective equivalence and therefore for cofibrant replacements $\tilde{f_\alpha}$ of f_α the coproduct $\amalg_\alpha \tilde{f_\alpha}$ is a cofibrant replacement of $\amalg_\alpha f_\alpha$.

Lemma 4.30 *Let C be a grainy category and $f : B \to B'$ be a morphism of bisimplicial objects in $Rad(C)$ whose rows are cofibrant. Then $\Delta(f) \in cl_l(\{f_i\}_{i\geq 0})$ where $f_i : B_i \to B'_i$ are the rows of f.*

Proof: Applying the wrapping functor to the columns of B and B' we get a commutative diagram of the form

$$\begin{tikzcd}
Wr_{col}(B) \arrow[r] \arrow[d] & Wr_{col}(B') \arrow[d] \\
B \arrow[r] & B'
\end{tikzcd}$$

where the vertical arrows are column-wise projective equivalences by Theorem 3.62. In view of Lemma 3.23 they define projective equivalences of the corresponding diagonal objects. Therefore it is sufficient to show that the morphism

$$\Delta(Wr_{col}(B)) \to \Delta(Wr_{col}(B')) \tag{18}$$

is in $cl_l(\{f_i\})$. By Corollary 3.41 the domain and codomain of this morphism are cofibrant. Therefore it is sufficient to show that for any Y which is local with respect to the set $\{f_i\}$ the map of simplicial sets

$$S(Y, \Delta(Wr_{col}(B'))) \to S(Y, \Delta(Wr_{col}(B)))$$

is a weak equivalence.

By Proposition 3.40 the morphism (18) is the colimit of the sequence of morphisms

$$\Delta(sk_n(Wr_{col}(B))) \to \Delta(sk_n(Wr_{col}(B')))$$

which arise from morphisms of elementary exact squares of the form

$$\left(\begin{array}{ccc} B_n \boxtimes \partial\Delta^n & \longrightarrow & \Delta(sk_{n-1}(Wr_{col}(B))) \\ \downarrow & & \downarrow \\ B_n \boxtimes \Delta^n & \longrightarrow & \Delta(sk_n(Wr_{col}(B))) \end{array}\right) \to \left(\begin{array}{ccc} B'_n \boxtimes \partial\Delta^n & \longrightarrow & \Delta(sk_{n-1}(Wr_{col}(B'))) \\ \downarrow & & \downarrow \\ B'_n \boxtimes \Delta^n & \longrightarrow & \Delta(sk_n(Wr_{col}(B'))) \end{array}\right)$$

Our result follows now from the fact a morphism of pull-back squares of simplicial sets

$$\left(\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}\right) \to \left(\begin{array}{ccc} B' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ A' & \longrightarrow & X' \end{array}\right)$$

where vertical arrows are fibrations and which is a weak equivalence outside of $B \to B'$ is also a weak equivalence on $B \to B'$.

Proposition 4.31 *Let C be a grainy category. Then for any admissible class E the class $cl_l(E)$ is $\bar{\Delta}$-closed.*

Proof: In view of Proposition 2.24 it is enough to check that it is closed under coproducts, contains $cl_{\bar{\Delta}}(\emptyset)$ and is Δ-closed. The first condition is Lemma 4.29. The second follows from Proposition 3.25. It clearly has the 2-out-of-3 property. It remains to show that if $f : B \to B'$ is a morphism of bisimplicial radditive functors with rows $f_{:}B_i \to B'_i$ then one has $\Delta(f) \in cl_l(\{f_i\})$. Applying the cofibrant replacement functor to the rows of B and B' we get a commutative square of the form

$$\begin{tikzcd} Cof_{rows}(B) \arrow[r] \arrow[d] & Cof_{rows}(B') \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

In view of Lemma 3.23 the vertical arrows define projective equivalences on the diagonal objects. It remains to show that the diagonal of the upper horizontal arrow is in $cl_l(\{f_i\})$ which follows from our assumption on C and Lemma 4.30.

Proposition 4.32 *Let E be a grainy category and E an admissible class of morphisms in $\Delta^{op}C^{\#}$. Then one has*

$$cl_{\bar{\Delta}}(E \cup W_{proj}) \cap C^{\#} = cl_{\bar{\Delta}}(E)$$

Proof: The right hand side is obviously contained in the left hand side. Let $f : X \to Y$ be a morphism in $\Delta^{op}Rad(C)$ which belongs to $cl_{\bar{\Delta}}(E \cup W_{proj})$ and whose domain and codomain are in $\Delta^{op}C^{\#}$. Then the square

$$\begin{tikzcd}
Lres(X) \arrow[r] \arrow[d] & Lres(Y) \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

together with Lemma 3.48 and Lemma 3.61 imply that f is in $cl_{\bar{\Delta}}(E)$.

Theorem 4.33 *Let C be a grainy category. Let E be a admissible set of morphisms in $\Delta^{op}C^{\#}$. Then one has:*

$$cl_l(E) = cl_{\bar{\Delta}}(E \cup W_{proj}) \tag{19}$$

$$cl_l(E) \cap C^{\#} = cl_{\bar{\Delta}}(E) \tag{20}$$

Proof: Proposition 4.31 implies that the right hand side of (19) is contained in the left hand side. The opposite inclusion is proved in Proposition 4.25. The second equality follows from the first one and Proposition 4.32.

Remark 4.34 The analog of the second part of Theorem 4.33 also holds for $[C]$ instead of $C^{\#}$.

Lemma 4.35 *Let $F : C \to C'$ be a functor such that C' is grainy. Let further E be an admissible class in $\Delta^{op}C^{\#}$ and E' and admissible class in $\Delta^{op}Rad(C')$ such that*

$$F^{rad}(E \amalg Id_C) \subset cl_l(E').$$

Then

$$\mathbf{L}F^{rad}(cl_l(E)) \subset cl_l(E')$$

and $\mathbf{L}F^{rad}$ defines a functor

$$H(C, E) \to H(C', E').$$

Proof: Let $f : X \to Y$ be in $cl_l(E)$. Then by Lemma 3.52 one has $Lres(f) \in cl_l(E) \cup [C]$ and by Proposition 4.26 we get

$$Lres(f) \in cl_{\bar{\Delta}}(E)$$

where the closure is in $\Delta^{op}C^{\#}$. By Proposition 2.16

$$F^{rad}(Lres(f)) \in cl_{\bar{\Delta}}(F^{rad}(E \amalg Id_C)).$$

By Proposition 4.31 the class $cl_l(E')$ is $\bar{\Delta}$-closed which finishes the proof.

Theorem 4.36 *Let $F : C \to C'$ be a functor between grainy categories which commutes with finite coproducts. Let E be an admissible class in $\Delta^{op}C^{\#}$ and E' an admissible class in $\Delta^{op}Rad(C')$. Assume further that one has:*

$$F^{rad}(E) \subset cl_l(E')$$

$$F_*(E' \amalg Id_{Rad(C')}) \subset cl_l(E)$$

Then

$$\mathbf{L}F^{rad}(cl_l(E)) \subset cl_l(E') \tag{21}$$

$$F_*(cl_l(E')) \subset cl_l(E) \tag{22}$$

and the resulting pair of functors between $H(C,E)$ and $H(C',E')$ is adjoint.

Proof: It is enough to prove the inclusions (21) and (22). The first one follows from Lemma 4.35 since under our assumptions F^{rad} commutes with $\amalg$ and $cl_l(E')$ is closed under finite coproducts by Lemma 4.29. For the second inclusion we have

$$F_*(cl_l(E')) = F_*(cl_{\bar{\Delta}}(E' \cup W_{proj})) \subset cl_{\bar{\Delta}}(F_*(E' \amalg Id_{Rad(C')}) \cup F_*(W_{proj})) \subset$$

$$\subset cl_l(E).$$

Where the first equality follows from Theorem 4.33(1), the second inclusion from Proposition 2.16 and the last one from Proposition 4.31.

Corollary 4.37 *Under the assumptions of the theorem suppose in addition that F is a full embedding. Then $\mathbf{L}F^{rad} : H(C,E) \to H(C',E')$ is a full embedding.*

Proof: Since we have a pair of adjoint functors it is enough to verify that the adjunction $Id \to F_*\mathbf{L}F^{rad}$ is an isomorphism in $H(C,E)$. It follows from the fact that $Id \to F_*F^{rad}$ is an isomorphism.

Lemma 4.38 *Under the assumptions of Theorem 4.36 the functor F_* takes E'-local objects to E-local objects.*

Proof: Since F_* obviously takes projectively fibrant object to projectively fibrant objects it is sufficient to show that for an E'-local Z, any $f : X \to Y$ in E and any simplicial set K the map

$$Hom_{H(C)}(Y \boxtimes K, F_* Z) \to Hom_{H(C)}(X \boxtimes K, F_* Z)$$

is bijective. By Lemma 3.56 this map is isomorphic to the map

$$Hom_{H(C)}(\mathbf{L}F^{rad}(Y \boxtimes K), Z) \to Hom_{H(C)}(\mathbf{L}F^{rad}(X \boxtimes K), Z)$$

On the other hand $f \boxtimes Id_K \in cl_l(E)$ and therefore $\mathbf{L}F^{rad}(f \boxtimes Id_K) \in cl_l(E')$ by Theorem 4.36. Therefore this map is bijective.

Proposition 4.39 *Under the assumptions of Theorem 4.36 assume in addition that F is surjective on the isomorphism classes of objects. Then one has*

$$cl_l(E') = F_*^{-1}(cl_l(E)).$$

Proof: Note first that our assumption that F is surjective on isomorphism classes of objects implies that $W_{proj} = F_*^{-1}(W_{proj})$. Theorem 4.36 implies the inclusion " $\subset$ ". Let $f : X \to Y$ be such that $F_*(f) \in cl_l(E)$. We need to show that $f \in cl_l(E')$. By Proposition 4.20 we get a commutative diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] & Cof(X) \arrow[l] \arrow[r] \arrow[d, "g"'] & Ex(Cof(X)) \arrow[d, "h"] \\
Y & Cof(Y) \arrow[l] \arrow[r] & Ex(Cof(Y))
\end{tikzcd}$$

where the arrows going to the right are in $cl_l(E)$ and the arrows going to the left are projective equivalences. Since $F_*(f) \in cl_l(E)$, Theorem 4.36 that $h \in cl_l(E)$. By Lemma 4.38, $F_*(h)$ is an E-local equivalence between E-local objects and therefore a projective equivalence. Since F_* reflects projective equivalences we conclude that h is a projective equivalence and therefore $f \in cl_l(E)$.

5 The pointed case

In this section we assume that C has finite coproducts and is pointed i.e. its initial object is also a final object. We will denote this object by pt.

For a pointed simplicial set A and $X \in \Delta^{op}C$ denote by $X \otimes A$ the object defined by the elementary push-out square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X \boxtimes A \arrow[d] \\
pt \arrow[r] & X \otimes A
\end{tikzcd}$$

where the upper arrow is defined by the distinguished point of A.

For a morphism $f : X \to X'$ define $cone(f)$ by the elementary push-out square

$$\begin{tikzcd}
X \amalg X \arrow[r] \arrow[d] & X \boxtimes \Delta^1 \arrow[d] \\
pt \amalg X \arrow[r] & cone(f)
\end{tikzcd}$$

Note that it is the K_Q of the corner

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X' \\
pt &
\end{tikzcd}$$

Set $cone(X) = cone(X \to pt)$. One can easily see that $cone(X) = X \otimes \Delta^1$. Set further $\Sigma^1 X = X \otimes S^1$.

Define term-wise coprojection sequences in $\Delta^{op}C^{\#}$ as pairs of morphisms

$$A \to B \to C$$

such that the first one is a term-wise coprojection and the square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
pt \arrow[r] & C
\end{tikzcd}$$

is cocartesian. Clearly, any term-wise coprojection $X \to Y$ extends canonically to a coprojection sequence $X \to Y \to Z$ where $Z = X/Y$. For any $f : X \to X'$ there is a coprojection sequence

$$X \to cyl(f) \to cone(f)$$

which extends in the obvious way to a sequence

$$X \to cyl(f) \to cone(f) \to \Sigma^1 X$$

Lemma 5.1 *For any coprojection sequence $X \xrightarrow{f} Y \to Z$ there is a commutative diagram*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & cyl(f) \arrow[r] \arrow[d, "1"'] & cone(f) \arrow[d, "2"'] \\
X \arrow[r] & Y \arrow[r] & Z
\end{tikzcd}$$

where the arrows (1) and (2) are in $cl_{\Delta}(\emptyset)$.

Proof: Our diagram is a part of the morphism of elementary push-out squares $Q_1 \to Q_2$ where

$$Q_1 = \left(\begin{tikzcd} X \arrow[r] \arrow[d] & Cyl(f) \arrow[d] \\ pt \arrow[r] & cone(f) \end{tikzcd}\right) \qquad Q_2 = \left(\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ pt \arrow[r] & Z \end{tikzcd}\right)$$

The morphism $Cyl(f) \to Y$ is in $cl_{\Delta,\amalg_{<\infty}}(\emptyset)$ by Lemma 2.8 and therefore the morphism $cone(f) \to Z$ is in $cl_{\Delta,\amalg_{<\infty}}(\emptyset)$ by Lemma 2.9, 2.10 and the 2-out-of-3 property of Δ-closure.

Lemma 5.2 *For any coprojection sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ one has:*

1. $g \in cl_{\Delta,\amalg_{<\infty}}(\{X \to pt\})$
2. $(pt \to Z) \in cl_{\Delta,\amalg_{<\infty}}(\{f\})$.

Proof: This is a particular case of Lemma 2.11

Definition 5.3 *A sequence of the form*

$$X \to Y \to Z \to \Sigma^1 X$$

in $H(C)$ is called a cofibration sequence if it is isomorphic to the image of a sequence of the form

$$X' \xrightarrow{i_f} cyl(f) \xrightarrow{j_f} cone(f) \xrightarrow{h_f} \Sigma^1 X'$$

for a morphism $f : X' \to Y'$ in $\Delta^{op}[C]$.

Lemma 5.4 *Any morphism $X \to Y$ in $H(C)$ can be extended to a cofibration sequence.*

Proof: It is sufficient to show that any morphism $g : X \to Y$ in $H(C)$ is isomorphic to the image of a morphism of the form $i_f : X' \to cyl(f)$ for $f : X' \to Y'$ in $\Delta^{op}[C]$. Since $cyl(f) \to Y'$ is a projective equivalence it is enough to show that g is isomorphic to the image of $f : X' \to Y'$ in $\Delta^{op}[C]$. This follows from the fact that the fibrant-cofibrant replacement $Ex(Cof(X))$ of any $X \in \Delta^{op}Rad(C)$ lies in $\Delta^{op}[C]$.

Lemma 5.5 *Any coprojection sequence $X \to Y \to Z$ in $\Delta^{op}[C]$ extends in $H(C)$ to a cofibration sequence*

$$X \to Y \to Z \to \Sigma^1 X$$

Proof: It follows from Lemma 5.1 and Proposition 3.25.

Let I_2 be the union of two copies of Δ^1 such that ∂_0 of the first copy is identified with ∂_1 of the second. Define $\Sigma^1_2 X$ by the elementary push-out square

$$\begin{tikzcd}
X \amalg X \arrow[r] \arrow[d] & X \boxtimes I_2 \arrow[d] \\
pt \amalg pt \arrow[r] & \Sigma^1_2 X
\end{tikzcd}$$

where the upper arrow corresponds to the two boundary points of I_2. We have two projections

$$p_1 : \Sigma^1_2 X \to \Sigma^1 X$$

$$p_2 : \Sigma^1_2 X \to \Sigma^1 X$$

given by the contractions of the first and the second copy of Δ^1 in I_2 respectively.

Lemma 5.6 *The morphisms p_1, p_2 are projective equivalences.*

Let $m : \Sigma^1 X \to \Sigma^1 X$ be the morphism in $H(C)$ given by $p_2 p_1^{-1}$.

Proposition 5.7 *Let $X \xrightarrow{p} Y \xrightarrow{q} Z \xrightarrow{r} \Sigma^1 X$ be a cofibration sequence in $H(C)$. Then $Y \xrightarrow{q} Z \xrightarrow{r} \Sigma^1 X \xrightarrow{m\Sigma^1 p} \Sigma^1 Y$ is a cofibration sequence.*

6 The additive case

In this section we consider the case of an additive category A and connect our constructions to the standard homological algebra.

Lemma 6.1 *Let A be an additive category. Then $Rad(A)$ is equivalent to the category of additive functors from A to the category of abelian groups.*

Proof: For the duration of the proof let $Add(A)$ be the category of additive contravariant functors from A to Ab. Any such functor considered as a functor to sets is radditive. Therefore we get

$$Add(A) \to Rad(A) \tag{23}$$

Let us show that it is an equivalence. A natural transformation between two additive functors is automatically additive since the addition on $F(X)$ is given by the morphism

$$F(X) \times F(X) = F(X \oplus X) \to F(X)$$

defined by the diagonal $X \to X \oplus X$. Therefore (23) is a full embedding. The same reasoning shows that any radditive functor is in the image of (23) i.e. it is an equivalence.

Lemma 6.2 *An additive category is grainy.*

Proof: By Lemma 6.1 the category $Rad(A)$ is additive i.e. finite coproducts of radditive functors coincide with finite products. Since the class of projective equivalences is closed under products we conclude that A is grainy.

Lemma 6.3 *Let $I \to Rad(A)$ be a diagram of radditive functors and $X \in A$ then*

$$colim(I)(X) = colim_{Ab}(I(X))$$

where the colimit on the right is in the category of abelian groups.

Proof: It follows from Lemma 6.1 since a similar result holds for additive functors from A to Ab.

Lemma 6.3 implies in particular that all the standard simplicial constructions on $Rad(A)$ commute with the global section functors provided that we consider these functors as taking values in the abelian groups and not sets.

In view of Lemma 6.1 the category $Rad(A)$ is abelian. Let

$$N : \Delta^{op} Rad(A) \to Compl_{-}(Rad(A))$$

be the normalization functor from simplicial objects to complexes[6].

Lemma 6.4 *A morphism f in $\Delta^{op} Rad(A)$ is a projective equivalence iff $N(f)$ is a quasi-isomorphism.*

Proof: Lemma 6.3 implies that for any $F \in \Delta^{op} Rad(A)$ and any $X \in A$ one has

$$N(F)(X) = N(F(X)).$$

The statement of the lemma follows now from the fact that a morphism of simplicial abelian groups is a weak equivalence iff its normalization is a quasi-isomorphism.

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[6] Here $Compl_{-}$ is the category of complexes which become zero after a finite number of terms in the direction of the differential.

Let

$$K : Compl_{-}(Rad(A)) \to \Delta^{op} Rad(A)$$

be the right adjoint to N which is defined for any abelian category and let D_{-} be the derived category of $Compl_{-}$. Combining Lemma 6.4 with the standard results about simplicial objects in abelian categories we get.

Proposition 6.5 *The functor N takes projective equivalences to quasi-isomorphisms and the resulting functor*

$$H(A) \to D_{-}(Rad(A)) \tag{24}$$

is a full embedding. Similarly K takes quasi-isomorphisms to projective equivalences and the resulting functor

$$D_{-}(Rad(A)) \to H(A)$$

is the localization right adjoint to (24).

Lemma 6.6 *For any $X \in \Delta^{op} Rad(A)$ there is a natural isomorphism*

$$N(\Sigma^1 X) = N(X)[1]$$

With respect to these isomorphisms the functor N takes cofibration sequences in $H(A)$ to distinguished triangles in $D_{-}(Rad(A))$.

Lemma 6.7 *Let $X \to Y \to Z \to X[1]$ be a distinguished triangle in $D_{-}(Rad(A))$ such that $X, Y, Z \in N(H(A))$. Then*

$$K(X) \to K(Y) \to K(Z) \to \Sigma^1 K(X)$$

is a coprojection sequence in $H(A)$.

Proof: One can always choose representatives for X, Y, Z which are in $Compl_{-}([A])$. Since objects of $[A]$ are projective in $Rad(A)$ any morphism $X \to Y$ in D_{-} is the image of a morphism in $Compl_{-}$ and $K(X), K(Y), K(Z) \in \Delta^{op}[A]$. It remains to extend $K(X) \to K(Y)$ to a cofibration sequence and use the fact that N is a full embedding and any morphism in a triangulated category has a unique, up to an isomorphism, extension to a distinguished triangle.

Remark 6.8 Note that the statement of Lemma 6.7 is false without the assumption that $X, Y, Z \in N(H(A))$.

Let E be a class of morphisms in $\Delta^{op}Rad(A)$. Let further L_0 be the class of objects Z in D_- such that there exists a distinguished triangle

$$X \xrightarrow{f} Y \to Z \to X[1]$$

with $f \in N(cl_{\bar{\Delta}}(E))$ and let L be the class of objects Z such that there exists $n \in \mathbf{Z}$ with $Z[n] \in L_0$.

Lemma 6.9 *For any E as above, L is a thick subcategory in D_-.*

Proof: Let us show first that $Z \in L$ iff there exists n such that $(Z[n] \to 0) \in N(cl_{\bar{\Delta}}(E))$. Indeed if $(Z[n] \to 0) \in N(cl_{\bar{\Delta}}(E))$ then $Z[n+1]$ is the cone of this morphism and therefore $Z \in L$. Suppose that Z fits into a distinguished triangle

$$X \xrightarrow{f} Y \to Z \to X[1] \tag{25}$$

Suspending everything we may assume that all the objects are in the image of N and therefore by Lemma 6.7 our triangle is an image of a cofibration sequence and therefore we may assume that there is a coprojection sequence in $\Delta^{op}[A]$ of the form

$$X' \xrightarrow{f'} Y' \to Z'$$

whose image in D_- gives, up to an isomorphism, the first three terms of (25). Therefore $Z' \to 0$ is in $cl_{\bar{\Delta}}(E)$ by Lemma 5.2(2).

To show that L is a triangulated subcategory it remains to show that if $(X \to 0), (Y \to 0) \in N(cl_{\bar{\Delta}}(E))$ and we are given a triangle of the form (25) then $(Z \to 0) \in N(cl_{\bar{\Delta}}(E))$. This follows by the same argument from Lemma 5.2(1).

Finally we need to check that if $Z = Z' \oplus Z'' \in L$ then $Z' \in L$. It follows from the fact that there is a distinguished triangle

$$\oplus_{n\geq 0} Z \to \oplus_{n\geq 0} Z \to Z' \to \oplus_{n\geq 0} Z[1]$$

and that for $Z \in L$ one has $\oplus_{n\geq 0} Z \in L$.

Remark 6.10 It is not clear in general that L is a localizing subcategory i.e. that it is closed under infinite direct sums.

For a class of morphisms E in $Compl_-$ let $cl_{vl}(E)$ be the class of morphisms whose cones belong to the localizing subcategory in D_- generated by the cones of elements of E.

Lemma 6.11 *One has:*

1. *$cl_{uvl}(E)$ has the 2-out-of-3 property*
2. *$cl_{uvl}(E)$ is closed under all direct sums*
3. *for a commutative diagram in D_-*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, "h"'] & X[1] \arrow[d, "f[1]"'] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

such that the rows are such that the rows are distinguished triangles one and $f, g \in cl_{uvl}(E)$ one has $h \in cl_{uvl}(E)$.

Proof: By the main theorem of the localization theory for triangulated categories the localization $D_- \to D_-[cl_{vl}(E)^{-1}]$ is a triangulated functor between triangulated categories and a morphism in D_- goes to an isomorphism iff it belongs to $cl_{vl}(E)$. In addition this projection commutes with direct sums. This implies the claims of the lemma.

Theorem 6.12 *For any class of morphisms E in $\Delta^{op}Rad(A)$ one has*

$$N(cl_{\bar{\Delta}}(E)) \subset cl_{vl}(N(E)).$$

Proof: We need to show that $L = N^{-1}cl_{uvl}(N(E))$ is $\bar{\Delta}$-closed. We do it using the characterization of $\bar{\Delta}$-closed classes given in Proposition 2.24. The class L has the the 2-out-of-3 property and is closed under coproducts by Lemma 6.11(1,2). By Lemma 6.4 it contains W_{proj} and therefore by Proposition 3.25 it contains $cl_{\bar{\Delta}}(\emptyset)$. It remains to check that it is closed under the diagonals. Let $f : B \to B'$ be a morphism of bi-simplicial objects whose rows $f_n : B_n \to B'_n$ are in L. By Lemma 6.2 the category A is grainy. Using the same reasoning as in the proof of Proposition 4.31 we reduce the problem to showing that

1. if $f : X \to X'$ is in L then $f \boxtimes \partial\Delta^n$ is in L,
2. given a morphism of elementary exact squares

$$\left(\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[d] \\ A \arrow[r] & X \end{tikzcd}\right) \to \left(\begin{tikzcd} Z' \arrow[r] \arrow[d] & Y' \arrow[d] \\ A' \arrow[r] & X' \end{tikzcd}\right)$$

such that $A \to A'$, $Z \to Z'$ and $Y \to Y'$ are in L one has $(X \to X') \in L$.

The first claim follows from the fact that for any $X \in \Delta^{op}Rad(A)$ there is a natural quasi-isomorphism

$$N(X \boxtimes \partial\Delta^n) = X[n] \oplus X$$

The second form the fact that for an elementary exact square

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & Y \arrow[d] \\
A \arrow[r] & X
\end{tikzcd}$$

the sequence $0 \to Z \to A \oplus Y \to X \to 0$ is exact and therefore extends to a distinguished triangle and from Lemma 6.11(3).

Corollary 6.13 *For any admissible class E the normalization functor defines a functor*

$$H(A, E) \to D_{-}(Rad(A))[cl_{vl}(N(E))^{-1}]$$

Proposition 6.14 *Let E be an admissible class in $\Delta^{op}[A]$ such that for any E-local X the suspension $\Sigma^1 X$ is E-local. Then the functor*

$$\Phi : H(A, E) \to D_{-}(Rad(A))[cl_{vl}(N(E))^{-1}]$$

is a full embedding.

Proof: To prove the proposition we will construct a right adjoint Ψ to Φ and show that the adjunction $Id \to \Psi\Phi$ is an isomorphism. We start with two lemmas.

Lemma 6.15 *Under the assumptions of the proposition one has*

$$L = cone(cl_{vl}(N(E)))$$

where L is the class considered in Lemma 6.9 based on $E \cup W_{proj}$ and $cone(-)$ is the class of cones of elements of $(-)$.

Proof: In view of Lemma 6.9 it is sufficient to show that under the assumptions of the proposition the class L is closed under direct sums in D_{-}. Let $(X_i)_{i \in I}$ be a family of objects in L. Note that the direct sum $\oplus L_i$ exists in D_{-} iff there exists n such that for all $i \in I$, $X_i[n]$ lies in the image of N. Shifting by this n we may suppose that all X_i are in the image of N. By Proposition 6.5 and Lemma 6.6, Σ^1 is a full embedding on $H(A)$. Therefore, our condition implies that if $\Sigma^n X \to 0$ is in $cl_l(E)$ for some n then $X \to 0$ is in $cl_l(E)$. Identifying $cl_l(E)$ with $cl_{\bar{\Delta}}(E \cup W_{proj})$ we conclude that $\oplus X_i$ is in L.

Lemma 6.16 *Under the assumptions of the proposition the projection*

$$\pi : D_{-}(Rad(A)) \to D_{-}(Rad(A))[cl_{vl}(N(E))^{-1}]$$

has a right adjoint ρ.

Proof: It is sufficient to show that for any $X \in D_-(Rad(A))$ there exists a morphism $Y \to Lc(Y)$ in $cl_{vl}(N(E))$ such that $Lc(Y)$ is right orthogonal to elements of $cl_{vl}(N(E))$. For X in $H(A)$ let $X \to L_E(X)$ be an E-local replacement of X. For any $Y \in D_-$ there exist $X \in H(A)$ and $n \geq 0$ such that $Y[n] = N(X)$. Set $Lc(Y) = N(L_E(X))[-n]$. We have a morphism $Y \to Lc(Y)$ which is in $cl_{vl}(N(E))$ by Theorem 6.12. It remains to verify that $Lc(Y)$ is right orthogonal to $cl_{vl}(N(E))$. It follows immediately from our assumption and Lemma 6.15.

To finish the proof of the proposition it remains to note that $\Psi = K\rho$ is the right adjoint to Φ as the composition of two right adjoints and that the construction of ρ implies that the adjunction $Id \to \Psi\Phi$ is an isomorphism.

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