Motivic Eilenberg-MacLane spaces

Vladimir Voevodsky[1,2]

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Contents

1 Introduction

This paper is the second one in a series of papers about operations in motivic cohomology. Here we show that in the context of smooth schemes over a field of characteristic zero all the bi-stable operations can be obtained in the usual way from the motivic reduced powers and the Bockstein homomorphism (Theorem 4.46). This paper has been delayed for many years because I tried to find a way to extend this result to all characteristics. Unfortunately, I am still unable to do it for reasons which will become clear by the end of this introduction.

Fix a perfect field k. A full subcategory C of Sch/k such that

[1]School of Mathematics, Institute for Advanced Study, Princeton NJ, USA. e-mail: vladimir@ias.edu
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1. $Spec(k)$ and $\mathbf{A}^1$ are in C
2. for X and Y in C the product $X \times Y$ is in C
3. if X is in C and $U \to X$ is etale then U is in C
4. for X and Y in C the coproduct $X \amalg Y$ is in C

will be called admissible. If in addition C is closed under the formation of quotients with respect to actions of finite groups it will be called f-admissible.

We will always consider C as a site with respect to the Nisnevich topology. The category Sm/k of smooth schemes over k is essentially the smallest admissible C since for any smooth X and any admissible C there exists a covering $\{U_i \to X\}$ with $U_i \in C$.

To any admissible C one can associate two homotopy category. The category $H_{\mathbf{A}^1}(C_+)$ which is equivalent to the pointed homotopy category of the site with interval $(C_{Nis}, \mathbf{A}^1)$ (see [6]) and the category $H_{\mathbf{A}^1}(Cor(C, R))$. The later category depends as well on a commutative ring of coefficients R and is called the $\mathbf{A}^1$-homotopy category of finite correspondences with coefficients in R over C. If $C = Sm/k$ then $H_{\mathbf{A}^1}(Cor(C))$ is the full subcategory of the triangulated category of motives introduced in [18] which consists of objects with no positive cohomology relative to the homotopy t-structure.

There is a pair of adjoint functors

$$\Lambda^r_R : H_{\mathbf{A}^1}(C_+) \to H_{\mathbf{A}^1}(Cor(C, R))$$

$$\mathbf{L}\Lambda^l_R : H_{\mathbf{A}^1}(Cor(C, R)) \to H_{\mathbf{A}^1}(C_+)$$

with Λ^r being the right adjoint. In the context of the topological analogy $\mathbf{L}\Lambda^l$ is the functor which takes a (pointed) homotopy type to its (reduced) homology complex and Λ^r is the adjoint Eilenberg-MacLane functor.

Let $K(A, p, q)_C$ be the object of $H_{\mathbf{A}^1}(C_+)$ which represents the motivic cohomology functor $H^{p,q}(-, A)$ on this category. Its motivic cohomology groups are responsible for the operations in motivic cohomology considered as functors on $H_{\mathbf{A}^1}(C_+)$.

Set $c(k) = 1$ if $char(k) =)$ and $c(k) = char(k)$ if $char(k) > 0$. Our first main theorem asserts the following.

Theorem 1.1 *Let A be a finitely generated abelian group, F a field where $c(k)$ is invertible and C an f-admissible category which is contained in the category SN of semi-normal schemes over k. Then one has:*

1. *For $p \geq q$ the object $\mathbf{L}\Lambda^l_F(K(A, p, q)_C)$ is a mixed Tate object.*

2. For $p \geq 2q$ the object $\mathbf{L}\Lambda^l_F(K(A,p,q)_C)$ is a pure Tate object.

This theorem should be sufficient in principle to show that all the bi-stable operations in the context of an f-admissible category are obtainable from the reduced powers and the Bockstein homomorphism. Unfortunately, the reduced powers have been introduced in [20] only in the context of smooth schemes. It should not be difficult to extend the constructions and results of [20] to semi-normal (or at least normal) schemes but so far it has not been done.

Given an embedding $i : C \to D$ we get two pairs of adjoint functors

$$i_* : H_{\mathbf{A}^1}(D_+) \to H_{\mathbf{A}^1}(C_+)$$

$$\mathbf{L}i^* : H_{\mathbf{A}^1}(C_+) \to H_{\mathbf{A}^1}(D_+)$$

and

$$i_* : H_{\mathbf{A}^1}(Cor(D)) \to H_{\mathbf{A}^1}(Cor(C))$$

$$\mathbf{L}i^* : H_{\mathbf{A}^1}(Cor(C)) \to H_{\mathbf{A}^1}(Cor(D)).$$

For more or less obvious reasons i_* are localizations which commute with Λ^r and $\mathbf{L}i^*$ are full embeddings which commute with $\mathbf{L}\Lambda^l$.

Theorem 1.2 *Under the resolution of singularities assumption i_* commutes with $\mathbf{L}\Lambda^l$.*

Combining Theorems 1.1 and 1.2 one easily gets the following result.

Theorem 1.3 *Let k be a field of characteristic zero, F any field, A a finitely generated abelian group and $p \geq 2q$. Then $\mathbf{L}\Lambda^l_F(K(A,p,q)_C)$ is a pure Tate object for any admissible C which is contained in SN. In addition this object does not depend on C, i.e. for an inclusion $i : C \to D$ one has*

$$\mathbf{L}i^*\mathbf{L}\Lambda^l_F(K(A,p,q)_C) = \mathbf{L}\Lambda^l_F(K(A,p,q)_D).$$

Combining Theorem 1.3 with results and constructions of [20] we prove in the last section the following result.

Theorem 1.4 *Let k be a field of characteristic zero. The one has*

$$colim_n H^{*+2n,*+n}(K(\mathbf{Z}/l,2n,n),\mathbf{Z}/l) = \mathcal{A}^{*,*}(k,\mathbf{Z}/l)$$

where $\mathcal{A}^{*,*}(k,\mathbf{Z}/l)$ is the motivic Steenrod algebra defined in [20].

If we could prove Theorem 1.2 in positive characteristic we could avoid the characteristic zero assumption in Theorems 1.3 and 1.4. In fact, it would be sufficient to prove that for a smooth scheme X with an action of a finite group G and a subgroup $H \subset G$ the map

$$\mathbf{L}\Lambda^l_R i_*(X/H) \to \mathbf{L}\Lambda^l_R i_*(X/G)$$

where $i : Sm/k \to SN/k$ and $[G : H]^{-1} \in R$, is a split epimorphism.

Let me add that we know almost nothing about objects of the form $\mathbf{L}\Lambda^l_F K(A, p, q)$ for $p < q$ and $q > 1$ even when $F = \mathbf{Q}$ and $A = \mathbf{Z}$. Their structure is related to the hard motivic conjectures inspired by the vision of an abelian category of motives with the weights filtration. For a related discussion see [14].

It took me ten years to write this paper and it numerous people provided useful comments on its intermediate versions. I would especially like to thank Chuck Weibel who has helped me a lot as a source of both expertise and willpower.

2 Motivic homology and homotopy

1 Main categories and functors

Let k be a field and C be an admissible subcategory in Sch/k. Since C contains $Spec(k)$ we may consider pointed objects in C. Let C_+ be the full subcategory in the category of pointed objects which consists of objects pointed by a disjoint point. We have the usual functor $C \to C_+$ which takes X to $X_+ = X \coprod Spec(k)$. As always we write $\vee$ for the coproduct in the pointed case and $\amalg$ for the coproduct in the free case. Note that $Rad(C_+)$ is equivalent to the category of all contravariant functors from the full subcategory of C_+ which consists of objects of the form X_+ where X is connected.

We get a sequence of categories and functors:

$$\Delta^{op}C \to \Delta^{op}C_+ \to \Delta^{op}C_+^{\#} \to \Delta^{op}Rad(C_+) \tag{1}$$

where $C_+^{\#}$ is the full subcategory of $Rad(C_+)$ consisting of filtering colimits of representable functors.

Consider now the category $Cor(Sch/k, R)$ of finite correspondences on Sch/k with coefficients in a commutative ring R. This category was described in detail in [21]. For C as above we let $Cor(C, R)$ denote the full subcategory in $Cor(Sch/k, R)$ which consists of objects lying in C. To distinguish objects of C from the corresponding objects of $Cor(C, R)$ we let

$$[-]_R : C \to Cor(C, R)$$

denote the functor which is the identity on objects and which takes morphisms to their graphs. When the ring of coefficients is not important or is clear from the context we will omit it from our notation.

The category $Cor(C)$ is additive and in particular has finite coproducts. Hence all the standard constructions of [26] apply again. In particular we get full embeddings of the form

$$\Delta^{op}Cor(C) \to \Delta^{op}Cor(C)^{\#} \to \Delta^{op}Rad(Cor(C)) \tag{2}$$

where the extended category of correspondences $Cor^{\#}$ is the closure of Cor with respect to infinite direct sums in $Rad(Cor)$. Since Cor is a pointed category the functor $[-] : C \to Cor(C)$ extends to a functor $C_+ \to Cor(C)$ which we denote by Λ. By definition we have $\Lambda(X_+) = [X]$.

This functor commutes with finite coproducts and therefore defines a pair of adjoint functors

$$\Lambda^l = \Lambda^{rad} : Rad(C_+) \to Rad(Cor(C))$$

$$\Lambda^r = \Lambda_* : Rad(Cor(C)) \to Rad(C_+)$$

where Λ^r is the right adjoint. Note that if we interpret $Rad(Cor(C))$ as the category of presheaves with transfers and $Rad(C_+)$ as the category of pointed radditive presheaves then Λ^r is just the "forgetting" functor which takes a presheaf with transfers to the same presheaf considered as a presheaf of pointed sets. We get a commutative diagram

$$\begin{array}{ccccccc}
\Delta^{op}C & \longrightarrow & \Delta^{op}C_+ & \longrightarrow & \Delta^{op}C_+^{\#} & \longrightarrow & \Delta^{op}Rad(C_+) \\
{\scriptstyle [-]}\downarrow & & {\scriptstyle \Lambda}\downarrow & & {\scriptstyle \Lambda^{\#}}\downarrow & & {\scriptstyle \Lambda^{l}}\downarrow \\
\Delta^{op}Cor(C) & = & \Delta^{op}Cor(C) & \longrightarrow & \Delta^{op}Cor(C)^{\#} & \longrightarrow & \Delta^{op}Rad(Cor(C))
\end{array}$$

connecting (1) with (2).

The following two classes of morphisms play an important role in all our constructions.

1. The class W_0 of elementary Nis-equivalences in $\Delta^{op}C$. Recall that a commutative square

$$\begin{array}{ccc}
B & \longrightarrow & Y \\
\downarrow & & \downarrow{\scriptstyle p} \\
A & \xrightarrow{j} & X
\end{array} \tag{3}$$

in C is called (upper) distinguished if p is etale, j is an open embedding, $B = p^{-1}(A)$ and $p : Y - B \to X_A$ is an isomorphism. Let Q be such a square, K_Q be the homotopy push-out of the corner

$$\begin{array}{ccc}
B & \longrightarrow & Y \\
\downarrow & & \\
A & &
\end{array}$$

in $\Delta^{op}C$ and $K_Q \to X$ the canonical morphism (see [19, 3.5.1, p.34] or [26, Section 2.1]). The class W_0 is the class of morphisms of this form for all upper distinguished squares in C.

2. The class W_1 of elementary $\mathbf{A}^1$-equivalences in C is the class of projections $X \times \mathbf{A}^1 \to X$ for all X in C.

Lemma 2.1 *One has:*

1. *The classes $(W_0)_+$ and $(W_1)_+$ are admissible i.e. the domains and codomains of the members of these classes are projectively cofibrant (see [26])*

2. *The classes $[W_0]$ and $[W_1]$ are admissible.*

Proof: We will only consider the case of $R(C_+)$. The case of $Cor(R)$ is similar. Objects of C_+ are obviously cofibrant in $\Delta^{op}R(C_+)$, therefore sources and targets of morphisms from $(W_1)_+$ as well as targets of morphisms from $(W_0)_+$ are cofibrant. Let us show that for a square of the form (3) the homotopy push-out $(K_Q)_+$ is cofibrant. By definition, $(K_Q)_+$ is given by the elementary push-out square

$$\begin{array}{ccc} B_+ \otimes \partial\Delta^1 & \longrightarrow & B_+ \otimes \Delta^1 \\ \downarrow & & \downarrow \\ (A \coprod Y)_+ & \longrightarrow & (K_Q)_+ \end{array}$$

Such squares remain push-out squares in $\Delta^{op}R(C_+)$. On the other hand by [26, Proposition 3.32] for any X from C_+ and any monomorphism of simplicial sets $K \to L$ the corresponding morphism $X \otimes K \to X \otimes L$ is a projective cofibration in $\Delta^{op}R(C_+)$. Since cofibrations are stable under push-outs we conclude that $(K_Q)_+$ is cofibrant.

For the definition of E-local objects and E-local equivalences used below see [26, Section 4.2].

1. An object in $\Delta^{op}Rad(C_+)$ is called Nis-local if it is $(W_0)_+$-local. A morphism in $\Delta^{op}Rad(C_+)$ is called a Nis-equivalence if it is an $(W_0)_+$-local equivalence.

2. An object in $\Delta^{op}Rad(C_+)$ is called $\mathbf{A}^1$-local if it is E-local with $E = (W_0)_+ \cup (W_1)_+$. A morphism in $\Delta^{op}Rad(C_+)$ is called an $\mathbf{A}^1$-equivalence if it is an E-local equivalence with $E = (W_0)_+ \cup (W_1)_+$.

3. An object in $\Delta^{op}Rad(Cor(C))$ is called Nis-local if it is $[W_0]$-local. A morphism in $\Delta^{op}Rad(Cor(C))$ is called a Nis-equivalence if it is an $[W_0]$-local equivalence.

4. An object in $\Delta^{op}Rad(Cor(C))$ is called $\mathbf{A}^1$-local if it is E-local with $E=[W_0]\cup[W_1]$. A morphism in $\Delta^{op}Rad(Cor(C))$ is called an $\mathbf{A}^1$-equivalence if it is an E-local equivalence with $E=[W_0]\cup[W_1]$.

The classes of Nis- and $\mathbf{A}^1$-local objects in $\Delta^{op}R(C_+)$ and $\Delta^{op}Cor(C)$ can be described in a more explicit way as follows.

Proposition 2.2 *An object S of $\Delta^{op}R(C_+)$ (resp. of $\Delta^{op}R(Cor(C))$) is Nis-local if and only if it is projectively fibrant and for any upper distinguished square*

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

in C, the diagram of simplicial sets

$$\begin{array}{ccc} S(X) & \longrightarrow & S(A) \\ \downarrow & & \downarrow \\ S(Y) & \longrightarrow & S(B) \end{array} \qquad (4)$$

is a homotopy pull-back square.

Proof: It follows from [26, Lemma 4.9], Lemma 2.1 and the obvious observation that for a projectively fibrant S the square (4) is homotopy pull-back iff

$$S(X,S)\to S(K_Q,S)$$

is a weak equivalence.

Proposition 2.3 *An object S of $\Delta^{op}R(C_+)$ (resp. of $\Delta^{op}R(Cor(C))$) is $\mathbf{A}^1$-local if and only if it is Nis-local and for any X in C the morphism $S(X)\to S(X\times\mathbf{A}^1)$ is a weak equivalence of simplicial sets.*

Proof: It follows immediately from [26, Lemma 4.9] and Lemma 2.1.

Let us denote the classes of Nis-equivalences and $\mathbf{A}^1$-equivalences in $\Delta^{op}Rad(Cor(C))$ by W^{tr}_{Nis} and $W^{tr}_{\mathbf{A}^1}$ respectively and the classes of Nis-equivalences and $\mathbf{A}^1$-equivalences in $\Delta^{op}Rad(C_+)$ by W_{Nis} and $W_{\mathbf{A}^1}$ respectively.

Let $H_{\mathbf{A}^1}(C_+)$ be the localization of $\Delta^{op}Rad(C_+)$ with respect to $W_{\mathbf{A}^1}$ and $H_{\mathbf{A}^1}(Cor(C))$ be the localization of $\Delta^{op}Rad(Cor(C))$ with respect to $W^{tr}_{\mathbf{A}^1}$. In the notation of [26] we have

$$H_{\mathbf{A}^1}(C_+)=H(C_+,(W_0)_+\cup(W_1)_+)$$

$$H_{\mathbf{A}^1}(Cor(C)) = H(Cor(C), [W_0] \cup [W_1]).$$

We will also consider $H_{Nis}(C_+)$ and $H_{Nis}(Cor(C))$ defined in the same way using Nis-equivalences instead of $\mathbf{A}^1$-equivalences.

Remark 2.4 To get the topological analog of this picture one needs to replace $H_{\mathbf{A}^1}(C_+)$ by the usual pointed homotopy category and $H_{\mathbf{A}^1}(Cor(C))$ by the homotopy category of simplicial abelian groups or more generally simplicial R-modules.

Proposition 2.5 *The categories C_+ and $Cor(C)$ are grainy i.e. the classes W_{proj} in the corresponding categories of simplicial radditive functors are $\bar{\Delta}$-closed (see [26, Definition 2.15]).*

Proof: For any admissible C one has $C = D^{\amalg}$ where D is the category of connected schemes from C. Therefore the case of C_+ follows from [26, Example 3.58(3)]. The case of $Cor(C)$ follows from [26, Example 3.58(1)].

Corollary 2.6 *The classes $W_{Nis}, W_{Nis}^{tr}, W_{\mathbf{A}^1}$ and $W_{\mathbf{A}^1}^{tr}$ are $\bar{\Delta}$-closed.*

Proof: It follows from Proposition 2.5, Lemma 2.1 and [26, Proposition 4.31].

Proposition 2.7 *The classes W_{Nis} and $W_{\mathbf{A}^1}$ are closed under finite products and smash-products. The classes W_{Nis}^{tr} and $W_{\mathbf{A}^1}^{tr}$ are closed under tensor products.*

Proof: We will consider only the case of $W_{\mathbf{A}^1}$ and smash-products. The other cases are similar. Note first that since $Rad(C_+)$ is equivalent to the category of pointed presheaves on the full subcategory of connected objects in C, the class W_{proj} is closed under smash-products. Together with the fact that $W_{\mathbf{A}^1}$ is $\bar{\Delta}$-closed and closed under filtering colimits this implies easily that it is enough to show that for $X \in C_+$ and $f \in W_{\mathbf{A}^1}$ one has $F \wedge Id_X \in W_{\mathbf{A}^1}$. Using [26, Proposition 2.16] applied to the functor $(-) \wedge Id_X$ one sees that it is enough to show that for $f \in (W_0)_+ \cup (W_1)_+$ and $F \in Rad(C)$ one has

$$(f \amalg Id_F) \wedge Id_X \in W_{\mathbf{A}^1}$$

Using again the identification of $Rad(C_+)$ with the category of pointed presheaves on the full subcategory of connected objects in C we see that

$$(f \amalg Id_F) \wedge Id_X = (f \wedge Id_X) \amalg (Id_F \wedge Id_X)$$

Our results follows since for $f \in (W_0)_+$ we have $f \wedge Id_X \in (W_0)_+$ and for $f \in (W_1)_+$ we have $f \wedge Id_X \in (W_1)_+$.

Define the class W_1' of $\mathbf{A}^1$-homotopy equivalences in $\Delta^{op}Rad(C_+)$ as follows. Let $i_0, i_1 : S^0 \to (\mathbf{A}^1)_+$ be the morphisms corresponding to the points 0 and 1 of $\mathbf{A}^1$ respectively. An elementary $\mathbf{A}^1$-homotopy between morphisms $f, g : X \to Y$ in $\Delta^{op}Rad(C_+)$ is a morphism $h : X \wedge (\mathbf{A}^1)_+ \to Y$ such that $h \circ (Id \wedge i_0) = f$ and $h \circ (Id \wedge i_1) = g$. Two morphisms are called elementary $\mathbf{A}^1$-homotopic if there exists an elementary $\mathbf{A}^1$-homotopy between them. Two morphisms are called $\mathbf{A}^1$-homotopic if they are equivalent with respect to the equivalence relation generated by the relation of being elementary $\mathbf{A}^1$-homotopic. A morphism $f : X \to Y$ is called an $\mathbf{A}^1$-homotopy equivalence if there exists a morphism $g : Y \to X$ such that the compositions $f \circ g$ and $g \circ f$ are $\mathbf{A}^1$-homotopic to the corresponding identity morphisms.

Lemma 2.8 *One has* $(W_1)_+ \amalg Id_{Rad(C_+)} \subset W_1'$.

Lemma 2.9 *One has* $W_1' \subset W_{\mathbf{A}^1}$.

Proof: By [26, 2007satr] it is enough to show that elements of W_1' become isomorphisms after localization with respect to $W_{\mathbf{A}^1}$. For any X the morphisms $Id_X \wedge i_0$ and $Id_X \wedge i_1$ become equal in the localized category since they are both sections of the isomorphism $X \wedge (\mathbf{A}^1)_+ \to X$. Therefore, after the localization any two $\mathbf{A}^1$-homotopic morphisms become equal. We conclude that an $\mathbf{A}^1$-homotopy equivalence f becomes an isomorphism since there exists g such that both compositions $f \circ g$ and $g \circ f$ are equal to the corresponding identities.

Let W_{Nis}' be the class of morphisms in $\Delta^{op}Rad(C_+)$ which are simplicial equivalences in the Nisnevich topology as morphisms of simplicial presheaves. For a detailed definition see e.g. [19].

Lemma 2.10 *One has* $(W_0)_+ \subset W_{Nis}'$.

Proof: See [19, Th. 3.6.1].

Proposition 2.11 *One has* $W_{Nis} = W_{Nis}'$.

Proof: "$\subset$" By Lemma 2.10 we have $(W_0)_+ \subset W_{Nis}$ and $W_{proj} \subset W_{Nis}$ for obvious reasons. It remains to check that W_{Nis} is $\bar{\Delta}$-closed. By [19, Prop. 3.1.5] the site C_{Nis} has enough points. Therefore a morphism $f : X \to Y$ is a Nis-equivalence if and only if for any point p of C_{Nis} the map of simplicial sets $p^*(f) : p^*(X) \to p^*(Y)$ is a weak equivalence. Our claim follows now from the fact that the class of weak equivalences of simplicial sets is $\bar{\Delta}$-closed.

"$\supset$" Let $f : X \to Y$ be in W_{Nis}'. The morphisms $Lres(X) \to X$ and $Lres(Y) \to Y$ are in W_{proj} and therefore in W_{Nis}'. By the first part of the proof W_{Nis}' is $\bar{\Delta}$-closed and in particular has the 2-out-of-3 property. We conclude that $Lres(f)$ is in W'_{Nis} and it is sufficient to prove that $Lres(f)$ is in W_{Nis}. By [19, Lemma 3.6.5] we have $Lref(f) \in cl_{\bar{\Delta}}((W_0)_+)$ which proves our claim.

Corollary 2.12 *Let X be an object of $\Delta^{op}Rad(C_+)$ and $a_{Nis}X$ the associated Nisnevich sheaf. Then the canonical morphism $X \to a_{Nis}X$ is in W_{Nis}.*

Lemma 2.13 *One has:*

1. $\Lambda^r([W_0]) \subset W_{Nis}$
2. $\Lambda^r([W_1]) \subset W'_1$

Proof: For the first inclusion we need to show that for any upper distinguished square Q of the form (3) the morphism $[K_Q] \to [X]$ is a local equivalence on C_{Nis} as a morphism of presheaves of sets or, equivalently, as a morphism of presheaves of abelian groups. It is further equivalent to the condition that the morphism of associated sheaves of abelian groups is a local equivalence on C_{Nis}. Consider the functor

$$\gamma : Rad(Cor(C)) \to ShvAb(C_{Nis})$$

which is the composition of the forgetting functor from $Rad(Cor(C))$ to presheaves of abelian groups on C with the associated sheaf functor. Clearly, γ respects finite coproducts and therefore it commutes with the K_Q construction. Hence the morphism we are interested in can be written as

$$\gamma([K_Q] \to [X]) = (K_{\gamma([Q])} \to \gamma([X]))$$

Consider a square S of pre-sheaves of abelian groups on C_{Nis} of the form

$$\begin{array}{ccc} S1 & \longrightarrow & S2 \\ \downarrow & & \downarrow \\ S3 & \longrightarrow & S4. \end{array}$$

Using the fact that C_{Nis} has enough points one verifies easily that the associated morphism $K_S \to S4$ is a local equivalence if and only if the sequence

$$0 \to a_{Nis}S1 \to a_{Nis}S2 \oplus a_{Nis}S3 \to a_{Nis}S4 \to 0$$

of associated Nisnevich sheaves is exact. Therefore the first assertion of the lemma would follow if we knew that for an upper distinguished square of the form (3) the sequence of sheaves

$$0 \to \gamma([B]) \to \gamma([A]) \oplus \gamma([Y]) \to \gamma([X]) \to 0$$

is exact. This is equivalent to [12, Proposition 4.3.9].

For the second inclusion need to show that for $X \in C$ the map of presheaves of sets $p : [X \times \mathbf{A}^1] \to [X]$ is an $\mathbf{A}^1$-homotopy equivalence. The inverse equivalence is given by $i : [X] \to [X \times \mathbf{A}^1]$ corresponding to the point 0 of $\mathbf{A}^1$. It remains to construct an $\mathbf{A}^1$-homotopy

$$[X \times \mathbf{A}^1] \times \mathbf{A}^1 \to [X \times \mathbf{A}^1]$$

between the identity and $p \circ i$. This is provided by the composition

$$[X \times \mathbf{A}^1] \times \mathbf{A}^1 \to [X \times \mathbf{A}^1 \times \mathbf{A}^1] \to [X \times \mathbf{A}^1]$$

where the first map is a particular case of a general map of the form

$$[X] \times Y \to [X \times Y]$$

and the second one is obtained from the multiplication map $\mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$.

Theorem 2.14 *The functor Λ satisfies the conditions of [26, Theorem 4.36] with respect to the pairs of classes*

$$(E, E') = ((W_0)_+, [W_0])$$

and

$$(E, E') = ((W_0)_+ \cup (W_1)_+, [W_0] \cup [W_1]).$$

Therefore the functors $\mathbf{L}\Lambda^l$ and Λ^r define adjoint pairs of functors between the corresponding Nis- and $\mathbf{A}^1$- homotopy categories:

$$\mathbf{L}\Lambda^l : H_{Nis}(C_+) \to H_{Nis}(Cor(C)) \quad \Lambda^r : H_{Nis}(Cor(C)) \to H_{Nis}(C_+)$$

and

$$\mathbf{L}\Lambda^l : H_{\mathbf{A}^1}(C_+) \to H_{\mathbf{A}^1}(Cor(C)) \quad \Lambda^r : H_{\mathbf{A}^1}(Cor(C)) \to H_{\mathbf{A}^1}(C_+)$$

Proof: The categories in question are grainy by Proposition 2.5 and the classes are admissible by Lemma 2.1. The first condition of [26, Theorem 4.36] is obviously satisfied. It remains to verify the second condition. Let us consider only the $\mathbf{A}^1$-case where we need to show that

$$\Lambda^r([W_0] \oplus Id_{Rad(Cor(C))}) \subset W^{tr}_{\mathbf{A}^1}$$

and

$$\Lambda^r([W_1] \oplus Id_{Rad(Cor(C))}) \subset W^{tr}_{\mathbf{A}^1}.$$

One can easily see that Λ^r takes finite coproducts (direct sums) to products. The first inclusion follows from Lemma 2.13(1) and Proposition 2.7. The second from Lemma 2.13(2) and again Proposition 2.7.

Corollary 2.15 *The functor Λ^l takes projective, Nis- and $\mathbf{A}^1$-equivalences in $\Delta^{op}C_+^{\#}$ to the corresponding equivalences in $\Delta^{op}Cor(C)^{\#}$.*

Remark 2.16 The functor Λ^l does not preserve projective equivalences between all objects of $\Delta^{op}Rad(C_+)$. We will give an example in the non-pointed case. Consider the morphism $p : Spec(\mathbf{C}) \to Spec(\mathbf{R})$ and let $\check{C}(p)$ be the corresponding Chech simplicial object. Then the morphism $\check{C}(p) \to Im(p)$ where $Im(p)$ is the image of p in $Rad(C)$, is a projective equivalence. On the other hand $\Lambda^l(\check{C}(p))$ takes $Spec(\mathbf{R})$ to the simplicial abelian group which computes homology of $\mathbf{Z}/2$ and therefore it is not equivalent to $\Lambda^l(Im(p))$ which is a single object in dimension zero.

Proposition 2.17 *One has*

1. $W_{proj} = (\Lambda^r)^{-1}(W_{proj})$
2. $W^{tr}_{Nis} = (\Lambda^r)^{-1}(W_{Nis})$
3. $W^{tr}_{\mathbf{A}^1} = (\Lambda^r)^{-1}(W_{\mathbf{A}^1})$

Proof: It follows immediately from [26, Proposition 4.39].

Proposition 2.18 *An object X of $\Delta^{op}Rad(Cor(C))$ is Nis-local (resp. $\mathbf{A}^1$-local) if and only if $\Lambda^r(X)$ is Nis-local (resp. $\mathbf{A}^1$-local) in $\Delta^{op}Rad(C_+)$.*

2 The category $H_{\mathbf{A}^1}(C_+)$ and the category $H_\bullet(C_{Nis}, \mathbf{A}^1)$

Let C_{Nis} be the category C considered as a site with the Nisnevich topology and $Shv_\bullet(C_{Nis})$ the category of pointed sheaves on it. Consider the following diagram of functors

$$\begin{array}{ccc} \Delta^{op}Rad(C_+) & \xrightarrow{a_\bullet} & \Delta^{op}Shv_\bullet(C_{Nis}) \\ {\scriptstyle p}\downarrow & & {\scriptstyle p'}\downarrow \\ H_{Nis}(C_+) & & H_{s,\bullet}(C_{Nis}) \\ {\scriptstyle q}\downarrow & & {\scriptstyle q'}\downarrow \\ H_{\mathbf{A}^1}(C_+) & & H_\bullet(C_{Nis}, \mathbf{A}^1) \end{array} \tag{5}$$

where $H_{s,\bullet}(C_{Nis})$ is the pointed homotopy category of simplicial sheaves on C_{Nis} and $H_\bullet(C_{Nis}, \mathbf{A}^1)$ is the pointed homotopy category of the site with interval $(C_{Nis}, \mathbf{A}^1)$ defined in [6]. The functor $a_\bullet$ here is the composition of the equivalence

$$Rad(C_+) \to Rad(C)_\bullet$$

from [26, Lemma 3.5] with the associated sheaf functor.

Theorem 2.19 *There are equivalences*

$$H_{Nis}(C_+) \stackrel{eq_0}{\to} H_{s,\bullet}(C_{Nis})$$

$$H_{\mathbf{A}^1}(C_+) \stackrel{eq_1}{\to} H_\bullet(C_{Nis}, \mathbf{A}^1)$$

which extend (5) to a commutative diagram.

Proof: Clearly, $a_\bullet$ is a strict localization and therefore $p'a_\bullet$ is a localization with respect to morphisms $f : X \to Y$ such that $a_\bullet(f)$ is a simplicial equivalence in $\Delta^{op}Shv_\bullet(C_{Nis})$. Proposition 2.11 implies now that there is an equivalence

$$H_{Nis}(C_+) \stackrel{eq_0}{\to} H_{s,\bullet}(C_{Nis})$$

which makes the corresponding square commutative.

Lemma 2.20 *An object $Z \in H_{Nis}(C_+)$ is the image of an $\mathbf{A}^1$-local object $\tilde{Z}$ in $\Delta^{op}Rad(C_+)$ iff $eq_0(Z)$ is $\mathbf{A}^1$-local in the sense of [6].*

Proof: By [6] an object Z is $\mathbf{A}^1$-local iff for any pointed simplicial sheaf F the map

$$Hom_{H_s}(F, Z) \to Hom_{H_s}(F \wedge (\mathbf{A}^1)_+, Z) \qquad (6)$$

is a bijection. By our definition Z is $\mathbf{A}^1$-local if it is projectively fibrant and for any $X \in C_+$ and any simplicial set K the map

$$Hom_H(X \boxtimes K, Z) \to Hom_H(X \wedge (\mathbf{A}^1)_+ \boxtimes K, Z)$$

is a bijection. Suppose that Z is local in the first sense. Then we can choose for it a representative $\tilde{Z}$ which is fibrant in the simplicial closed model structure on $\Delta^{op}Shv_\bullet$. Such a representative is projectively fibrant and one has

$$Hom_{H(C_+)}(F, \tilde{Z}) = Hom_{H_s}(F, Z)$$

for any F. Since $F \boxtimes K = F \wedge (K_+)$ where on the right K is considered as a constant simplicial sheaf we conclude that $\tilde{Z}$ is local in our sense.

If $\tilde{Z}$ is $\mathbf{A}^1$-local in our sense then it is Nis-local and by [26, Lemma 4.22] the map (6) is isomorphic to a similar map of Hom-sets in $H(C_+)$. Our result follows since by Proposition 2.7 the map $F \wedge (\mathbf{A}^1_+) \to F$ is an $\mathbf{A}^1$-equivalence.

By definition, q' is the localization with respect to morphisms $f : X \to Y$ such that for any $\mathbf{A}^1$-local Z the map

$$Hom_{H_s}(Y, Z) \to Hom_{H_s}(X, Z)$$

is bijective where $\mathbf{A}^1$-locality is understood in the sense of [6]. By [26, Lemma 4.24], q is the localization with respect to maps which define bijections on the sets of morphisms to objects Z which are $\mathbf{A}^1$-local in our sense. The proof of the theorem follows by Lemma 2.20.

3 The category $H_{\mathbf{A}^1}(Cor(Sm/k))$ and the category $DM_-^{eff}(k)$

In this section we construct a full embedding from the category $H_{\mathbf{A}^1}(Cor(Sm/k))$ defined in Section 1 to the triangulated category of motives $DM = DM_-^{eff}$. The category DM has a number of equivalent definitions. For our purposes it will be convenient to take the following one which we give in the case of a general admissible category C.

Let $Compl_-(R(Cor(C))$ be the category of complexes over $R(Cor(C))$ which are bounded from the homological below i.e. complexes such that going in the direction of the differential we eventually reach a point after which all the terms are zero. Let $D_-(R(Cor(C)))$ be the corresponding derived category. Consider two classes of morphisms

1. The class E_0 consists of morphisms of the form $([B] \to [A] \oplus [Y]) \to [X]$ for all upper distinguished squares in C of the form (3).
2. The class E_1 consists of morphisms of the form $[X \times \mathbf{A}^1] \to [X]$ for all X in C.

For a class E of morphisms in a triangulated category let $cl_{vl}(E)$ denote the class of morphisms whose cones belong to the localizing subcategory generated by cones of morphisms from E. Then

$$DM_-^{eff}(C) = D_-(R(Cor(C)))[cl_{vl}(E_0 \cup E_1)^{-1}].$$

Lemma 2.21 *When $C = Sm/k$ the definition of DM_-^{eff} given above is equivalent to the usual one.*

Consider the normalization functor $N : \Delta^{op}R(Cor(C)) \to Compl_-(R(Cor(C)))$ and its right adjoint K which takes a complex to the simplicial object corresponding to the canonical truncation of this complex at level zero. The following results are proved in [26, Section 6] in the context of any additive category.

Proposition 2.22 *The functor N takes projective equivalences to quasi-isomorphisms and the resulting functor*

$$H(Cor(C)) \to D_-(R(Cor(C))) \tag{7}$$

is a full embedding. Similarly K takes quasi-isomorphisms to projective equivalences and the resulting functor

$$D_-(R(Cor(C))) \to H(Cor(C))$$

is the localization right adjoint to (7).

Proposition 2.23 *For any class of morphisms E in $\Delta^{op}R(Cor(C))$ one has*

$$N(cl_{\bar{\Delta}}(E)) \subset cl_{vl}(N(E)).$$

Proposition 2.24 *Functor (7) takes elements of* W_{tr}^{Nis} *(resp.* $W_{\mathbf{A}^1}^{tr}$*) to elements of* $cl_{vl}(E_0)$ *(resp.* $cl_{vl}(E_0 \cup E_1)$*) and therefore defines functors*

$$H_{Nis}(Cor(C)) \to D_-(R(Cor(C)))[cl_{vl}(E_0)^{-1}]. \tag{8}$$

and

$$H_{\mathbf{A}^1}(Cor(C)) \to DM_-^{eff}(C). \tag{9}$$

Proof: By [26, Theorem 4.33(1)], Lemma 2.1 and Proposition 2.5 we have:

$$W_{\mathbf{A}^1}^{tr} = cl_{\bar{\Delta}}([W_0] \cup [W_1] \cup W_{proj}).$$

Therefore, by Proposition 2.23 it is sufficient to check that N maps W_{proj} to isomorphisms in $D_-(R(Cor(C)))$, $[W_0]$ to E_0 and $[W_1]$ to E_1. For W_{proj} it is a part of Proposition 2.22. For $[W_1]$ it is obvious. Finally $[W_0]$ is mapped to E_0 since for an upper distinguished square Q of the form (3) one has

$$N(K_Q) = ([B] \to [A] \oplus [Y]).$$

Proposition 2.25 *The functor (8) is a full embedding.*

Proof: If R, L form an adjoint pair of functors then L is a full embedding iff the adjunction $Id \to RL$ is an isomorphism. In view of Proposition 2.22 this implies that in order to prove the proposition it is sufficient to show that the right adjoint to N takes $cl_{vl}(E_0)$ to W_{Nis}^{tr}. Elements of E_0 are quasi-isomorphisms in the Nisnevich topology by Lemma 2.13(1). Since the class of quasi-isomorphisms is closed in the vl-sense we conclude that any element f of $cl_{vl}(E_0)$ is a quasi-ismorphism in the Nisnevich topology. Therefore our result follows from Proposition 2.17(2).

Let $Shv_{tr}(C)$ be the category of sheaves with transfers on C i.e. the full subcategory of $R(Cor(C))$ which consists of objects which are sheaves in the Nisnevich topology. Let us recall the following result.

Proposition 2.26 *The category* $Shv_{tr}(C)$ *is abelian and the associated sheaf functor defines an equivalence*

$$D_-(R(Cor(C)))[cl_{vl}(E_0)^{-1}] \to D_-(Shv_{tr}(C)).$$

Theorem 2.27 *For* $C = Sm/k$ *the functor (9) is a full embedding.*

Proof: By [26, Proposition 4.21] the category $H_{\mathbf{A}^1}$ can be identified with the full subcategory of $\mathbf{A}^1$-local objects in H_{Nis}. The Verdier localization theory for triangulated categories implies that the projection

$$D_-(R(Cor(C)))[cl_{vl}(E_0)^{-1}] \to D_-(R(Cor(C)))[cl_{vl}(E_0 \cup E_1)^{-1}]$$

to objects which are right orthogonal to elements of E_1 is a full embedding. These observations show that in order to prove the theorem it s sufficient to show that for any $\mathbf{A}^1$-local S the complex $N(S)$ is right orthogonal to elements of E_0. In the view of Proposition 2.26 this means that for all $X \in C$ the maps on hypercohomology

$$\mathbf{H}^*_{Nis}(X, N(S)) \to \mathbf{H}^*_{Nis}(X \times \mathbf{A}^1, N(S))$$

are isomorphisms. By Proposition 2.3, $N(S)$ is a complex of presheaves with transfers with homotopy invariant cohomology presheaves and our claim becomes equivalent to the main theorem of [16].

Remark 2.28 I do not know whether or not the functor (9) is a full embedding for a general admissible C. The problem is that the main theorem of [16] is only known for smooth schemes. On the other hand it should be possible to prove using general arguments that (9) becomes a full embedding after $H_{\mathbf{A}^1}$ is stabilized with respect to the simplicial suspension. From this point of view the main theorem of [16] may be stated by saying that $H_{\mathbf{A}^1}(Cor(Sm/k))$ is Σ_s-stable.

4 Change of the underlying category C

Let C, D be two admissible subcategories such that $C \subset D$.

Proposition 2.29 *The inclusion $i : C_+ \to D_+$ satisfies the conditions of [26, Theorem 4.36] with respect to the classes*

$$(E, E') = ((W_0)_+, (W_0)_+)$$

and

$$(E, E') = ((W_0)_+ \cup (W_1)_+, (W_0)_+ \cup (W_1)_+)$$

and therefore i_* and $\mathbf{L}i^{rad}$ define a pair of adjoint functors between the corresponding homotopy categories:

$$\mathbf{L}i^{rad} : H_{Nis}(C_+) \to H_{Nis}(D_+) \quad i_* : H_{Nis}(D_+) \to H_{Nis}(C_+)$$

and

$$\mathbf{L}i^{rad} : H_{\mathbf{A}^1}(C_+) \to H_{\mathbf{A}^1}(D_+) \quad i_* : H_{\mathbf{A}^1}(D_+) \to H_{\mathbf{A}^1}(C_+)$$

Proof: The functor i clearly commutes with the coproducts. The categories in question are grainy and the classes are admissible. The first condition of [26, Theorem 4.36] is obviously satisfied. It remains to check that

$$i_*((W_0)_+ \coprod Id_{Rad(C_+)}) \subset W_{Nis}$$

and

$$i_*((W_1)_+ \coprod Id_{Rad(C_+)}) \subset W_{\mathbf{A}^1}.$$

The first inclusion follows from the obvious fact that $i_*(W'_{Nis}) \subset W'_{Nis}$ and Lemma 2.11. The second inclusion from the obvious fact that $i_*(W'_1) \subset W'_1$, Lemma 2.8 and Lemma 2.9.

Corollary 2.30 *The functror $\mathbf{L}i^{rad}$ between the Nis- and $\mathbf{A}^1$- homotopy categories is a full embedding.*

Proof: It follows from the proposition and [26, Corollary 4.37].

Proposition 2.31 *The inclusion $i : Cor(C) \to Cor(D)$ satisfies the conditions of [26, Theorem 4.36] with respect to the classes*

$$(E, E') = ([W_0], [W_0])$$

and

$$(E, E') = ([W_0] \cup [W_1], [W_0] \cup [W_1])$$

and therefore i_* and $\mathbf{L}i^{rad}$ define a pair of adjoint functors between the corresponding homotopy categories.

Proof: The functor i clearly commutes with the coproducts. The categories in question are grainy and the classes are admissible. The first condition of [26, Theorem 4.36] is obviously satisfied. It remains to check that

$$i_*([W_0] \oplus Id_{Rad(Cor(C))}) \subset W^{tr}_{Nis}$$

and

$$i_*([W_1] \oplus Id_{Rad(Cor(C))}) \subset W^{tr}_{\mathbf{A}^1}.$$

The first inclusion follows from the corresponding non-additive result, Lemma 2.13(1) and Proposition 2.17(2). Similarly, the second one follows from Lemma 2.13(2) and Proposition 2.17(3).

Corollary 2.32 *The functror $\mathbf{L}i^{rad}$ between the Nis- and $\mathbf{A}^1$- homotopy categories of finite correspondences is a full embedding.*

Proof: It follows from the proposition and [26, Corollary 4.37].

Let us analyze now how the functors i_* and $\mathbf{L}i^{rad}$ for schemes and finite correspondences are related to each other. Consider the following diagram

$$\begin{array}{ccccc} H_{\mathbf{A}^1}(C_+) & \xrightarrow{\mathbf{L}i^{rad}} & H_{\mathbf{A}^1}(D_+) & \xrightarrow{i_*} & H_{\mathbf{A}^1}(C_+) \\ {\scriptstyle \mathbf{L}\Lambda^l}\downarrow & & {\scriptstyle \mathbf{L}\Lambda^l}\downarrow & & {\scriptstyle \mathbf{L}\Lambda^l}\downarrow \\ H_{\mathbf{A}^1}(Cor(C)) & \xrightarrow{\mathbf{L}i^{rad}} & H_{\mathbf{A}^1}(Cor(D)) & \xrightarrow{i_*} & H_{\mathbf{A}^1}(Cor(C)) \\ {\scriptstyle \Lambda^r}\downarrow & & {\scriptstyle \Lambda^r}\downarrow & & {\scriptstyle \Lambda^r}\downarrow \\ H_{\mathbf{A}^1}(C_+) & \xrightarrow{i^{rad}} & H_{\mathbf{A}^1}(D_+) & \xrightarrow{i_*} & H_{\mathbf{A}^1}(C_+) \end{array} \tag{10}$$

The commutative square

$$\begin{array}{ccc} C_+ & \xrightarrow{i} & D_+ \\ {\scriptstyle \Lambda^l}\downarrow & & \downarrow{\scriptstyle \Lambda^l} \\ Cor(C) & \xrightarrow{i} & Cor(D) \end{array}$$

immediately shows that the lower right square of (10) commutes. The upper left square is left adjoint to the lower right one and therefore it is commutative as well. The lower left corner is unlikely to commute. We do not know whether the upper right one commutes in general but there is the following important partial result.

Theorem 2.33 *Let k be a field with resolution of singularities and $C \subset Sm/k$. Then for any admissible D which contains C the upper right square of (10) commutes i.e. one has*

$$\mathbf{L}\Lambda^l i_* = i_* \mathbf{L}\Lambda^l.$$

Proof: It is easy to see that it is sufficient to consider the case $C = Sm/k$ and $D = Sch/k$ i.e. we need to check the commutativity of the square

$$\begin{array}{ccc} H_{\mathbf{A}^1}((Sch/k)_+) & \xrightarrow{i_*} & H_{\mathbf{A}^1}((Sm/k)_+) \\ {\scriptstyle \mathbf{L}\Lambda^l}\downarrow & & {\scriptstyle \mathbf{L}\Lambda^l}\downarrow \\ H_{\mathbf{A}^1}(Cor(Sch/k)) & \xrightarrow{i_*} & H_{\mathbf{A}^1}(Cor(Sm/k)) \end{array} \tag{11}$$

We have a natural transformation

$$\mathbf{L}\Lambda^l i_* \to i_* \mathbf{L}\Lambda^l \tag{12}$$

arising from the adjunctions and the commutativity of the lower right square of (10) and we need to show that it is an isomorphism. Consider the square

$$\begin{array}{ccc} \mathbf{L}\Lambda^l\, i_*\, \mathbf{L}i^{rad}\, i_* & \longrightarrow & \mathbf{L}\Lambda^l\, i_* \\ \downarrow & & \downarrow \\ i_*\, \mathbf{L}\Lambda^l\, \mathbf{L}i^{rad}\, i_* & \longrightarrow & i_*\, \mathbf{L}\Lambda^l \end{array}$$

where the vertical arrows come from (12) and the horizontal ones from the adjunction $\mathbf{L}i^{rad}i_* \to Id$. We need to prove that the right vertical arrow is an iso. We will do it by showing that the other three arrows are isos.

The upper horizontal arrow is an iso since the composition

$$i_* \to i_* \, \mathbf{L}i^{rad} \, i_* \to i_* \tag{13}$$

is identity (by definition of adjunction) and the first arrow is an iso by Corollary 2.30. To see that the left vertical arrow is an iso one first exchanges $\mathbf{L}\Lambda^l$ and $\mathbf{L}\Lambda^l$ by commutativity of the upper left square of the main diagram and them uses that the analog of (13) for functors on $H_{\mathbf{A}^1}(Cor)$'s consists of isos by Corollary 2.32. It remains to show that under our assumptions the lower horizontal arrow is an iso. It follows from Lemmas 2.34-2.36 given below.

Lemma 2.34 *Let k be a field with resolution of singularities, then the morphism*

$$\mathbf{L}i^{rad}i_*(F) = i^{rad} \, Lres \, i_*(F) \to i^{rad}i_*(F) \to F$$

is a local equivalence in the cdh-topology for any F in $\Delta^{op}Rad(Sch/k_+)$.

Proof: Recall from [17] that Sm/k can be equipped with scdh-topology such that the natural functor $Sm/k \to Sch/k$ defines morphism of sites

$$\pi : (Sch/k)_{cdh} \to (Sm/k)_{scdh}$$

and that when k admits resolution of singularities the corresponding functors π_*, π^* of direct and inverse images on sheaves are equivalences (see [17, Lemma 4.6]). Let a be the associated sheaf functor. It is clearly sufficient to check that

$$ai^{rad} \, Lres \, i_*(F) = \pi^* a \, Lres \, i_*(F) \to \pi^* ai_*(F) \to aF$$

is an equivalence in the cdh-topology. The functor a takes projective equivalences to local equivalences because it commutes with the formation of (pre-)sheaves of homotopy groups. The functor π^* takes local equivalences to local equivalences because it is an equivalence. Finally the morphism $\pi^* ai_*(F) \to aF$ is an isomorphism again because π^* and π_* are mutually inverse equivalences.

Lemma 2.35 *Let $f : X \to Y$ be a morphism in $\Delta^{op}(Sch/k)^{\#}_{+}$ which is a local equivalence in the cdh-topology. Then $\Lambda^l(f)$ is a local equivalence in the cdh-topology.*

Proof: By [17] the class of cdh-equivalences on $\Delta^{op}(Sch/k)^{\#}_{+}$ is $cl_{\bar{\Delta}}((W_0)_+ \cup (W_2)_+)$ where W_2 is defined in the sam way as W_0 but with respect to the lower distinguished squares (i.e. abstract blow-up squares). Therefore it is sufficient to verify that both for upper and for lower distinguished Q the morphism $[K_Q] \to [X]$ is an equivalence in the cdh-topology. For upper distinguished ones we know it from Lemma 2.13(1). To deal with the lower distinguished ones we may use the same reasoning as in the proof of that lemma. It remains to show that for a lower distinguished Q the sequence of cdh-sheaves

$$0 \to a[B] \to a[A] \oplus a[Y] \to a[X] \to 0$$

is exact n the cdh-topology. This is the statement of [11, Prop. 4.3.3].

Lemma 2.36 *Let k be a field with resolution of singularities and $f : X \to Y$ be a local equivalence in the cdh-topology in $\Delta^{op}Rad(Cor(Sm/k))$. Then the image of f in $H_{\mathbf{A}^1}(Cor(Sm/k))$ is an isomorphism.*

Proof: By Theorem 2.27 it is sufficient to check that the corresponding morphism is an isomorphism in DM. It follows from [1, Theorem 5.5(2)].

3 Symmetric powers

1 Generalized symmetric powers on $H_{\mathbf{A}^1}(C_+)$

In this section we assume that the underlying category C is f-admissible. Let $\Phi = (G, \phi : G \to S_n)$ be a permutation group i.e. a group together with an embedding into the symmetric group. Consider the functor S^Φ (resp. $\tilde{S}^\Phi$) from C_+ to itself of the form

$$X_+ \mapsto (X_+)^n/G$$

(resp. of the form $X_+ \mapsto (X_+)^{\wedge n}/G = (X^n/G)_+$). Let $\Phi_i = (G_i, \phi_i : G_i \to S_{n_i})$, $i = 1, 2$ be two permutation groups. Define their wreath product $\Phi_1 * \Phi_2$ as follows. The direct power $G_1^{n_2}$ acts on $\{n_1\} \times \{n_2\}$ in the obvious way. Consider the action of G_2 on the same set which is the product of the action defined by ϕ_2 on $\{n_2\}$ and the trivial action on $\{n_1\}$. Let $\phi : G \to S_{n_1 n_2}$ be the subgroup generated by the images of $G_1^{n_2}$ and G_2. We set

$$\Phi_1 * \Phi_2 = (G, \phi : G \to S_{n_1 n_2})$$

(One can see that in fact G is the semi-direct product of $G_1^{n_1}$ and G_2 with respect to the obvious action of the later on the former.) The following result is straightforward.

Proposition 3.1 *For any X in C_+ and any Φ_1 and Φ_2 as above there is an isomorphism*

$$S^{\Phi_1 * \Phi_2}(X) = S^{\Phi_2}(S^{\Phi_1}(X)) \tag{14}$$

and similarly for $\tilde{S}$'s. These isomorphisms are natural in X.

Define $\Phi_1 \times \Phi_2$ by the formula

$$\Phi_1 \times \Phi_2 = (G_1 \times G_2, \phi : G_1 \times G_2 \to S_{n_1+n_2})$$

where ϕ is the composition of $\phi_1 \times \phi_2$ with the obvious embedding $S_{n_1} \times S_{n_2} \to S_{n_1+n_2}$. We have another straightforward result.

Proposition 3.2 *For any X in C_+ and any Φ_1 and Φ_2 as above there is an isomorphism*

$$S^{\Phi_1 \times \Phi_2}(X) = S^{\Phi_1}(X) \times S^{\Phi_2}(X) \tag{15}$$

and

$$\tilde{S}^{\Phi_1 \times \Phi_2}(X) = \tilde{S}^{\Phi_1}(X) \wedge \tilde{S}^{\Phi_2}(X). \tag{16}$$

These isomorphisms are natural in X.

The inverse image functor on the category of presheaves of sets on C_+ defined by $\tilde{S}^\Phi$ and S^Φ takes radditive functors to radditive functors by [26, Proposition 3.16] and we keep the notation S^Φ (resp. $\tilde{S}^\Phi$) for the resulting functor from $Rad(C_+)$ to itself.

As a corollary of the fact that the isos of Proposition 3.1 and 3.2 are natural we conclude that they extends to $C_+^{\#}$ and further to $\Delta^{op}C_+^{\#}$. To prove Proposition 3.4 and Theorem 3.6 we need to connect our constructions to the constructions of [19]. Let C/G be the category f G-objects in C considered as a site with the equivariant Nisnevich topology.

Lemma 3.3 *The functor $a_{Nis}\tilde{S}^\Phi$ is isomorphic to the functor*

$$F \mapsto \eta_\#(a_{Nis}F)^{\wedge I}$$

where I is the disjoint union of n copies of $Spec(k)$ with the obvious action of G and $\eta_\#$ is defined by the morphism of sites $\eta : C/G \to C$

Proof: Observe first that the equality $\tilde{S}^\Phi(F) = \eta_\# F^{\wedge I}$ is obvious for $F \in C_+$ from the definitions. Therefore it also holds for $F \in C_+^{\#}$ since al the functors n question commute with filtering colimits. We have further:

$$a_{Nis}\tilde{S}^\Phi(F) = a_{Nis}\pi_0(\tilde{S}^\Phi(Lres(F))) = a_{Nis}\pi_0(\eta_\#(Lres(F))^{\wedge I}) = \pi_0^{Nis}(\eta_\#(Lres(F))^{\wedge I}) =$$

$$= \eta_\#\pi_0^{Nis}((Lres(F))^{\wedge I}).$$

where π_0^{Nis} is the π_0 functor for sheaves in the Nisnevich topology. The last equality holds since $\eta_\#$ is a left adjoint it and as such commutes with colimits and in particular with the formation of π_0. The map $Lres(F) \to a_{Nis}F$ is a local equivalence in the Nisnevich topology and we conclude by [19, Proposition 5.2.11] that

$$\pi_0^{Nis}((Lres(F))^{\wedge I}) = (a_{Nis}F)^{\wedge I}$$

which finishes the proof.

If $Z \subset X$ is a closed subset of X we will write $\tilde{S}^{\Phi}(Z)$ for the closed subset in $\tilde{S}^{\Phi}(X)$ which is the image of $Z^n \subset X^n$.

Proposition 3.4 *Let Z be a closed subset of X. Consider $X/(X-Z)$ as a radditive functor on C_+ and let a_{Nis} be the functor of associated Nisnevich sheaf. Then one has*

$$a_{Nis}\tilde{S}^{\Phi}(X/(X-Z)) = a_{Nis}(\tilde{S}^{\Phi}(X)/(\tilde{S}^{\Phi}(X) - \tilde{S}^{\Phi}(Z)).$$

Proof: It follows from Lemma 3.3, [19, Example 5.2.8] and the fact that $\eta_{\#}$ commutes with colimits.

Remark 3.5 The statement of the proposition is false on the level of radditive functors i.e. $\tilde{S}^{\Phi}(X/(X-Z))$ and $\tilde{S}^{\Phi}(X)/(\tilde{S}^{\Phi}(X) - \tilde{S}^{\Phi}(Z)$ are different as radditive functors and only the associated Nisnevich sheaves are isomorphic.

The general definition of pointed (ind-)solid sheaves is given in [19, Def. 4.1.5]. For us it will only be important that pointed sheaves of the form $a_{NIs}(X/(X-Z))$ (and in objects of C_+) are solid. A simplicial sheaf is called (ind-)solid if each of its terms is.

Theorem 3.6 *Let $f : X \to Y$ be a Nis- (resp. an $\mathbf{A}^1$-) equivalence between simplicial (pointed) ind-solid sheaves. Then $\tilde{S}^{\Phi}(f)$ is a Nis- (resp. $\mathbf{A}^1$-) equivalence. In particular, for an ind-solid F the obvious morphism*

$$\tilde{S}^{\Phi}(Lres(F)) \to \tilde{S}^{\Phi}(F)$$

is a Nis-equivalence.

Proof: By Lemma 3.3 we may replace the functor $\tilde{S}^{\Phi}$ by $\eta_{\#}()^{\wedge I}$. By [19, Proposition 5.2.11] the functor $(-)^{\wedge I}$ preserves Nis- and $\mathbf{A}^1$-equivalences. By [19, Proposition 5.2.9] this functor takes pointed (ind-)solid sheaves to pointed (ind-)solid sheaves. Any pointed (ind-)solid sheaf is (ind-)solid therefore [19, Proposition 5.1.4] applies to $f^{\wedge I}$ and we conclude that $\tilde{S}^{\Phi}(f)$ is a free equivalence of pointed sheaves and therefore a pointed equivalence[3].

[3]We never used free equivalences in this paper but it should be clear that a pointed morphism which is a free Nis- r $\mathbf{A}^1$-equivalence is the corresponding pointed equivalence.

Corollary 3.7 *There exist unique up to a canonical isomorphism functors $\tilde{S}^{\Phi} : H_{\mathbf{A}^1}(C_+) \to H_{\mathbf{A}^1}(C_+)$ such that the squares*

$$\begin{array}{ccc} \Delta^{op}C_+^{\#} & \xrightarrow{\tilde{S}^{\Phi}} & \Delta^{op}C_+^{\#} \\ \downarrow & & \downarrow \\ H_{\mathbf{A}^1} & \xrightarrow{\tilde{S}^{\Phi}} & H_{\mathbf{A}^1} \end{array}$$

commute.

In the case of the ordinary symmetric products associated with the permutation group (S_n, Id) we will use the simplified notations S^n and $\tilde{S}^n$. For small values of n one has

$$\begin{array}{ll} S^0(X_+) = pt & \tilde{S}^0(X_+) = S^0 \\ S^1(X_+) = X_+ & \tilde{S}^1(X_+) = X_+ \\ S^2(X_+) = (X^2/S_2)_+ \vee X_+ & \tilde{S}^2(X_+) = (X^2/S_2)_+ \end{array}$$

Lemma 3.8 *For any $X, Y \in C_+^{\#}$ there are natural isomorphisms*

$$\tilde{S}^n(X \vee Y) = \vee_{n\ge i\ge 0}(\tilde{S}^i X \wedge \tilde{S}^{n-i} Y).$$

Proof: Straightforward.

Lemma 3.9 *For any X in C_+ and any $n > 0$ there is a coprojection sequence*

$$S^{n-1}(X) \to S^n(X) \to \tilde{S}^n(X). \tag{17}$$

These sequences are natural in X.

As a corollary of the fact that the sequences of the lemma are natural we conclude that it extends to $C_+^{\#}$ and further to $\Delta^{op}C_+^{\#}$.

The colimit of the sequence (17) clearly exists in $C_+^{\#}$ and we denote it by S^{∞}. Note that for $X \in C$ one has

$$S^{\infty}(X_+) = (\coprod_{n\ge 1} S^n X)_+ = \vee_{n\ge 1}\tilde{S}^n(X_+).$$

We will also consider the functors $S^{\infty}[1/d]$ for integers $d > 0$ defined by the rule

$$S^{\infty}[1/d](X) = colim(S^{\infty}(X) \xrightarrow{\times d} S^{\infty}(X) \xrightarrow{\times d} \ldots)$$

where $\times d$ is the multiplication by d map with respect to the abelian monoid structure of S^{∞}.

Lemma 3.10 *For any $X, Y \in C_+^{\#}$ there is a natural isomorphism*

$$S^{\infty}[1/d](X \vee Y) = S^{\infty}[1/d](X) \times S^{\infty}[1/d](Y). \tag{18}$$

Proof: The maps $X \vee Y \to X$ and $X \vee Y \to Y$ define the map from the left to the right hand side of (18). To verify that it is an iso we may assume that $X = U_+$ and $Y = V_+$ for $U, V \in C$. The case $d = 1$ follows then immediately from Lemma 3.8. The case $d > 1$ follows from the case $d = 1$ and the fact that finite products commute with filtering colimits.

We have the following analog of Corollary 3.7.

Corollary 3.11 *For any $d > 0$ there exist a unique up to a canonical isomorphism functor $S^{\infty}[1/d] : H_{\mathbf{A}^1}(C_+) \to H_{\mathbf{A}^1}(C_+)$ such that the squares*

$$\begin{array}{ccc} \Delta^{op}C_+^{\#} & \xrightarrow{S^{\infty}[1/d]} & \Delta^{op}C_+^{\#} \\ \downarrow & & \downarrow \\ H_{\mathbf{A}^1} & \xrightarrow{S^{\infty}[1/d]} & H_{\mathbf{A}^1} \end{array}$$

commutes.

2 Generalized symmetric powers on $H_{\mathbf{A}^1}(Cor(C))$

We continue to assume that the underlying category C is f-admissible. Set

$$c = c(k) = \begin{cases} 1 & \text{if } char(k) = 0 \\ char(k) & \text{otherwise} \end{cases}$$

In what follows we assume that c is invertible in our ring of coefficients R. We start with the following result.

Proposition 3.12 *Let $X, Y \in C$ and let G be a finite group acting on X. Then*

$$Hom_{Cor(C,R)}([X/G], [Y]) = Hom_{Cor(C,R)}([X], [Y])^G$$

i.e. $[X/G]$ is the categorical quotient for the action of G on $[X]$.

Proof: It follows from the fact that the functor represented by $[Y]$ on Sch/k is a qfh-sheaf by [12, Proposition 4.2.7] and that for qfh-sheaves F one has $F(X/G) = F(X)^G$.

Proposition 3.13 *There exist (unique) functors* $S^{\Phi}_{tr} : Cor(C) \to Cor(C)$ *such that the squares*

$$\begin{array}{ccc} C_+ & \xrightarrow{\tilde{S}^{\Phi}} & C_+ \\ {\scriptstyle \Lambda^l}\downarrow & & \downarrow{\scriptstyle \Lambda^l} \\ Cor(C) & \xrightarrow{S^{\Phi}_{tr}} & Cor(C) \end{array}$$

commute.

Proof: Proposition 3.12 shows that categorical quotients with respect to finite group actions exist in $Cor(C, R)$. Therefore we may define $S^{\Phi}_{tr}(X)$ as $X^{\otimes n}/G$. One verifies easily that the required squares commute.

Again, the functors S^{Φ}_{tr} extend canonically to functors from $Cor(C)^{\#}$ to itself which commute with filtering colimits of coprojections.

Let Φ be as above, $i : H \to G$ a subgroup of G and Ψ the permutation group $(H, \psi = \phi \circ i)$. Assume for a moment that H is normal in G and consider finite correspondences with coefficients in a commutative ring R such that $d = [G : H]$ is invertible in R. Then for any X in $Cor(C, R)^{\#}$ there is an action of G/H on $S^{\Psi}_{tr}(X)$ and it is more or less obvious that $S^{\Phi}_{tr}(X)$ is the direct summand of G/H-invariants in $S^{\Psi}_{tr}(X)$. We will need an analog of this observation in the case when H is not necessarily normal in G.

For any $g \in G$ let Ψ_g be the permutation group corresponding to the subgroup $H \cap gHg^{-1}$. Then for any X in C_+ there are two morphisms

$$p : S^{\Psi_g}_{tr}(X) \to S^{\Psi}_{tr}(X)$$

$$p' : S^{\Psi_g}_{tr}(X) \to S^{\Psi}_{tr}(X)$$

where

$$p : X^{\otimes n}/(H \cap gHg^{-1}) \to X^{\otimes n}/H$$

is the projection and p' is the map whose composition with $X^{\otimes n} \to X^{\otimes n}/(H \cap gHg^{-1})$ is $x \mapsto gx$ followed by the projection.

Theorem 3.14 *Let* $d = [G : H]$ *be invertible in the ring of coefficients* R*. Then for any* X *in* $Cor(C, R)$ *there is a split co-equalizer sequence:*

$$\oplus_{g \in G} S^{\Psi_g}_{tr}(X) \rightrightarrows S^{\Psi}_{tr}(X) \to S^{\Phi}_{tr}(X)$$

where the two arrows are given by p *and* p' *on each summand. Equivalently, for any* F *in* $Rad(Cor(C, R))$ *there is a split equalizer sequence of* R*-modules of the form*

$$F(S^{\Phi}_{tr}(X)) \to F(S^{\Psi}_{tr}(X)) \rightrightarrows \oplus_{g \in G} F(S^{\Psi_g}_{tr}(X)).$$

In the particular case of Theorem 3.14 we get.

Corollary 3.15 *Under the assumptions of the theorem assume in addition that H is normal in G. Then*

$$S^{\Phi}_{tr}(X) = (S^{\Psi}_{tr}(X))^{G/H}$$

i.e. $S^{\Phi}_{tr}(X)$ is the image of the projector $d^{-1}\sum_{u\in G/H} u$ acting on $S^{\Psi}_{tr}(X)$.

As a particular case of Corollary 3.15 we get.

Corollary 3.16 *Let n be an integer and R be a ring where $n!$ is invertible. Then for any X in $Cor(C,R)$ the obvious morphism $X^{\otimes n} \to S^n_{tr}(X)$ defines an isomorphism between $S^n_{tr}(X)$ and the image of the projector $(1/n!)\sum_{\sigma\in S_n}\sigma$ on $X^{\otimes n}$.*

Proof: In view of Proposition 3.12 the theorem and its corollaries are particular cases of Proposition 5.10.

Proposition 3.17 *Let n be an integer, l a prime and $n = \sum n_i l^i$ the l-primary decomposition of n. Let further R be an l-local ring. Then for any X in $Cor(C,R)$ there is a split epimorphism*

$$p_{n,l} : \otimes_i((S^l_{tr})^{\circ i}(X))^{\otimes n_i} \to S^n_{tr}(X) \tag{19}$$

such that both $p_{n,l}$ and its section are natural in X.

Proof: Denote temporarily by Φ_l the permutation group (S_l, Id). For a sequence of non-negative integers $\underline{q} = (q_1,\ldots,q_k)$ consider the permutation group

$$\Phi_{l,\underline{q}} = \prod_i (\Phi_l^{*i})^{\times q_i} = (G_{l,\underline{q}}, \phi_{l,\underline{q}} : G_{l,\underline{q}} \to S_n)$$

where $n = \sum q_i l^i$.

By Propositions 3.1 and 3.2 the left hand side of (19) is canonically isomorphic to $\tilde{S}^{\Phi}(X)$ where $\Phi = \Phi_{l,\underline{n}}(X)$ for $\underline{n} = (n_0,\ldots,n_i,\ldots)$. The morphism $p_{n,l}$ is associated with the embedding $G_{l,\underline{n}} \to S_n$. Using the fact that $\underline{n}$ is the l-primary decomposition of n and computing how many times l divides $n!$ one concludes that $[S_n : G_{l,\underline{n}}]$ is prime to l and therefore invertible in R. Our result follows now from Theorem 3.14.

Proposition 3.18 *For any $n \geq 0$ and any X, Y in $Cor(C,R)$ there are isomorphisms*

$$S^n_{tr}(X\oplus Y) = \oplus_{i\geq 0} S^i_{tr}(X)\otimes S^{n-i}_{tr}(Y)$$

which are natural in X and Y.

Proof: We have morphisms

$$S^i_{tr}(X) \otimes S^{n-i}_{tr}(Y) \to S^n_{tr}(X \oplus Y)$$

which are obvious from the definition of S^i_{tr} as $(-)^{\otimes i}/S_i$. These morphisms are clearly natural in X and Y. On the other hand they are compatible with the morphisms which define the isomorphism of Lemma 3.8 and therefore their sum gives an isomorphism.

Corollary 3.19 *Let $(X_\alpha)_{\alpha \in A}$ be a family of objects in $Cor^{\#}$. Then one has*

$$S^n_{tr}(\bigoplus_{\alpha \in A} X_\alpha) = \bigoplus_{k_1\alpha_1 + \ldots + k_n\alpha_n \in S^n A} S^{k_1}_{tr} X_{\alpha_1} \otimes \ldots \otimes S^{k_n}_{tr} X_{\alpha_n}.$$

Let $X \xrightarrow{i} Y \xrightarrow{p} Z$ be a coprojection sequence in Cor together with a choice of a section s for p. Then $f = i \oplus s : X \oplus Z \to Y$ is an isomorphism and applying Proposition 3.18 we get isomorphisms

$$f_n : \oplus_{i \geq 0} S^i_{tr}(X) \otimes S^{n-i}_{tr}(Z) \to S^n_{tr}(Y).$$

Let

$$S^n_{i,j}(X, Y) = \oplus_{i \leq a \leq j} S^a_{tr}(X) \otimes S^{n-a}_{tr}(Z). \tag{20}$$

Note that we have:

$$S^n_{i,i}(X, Y) = S^i_{tr} X \otimes S^{n-i}_{tr} Z,$$

in particular:

$$S^n_{n,n} = S^n X, \quad S^n_{0,0} = S^n Z, \quad S^n_{0,n}(X, Y) = S^n_{tr} Y,$$

For any $m, k, j \geq 0$ such that $m \leq j + 1$ and $k \leq j$ we get a coprojection sequence

$$S^n_{k+1,j}(X, Y) \to S^n_{m,j}(X, Y) \to S^n_{m,k}(X, Y) \tag{21}$$

An elementary but tedious computation proves the following lemma.

Lemma 3.20 *Objects (20) and sequences (21) are natural with respect to commutative squares of the form*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

where horizontal arrows are coprojections and vertical ones are general morphisms.

In particular, both the objects and the sequences do not depend (up to a canonical isomorphism) on the choice of the section s.

Since all the elements of the results 3.12-3.20 are natural in X they extends to objects of $Cor^{\#}$ and further to objects of $\Delta^{op} Cor^{\#}$.

Theorem 3.21 *The functors S^{Φ}_{tr} take $\mathbf{A}^1$-equivalences between objects of $\Delta^{op}Cor^{\#}$ to $\mathbf{A}^1$-equivalences. In particular there exist unique up to a canonical isomorphism functors*

$$S^{\Phi}_{tr}: H_{\mathbf{A}^1}(Cor(C)) \to H_{\mathbf{A}^1}(Cor(C))$$

such that the squares

$$\begin{array}{ccc} \Delta^{op}Cor^{\#} & \xrightarrow{S^{\Phi}_{tr}} & \Delta^{op}Cor^{\#} \\ \downarrow & & \downarrow \\ H_{\mathbf{A}^1} & \xrightarrow{S^{\Phi}_{tr}} & H_{\mathbf{A}^1} \end{array}$$

commute.

Proof: We will omit Φ from our notation. Applying [26, Lemma 4.35] to the functor S^{Φ}_{tr} we see that it is sufficient to verify that

$$S^{\Phi}_{tr}(([W_0] \cup [W_1]) \oplus Id_{Cor(C)}) \subset W^{tr}_{\mathbf{A}^1} \tag{22}$$

We have

$$S^{\Phi}_{tr}(([W_0] \cup [W_1]) \oplus Id_{Cor(C)}) = S^{\Phi}_{tr}(\Lambda^l(((W_0)_+ \cup (W_1)_+) \amalg Id_C)) = \Lambda^l(\tilde{S}^{\Phi}(((W_0)_+ \cup (W_1)_+) \amalg Id_C))$$

We have

$$\tilde{S}^{\Phi}(((W_0)_+ \cup (W_1)_+) \amalg Id_C) \subset W_{\mathbf{A}^1}$$

by Theorem 3.6 and therefore (22) holds by Corollary 2.15.

Proposition 3.22 *For any $n > 0$ and any cofibration sequence*

$$X \to Y \to Z \to \Sigma^1 X$$

in $H_{\mathbf{A}^1}(Cor(C))$ there are objects $S^n_{i,j}(X,Y)$ such that:

$$S^n_{i,i}(X,Y) = S^i_{tr}X \otimes S^{n-i}_{tr}Y,$$

in particular:

$$S^n_{n,n} = S^n X, \quad S^n_{0,0} = S^n Z, \quad S^n_{0,n}(X,Y) = S^n_{tr}Y,$$

and cofibration sequences

$$S^n_{k+1,j}(X,Y) \to S^n_{m,j}(X,Y) \to S^n_{m,k}(X,Y) \to \Sigma^1 S^n_{k+1,j}(X,Y). \tag{23}$$

Note that while the sequences (21) split the corresponding sequences of Proposition 3.22 need not have splittings.

Corollary 3.23 *Under the assumptions of the proposition there is a tower of cofibration sequences*

$$S^n_{tr}X \to S^n_{tr}Y \to S^n_{0,n-1}(X,Y) \to \Sigma^1 S^n_{tr}X$$

$$S^i_{tr}X \otimes S^{n-i}_{tr}Z \to S^n_{0,i}(X,Y) \to S^n_{0,i-1}(X,Y) \to \Sigma^1(S^i_{tr}X \otimes S^{n-i}_{tr}Z) \quad i = n-1, \dots, 2$$

$$X \otimes S^{n-1}_{tr}Z \to S^n_{0,1}(X,Y) \to S^n_{tr}Z \to \Sigma^1(X \otimes S^{n-1}_{tr}Z)$$

Proof: The proposition follows immediately from the fact that any cofibration sequence is the image of a coprojection sequence and Lemma 3.20. The corollary is obtained by taking a subset of sequences constructed in the proposition.

Let us introduce the following notion of the "Verdier closure" in the unstable categories.

Definition 3.24 *Let A be a class of objects in $H_{\mathbf{A}^1}(Cor(C))$. Define $Cl_{vl}(A)$ as the smallest class which contains A and has the following properties:*

1. *if $\Sigma^1 X \in Cl_{vl}(A)$ then $X \in Cl_{vl}(A)$*
2. *if X_α is a family of objects from $Cl_{vl}(A)$ then $\oplus_\alpha X_\alpha \in Cl_{vl}(A)$*
3. *if in a cofibration sequence $X \to Y \to Z \to \Sigma^1 X$ one has $X, Y \in Cl_{vl}(A)$ then $Z \in Cl_{vl}(A)$*

Lemma 3.25 *If in a cofibration sequence $X \to Y \to Z \to \Sigma^1 X$ two of the three terms are in $Cl_{vl}(A)$ then so is the third.*

Proof: It follows from properties (1) and (3) of the definition of Cl_{vl} and [26, Proposition 5.7].

Lemma 3.26 *If $X_1 \to X_2 \to \dots \to X_n \to \dots$ is a sequence of objects from $Cl_{vl}(A)$ then $hocolim\, X_i \in Cl_{vl}(A)$.*

Proof: Follows by (2) and (3) of Definition 3.24 because there is a cofibration sequence

$$\oplus_i X_i \to \oplus_i X_i \to hocolim\, X_i \to \Sigma^1 \oplus_i X_i.$$

Lemma 3.27 *The subcategory $Cl_{vl}(A)$ is closed under direct summands.*

Proof: Let $Z = X \oplus Y \in cl_v(A)$ and let p be the projector whose image is X. Then X is the homotopy colimit of the sequence $Z \stackrel{p}{\to} Z \stackrel{p}{\to} \ldots$ and our result follows from Lemma 3.26.

Theorem 3.28 *Let A be a class which is closed under finite direct sums and under S^n_{tr} for all $n \geq 1$. Then $Cl_{vl}(A)$ is closed under tensor products and under S^n_{tr} for all $n \geq 1$.*

Proof: Let us show first that $cl_v(A)$ is closed under tensor products. Let $X, Y \in A$. Then $X \otimes Y$ is a direct summand of $S^2_{tr}(X \oplus Y)$ and therefore it is in $Cl_{vl}(A)$ by our assumption on A and Lemma 3.27. Observe now that for any X the class of all Y such that $X \otimes Y \in Cl_{vl}(A)$ is vl-closed. For $X \in A$ it contains A and therefore $Cl_{vl}(A)$. Consider now the class of all X such that for any $Y \in Cl_{vl}(A)$ one has $X \otimes Y \in Cl_{vl}(A)$. This class is clearly v-closed and by the previous comment contains A. We have shown that $Cl_{vl}(A)$ is closed under tensor products.

We have to show now that the class of all X such that $S^n_{tr}(X) \in Cl_{vl}(A)$ is v-closed. We proceed by induction on n. The case $n = 1$ is obvious. Assume that the statement is proved for $n - 1$ and therefore $Cl_{vl}(A)$ is closed under all symmetric products S^i_{tr} with $i < n$.

Let X be such that $S^n_{tr}Z = S^n_{tr}(\Sigma^1 X) \in cl_v(A)$. We need to show that $S^n_{tr}X \in Cl_{vl}(A)$. We have a coprojection sequence $X \to Cone(X) \to Z$. Our claim follows now from Corollary 3.23 and Lemma 3.25. Similar argument implies that our class has the third property of Definition 3.24. The second property follows from Corollary 3.19.

Definition 3.29 *An object X of $H_{\mathbf{A}^1}(Cor(C))$ is called even (resp. odd) if the permutation isomorphism $\sigma : X \otimes X \to X \otimes X$ is the identity (resp. the multiplication by -1).*

We have the following obvious fact.

Lemma 3.30 *1. Tensor product of two 1-odd or two 1-even objects is 1-even.*

2. Tensor product of a 1-odd and a 1-even object is 1-odd.

Lemma 3.31 *Let X be an even (resp. odd) object. Then $\Sigma^1_s X$ is odd (resp. even).*

Proof: It follows from the fact that $\Sigma^1_s X = X \otimes S^1_s$ and that S^1_s is odd.

Lemma 3.32 *Let R be an l-local ring, X an object of $H_{\mathbf{A}^1}(Cor(C))$ and $1 < n < l$ an integer. Then one has:*

1. *If X is odd then the map $S^n_{tr}(X) \to pt$ is an isomorphism.*

2. *If X is even then the map $X^{\otimes n} \to S^n_{tr}(X)$ is an isomorphism.*

Proof: Since the projection $\Delta^{op}Cor(C)^{\#} \to H_{\mathbf{A}^1}$ is an additive functor Proposition 3.16 implies hat $S^n_{tr}(X)$ as an object of $H_{\mathbf{A}^1}$ is the image of the averaging projector. Therefore for an even X we get X. The number of elements in S_n is even for $n > 1$ and therefore for an odd X and $n > 1$ we get zero.

Proposition 3.33 *Let R be an l-local ring and $n \leq l$. Let*

$$X \to Y \to Z \to \Sigma^1 X$$

be a cofibration sequence in $H_{\mathbf{A}^1}(Cor(C))$. Then one has:

1. *If X is odd then there are cofibration sequences of the form*

$$X \otimes S^{n-1}_{tr} Z \to S^n_{0,1}(X, Y) \to S^n_{tr} Z \to \Sigma^1 X \otimes S^{n-1}_{tr} Z \tag{24}$$

$$S^n_{tr} X \to S^n_{tr} Y \to S^n_{0,1}(X, Y) \to \Sigma^1 S^n_{tr} X \tag{25}$$

2. *If Z is odd then there are cofibration sequences in of the form*

$$S^{n-1}_{tr} X \otimes Z \to S^n_{0,n-1}(X, Y) \to S^n_{tr} Z \to \Sigma^1 S^{n-1}_{tr} X \otimes Z \tag{26}$$

$$S^n_{tr} X \to S^n_{tr} Y \to S^n_{0,n-1}(X, Y) \to \Sigma^1 S^n_{tr} X \tag{27}$$

Proof: We will only analyze sequences (24), (25). The remaining cases are similar.

The sequence (24) is the last one of the family of sequences in Corollary 3.23. The first sequence of the same family gives us

$$S^n_{tr} X \to S^n_{tr} Y \to S^n_{0,n-1}(X, Y) \to \Sigma^1 S^n_{tr} X.$$

The remaining sequences of the family give us morphisms

$$S^n_{0,n-1}(X, Y) \to \ldots \to S^n_{0,1}(X, Y).$$

By our assumption on X, Lemma 3.32(1) and Proposition [26, Lemma 5.2(1)] these morphisms are isomorphisms in $H_{\mathbf{A}^1}(Cor(C))$ which implies our result.

Proposition 3.34 *Let l be a prime, R be an l-local ring and X is an object of $H_{\mathbf{A}^1}(Cor(C))$. Then one has:*

1. *If $l = 2$ or X is odd then there is a cofibration sequence of the form*

$$\Sigma_s^{l-1} X^{\otimes l} \to \Sigma^1 S^l_{tr}(X) \to S^l_{tr}(\Sigma^1_s X) \to \Sigma^l_s X^{\otimes l}. \tag{28}$$

2. *If $l = 2$ or X is even then there is a cofibration sequence of the form*

$$\Sigma_s^{1} X^{\otimes l} \to \Sigma^1 S^l_{tr}(X) \to S^l_{tr}(\Sigma^1_s X) \to \Sigma^2_s X^{\otimes l}. \tag{29}$$

Proof: There is an obvious coprojection sequence

$$X \to Cone(X) \to \Sigma^1_s X \tag{30}$$

where $Cone(X)$ is the simplicial cone of X. Applying to this sequence Proposition 3.33 and using Lemma 3.32(2) and Lemma 3.31 we get the required sequences.

We will now consider the finite correspondence versions of the infinite symmetric powers S^∞ and $S^\infty[1/d]$.

Proposition 3.35 *For any X in C_+, any $i \geq 1$ and any R the sequences*

$$\Lambda^l_R S^{i-1}(X) \to \Lambda^l_R S^i(X) \to \Lambda^l_R \tilde{S}^i(X) = S^i_{tr}(X)$$

in are split exact in a manner natural in X. In particular, there is an isomorphism

$$\Lambda^l_R(S^\infty(X)) \cong \oplus_{i\geq 1} S^i_{tr}(X).$$

Proof: Let $* : Spec(k) \to X$ denote the distinguished point of X and $S^n(X) \to S^{n+1}(X)$ the the inclusions given on the level of the products by $(x_1, \ldots, x_n) \mapsto (*, x_1, \ldots, x_n)$. We need to construct maps

$$\Lambda^l S^{n+1}(X) \to \Lambda^l S^n(X)$$

which split the morphisms in Cor defined by these inclusions. Since

$$\Lambda^l S^{n+1}(X) = \Lambda^l(X^{n+1}/S_{n+1}) = \Lambda^l(X^{n+1})/S_{n+1}$$

(by Proposition 3.12) it suffice to construct maps $s_{n+1} : \Lambda^l X^{n+1} \to \Lambda^l X^n$ such that

1. $\Lambda^l X^{n+1} \stackrel{s_{n+1}}{\to} \Lambda^l X^n \to (\Lambda^l X^n)/S_n$ is invariant under the action of S_{n+1} on X^{n+1},

2. $\Lambda^l X^n \stackrel{*\times Id}{\to} \Lambda^l X^{n+1} \stackrel{s_{n+1}}{\to} \Lambda^l X^n$ is the identity.

Let $[n]$ be the set $\{0, \ldots, n\}$. Any map $f : [m] \to [n]$ defines in the obvious way a map $f_X : X^{n+1} \to X^{m+1}$. Set

$$s_n = \sum_{0 \le m < n} \sum_{i:[m]\to[n]} (-1)^{n-m-1} [(*)^{(n-m-1)} \times i_X]$$

where i runs through all order preserving monomrphisms $[m] \to [n]$. For example for $n = 2$ we get (ignoring the [-] i the notation)

$$s_2 = pr_{12} + pr_{23} + pr_{13} - * \times pr_1 - * \times pr_2 - * \times pr_3 + * \times *.$$

One verifies easily that the maps s_n defined in this way satisfy the two conditions stated above.

Set $S^{\infty}_{tr} = \oplus_{n \ge 1} S^n_{tr}$. One verifies easily that for $X \in Cor(C, R)$, $S^{\infty}_{tr}(X)$ is an abelian monoid in $Cor(C, R)$ and that the monoid structure is natural in X. Therefore for $d > 0$ we may define the functor

$$S^{\infty}_{tr}[1/d] : Cor(C, R) \to Rad(Cor(C, R))$$

as the colimit of the sequence

$$S^{\infty}_{tr} \xrightarrow{\times d} S^{\infty}_{tr} \xrightarrow{\times d} \ldots$$

Combining Proposition 3.13 with Proposition 3.35 we get the following result.

Proposition 3.36 *For any $d > 0$ there is a commutative square of functors*

$$\begin{array}{ccc} C^{\#}_{+} & \xrightarrow{S^{\infty}[1/d]} & C^{\#}_{+} \\ {\scriptstyle \Lambda^l_R}\downarrow & & \downarrow{\scriptstyle \Lambda^l_R} \\ Cor^{\#} & \xrightarrow{S^{\infty}_{tr}[1/d]} & Cor^{\#} \end{array}$$

Proposition 3.37 *For any $d > 0$ and any X, Y in $Cor(C, R)$ there is an isomorphism*

$$S^{\infty}_{tr}[1/d](X \oplus Y) = [S^{\infty}_{tr}[1/d](X) \otimes S^{\infty}_{tr}[1/d](Y)] \oplus S^{\infty}_{tr}[1/d](X) \oplus S^{\infty}_{tr}[1/d](Y)$$

which is natural in X and Y.

Proof: Consider first the case of $d = 1$. We have

$$[S^{\infty}_{tr}(X) \otimes S^{\infty}_{tr}(Y)] \oplus S^{\infty}_{tr}(X) \oplus S^{\infty}_{tr}(Y) = \sum_{i,j \ge 0, i+j>0} S^i_{tr}(X) \otimes S^j_{tr}(Y) =$$

$$= \sum_{n>0} \sum_{i \ge 0} S^i_{tr}(X) \otimes S^{n-i}_{tr}(Y) = S^{\infty}_{tr}(X \oplus Y)$$

where the last equality holds by Proposition 3.18. One verifies easily that the multiplication by d preserves the decomposition which we established and therefore our result extends to $d > 0$.

Proposition 3.38 *The functors* $S^{\infty}_{tr}[1/d]$ *take* $\mathbf{A}^1$*-equivalences between objects of* $\Delta^{op}Cor^{\#}$ *to* $\mathbf{A}^1$*-equivalences. In particular there exist unique up to a canonical isomorphism functors*

$$S^{\infty}_{tr}[1/d] : H_{\mathbf{A}^1}(Cor(C)) \to H_{\mathbf{A}^1}(Cor(C))$$

such that the squares

$$\begin{array}{ccc} \Delta^{op}Cor^{\#} & \xrightarrow{S^{\infty}_{tr}[1/d]} & \Delta^{op}Cor^{\#} \\ \downarrow & & \downarrow \\ H_{\mathbf{A}^1} & \xrightarrow{S^{\infty}_{tr}[1/d]} & H_{\mathbf{A}^1} \end{array}$$

commute.

Proof: Follows immediately from Theorem 3.21 and the fact that $\mathbf{A}^1$-equivalences are closed under direct sums and filtering colimits.

Lemma 3.39 *Under the assumption Theorem 3.28 the class* $Cl_{vl}(A)$ *is closed under* $S^{\infty}_{tr}[1/d]$ *for all* $d > 0$.

Proof: Follows from the theorem and Lemma 3.26.

Remark 3.40 All the results proved in this section for $H_{\mathbf{A}^1}(Cor(C))$ also hold for the intermediate homotopy categories $H(Cor(C))$ and $H_{Nis}(Cor(C))$.

3 The motive of $\tilde{S}^l(T^n)$

In this section we consider finite correspondences with coefficients in the finite field $\mathbf{F}_l$ where l is a prime $l \neq char(k)$. The underlying category C will be the category of quasi-proijective schemes. Let $T^n = a_{Nis}(\mathbf{A}^n/(\mathbf{A}^n - \{0\}))$ be the standard model of the motivic n-sphere and

$$L_n = \mathbf{L}\Lambda^l_F(T^n)$$

its image in $H_{\mathbf{A}^1}(Cor(C))$. The goal of this section is to compute the isomorphism class of $S^l_{tr}(L_n)$.

Recall that for a linear representation $\rho : G \to Aut(V)$ of a finite group V we let $Th(\rho)$ denote the quotient sheaf (in the Nisnevich topology) $V/V - \{0\}$ where V and $V - \{0\}$ are considered as representable sheaves on quasi-projective G-schemes (see [22]). Applying the functor $Quot_G$ on sheaves we get a sheaf $Quot_G(Th(\rho))$.

We start by computing the isomorphism class of $\mathbf{L}\Lambda^l Quot_G(Th(\rho))$ in $H_{\mathbf{A}^1}(Cor(C))$ in the case when ρ is a representation of the cyclic group $\mathbf{Z}/l$. Similar computations were done independently by Nie (see [7]). In view of Corollary 2.12 we may work with Nisnevich sheaves their isomorphisms etc. instead of radditive functors as long as we are interested in computations up to Nis-equivalences.

Let us say that a linear representation $\rho : \mathbf{Z}/l \to Aut(V)$ is free if the corresponding action of $\mathbf{Z}/l$ on $V - \{0\}$ is free. Since $char(k) \neq l$, any ρ has a canonical decomposition into a direct sum $\rho = \lambda \oplus \tau$ where λ is free and τ is a trivial representation. As was shown in [22] one has an isomorphism of sheaves

$$Quot_{\mathbf{Z}/l}(Th(\lambda \oplus \tau)) = Quot_{\mathbf{Z}/l}(Th(\lambda)) \wedge T^d$$

where $d = dim(\tau)$. Therefore it is sufficient to consider free representations ρ. For a linear representation of any G we have

$$Quot_G(Th(\rho)) = Quot_G(V)/Quot_G(V - \{0\})$$

because $Quot_G$ is a left adjoint and therefore commutes with colimits (see [19, 5.1]). Set

$$X_\rho = Quot_G(V - \{0\}) = (V - \{0\})/G.$$

Since V is G-equivariantly $\mathbf{A}^1$-contractible, $Quot_G(V)$ is $\mathbf{A}^1$-contractible and therefore $Quot_G(Th(\rho))$ is the unreduced suspension of X_ρ i.e. there is a cofibration sequence in $H_{\mathbf{A}^1}(C_+)$ of the form

$$(X_\rho)_+ \to S^0 \to Quot_G(Th(\rho)) \to \Sigma^1(X_\rho)_+. \tag{31}$$

If $dim(V) \neq 0$ there exists a rational point $*$ in X_ρ and a choice of such a point defines a splitting of the sequence (31) and therefore an isomorphism in $H_{\mathbf{A}^1}(C_+)$ of the form

$$Quot_G(Th(V)) = \Sigma^1(X_\rho, *).$$

Collecting these arguments together we get the following result.

Proposition 3.41 *Let $\lambda : \mathbf{Z}/l \to Aut(V)$ be a linear representation of the cyclic group and $\lambda = \rho \oplus \tau$ be its decomposition into the direct sum of a free and a trivial representation. Assume that $dim(\rho) > 0$ and let $*$ be a rational point in X_ρ. Then there is a canonical isomorphism in $H_{\mathbf{A}^1}(C_+)$ of the form*

$$Quot_G(Th(\lambda)) = \Sigma^1(X_\rho, *) \wedge T^d$$

where $d = dim(\tau)$.

The scheme X_ρ is smooth and Corollary 2.32 together with Section 3 imply that we may do our computation in the more familiar context of the triangulated category DM_-^{eff}.

Consider the motivic cohomology group $H^{1,1}(X_\rho, \mathbf{F}_l)$. It coincides with the etale cohomology group $H^1_{et}(X_\rho, \mu_l)$. Since the action of $\mathbf{Z}/l$ on $V - \{0\}$ is free the projection $V - \{0\} \to X_\rho$ is an etale Galois covering with the Galois group $\mathbf{Z}/l$ which defines a map

$$u_\rho : \mu_l(k) = Hom(\mathbf{Z}/l, \mu_l) \to H^1_{et}(X_\rho, \mu_l) = H^{1,1}(X_\rho, \mathbf{Z}/l)$$

where $\mu_l(k)$ is the group of l-th roots of unity in k.

This construction is clearly natural i.e. the following lemma holds.

Lemma 3.42 *Let $\rho : \mathbf{Z}/l \to Aut(V)$, $\alpha : \mathbf{Z}/l \to Aut(W)$ be two free representations and $f : W - \{0\} \to V - \{0\}$ be an equivariant morphism. Then one has*

$$u_\alpha = Quot_{\mathbf{Z}/l}(f)^* u_\rho.$$

Assume now that k has an l-root of unity ξ and set $u_\rho = u_\rho(\xi)$. Let $v = \beta(u)$ where β is the Bockstein homomorphism. We now pass to the category DM and use the standard motivic notation. Classes v^i and uv^i define morphisms

$$v^i : M(X_\rho) \to \mathbf{F}_l(i)[2i]$$

$$uv^i : M(X_\rho) \to \mathbf{F}_l(i+1)[2i+1].$$

Consider the morphism

$$I(\rho,\xi) = \oplus_{i=0}^{n-1}(v^i \oplus uv^i) : M(X_\rho) \to \bigoplus_{i=0}^{n-1}(\mathbf{F}_l(i)[2i] \oplus \mathbf{F}_l(i+1)[2i+1]) \qquad (32)$$

where $n = dim(\rho)$.

Proposition 3.43 *The morphism $I(\rho,\xi)$ is an isomorphism.*

Proof: Our representation V can be written canonically as a direct sum $\oplus V_m$ where the restriction of ρ to V_m takes 1 to the multiplication by ξ^m. The condition that ρ is free means that $n > 0$ and $V_0 = 0$.

Consider first the case when $V = V_1$. Then [20, Lemma 6.3] implies that X_ρ is canonically isomorphic to the complement to the zero section of the line bundle $\mathcal{O}(-l)$ on $\mathbf{P}^{n-1}$ and our result follows easily by computations similar to the one in [20, pp.18-19].

Consider now the general $V = \oplus V_m$. Denote by W the same space as V but considered as a $\mathbf{Z}/l$-scheme with respect to the representation α where $\alpha(1) = (w \mapsto \xi w)$. Let us choose a basis (e_{ij}) in each V_j. Let $p : W \to V$ be the morphism which takes $\sum x_{ij}e_{ij}$ to $\sum x_{ij}^j e_{ij}$. This morphism is clearly $\mathbf{Z}/l$-equivariant and maps $W - \{0\}$ to $V - \{0\}$. Since m runs from 1 to $l-1$, the resulting morphism $q : W - \{0\} \to V - \{0\}$ is finite and surjective and of degree $\prod m^{dimV_m}$ which is prime to l. The same is then true for the morphism $p = Quot_{\mathbf{Z}/l}(q) : X_\alpha \to X_\rho$. Since we work with $\mathbf{F}_l$ coefficients the morphism of motives $M(p)$ is a split epimorphism.

Consider now the class $u_\rho = u_\rho(\xi) \in H^{1,1}(X_\rho, \mathbf{F}_l)$. By Lemma 3.42 we have $p^*(u_\rho) = u_\alpha$ and therefore we have $I(\rho, \xi) \circ M(p) = I(\alpha, \xi)$. Since $M(p)$ is a split epimorphism and $I(\alpha, \xi)$ is an isomorphism we conclude that both $M(p)$ and $I(\rho, \xi)$ are isomorphisms.

Let now k be a general field of characteristic $\neq l$, f_l be the polynomial $(x^l - 1)/(x - 1)$ and $E = k[x]/(f)$. If f_l is irreducible then E is a finite extension of k obtained by adjoining l-th roots of unity. In general E is a product of fields and in particular it is the product of $l - 1$ copies of k if k contains a non-trivial l-th root of unity. In any case the group $G = (\mathbf{Z}/l)^*$ acts on E over k and $Spec(E)/G = Spec(k)$. In the irreducible case G is the Galois group of E over k but in general it is a proper subgroup of $Aut(Spec(E)/Spec(k))$. In what follows we will use the terminology which may assume the irreducible case but the arguments actually do not depend on such an assumption. For simplicity of notations we will write $[E]$ instead of $[Spec(E)]$.

Let $\mathbf{F}_l\langle n\rangle$ be the unit object $\mathbf{F}_l$ of $H_{\mathbf{A}^1}(Cor(C))$ equipped with the action of $U = (Z/l)^*$ given by $i \mapsto i^n \cdot Id$. Since the order of $(\mathbf{Z}/l)^*$ is invertible in $\mathbf{F}_l$ the quotients with respect to actions of $(\mathbf{Z}/l)^*$ exist as direct summands in $H_{\mathbf{A}^1}$. Set

$$m_l = Quot_U(\mathbf{F}_l\langle 1\rangle \otimes [E])$$

Using the obvious projection formula one verifies easily that for any X one has

$$m_l \otimes X = Quot_U(\mathbf{F}_l\langle 1\rangle \otimes X \otimes [E])$$

and

$$m_l^{\otimes n} = Quot_U(\mathbf{F}_l\langle n\rangle \otimes [E])$$

Below we will write m_l^n instead of $m_l^{\otimes n}$. Note that since $\mathbf{F}_l\langle n\rangle$ depends only on the class of n in $(\mathbf{Z}/l)^*$ we have

$$m_l^{l-1} = m_l^0 = \mathbf{F}_l. \tag{33}$$

Remark 3.44 If k has a non-trivial l-th root of unity then $Spec(E)$ is isomorphic to the sum of $l-1$ copies of $Spec(k)$. A choice of such a root defines an isomorphism $[E] \to \mathbf{F}_l\langle 1\rangle \otimes [E]$ in U-$H_{\mathbf{A}^1}$ and therefore an isomorphism $\mathbf{F}_l \to m_l$ in $H_{\mathbf{A}^1}$. If k has no non-trivial l-th roots of unity then m_l is a non-trivial object which is not even a Tate motive. It is however an Artin motive (or "0-motive") since m_l can be shown to be a direct summand of $[E]$.

Remark 3.45 For X over k let $\mu_l(X)$ be the group of l-th roots of unity in $\mathcal{O}^*(X)$. Since $\mu_l(-)$ is functorial for finite correspondences as well as for morphisms we may consider it as an object of $Rad(Cor)$ and hence as an object of $H_{\mathbf{A}^1}$. One verifies easily this object coincides with our m_l.

Let ρ be a free representation. Applying the construction of $I(\rho, \xi)$ to the lifting of ρ to E/k and the canonical l-th root of unity in E we get an isomorphism of the form (32). Restricting scalars back to k we get an isomorphism

$$M(X_\rho)\otimes[E]\to\bigoplus_{i=0}^{n-1}(\mathbf{F}_l(i)[2i]\oplus\mathbf{F}_l(i+1)[2i+1])\otimes[E] \tag{34}$$

This isomorphism is not equivariant with respect to the action of $G=(\mathbf{Z}/l)^*$ but it defines an equivariant isomorphism of the form

$$J(\rho): M(X_\rho)\otimes[E]\to\bigoplus_{i=0}^{n-1}(\mathbf{F}_l\langle i\rangle(i)[2i]\oplus\mathbf{F}_l\langle i+1\rangle(i+1)[2i+1])\otimes[E] \tag{35}$$

Applying $Quot_U$ to $J(\rho)$ we get an isomorphism

$$I(\rho): M(X_\rho)\to\bigoplus_{i=0}^{n-1}(m_l^i(i)[2i]\oplus m_l^{i+1}(i+1)[2i+1]). \tag{36}$$

Let us look now at the reduced motive $\tilde{M}(X_\rho,*)$ for a point $*$ of X_ρ. Let us say that $*$ is a split point if it is the image of a rational point in $V-\{0\}$. In that case the restriction of u to $Spec(k)$ with respect to $*$ is trivial and we get from (36) an isomorphism

$$\tilde{M}(X_\rho,*)=m_l^n(n)[2n-1]\oplus\bigoplus_{i=1}^{n-1}(m_l^i(i)[2i-1]\oplus m_l^i(i)[2i])$$

since passing to the reduced motive eliminates the direct summand $\mathbf{F}_l=m_l^0(0)[0]$.

Combining this computation with Proposition 3.41 we get the following result where we use the motivic notation in $H_{\mathbf{A}^1}(Cor(C))$.

Theorem 3.46 *Let $\lambda:\mathbf{Z}/l\to Aut(V)$ be a linear representation of the cyclic group and $\lambda=\rho\oplus\tau$ be its decomposition into the direct sum of a free and a trivial representation. Assume that $d=dim(\rho)>0$ and let $*$ be a split rational point in X_ρ. Then there is a canonical isomorphism in $H_{\mathbf{A}^1}(Cor(C))$ of the form*

$$\tilde{M}(Quot_{\mathbf{Z}/l}(Th(\lambda)))=m_l^d(d+n)[2d+2n]\oplus\bigoplus_{i=1}^{d-1}(m_l^i(i+n)[2i+2n]\oplus m_l^i(i+n)[2i+1+n]) \tag{37}$$

where $n=dim(\tau)$.

We will consider now a special case when $\lambda:\mathbf{Z}/l\to Aut(V)$ is the direct sum of n copies of the regular representation of $\mathbf{Z}/l$. The additional feature which appears in this case is the action of the automorphism group $U=(\mathbf{Z}/l)^*$ of $\mathbf{Z}/l$ on V. Let us denote this action by

$s : (\mathbf{Z}/l)^* \to Aut(V)$. This action does not commute with the action defined by λ i.e. the morphisms $s(m)$ are not $\mathbf{Z}/l$-equivariant but for any $a \in (\mathbf{Z}/l)^*$ the square

$$\begin{array}{ccc} V - \{0\} & \xrightarrow{s(m)} & V - \{0\} \\ {\scriptstyle \lambda(1)}\downarrow & & \downarrow{\scriptstyle \lambda(a)} \\ V - \{0\} & \xrightarrow{s(m)} & V - \{0\} \end{array}$$

commutes. Therefore s defines an action of $(\mathbf{Z}/l)^*$ on the related quotients and in particular on $Quot_{\mathbf{Z}/l}(Th(V))$ and we write

$$r(a) : Quot_{\mathbf{Z}/l}(Th(V)) \to Quot_{\mathbf{Z}/l}(Th(V))$$

for the automorphism corresponding to $a \in (\mathbf{Z}/l)^*$.

Proposition 3.47 *Let ρ be the direct sum of $n > 0$ copies of the regular representation of $\mathbf{Z}/l$. Then one has a canonical isomorphism*

$$\tilde{M}(Quot_{\mathbf{Z}/l}(Th(\rho))) = m_l^d(nl)[2nl] \oplus \bigoplus_{i=1}^{n(l-1)-1} (m_l^i(i+n)[2i+2n] \oplus m_l^i(i+n)[2i+1+n]) \quad (38)$$

With respect to this isomorphism the morphism $\tilde{M}(r(a))$ is of the form

$$M(r(a)) = a^{-n(l-1)} Id \oplus \bigoplus_{i=1}^{n(l-1)-1} (a^{-i} Id \oplus a^{-i} Id) \quad (39)$$

Proof: In the decomposition

$$V_\rho = V_\lambda \oplus V_\tau \quad (40)$$

of V into the free and trivial parts we have $dim(V_\lambda) = n(l-1)$ and $dim(V_\tau) = n$ which together with Theorem 3.46 implies the first part of the lemma. To prove the second part observe first that the decomposition (40) is invariant under the action of $(\mathbf{Z}/l)^*$ and that the corresponding action on V_τ is trivial. Therefore, the action of $(\mathbf{Z}/l)^*$ on (38) is determined by its action on $M(X_\lambda)$. To verify (39) we may extend our base field such that it acquires a non-trivial l-th root of unity ξ. Then the action of any endomorphism of X_λ on its motive is determined by its action on the motivic cohomology class $u_\lambda = u_\lambda(\xi)$.

Note now that we may consider $s(m)$ as an equivariant morphism assuming that the action on the first copy of $V_\rho - \{0\}$ is given by ρ and on the second copy by ρ_m where $\rho_m(1) = \rho(m)$. Applying Lemma 3.42 we conclude that $s(m)^*(u_\rho) = m^{-1} \cdot u_\rho$ which implies our result.

We are ready now to prove the main theorem of this section. Recall that we let L_n denote the image of T^n in $H_{\mathbf{A}^1}(Cor(C))$. Note that in the motivic notation we have $L_n = \mathbf{F}_l(n)[2n]$.

Theorem 3.48 *Let l be a prime and k be a perfect field of characteristic $\neq l$. Let C be the category of quasi-projective schemes over k. Then for any $n > 0$ there is an isomorphism in $H_{\mathbf{A}^1}(Cor(C, \mathbf{F}_l))$ of the form*

$$S^l_{tr}(L_n) = L_{ln} \oplus \bigoplus_{i=1}^{n-1} (L_{i(l-1)+n} \oplus \Sigma^1_s L_{i(l-1)+n}).$$

Proof: Let Ψ be the permutation group $(G, i : G \to S_l)$ where $G = \mathbf{Z}/l$ is embedded into S_l as the subgroup generated by the cycle $(1 \dots l)$. The symmetric power $\tilde{S}^\Psi$ associated with Ψ is the l-th cyclic power. Let us apply Theorem 3.14 to the natural transformation of functors $S^\Psi_{tr} \to S^l_{tr}$. Since $\mathbf{Z}/l$ has no non-trivial subgroups for any $g \in S_l$ one has $gGg^{-1} = G$ or $G \cap gGg^{-1} = \{e\}$. Therefore, by Theorem 3.14 for any F in $Rad(Cor(C, R))$ the submodule $F(S^l_{tr}(X))$ of $F(S^\Psi_{tr}(X))$ consists of elements which are invariant under the action of

$$(\mathbf{Z}/l)^* = Norm(G)/G = Aut(G)$$

and such that their image in $F(X^{\otimes l})$ is invariant under the action of S_l.

We have

$$S^\Psi_{tr}(L_n) = \mathbf{L}\Lambda^l \tilde{S}^\Psi(Lres(T^n)).$$

As was noted in the previous section T^n is a solid sheaf. Therefore up to a Nis-equivalence one has

$$\tilde{S}^\Psi(Lres(T^n)) = \tilde{S}^\Psi(T^n) = \tilde{S}^\Psi \mathbf{A}^n / (\tilde{S}^\Psi \mathbf{A}^n - \{0\}) = Quot_G(Th(\rho_n))$$

where ρ_n is the direct sum of n copies of the regular representation of $\mathbf{Z}/l$. Here the first equality holds by Theorem 3.6 and the second by Proposition 3.4. By Proposition 3.47 we have

$$S^\Psi_{tr}(L_n)^{(\mathbf{Z}/l)^*} = (\mathbf{L}\Lambda^l Quot_G(Th(\rho_n)))^{(\mathbf{Z}/l)^*} = L_{ln} \oplus \bigoplus_{i=1}^{n-1} (L_{i(l-1)+n} \oplus \Sigma^1_s L_{i(l-1)+n}). \qquad (41)$$

It remains to verify that

$$S^\Psi_{tr}(L_n)^{(\mathbf{Z}/l)^*} = S^l_{tr}(L_n)$$

which according to the previous remarks is equivalent to checking that the restrictions of the motivic cohomology classes of $S^\Psi_{tr}(L_n)^{(\mathbf{Z}/l)^*}$ corresponding to the direct summands of the right hand side of (41) to $L_n^{\otimes l} = L_{nl}$ are invariant under the action of S_l. Since the motivic cohomology of L_{nl} of weights strictly less than nl are zero it is sufficient to verify the invariance of the top class given by the projection to L_{nl}. The corresponding motivic cohomology group of L_{nl} is $\mathbf{F}_l$ and the action of permutations on this group is trivial since L is an even object.

4 Proper Tate objects

In this section we continue to assume that the coefficients ring used to define finite correspondences is a field F.

Definition 3.49 *An object X in $H_{\mathbf{A}^1}(Cor(C))$ is called a proper Tate object if it is isomorphic to a coproduct (direct sum) of objects of the form $\Sigma^i L_j$ for $i \geq 0$.*

We denote the full subcategory of proper Tate objects by PT. Let further PT_n (resp. $PT_{\leq n}$) be the full subcategory in PT which consists of direct sums of objects of the form $\Sigma^k L_n$ (resp. $\Sigma^k L_m$ for $m \leq n$). All these subcategories are clearly closed under direct sums and tensor products.

Since objects L_j belong to $H_{\mathbf{A}^1}(Cor(Sm/k))$, Theorem 2.27 is applicable and we may consider PT as a subcategory of DM. In the standard DM notation this subcategory consists of direct sums of objects of the form $F(n)[m]$ with $m \geq 2n$. To proceed further it will be convenient for us to distinguish some other subcategories in $DM = DM_-$:

1. We let $\overline{DT}$ (resp. $\overline{DT}_{\leq n}$ or $\overline{DT}_n$) denote the full subcategory of mixed Tate objects in DM i.e. the smallest subcategory which contains elementary Tate objects $F(i)[j]$ for all $i \geq 0$ (resp. for all $n \geq i \geq 0$ or for $i = n$) and all $j \in \mathbf{Z}$ and is closed under direct sums, direct summands and distinguished triangles.
2. We let $\overline{SDT}$ (resp. $\overline{SDT}_{\leq n}$ or $\overline{SDT}_n$) denote the full subcategory pure Tate objects i.e. the full subcategory of $\overline{DT}$ (resp. of $\overline{DT}_{\leq n}$ or $\overline{DT}_n$) which consists of direct sums of elementary Tate objects.

Proposition 3.50 *The category $\overline{DT}_n$ is equivalent to the category of graded F-vector spaces $(V_i)_{i\in\mathbf{Z}}$ for which there exists j such that $V_i = 0$ for all $i < j$. In particular it is abelian and semi-simple.*

Proof: The cancellation theorem shows that the functor $\overline{DT}_0 \to \overline{DT}_n$ given by $X \mapsto X(n)$ is an equivalence. The statement of the proposition for $n = 0$ follows immediately from the fact that $H^{0,0}(Spec(k), F) = F$ and $H^{p,0}(Spec(k), F) = 0$ for $p \neq 0$.

Corollary 3.51 *One has $\overline{DT}_n = \overline{SDT}_n$.*

Corollary 3.52 *For any $n \geq 0$ the category PT_n is equivalent to the full subcategory of the category of graded F-vector spaces which consists of objects (V_i) such that $V_i = 0$ for $i < 2n$.*

Corollary 3.53 *The categories PT_n are abelian semi-simple categories.*

Corollary 3.54 *The subcategory $\overline{SDT}_n$ (resp. PT_n) is closed under cones.*

Proof: Let $X \stackrel{f}{\to} Y \to Z \to X[1]$ be a distinguished triangle with $X, Y \in PT_n$. In view of Proposition 3.52 the morphism f is isomorphic to a morphism of the form

$$ker(f) \oplus Im(f) \stackrel{0\oplus Id}{\to} coker(f) \oplus Im(f)$$

Since the cone of a direct sum of two morphisms is the direct sum of cones we reduced the problem to the cases $f = 0$ and $f = Id$ which are obvious.

In view of Proposition 3.50 there are well defined functors

$$d_i : \overline{DT}_n \to F - vect$$

such that $d_i(F(n)[i]) = F$ and $d_i(F(n)[i]) = 0$ for $i \neq j$.

We clearly have

$$PT \subset \overline{SDT} \subset \overline{DT}.$$

For any X which belongs to $\overline{DT}$ and any n let

$$s_{>n}X \to X \stackrel{p_n}{\to} s_{\leq n}X \to s_{>n}X[1] \tag{42}$$

and

$$s_nX \to s_{\leq n}X \stackrel{q_n}{\to} s_{\leq n-1}X \to s_nX[1] \tag{43}$$

be the corresponding members of the slice tower of X (see [23])[4].

Lemma 3.55 *The functor $\oplus_{m\leq n} s_m$ on $\overline{DT}_{\leq n}$ reflects isomorphisms.*

Proof: The towers (42), (43) are finite for objects of $\overline{DT}_{\leq n}$ since they are finite for the generating objects of this category. This immediately implies that for any $X \in \overline{DT}_{\leq n}$ such that $\oplus_{m\leq n} s_m(X) = 0$ we have $X = 0$. Since the functor $\oplus_{m\leq n} s_m$ is triangulated we conclude that it reflects isomorphisms.

Remark 3.56 As one can see from Remark 3.67 below that the analog of Lemma 3.55 for the whole $\overline{DT}$ and $\oplus_{n\geq 0} s_n$ is false.

Recall that an object X is called compact (resp. countably compact) if for any (resp. any countable) set of objects $(Y_\alpha)_{\alpha\in A}$ one has

$$Hom(X, \oplus_{\alpha\in A} Y_\alpha) = \oplus_\alpha Hom(X, Y_\alpha).$$

Recall also the following results.

[4] In [23] we used the notations $\Pi_{\geq n}$ and $\Pi_{<n}$ instead of $s_{\geq n}$ and $s_{<n}$.

Lemma 3.57 *For any smooth scheme U the object $M(U)$ is compact in DM.*

Corollary 3.58 *The Tate objects $F(i)[j]$ are compact.*

Lemma 3.59 *Objects of the form $M(U)$ for smooth schemes U generate DM in the sense that for any X such that $Hom(M(U)[n], X) = 0$ for all U and all $n \in \mathbf{Z}$ one has $X = 0$.*

Proposition 3.60 *Let $X \in \overline{DT}$. Then $X \in \overline{SDT}$ iff the following two conditions hold:*

1. *All the members of the slice tower (42) for X split.*
2. *For any scheme U and any $i \in \mathbf{Z}$ one has*

$$\cap_{n\geq 0} Im(Hom(M(U)[i], s_{>n}X) \to Hom(M(U)[i], X)) = 0.$$

Proof: If X is a direct sum of Tate objects then the first condition holds for obvious reasons and the second holds since $M(U)$ is compact. Let us show that the opposite implication holds.

By Corollary 3.51 we have $s_nX \in \overline{SDT}$ for any X. Let X be such that the morphisms p_n in triangles (42) split. Let us choose their sections $t_n : s_{\leq n}X \to X$ and let $i_n : s_nX \to s_{\leq n}X$ be the canonical morphism from the corresponding triangle (43). Set

$$r_n = t_n i_n : s_nX \to X$$

and

$$r = \oplus_{n\geq 0} r_n : \oplus_n s_nX \to X.$$

Since $Hom(s_mX, s_{\leq n}X) = 0$ for $m > n$ there exists a unique morphism $u_n : \oplus_{m\leq n} s_mX \to s_{\leq n}X$ such that the square

$$\begin{array}{ccc} \oplus_{m\geq 0} s_mX & \xrightarrow{r} & X \\ \downarrow & & \downarrow p_n \\ \oplus_{n\geq m\geq 0} s_mX & \xrightarrow{u_n} & s_{\leq n}X \end{array} \tag{44}$$

commutes. Let us show that u_n is an isomorphism. In view of Lemma 3.55 it is sufficient to show that $s_j(u_n)$ is an isomorphism for all $j \leq n$. Note that for any $j \leq n$ the morphisms u_n and u_j fit into a commutative square

$$\begin{array}{ccc} \oplus_{n\geq m\geq 0} s_mX & \xrightarrow{u_n} & s_{\leq n}X \\ \downarrow & & \downarrow \\ \oplus_{j\geq m\geq 0} s_mX & \xrightarrow{u_j} & s_{\leq j}X \end{array} \tag{45}$$

and that for any j the composition $s_jX \to \oplus_{j\geq m\geq 0} \stackrel{u_m}{\to} s_{\leq j}X$ equals i_j. Applying the functor s_j to the square and using the fact that $s_j(i_j)$ is an isomorphism we conclude that all $s_j(u_n)$ and therefore u_n itself are isomorphisms.

Extending the columns of (44) to distinguished triangles and letting *cone* denote the cone of r we conclude that there are distinguished triangles of the form

$$\oplus_{m>n} s_m X \to s_{>n}X \to cone \to \oplus_{m>n} s_m X[1]$$

for all n. So far we have only used the first condition of the proposition. Let us show now that under the second condition we have $cone = 0$ which implies that r is an isomorphism and therefore $X \in \overline{SDT}$.

By Lemma 3.59 it is enough to show that $Hom(M(U)[i], cone) = 0$ for all smooth U and all $i \in \mathbf{Z}$. One verifies easily using Lemma 3.57 that for any m, any morphism $M(U)[i] \to cone$ lifts to a morphism $M(U)[i] \to s_{>m}X$ which immediately implies our claim.

Corollary 3.61 *An object X of $\overline{DT}$ lies in PT iff the conditions of Proposition 3.60 hold and in addition $d_i s_n(X) = 0$ for $i < 2n$.*

Proposition 3.62 *The subcategories $\overline{SDT}$ and PT are closed under direct summands.*

Proof: The conditions of Proposition 3.60 and Corollary 3.61 are obviously stable under passing to direct summands.

Lemma 3.63 *Let $X \to Y \to Z \to X[1]$ be a distinguished triangle with all the objects being in $\overline{DT}$. Then for all n there exist a commutative diagrams of the form*

$$\begin{array}{ccccccc}
s_{>n}X & \longrightarrow & s_{>n}Y & \longrightarrow & s_{>n}Z & \longrightarrow & s_{>n}X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
s_{\leq n}X & \longrightarrow & s_{\leq n}Y & \longrightarrow & s_{\leq n}Z & \longrightarrow & s_{\leq n}X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
s_{>n}X[1] & \longrightarrow & s_{>n}Y[1] & \longrightarrow & s_{>n}Z[1] & \longrightarrow & s_{>n}X[2]
\end{array}$$

and

$$\begin{array}{ccccccc}
s_n X & \longrightarrow & s_n Y & \longrightarrow & s_n Z & \longrightarrow & s_n X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
s_{\leq n} X & \longrightarrow & s_{\leq n} Y & \longrightarrow & s_{\leq n} Z & \longrightarrow & s_{\leq n} X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
s_{<n} X & \longrightarrow & s_{<n} Y & \longrightarrow & s_{<n} Z & \longrightarrow & s_{<n} X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
s_n X[1] & \longrightarrow & s_n Y[1] & \longrightarrow & s_n Z[1] & \longrightarrow & s_n X[2]
\end{array}$$

where rows are distinguished triangles and columns are the distinguished triangles of (42) and (43) respectively.

Proof: Straightforward.

Lemma 3.64 *Consider a cofibration sequence in $H_{\mathbf{A}^1}(Cor(C))$ of the form*

$$X \to Y \to Z \to \Sigma^1 X \tag{46}$$

Assume further that $X \in \overline{SDT}_{\leq n}$ (resp. in $X \in PT_{\leq n}$) and $Z \in \overline{SDT}_n$ (resp. $Z \in PT_n$). Then $Y \in \overline{SDT}_{\leq n}$ (resp. $Y \in PT_{\leq n}$).

Proof: We only consider the case of $\overline{SDT}$, the case of PT is similar. In view of lemma 3.63 there is a commutative diagram

$$\begin{array}{ccccccc}
s_n X & \longrightarrow & s_n Y & \longrightarrow & Z & \longrightarrow & s_n X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
s_{<n} X & \xrightarrow{3} & s_{<n} Y & \longrightarrow & 0 & \longrightarrow & s_{<n} X[1] \\
{\scriptstyle 2}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow & & \downarrow \\
s_n X[1] & \longrightarrow & s_n Y[1] & \longrightarrow & s_n Z[1] & \longrightarrow & s_n X[2]
\end{array}$$

where both rows and columns are distinguished triangles. To prove the lemma it is sufficient to show that morphism (1) is zero since then

$$Y \cong s_{<n} X \oplus Z \in \overline{SDT}_{\leq n}.$$

This follows immediately from the fact that (3) is an isomorphism and (2) is zero since $X \cong s_n X \oplus s_{<n} X$.

Proposition 3.65 *The subcategories $PT_{\geq q}$ are closed under homotopy colimits of sequences.*

Proof: It is clearly sufficient to consider the category $PT = PT_{\geq 0}$. Let $X_1 \stackrel{s_1}{\to} \ldots \stackrel{s_{n-1}}{\to} X_n \stackrel{s_n}{\to} \ldots$ be a sequence of morphisms in $H_{\mathbf{A}^1}(Cor(C))$. Recall that

$$hocolim_{i\geq 1}X_i = cone(\oplus_{i\geq 1}X_i \stackrel{Id-s}{\to} \oplus_{\geq 1} iX_i)$$

where $s = \oplus_{i\geq 1}s_i$ so that in DM we have a distinguished triangle

$$\oplus_{i\geq 1}X_i \to \oplus_{i\geq 1}X_i \to hocolim_{i\geq 1}X_i \to \oplus_{i\geq 1}X_i[1]. \tag{47}$$

One sees immediately that for any countably compact X, a sequence X_i as above and any morphism $X \to hocolim_{i\geq 1}X_i$ the composition

$$X \to hocolim_{i\geq 1}X_i \to \oplus_{i\geq 1}X_i[1]$$

is zero. Therefore, the same holds for any X which is a direct sum of compact objects. Set

$$X_\infty = hocolim_{i\geq 1}X_i$$

Let us show that X_∞ satisfies the conditions of Corollary 3.61. It is clearly sufficient to check that it satisfies the conditions of Proposition 3.60. To verify the second condition observe that $s_{>n}X_\infty$ fits into a distinguished triangle

$$\oplus_i s_{>n}X_i \to \oplus_i s_{>n}X_i \to s_{>n}X_\infty \to \oplus_i s_{>n}X_i[1]$$

and that since $X_i \in PT$ one has

$$Hom(M(U)[j], s_{>n}X_i) = 0$$

for all $n > j + dim(U)$.

It remains to verify that the triangles (42) split for X_∞. Let us show first that the triangles (43) split and therefore $s_{\leq n}X_\infty \in PT$ for all n.

We proceed by induction on n. For $n = -1$ the statement is trivial. Consider the second diagram of Lemma 3.63 corresponding to (47):

$$\begin{array}{ccccccc}
\oplus_i s_n X_i & \longrightarrow & \oplus_i s_n X_i & \longrightarrow & s_n X_\infty & \longrightarrow & \oplus_i s_n X_i[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\oplus_i X_i & \longrightarrow & \oplus_i X_i & \longrightarrow & X_\infty & \longrightarrow & \oplus_i X_i[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\oplus_i s_{<n} X_i & \longrightarrow & \oplus_i s_{<n} X_i & \stackrel{3}{\longrightarrow} & s_{<n} X_\infty & \stackrel{4}{\longrightarrow} & \oplus_i s_{<n} X_i[1] \\
\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 2}\downarrow & & \downarrow \\
\oplus_i s_n X_i[1] & \longrightarrow & \oplus_i s_n X_i[1] & \longrightarrow & s_n X_\infty[1] & \longrightarrow & \oplus_i s_n X_i[2]
\end{array}$$

whose rows and columns are distinguished triangles. By Corollary 3.54 we have

$$s_n X_\infty \in PT_n$$

and by the inductive assumption

$$s_{<n} X_\infty \in PT_{<n}. \tag{48}$$

By (48) and Lemma 3.58, $s_{<n}X_\infty$ is a direct sum of compact objects therefore morphism (4) is zero and (3) is a split epimorphism. On the other hand (1) is zero since $X_i \in PT$ and we conclude that (2) is zero and there is an isomorphism

$$X_\infty = s_n X_\infty \oplus s_{<n} X_\infty$$

and in particular $X_\infty \in PT_{\leq n}$.

Consider now the diagram

$$\begin{array}{ccccccc}
\oplus_i s_{>n} X_i & \longrightarrow & \oplus_i s_{>n} X_i & \longrightarrow & s_{>n} X_\infty & \longrightarrow & \oplus_i s_{>n} X_i[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\oplus_i X_i & \longrightarrow & \oplus_i X_i & \longrightarrow & X_\infty & \longrightarrow & \oplus_i X_i[1] \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle 5} & & \downarrow \\
\oplus_i s_{\leq n} X_i & \longrightarrow & \oplus_i s_{\leq n} X_i & \xrightarrow{3} & s_{\leq n} X_\infty & \xrightarrow{4} & \oplus_i s_{\leq n} X_i[1] \\
\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 2}\downarrow & & \downarrow \\
\oplus_i s_{>n} X_i[1] & \longrightarrow & \oplus_i s_{>n} X_i[1] & \longrightarrow & s_{>n} X_\infty[1] & \longrightarrow & \oplus_i s_{>n} X_i[2]
\end{array}$$

with all rows and columns being distinguished triangles. By the first part of the proof we know that $s_{\leq n}X_\infty$ is a direct sum of compact object and therefore (3) is a split epimorphism. Since (1) is zero we conclude that (2) is zero and therefore (5) is a split epimorphism.

As an immediate corollary of the proof of Proposition 3.65 we get the following result.

Corollary 3.66 *Let* $X_1 \xrightarrow{s_1} \ldots \xrightarrow{s_{n-1}} X_n \xrightarrow{s_n} \ldots$ *be a sequence of morphisms in* PT. *Then the distinguished triangle*

$$\oplus_{i\geq 1} X_i \to \oplus_{i\geq 1} X_i \to hocolim_{i\geq 1} X_i \to \oplus_{i\geq 1} X_i[1]. \tag{49}$$

splits.

Remark 3.67 The whole category of pure Tate objects is clearly not closed under homotopy colimits of sequences. Let for example $k = \mathbf{R}$ be the field of real numbers. Let further $\mu \in H^{1,1}(Spec(k), \mathbf{Z}/2)$ be the class corresponding to -1. Then $\mu^{\wedge n} \neq 0$ for all n and the sequence

$$\mathbf{F}_2 \xrightarrow{\mu} \mathbf{F}_2(1)[1] \xrightarrow{\mu^{\wedge 2}} \mathbf{F}_2(2)[2] \to \ldots$$

has a homotopy colimit X which is not a direct sum of Tate objects since $s_{\leq n}X = 0$ for all n. We do not know if the similar phenomenon may occur in $H_{\mathbf{A}^1}(Cor(C))$ (or DM_-) over a field whose etale cohomological dimension with coefficients in F-modules is finite. If we consider the larger category DM which allows for complexes infinite in both directions and $char(F) = l$ then we get an example from the sequence

$$F \xrightarrow{a} F(l-1) \xrightarrow{a^{\wedge 2}} F(2(l-1)) \to \dots$$

where a is the canonical element of $H^{l-1,0}(Spec(k), F)$. The homotopy colimit of this sequence is the object representing the etale cohomology with coefficients in F.

Our next goal is to prove Theorem 3.70 which shows that PT is closed under (standard) symmetric powers. Note that symmetric powers have only been defined on $H_{\mathbf{A}^1}(Cor(C, R))$ where C is an f-admissible category $char(k) = 0$ or $char(k)$ is invertible in R. Therefore, for the purposes of Theorem 3.70 we must consider PT as a full subcategory in $H_{\mathbf{A}^1}(Cor(C, F))$ where C is the category of quasi-projective schemes and if $char(k) > 0$ then F is a field such that $char(F) \neq char(k)$.

Proposition 3.68 *The object L is 1-even.*

Proof: See [4, Corollary 15.8].

Lemma 3.69 *One has $S^l_{tr}(\Sigma^k L^m) \in PT_{\leq lm}$ for any $k, m \geq 0$.*

Proof: One proceeds by induction on k. For $k = 0$ the result follows from Theorem 3.48. The inductive step follows immediately from Proposition 3.34 and Lemma 3.64.

Theorem 3.70 *Let k be a field such that $char(k) = 0$ or $char(k) \neq char(F)$. Then for any $q \geq 0$ the category $PT_{\geq q}$ is closed under symmetric powers i.e. for $X \in PT_{\geq q}$ and any $n \geq 0$ one has $S^n_{tr}(X) \in PT_{\geq q}$.*

Proof: We will only consider the case $char(F) = l > 0$. The case of $char(F) = 0$ is obvious. In view of Corollary 3.19 it is sufficient to show that $S^n_{tr}(\Sigma^k L^q)$ is in $PT_{\geq q}$ for all $k, q, n \geq 0$. This follows from Proposition 3.17, Proposition 3.62 and Lemma 3.69.

Corollary 3.71 *Under the assumptions of the theorem one has $S^{\infty}_{tr}[1/c](X) \in PT_{\geq q}$ for all $c > 0$.*

Proof: For $c = 1$ it follows from the theorem since by definition one has

$$S^{\infty}_{tr}(X) = \oplus_{n\geq 1} S^n_{tr}(X).$$

For $c > 0$ it follows from the case $c = 1$ and Proposition 3.65.

Remark 3.72 Note that out computation of the l-th cyclic power off T^n in Section 3 shows that PT is not closed under the generalized symmetric products S^{Φ}_{tr}.

Remark 3.73 We can make the computations done in the proof of Theorem 3.70 more precise as follows. First observe that Proposition 3.34 implies that there are cofibration sequences in $H_{\mathbf{A}^1}(Cor(C))$ of the form

$$\Sigma^1 S^l_{tr}(\Sigma^{2i+1} L^m) \to S^l_{tr}(\Sigma^{2i+2} L^m) \to \Sigma^{2(i+1)l} L^{lm} \tag{50}$$

and

$$\Sigma^1 S^l_{tr}(\Sigma^{2i} L^m) \to S^l_{tr}(\Sigma^{2i+1} L^m) \to \Sigma^{2il+2} L^{lm} \tag{51}$$

for all $k \geq 0$. Using the topological realization functor together with the fact that we know the topological homology of $\tilde{S}^l(S^n)$ with coefficients in $\mathbf{F}_l$ one can show that these sequences split so that we have isomorphisms

$$S^l_{tr}(\Sigma^{2i} L^m) = \Sigma^1 S^l_{tr}(\Sigma^{2i-1} L^m) \oplus \Sigma^{2il} L^{lm}$$

and

$$S^l_{tr}(\Sigma^{2i+1} L^m) = \Sigma^1 S^l_{tr}(\Sigma^{2i} L^m) \oplus \Sigma^{2il+2} L^{lm}$$

Using obvious induction on i one can now get explicit formulas for the isomorphism classes of $S^l_{tr}(\Sigma^i L^m)$.

4 Eilenberg-MacLane spaces and their motives

1 Generalities

Let us fix an admissible category C. Consider the motivic spheres $S^1_t = (\mathbf{A}^1 - \{0\}, 1)$ and $S^q_t = (S^1_t)^{\wedge q}$ as radditive functors on C_+. Since the distinguished point of $S^1_t = (\mathbf{A}^1 - \{0\}, 1)$ is not disjoint the radditive functor on C_+ which it represents is not in $C^{\#}_+$ and in what follows we have to use $\mathbf{L}\Lambda^l_R = \Lambda^l_R \circ Lres$ and similarly the derived versions of other functors when we apply them to S^q_t. Set

$$l_q = \mathbf{L}\Lambda^l_R(S^q_t) = \Lambda^l_R(Lres(S^q_t)).$$

Let us define *unstable* motivic cohomology of $X \in \Delta^{op}C^{\#}_{+}$ with coefficients in an abelian group A by the formula

$$H^{p,q}(X,A) = \begin{cases} Hom_{H_{\mathbf{A}^1}}(\Lambda^l X,\ \Sigma^{p-q}(A \otimes_{\mathbf{L}} l_q)) & \text{for} \quad p \geq q \\ Hom_{H_{\mathbf{A}^1}}(\Sigma^{p-q}\Lambda^l X,\ A \otimes_{\mathbf{L}} l_q) & \text{for} \quad p \leq q \end{cases} \tag{52}$$

where $\otimes_{\mathbf{L}}$ is the derived tensor product given by

$$A \otimes_{\mathbf{L}} l_q = Lres(A) \otimes l_q.$$

If A is an S-module for a commutative ring S we may consider in this definition the category $H_{\mathbf{A}^1}(Cor(C,S))$ with coefficients in S and take the tensor product over S.

If $C = Sm/k$ then the full embedding theorem 2.27 shows that this definition agrees with the definition

$$H^{p,q}(X,A) = Hom_{DM}(M(X), A(q)[p]) \tag{53}$$

which goes back to [27].

By the adjunction between Λ^r and $\mathbf{L}\Lambda^l$ the functors $H^{p,q}(-,A)$ are represented on $H_{\mathbf{A}^1}(C_+)$ by the spaces

$$K(A,p,q)_C = \begin{cases} \Lambda^r \Sigma^{p-q}(A \otimes_{\mathbf{L}} l_q) & \text{for} \quad p \geq q \\ \Omega^{p-q}_{\mathbf{A}^1}\Lambda^r(A \otimes_{\mathbf{L}} l_q) & \text{for} \quad p \leq q \end{cases} \tag{54}$$

In what follows we will only consider the case $p \geq q$. The case $p < q$ is much more complicated.

Let $c = 1$ if $char(k) = 0$ and $c = char(k)$ if $char(k) > 0$. For the reminder of this section set

$$S = \mathbf{Z}[1/c].$$

For an abelian group A we will sometimes write $A[1/c]$ instead of $A \otimes S$. If M is an abelian monoid we let M^+ denote its group completion and $M[1/c]$ the colimit of the sequence

$$M \stackrel{x \mapsto cx}{\longrightarrow} M \stackrel{x \mapsto cx}{\longrightarrow} \dots$$

Clearly, for any M one has $(M[1/c])^+ = M^+[1/c]$. Let us recall the following definitions and results (see [13], [2]).

Definition 4.1 *A scheme U is called semi-normal if it is reduced and any finite morphism $U' \to U$ such that U' is reduced and for any field K the map $U'(K) \to U(K)$ is bijective is an isomorphism.*

Lemma 4.2 *Let U be a semi-normal affine scheme of finite type over k and $f : U' \to U$ be a universal homeomorphism of finite type. Then there exists $n \geq 0$ such that $\mathcal{O}(U)$ contains $\mathcal{O}(U')^{c^n}k$. In particular, if $char(k) = 0$ then f is an isomorphism.*

Proof: Note first that a semi-normal scheme is necessarily reduced and therefore $\mathcal{O}(U) \to \mathcal{O}(U')$ is a monomorphism. Let K be a field and $x : Spec(K) \to U$ a K-point of U. Since f is a universal homeomorphism there exists a purely inseparable field extension $K \subset K'$ and a K' point x' of U' lying over x. Moreover there exists n such that for any x as above one may choose K' of degree c^n over K.

Let $R = \mathcal{O}(U)\mathcal{O}(U')^{c^n}$. We claim that $R = \mathcal{O}(U)$. Indeed the morphism $Spec(R) \to U$ is finite and since for a purely inseparable $K \subset K'$ one has $(K')^{degK'/K} \subset K$, our choice of n implies that for any K the map $Spec(R)(K) \to U(K)$ is a bijection.

Lemma 4.3 *Let U be an affine scheme of finite type over k and $d_n : \mathcal{O}(S^{c^n}U) \to \mathcal{O}(U)$ be the map defined by the diagonal. Then one has*

$$Im(d_n) \subset \mathcal{O}(U)^{c^n}k.$$

Proof: One observes first that $\mathcal{O}(S^iU)$ is generated by elements of the form $\sum a_1 \otimes \ldots \otimes a_i$ where $a_j \in \mathcal{O}(U)$ and the sum is over a set of representatives in S^i of cosets of S^i with respect to the stabilizer of the tuple $(a_1, \ldots, a_i)$ as an element of $\mathcal{O}(U)^{\times i}$. This observation together with the divisibility properties of binomial coefficients implies our claim.

Let SN/k be the category of semi-normal schemes over k. By Lemma 5.5 it is f-admissible.

Proposition 4.4 *For any X the functor $U \mapsto Hom(U, \amalg_{n\geq 0}S^nX)[1/c]$ is a sheaf in the qfh-topology on SN/k.*

Proof: Our functor is a filtering colimit of representable functors and the results of [15, Section 3.2] imply that the associated qfh-sheaf is of the form

$$U \mapsto colimit_{U' \to U} Hom(U', \amalg_{n\geq 0}S^nX)[1/c]$$

over all universal homeomorphisms $U' \to U$. To prove the proposition it remains to show that for a universal homeomorphism $U' \to U$ such that both U and U' are semi-normal, the map

$$Hom(U, \amalg_{n\geq 0}S^nX)[1/c] \to Hom(U', \amalg_{n\geq 0}S^nX)[1/c]$$

is bijective. Since semi-normal schemes are reduced it is injective. It remains to show that for a map $U' \to S^iX$ there exists n and a map $U \to S^{ic^n}X$ such that the diagram

$$\begin{array}{ccc} U' & \longrightarrow & S^iX \\ \downarrow & & \downarrow \\ U & \longrightarrow & S^{ic^n}X \end{array}$$

commutes. It is sufficient to consider the case of affine U, U' and X. Then the claim follows from Lemmas 4.3 and 4.2.

Proposition 4.5 *Let X be such that $S^n X$ exist and $Cor^{eff}(U, X)$ be the monoid of effective finite correspondences from U to X. Then for any semi-normal U one has:*

$$Hom(U, \amalg_{n\geq 0} S^n(X))[1/c] = Cor^{eff}(U, X)[1/c].$$

Proof: By [12, Proposition 4.2.7] $Cor(-, X)^{eff}[1/c]$ is a qfh-sheaf and by Proposition 4.4 the same holds for $S^\infty[1/c](X_+)$. On the other hand by [10, Theorem 6.8] we have

$$Cor^{eff}(U, X)[1/c] = Hom(U, S^\infty(X))[1/c].$$

for any normal U. Since any scheme has a qfh-covering by normal schemes we conclude that this equality holds for all semi-normal U.

Remark 4.6 It is not hard to see that there are natural maps

$$Hom(U, \amalg_{n\geq 0} S^n(X))[1/c] = Cor^{eff}(U, X)[1/c]$$

for all U and it might be the case that for $char(k) > 0$ these maps are bijective for all U. For $char(k) = 0$ i.e. $c = 1$ they are not necessarily surjective if U is not semi-normal. For example let U be the cuspidal cubic and $X = \mathbf{A}^1$. Then the graph of the normalization map $X \to U$ is in $Cor^{eff}(U, X)$ but clearly not in the image of $Hom(U, \amalg_{n\geq 0} S^n(X))$.

Theorem 4.7 *Let k be a perfect field and C an f-admissible category which is contained in the category of semi-normal schemes. Consider $\Lambda^r_S \Lambda^l_S$ as a functor from $C_+^\#$ to $Rad(C_+)$. Then one has*

$$\Lambda^r_S \Lambda^l_S = a_{Nis}(S^\infty[1/c])^+ \tag{55}$$

Proof: Since both sides of (55) commute with filtering colimits it is enough to very that for $X \in C$ one has:

$$\Lambda^r_S \Lambda^l_S(X_+) = a_{Nis}(S^\infty[1/c](X_+))^+$$

By definition, $\Lambda^r_S \Lambda^l_S$ takes X_+ to the functor

$$U \mapsto Cor(U, X)[1/c] \tag{56}$$

where $Cor(U, X)$ is the group of finite correspondences from U to X and by Proposition 4.5 we have

$$S^\infty[1/c](X_+) = Cor^{eff}(-, X)$$

It remains to show that $Cor(-, X) = a_{Nis}(Cor^{eff}(-, X)^+)$. This follows from the lemma below.

Lemma 4.8 *Let U be a henselian local scheme. Then*

$$Cor(U, X) = Cor^{eff}(U, X)^+$$

Proof: Note first that the statement is obvious fro normal schemes U. For a general U and a relative cycle $\mathcal{Z} = \sum n_i z_i$ on $X \times U$ over U the individual points z_i need not be relative cycles over U and a more sophisticated argument is required. By definition of a finite correspondence (cf. [21]) we know that the points z_i lie over the generic points of U and that the closure of each z_i is finite over U. Therefore, in order to show that $\mathcal{Z}$ is the difference of two effective relative cycles it is enough to show that for any z which lies over a generic point of U and such that its closure Z is finite over U there exists a relative finite cycle of the form $z + \sum m_j z_j$ where $m_j \geq 0$. To use induction we will consider a slightly more general situation namely that of a flat equi-dimensional morphism $X \to U$ and z in X. Let us proceed by induction on the dimension of X over U. If $dim(X/U) = 0$ then we have a quasi-finite morphism over a henselian scheme and by removing the components which are not finite over U we may assume that it is finite. Then the fundamental cycle of X satisfies our conditions by [12, Section 3.2]. If $dim(X/U) > 0$ then our assumption that Z is finite and irreducible implies that it has only one point over the closed point of U. Taking an open neighborhood of this point we may assume that X is affine. Choose closed points $x_1, \ldots, x_k$ in each of the irredusible components of the closed fiber of X which do not lie in Z. Since X is affine there exists a regular function on X which is 0 on Z and 1 in each of these points. By [5, I.2.5] the divisor of this function is flat over U which completes the inductive step.

Remark 4.9 The previous remark (4.6) shows that if we consider all schemes instead of semi-normal ones then (55) stops being an isomorphism at least in characteristic zero. It is possible that there is still an $\mathbf{A}^1$-equivalence of the form (4.6) for all schemes. It is also clear that there is an equivalence of the same form with respect to a modification of the Nisnevich topology which allows for semi-normalizations as coverings.

Proposition 4.10 *The morphism*

$$S^{\infty}[1/c](Lres(S^q_t)) \to S^{\infty}[1/c]^+(Lres(S^q_t))$$

is an $\mathbf{A}^1$-equivalence.

Proof: Let $C_* = Sing_*$ be the functor of [6, p.87] which we consider as a functor from $Rad(C_+)$ to $\Delta^{op}Rad(C_+)$ and which we extend to simplicial radditive functors using the diagonal. We have a sequence of morphisms:

$$S^{\infty}[1/c](Lres(S^q_t)) \leftarrow C_*S^{\infty}[1/c](Lres(S^q_t)) \to (C_*S^{\infty}[1/c])^+(Lres(S^q_t)) \cong$$
$$\cong C_*(S^{\infty}[1/c]^+(Lres(S^q_t))) \to S^{\infty}[1/c]^+(Lres(S^q_t)).$$

Let us show that they are all $\mathbf{A}^1$-equivalences. For the first and the last ones it follows from the fact that for any F the morphism $C_*(F) \to F$ is an $\mathbf{A}^1$-equivalence (see [6, Corollary 3.8]). It remains to verify that the second one is an $\mathbf{A}^1$-equivalence.

Lemma 4.11 *Let X be a simplicial abelian monoid such that $\pi_0(X)$ is a group. Then the natural map $X \to X^+$ is a weak equivalence.*

Proof: See [8, p. 381].

In view of Lemma 4.11 it is enough check that for any $U \in C$, π_0 of the simplicial set $C_*S^\infty[1/c](Lres(S_t^q))(U)$ is a group. We have

$$\pi_0 C_*S^\infty[1/c](Lres(S_t^q))(U) = \pi_0 C_*S^\infty[1/c](S_t^q)(U).$$

This set is the set of $\mathbf{A}^1$-homotopy classes of maps from $[U]$ to $[\mathbf{A}^1 - \{0\}]^{\otimes q}$ in the subcategory of $Cor(C,S)$ where morphisms are effective finite correspondences modulo those classes which contain correspondences landing in the part of $[\mathbf{A}^1 - \{0\}]^{\otimes q}$ which contacts to the point in S_t^q. Let $\phi : \mathbf{A}^1 - \{0\} \to \mathbf{A}^1 - \{0\}$ be the morphism $z \mapsto z^{-1}$. For $f : U \to [\mathbf{A}^1 - \{0\}]^{\otimes q}$ in our category set $f^- = (\phi \otimes Id^{\otimes(q-1)}) \circ f$. Let us show that $f + f^{-1}$ is $\mathbf{A}^1$-homotopic to a correspondence which goes to zero. Note first that we have

$$f + (\phi \otimes Id^{q-1}) \circ f = (Id \otimes Id^{\otimes(q-1)}) \circ f + (\phi \otimes Id^{q-1}) \circ f = ((Id + \phi) \otimes Id^{\otimes(q-1)}) \circ f.$$

therefore it is enough to show that the finite correspondence $\phi + Id$ from $\mathbf{A}^1 - \{0\}$ to itself is $\mathbf{A}^1$-homotopic in the category of effective finite correspondences to a finite correspondences which lands in 1. This is easily done by an explicit construction.

Corollary 4.12 *Let $n > 0$ and $X = cone(Lres(S_t^q) \xrightarrow{f} Lres(S_t^q))$ where f is given by the n-th power map $\mathbf{A}^1 - \{0\} \to \mathbf{A}^1 - \{0\}$ on the first S_t^1. Then*

$$S^\infty[1/c](X) \to S^\infty[1/c]^+(X)$$

is an $\mathbf{A}^1$-equivalence

Proof: Using the same reasoning as in the proof of the proposition one reduces the problem to showing that

$$\pi_0 C_*S^\infty[1/c](X)(U) = \pi_0 C_*S^\infty[1/c](\pi_0(X))(U)$$

is a group. In what follows we drop U to simplify the notation. For any map $f : X \to Y$ one has

$$\pi_0(cone(f)) = coeq(\pi_0(X) \vee \pi_0(Y) \rightrightarrows \pi_0(Y))$$

where the first map is f on X and Id on Y and the second map is 0 on X and Id on Y. Therefore

$$\pi_0 C_*S^\infty[1/c](\pi_0(X)) = \pi_0 C_*S^\infty[1/c](coeq((S_t^q \vee S_t^q) \rightrightarrows S_t^q).$$

Set

$$M = \pi_0 C_*S^\infty[1/c](S_t^q)$$

Since $S^{\infty}[1/c]$ and C_* commute with reflexive coequalizers,

$$S^{\infty}[1/c](X \vee Y) = S^{\infty}[1/c](X) \times S^{\infty}[1/c](Y)$$

by Lemma 18 and C_* commutes with products we get

$$\pi_0 C_* S^{\infty}[1/c](\pi_0(X)) = coeq(M \times M \rightrightarrows M)$$

It remains to note that by the proof of the proposition M is a group and a reflexive coequalizer of a diagram of groups is a group.

Lemma 4.13 *Let $n > 0$ and $X = cone(Lres(S_t^q) \xrightarrow{f} Lres(S_t^q))$ where f is given by the n-th power map $\mathbf{A}^1 - \{0\} \to \mathbf{A}^1 - \{0\}$ on the first S_t^1. Then in one has*

$$\Lambda_S^l(X) = \mathbf{Z}/n \otimes_{L,S} l_q.$$

Proof: One can show easily that in $H_{\mathbf{A}^1}(Cor(C))$ the map $l_1 \to l_1$ given by $z \mapsto z^n$ coincides with the multiplication by n map with respect to the additive structure which implies the statement of the lemma.

Lemma 4.14 *Let C be an f-admissible category. Then for any finitely generated abelian group A there exists an object $X = X(A) \in \Delta^{op} C_+^{\#}$ such that the following conditions hold:*

1. *$\Lambda_S^l(X) = A \otimes_{L,S} l_q$ in $H_{\mathbf{A}^1}(Cor(C))$*
2. *$S^{\infty}[1/c](X) \to S^{\infty}[1/c](X)^+$ is an $\mathbf{A}^1$-equivalence.*

Proof: A finitely generated abelian group A is a finite product of copies of $\mathbf{Z}$ and of cyclic groups $\mathbf{Z}/n$. Set

$$X(\mathbf{Z}) = Lres(S_t^q)$$

$$X(\mathbf{Z}/n) = cone(Lres(S_t^q) \xrightarrow{f} Lres(S_t^q))$$

where f is as in Lemmas 4.12 and 4.13 and

$$X(A \times B) = X(A) \vee X(B).$$

The first condition follows easily from Lemma 4.13. The second condition follows from Lemmas 4.10 and 4.12, Lemma 3.10 and the fact that the product of $\mathbf{A}^1$-equivalences is an $\mathbf{A}^1$-equivalence.

Remark 4.15 While any abelian group is a filtering colimit of finitely generated ones our construction of $X(A)$ is not natural in A and therefore we can not immediately extend Lemma 4.14 and the results proved below with the help of this lemma to all abelian groups. The usual functorial construction of the Moore space corresponding to an abelian group does not work in our case since there are no morphisms of the form $\vee_{i\in I}S^1_t \to \vee_{j\in J}S^1_t$ in $H_{\mathbf{A}^1}(C_+)$ corresponding to general homomorphisms of free abelian groups $\oplus_{i\in I}\mathbf{Z} \to \oplus_{j\in J}\mathbf{Z}$.

The main objects of our investigation, the images of the spaces $K(A,p,q)$ in $H_{\mathbf{A}^1}(Cor(C,R))$ can be written as

$$M(A,p,q;R) = \mathbf{L}\Lambda^l_R(K(A,p,q)) = \mathbf{L}\Lambda^l_R\Lambda^r_S\Sigma^{p-q}(A\otimes_{\mathbf{L}} l_q).$$

Theorem 4.16 *Let R be a commutative S-algebra, A be a finitely generated S-module, C an f-admissible category which is contained in semi-normal schemes and $p \geq q$ two integers such that $p > 0$. Then one has*

$$M(A,p,q;R) = S^\infty_{tr}[1/c]\Sigma^{p-q}((A\otimes_{\mathbf{L},S} R)\otimes_{\mathbf{L},R} l_q)$$

where $A\otimes_{\mathbf{L},S}R$ is a simplicial R-module whose quasi-isomorphism class represents the derived tensor product of A and R over S.

Proof: Let us start with the following lemma.

Lemma 4.17 *Let X, Y be objects of $\Delta^{op}C^{\#}_+$ such that there is a projective equivalence*

$$\Lambda^l_S(X) = A\otimes_{\mathbf{L},S}\Lambda^l_S(Y),$$

then there is a projective equivalence

$$\Lambda^l_R(X) = (A\otimes_{\mathbf{L},S} R)\otimes_{\mathbf{L},R}\Lambda^l_R(Y).$$

Proof: It follows from the general "change of coefficients" formula which which has the same form and proof in our context as in the topological one.

By Lemma 4.14 and an obvious remark for $p > q$ there exists $X \in \Delta^{op}C^{\#}_+$ such that

1. $\Lambda^l_S(X) = \Sigma^{p-q}(A\otimes_{L,S} l_q)$ in $H_{\mathbf{A}^1}(Cor(C))$
2. $S^\infty[1/c](X) \to S^\infty[1/c](X)^+$ is an $\mathbf{A}^1$-equivalence.

Then

$$K(A,p,q)=\Lambda^r_S(A\otimes_{L,S} l_q)=\Lambda^r_S\Lambda^l_S(X)=S^{\infty}[1/c]^{+}(X)=S^{\infty}[1/c](X).$$

and therefore

$$M(A,p,q;R)=\Lambda^l_R S^{\infty}[1/c](X)=S^{\infty}_{tr}[1/c](\Lambda^l_R(X)).$$

Applying Lemma 4.17 we get

$$M(A,p,q;R)=S^{\infty}_{tr}[1/c]\Sigma^{p-q}((A\otimes_{\mathbf{L},S}R)\otimes_{\mathbf{L},R} l_q).$$

Corollary 4.18 *Under the assumptions of the theorem assume in addition that $R=F$ is a field. Then one has:*

$$M(A,p,q;R)=(M(A,p,q;F)_0\otimes M(A,p,q;F)_1)\oplus M(A,p,q;F)_0\oplus M(A,p,q;F)_1$$

where

$$M(A,p,q;F)_0=S^{\infty}_{tr}[1/c]\Sigma^{p-q}((A\otimes_{\mathbf{Z}}F)\otimes_F l_q)$$

and

$$M(A,p,q;F)_1=S^{\infty}_{tr}[1/c]\Sigma^{p-q+1}(Tor^1_{\mathbf{Z}}(A,F)\otimes_F l_q).$$

Proof: It follows from the general case by Corollary 3.37 since

$$A\otimes_{\mathbf{L}}F=(A\otimes_{\mathbf{Z}}F)\oplus\Sigma^1 Tor^1_{\mathbf{Z}}(A\otimes F).$$

Corollary 4.19 *Let k be a field, F a field of characteristic $l\neq char(k)$ and C an f-admissible category. Then for $p\geq q$ and $p>0$ one has:*

$$M(\mathbf{Z}/l,p,q;F)=(M(\mathbf{Z},p,q;F)\otimes M(\mathbf{Z},p+1,q;F))\oplus M(\mathbf{Z},p,q;F)\oplus M(\mathbf{Z},p+1,q;F)$$

Define the subcategory $\overline{DT}$ in $H_{\mathbf{A}^1}(Cor(C))$ as $Cl_{vl}(PT)$. For $C=Sm/k$ it is easy to see that $\overline{DT}$ is the intersection of $H_{\mathbf{A}^1}$ with the subcategory $\overline{DT}$ of DM which was introduced in Section 4.

Theorem 4.20 *Let k be a field and F a field such that $c(k)$ is invertible in F. Then for any finitely generated abelian group A and an f-admissible C which is contained in semi-normal schemes one has:*

1. *For any $p\geq q\geq 0$*

$$M(A,p,q;F)_C\in\overline{DT}_{\geq q}$$

2. For any $p \geq 2q \geq 0$

$$M(A,p,q;F)_C \in PT_{\geq q}.$$

Proof: If $p = q = 0$ then $K(A,p,q)$ is the constant radditive functor corresponding to A and $M(A,p,q;F)$ is the direct sum of as many copies of the unit object as there are elements in A. In particular it is in PT. Hence we may assume that $p > 0$.

By Corollary 4.18 it is sufficient to verify that the objects

$$S^{\infty}_{tr}[1/c]\Sigma^{p-q}((A \otimes_{\mathbf{Z}} F) \otimes_F l_q)$$

and

$$S^{\infty}_{tr}[1/c]\Sigma^{p-q+1}(Tor^1_{\mathbf{Z}}(A,F) \otimes_F l_q)$$

are in $\overline{DT}_{\geq q}$ for $p \geq q$ and in $PT_{\geq q}$ for $p \geq 2q$. Since F is a field $A \otimes_{\mathbf{Z}} F$ and $Tor^1_{\mathbf{Z}}(A,F)$ are direct sums of copies of F. Applying Corollary 3.37 plus an additional standard argument to deal with infinite direct sums we see that it is sufficient to show that $S^{\infty}_{tr}[1/c](\Sigma^{p-q}l_q)$ belongs to the appropriate category. For $p \geq 2q$ it follows from from Corollary 3.71. For $2q > p \geq q$ it follows from Corollary 3.39 and the case $p \geq 2q$ since $\Sigma^i l_q \in Cl_{vl}(PT_{\geq q})$.

Remark 4.21 For $p < q$ the objects $M(A,p,q;F)$ are not necessarily mixed Tate motives. As an example one may take $K(\mathbf{Z}/l,0,1)$ which is the scheme of l-roots of unity and its motive is an Artin motive.

Remark 4.22 The motives $M(A,p,q;F)$ are not pure Tate motives for $2q > p \geq q$. Let us show for example that $S^l_{tr}(l_2)$ is not a direct sum of Tate motives. Let $X = \Sigma^1 l_2$. Then X is odd and by Proposition 3.34 we get a cofibration sequence

$$\Sigma^{l-1}X^{\otimes l} \to \Sigma^1 S^l_{tr}(X) \to S^l_{tr}(\Sigma^1 X) \to \Sigma^l X^{\otimes l}.$$

Since $\Sigma^1 X = L_2$ we can rewrite it as

$$\Sigma^{2l-1}l_{2l} \to \Sigma^1 S^l_{tr}(X) \to S^l_{tr}(L_2) \to \Sigma^l l_{2l} \qquad (57)$$

and Theorem 3.48 shows that in the motivic notation (an up to suspensions) the sequence (57) is of the form

$$\mathbf{F}_l(2l)[4l-2] \to S^l_{tr}(X) \to \mathbf{F}_l(2l)[4l-1] \oplus \mathbf{F}_l(l+1)[2l+1] \oplus \mathbf{F}_l(l+1)[2l+2] \to \mathbf{F}_l(2l)[4l-1]$$

Since the topological realization of $S = \Sigma^1 S^2_t$ is the 3-sphere we know that the ordinary homology of $S^l_{tr}(X)$ are of the form $H_{2l+1} = H_{2l+2} = \mathbf{F}_l$ and the rest of the homology groups are zero. For $l > 2$ it implies that the right hand side morphism can only be non-zero on the first summand and therefore we have

$$S^l_{tr}(X) \cong \mathbf{F}_l(l+1)[2l+1] \oplus \mathbf{F}_l(l+1)[2l+2]. \qquad (58)$$

Let us now use Proposition 3.34 for the relation $\Sigma^1 l_2 = X$. We get a cofibration sequence

$$\Sigma^1 l_{2l} \to \Sigma^1 S^l_{tr}(l_2) \to S^l_{tr}(X) \to \Sigma^2 l_{2l}$$

or in the motivic notation

$$\mathbf{F}_l(2l)[2l+1] \to S^l_{tr}(l_2) \to S^l_{tr}(X) \to \mathbf{F}_l(2l)[2l+2].$$

Using (58) we further rewrite it as

$$\mathbf{F}_l(2l)[2l+1] \to S^l_{tr}(l_2) \to \mathbf{F}_l(l+1)[2l+1] \oplus \mathbf{F}_l(l+1)[2l+2] \to \mathbf{F}_l(2l)[2l+2].$$

Topologically S^2_t is a 2-sphere and therefore $S^l_{tr}(l_2)$ has only one non-trivial ordinary homology group in dimension $2l$. If the base field is algebraically closed then the right hand side morphism is zero on the first summand and we conclude that

$$S^l_{tr}(l_2) \cong \mathbf{F}_l(l+1)[2l] \oplus C$$

where C is the fiber of the canonical morphism $\mathbf{F}_l(l+1)[2l+1] \to \mathbf{F}_l(2l)[2l+1]$. The topological realization of C is trivial and it does not affect the ordinary homology of $S^l_{tr}(l_2)$ but the object itself is non-trivial. We conclude that $S^l_{tr}(l_2)$ and therefore $M(\mathbf{Z}/l, 2, 2; \mathbf{F}_l)$ is never a direct sum of Tate motives.

To prove an analog of Theorem 4.20 for $C = Sm/k$, we need additional arguments which depend on resolution of singularities.

Let $i : C \subset D$ be two admissible subcategories. Set $K_C = K(A,p,q)_C$ and $K_D = K(A,p,q)_D$ and let M_C and M_D denote the corresponding objects of $H_{\mathbf{A}^1}(Cor(-,S))$. In what follows we omit S from our notations. By Section 4 we have two pairs of adjoint functors connecting the categories $H_{\mathbf{A}^1}(Cor(-))$ and $H_{\mathbf{A}^1}((-)_+)$ over C and D respectively. We denote functors of both pairs by $\mathbf{L}i^{rad}$ and i_* where $\mathbf{L}i^{rad}$ is the left adjoint and goes from the category over C to the category over D. For simplicity of notation we will let $=$ denote canonical projective equivalences.

Lemma 4.23 *For any k, any $i : C \to D$, any A and any p, q one has*

$$K_C = i_* K_D$$

Proof: This follows immediately from the definition of $K(A,p,q)$ as a representing object and the adjunction between i_* and $\mathbf{L}i^{rad}$.

Proposition 4.24 *Let k be a field with resolution of singularities, $C \subset Sm/k$, D any admissible subcategory which contains C. Then one has:*

$$M_C = i_* M_D$$

Proof: We have

$$i_*M_D = i_*\mathbf{L}\Lambda^l K_D = \mathbf{L}\Lambda^l i_* K_D = \mathbf{L}\Lambda^l K_C = M_C$$

where the second equality holds by Theorem 2.33 and the third by Lemma 4.23.

Theorem 4.25 *Let k be a field which admits resolution of singularities and F a field such that $char(F) = 0$ or $char(F) \neq char(k)$. Then for any finitely generated abelian group A and any admissible C contained in the category of smooth schemes one has:*

1. *For any $p \geq q \geq 0$*
$$M(A,p,q;F)_C \in \overline{DT}_{\geq q}$$
2. *For any $p \geq 2q \geq 0$*
$$M(A,p,q;F)_C \in PT_{\geq q}.$$

Proof: Note first that we may assume that C is contained in the category of smooth quasi-projective schemes. Let $i : C \to D$ be the embedding of C into the category D of all quasi-projecive schemes. By Lemma 5.3 D is f-admissible. Our result follows now from Theorem 4.20 and Proposition 4.24 since i_* clearly commutes with direct sums, suspensions and cofibration sequences and therefore takes $\overline{DT}_{\geq q}$ to $\overline{DT}_{\geq q}$ and $PT_{\geq q}$ to $PT_{\geq q}$.

Remark 4.26 Our methods imply easily that under the resolution of singularities assumption and for $p \geq 2q$ the motive $M(A,p,q;F)$ does not depend on the choice of C. For $p < 2q$ it is unclear since for $C \subset D$ the category $\overline{DT}(D)$ can be larger than $\overline{DT}(C)$ when both are considered as subcategories in $H_{\mathbf{A}^1}(Cor(D))$. The problem is that it is unclear that an object $X \in H_{\mathbf{A}^1}(Cor(D))$ such that $\Sigma^1 X$ is in $H_{\mathbf{A}^1}(Cor(C))$ is itself in $H_{\mathbf{A}^1}(Cor(C))$.

2 Topological realization functors

In this section we assume that the base field is $\mathbf{C}$ and R is the ring of coefficients for homology and correspondences. We set $C = QP/\mathbf{C}$ to be the category of quasi-projective schemes over $\mathbf{C}$.

Let Top be the category of "nice" topological spaces e.g. of spaces which admit the structure of a finite CW-complex. Sending $X \in C$ to the topological space of its $\mathbf{C}$-points we get a functor

$$\pi : C \to Top$$

We will need the following classical properties of this functor.

Lemma 4.27 *1. Functor π commutes with disjoint unions.*

2. *Functor π commutes with fiber products.*

3. *Functor π commutes with finite group quotients.*

Proof: See [].

Lemma 4.28 *Consider C as a site with the Nisnevich topology and Top as a site where coverings are surjective local homeomorphisms. Then π is a morphism of sites.*

Proof: It is a continuous map of sites since it clearly takes coverings to coverings. It is a morphism of sites by Lemma 4.27(2).

The inverse image functor together with the associated sheaf functor from $Rad(C_+)$ to $Shv_\bullet(C_{Nis})$ define a functor

$$\pi^* : \Delta^{op} Rad(C_+) \to \Delta^{op} Shv_\bullet(Top)$$

In view of Proposition 2.11 and Lemma 4.28 this functor takes W_{Nis} to local equivalences of simplicial sheaves.

Let Λ_0^r be the "forgetting of transfers" functor from $Rad(Cor(C, R))$ to presheaves of R-modules on C. Then $\pi^*\Lambda_0^r(X)$ is a functor

$$\Delta^{op} Rad(Cor(C, R)) \to \Delta^{op} Shv_{R-mod}(Top)$$

Evaluating a pointed sheaf (resp. a sheaf of R-modules) X on the standard cosimplicial object $\Delta_{top}^\bullet$ in Top we get a pointed simplicial set (resp. a simplicial R-module) $Sing_*(X)$. Composing π^* (resp. $\pi^*\Lambda_0^r$) with $Sing_*$ followed by the diagonal we get two functors

$$T_{\mathbf{C}} = \Delta Sing_* \pi^* : \Delta^{op} Rad(C_+) \to \Delta^{op} Sets_\bullet$$

and

$$T_{\mathbf{C}}^{tr} = \Delta Sing_* \pi^* \Lambda_0^r : \Delta^{op} Rad(Cor(C, R)) \to \Delta^{op} R - mod$$

Lemma 4.29 *The functor $T_{\mathbf{C}}$ takes $\mathbf{A}^1$-equivalences to weak equivalences of simplicial sets and therefore defines a functor*

$$t_{\mathbf{C}} : H_{\mathbf{A}^1}(C_+) \to H_\bullet^{top}$$

Proof: Straightforward using the description of $W_{\mathbf{A}^1}$ in terms of $\bar{\Delta}$-closure.

Lemma 4.30 *Functor $t_{\mathbf{C}}$ commutes with the smash products.*

Proof: One can easily see that for any $X, Y \in \Delta^{op} Rad(C_+)$ there is a natural map $T_{\mathbf{C}}(X) \wedge T_{\mathbf{C}}(Y) \to T_{\mathbf{C}}(X \wedge Y)$. Since the class of weak equivalences of pointed simplicial sets is $\bar{\Delta}$-closed it is sufficient to check that it is a weak equivalence for $X = X'_+$, $Y = Y'_+$ where $X', Y' \in C$. In this case our map is an isomorphism because both π^* and $Sing_*$ commute with direct products.

One can easily see that there are canonical isomorphisms:

$$t_{\mathbf{C}}(S^1_s) = S^1 \tag{59}$$

$$t_{\mathbf{C}}(S^1_t) = S^1 \tag{60}$$

Let $H(R - mod)$ be the homotopy category of simplicial R-modules.

Lemma 4.31 *The functor $T^{tr}_{\mathbf{C}}$ takes $\mathbf{A}^1$-equivalences to weak equivalences of simplicial R-modules and therefore defines a functor*

$$t^{tr}_{\mathbf{C}} : H_{\mathbf{A}^1}(Cor(C, R)) \to H(R - mod) \tag{61}$$

Proof: Straightforward using the description of $W^{tr}_{\mathbf{A}^1}$ in terms of $\bar{\Delta}$-closure.

Lemma 4.32 *The square*

$$\begin{array}{ccc} H_{\mathbf{A}^1}(Cor(C,R)) & \xrightarrow{t^{tr}_{\mathbf{C}}} & H(R - mod) \\ {\scriptstyle \Lambda^r}\downarrow & & {\scriptstyle \phi}\downarrow \\ H_{\mathbf{A}^1}(C_+) & \xrightarrow{t_{\mathbf{C}}} & H^{top}_{\bullet} \end{array}$$

where ϕ is the forgetting functor, commutes up to a natural isomorphism.

Proof: One can easily see that this square actually commutes on the level of simplicial objects i.e. even before one passes to the homotopy categories.

Let

$$H_R : \Delta^{op} Sets_{\bullet} \to \Delta^{op} R - mod$$

be the functor which takes a pointed simplicial set X to the free R-module $H_R(X)$ generated X.

Lemma 4.33 *The square*

$$\begin{array}{ccc} H_{\mathbf{A}^1}(C_+) & \xrightarrow{t_{\mathbf{C}}} & H_{\bullet}^{top} \\ {\scriptstyle \mathbf{L}\Lambda^l}\downarrow & & {\scriptstyle H_R}\downarrow \\ H_{\mathbf{A}^1}(Cor(C,R)) & \xrightarrow{t_{\mathbf{C}}^{tr}} & H(R-mod) \end{array} \tag{62}$$

commutes up to a natural isomorphism.

Proof: We may interpret $H_{\mathbf{A}^1}(C_+)$ as a localization of $\Delta^{op}C_+^{\#}$ and consider Λ^l instead of $\mathbf{L}\Lambda^l$. By definition, $H_R T_{\mathbf{C}}(X)$ is the free R-module generated by the pointed simplicial set $\Delta Sing_*(X(\mathbf{C}))$ and $T_{\mathbf{C}}^{tr}\Lambda_R^l(X)$ is the simplicial R-module $\Delta Sing_*(\pi^*\Lambda_0^r\Lambda_R^l(X))$. The natural transformation $X(\mathbf{C}) \to \pi^*\Lambda_0^r\Lambda_R^l(X)$ together with the universal property of free R-modules provide us with a natural transformation of the form

$$H_R T_{\mathbf{C}} \to T_{\mathbf{C}}^{tr}\mathbf{L}\Lambda_R^l \tag{63}$$

Since the forgetting functor ψ from R-modules to abelian groups reflects equivalences and one has

$$\psi H_R T_{\mathbf{C}}(X) = (\psi H_{\mathbf{Z}} T_{\mathbf{C}}(X)) \otimes R$$

and

$$\psi T_{\mathbf{C}}^{tr}\Lambda_R^l(X) = (\psi T_{\mathbf{C}}^{tr}\Lambda_{\mathbf{Z}}^l(X)) \otimes R$$

its is enough to consider the case $R = \mathbf{Z}$. Since both sides of (63) commute with the simplicial suspension (because of Lemma 4.27(1)) it is enough to verify that

$$\phi H_{\mathbf{Z}}\Delta Sing_*\pi^*(X) \to \Delta Sing^*\pi^*\Lambda^r\Lambda^l(X)$$

is a weak equivalence for X which is a suspension of an object of $\Delta^{op}C_+^{\#}$.

Lemma 4.34 *For any $X \in \Delta^{op}C_+^{\#}$ there is a natural isomorphism*

$$\pi^*S^\infty(X)^+ \to \pi^*\Lambda^r\Lambda^l(X).$$

Proof: It follows easily from Theorem 4.7 together with the fact that a Nisnevich covering defines a local homeomorphism on $\mathbf{C}$-points and semi-normalization is a homeomorphism on $\mathbf{C}$-points.

By Lemma 4.34 we can replace $\Lambda^r\Lambda^l$ by $(S^\infty)^+$ and by Lemma 4.11 further replace it by S^∞. By Lemma 4.27(2,3) the functor π^* takes S^∞ to S^∞ and our result follows from the Dold-Thom theorem in the form which asserts that for a simplicial topological space X the maps

$$H_{\mathbf{Z}}\Delta Sing_*(\Sigma_s^1 X) \leftarrow H_{\mathbf{N}}\Delta Sing_*(\Sigma_s^1 X) \to \Delta Sing_*(S^\infty(\Sigma_s^1 X))$$

are weak equivalences.

Proposition 4.35 *The functor $t^{tr}_{\mathbf{C}}$ commutes with (derived) tensor products.*

Proof: Note first that there is a natural transformation on the level of simplicial objects of the form

$$T^{tr}_{\mathbf{C}}(X) \otimes_R T^{tr}_{\mathbf{C}}(Y) \to T^{tr}_{\mathbf{C}}(X \otimes Y)$$

Since the class of weak equivalences of simplicial R-modules is $\bar{\Delta}$-closed it is enough to verify that it is a weak equivalence for $X, Y \in Cor(C, R)$ i.e. for $X = \Lambda^l(X'_+)$, $Y = \Lambda^l(Y'_+)$ where $X', Y' \in C$. In this case the claim follows from Lemma 4.33 and the Kunnet isomorphism theorem for topological homology.

Lemma 4.33 together with isomorphisms (59), (60) implies that there is a canonical isomorphism

$$t^{tr}_{\mathbf{C}}(L_n) = R[2n] \tag{64}$$

and similarly

$$t^{tr}_{\mathbf{C}}(l_n) = R[n] \tag{65}$$

In particular for $R = \mathbf{Z}/l$ we get

$$t^{tr}_{\mathbf{C}}(l_n) = \mathbf{Z}/l[n] \tag{66}$$

Note that it would be more natural to get $\mu_l[n]$ where μ_l is the group of l-roots of unity in $\mathbf{C}$ on the right hand side of (66). We get $\mathbf{Z}/l$ there because the isomorphism (60) defines an identification of μ_l which is the fiber of the l-th power map on S^1_t with $\mathbf{Z}/l$ which is the fiber of the l-th power map on the circle. If the circle is oriented counter clock-wise then this identification corresponds to the choice of the l-th root of unity with the smallest argument.

For an R-module A let $K(A, p)$ and $K(A, p, q)$ be the topological and the motivic Eilenberg-MacLane spaces representing the functors $H^p(-, A)$ and $H^{p,q}(-, A)$ respectively.

Lemma 4.36 *For $p \geq q$ there is a canonical isomorphism $t_{\mathbf{C}}(K(A, p, q)) \to K(A, p)$.*

Proof: We have

$$t_{\mathbf{C}}(K(A, 2n, n)) = t_{\mathbf{C}}\Lambda^r(A \otimes \Sigma^{p-q} l_q) = \phi t^{tr}_{\mathbf{C}}(A \otimes \Sigma^{p-q} l_q) = \phi(A[p]) = K(A, p)$$

where the second isomorphism is from Lemma 4.32 and the third one from Lemma 4.33.

Remark 4.37 The statement of Lemma 4.36 is false for $p < q$ at least when A is not a torsion abelian group. For example, $K(\mathbf{Z}, 0, 1) = pt$ while $K(\mathbf{Z}, 0) = \mathbf{Z}$.

Using the definition of motivic cohomology given at the beginning of Section 4 in combination with Lemma 4.33 and isomorphisms (65) we get canonical maps:

$$H^{p,q}(X, A) \to H^p(t_{\mathbf{C}}(X), A)$$

3 Application to stable operations.

In this section we will show that over a field of characteristic zero the algebra of all stable operations in motivic cohomology with coefficients in $\mathbf{F}_l$ coincides with the motivic Steenrod algebra. We work with smooth schemes over k. Let K_n be the motivic Eilenberg-MacLane space $K(\mathbf{Z}/l, 2n, n)$ and K_{2n}^{top} be its topological counterpart $K(\mathbf{Z}/l, 2n)$. The abelian group of all stable operations is given by

$$M^{*,*} = \lim_{\leftarrow n} H^{*+2n,*+n}(K_n, \mathbf{F}_l)$$

where the homomorphisms of the system are defined by the morphisms

$$\Sigma_T^1 K_n \to K_{n+1} \tag{67}$$

Lemma 4.38 *As a $H^{*,*}$ module, $M^{*,*}$ is free.*

Proof: Let

$$M_n = \mathbf{L}\Lambda^l(K_n)$$

be the class of K_n in $H_{tr}(Sm/k, \mathbf{F}_l)$. In view of Theorem 4.25(2) there exist objects M'_n in PT such that $M_n = \Sigma_T^n M'_n$. Morphisms (67) define a sequence

$$M'_0 \to M'_1 \to M'_2 \to \dots$$

Let M be its homotopy colimit. By Proposition 3.65 we have $M \in PT$ and in particular the motivic cohomology of M is a free $H^{*,*}$-module. On the other hand Corollary 3.66 implies that the motivic cohomology of M coincide with $M^{*,*}$.

Let $\mathcal{A}^{*,*}$ be the motivic Steenrod algebra. Since operations from $\mathcal{A}^{*,*}$ are stable with respect to Σ_T they act on $M^{*,*}$. Denote by ι the element in $M^{*,*}$ whose restriction to K_n is the canonical class ι_n. Acting by elements of $\mathcal{A}^{*,*}$ on ι we get a map

$$u : \mathcal{A}^{*,*} \to M^{*,*}$$

Assume that k is an algebraically closed field of characteristic $\neq l$. Let us choose a primitive l-th root of unity in k and let τ be the corresponding element of $H^{0,1} = H^{0,1}(Spec(k))$. Then by [,] we have

$$H^{*,*} = \mathbf{F}_l[\tau].$$

For a module A over $F[\tau]$ we will write $A \otimes_0 F$ (resp. $A \otimes_1 F$) for the tensor product of A with $\mathbf{F}_l$ with respect to the homomorphism $F[\tau] \to F$ which takes τ to 0 (resp. to 1).

Lemma 4.39 *Let $X \in \Delta^{op}Rad(Sch/\mathbf{C}_+)$ be such that $\mathbf{L}\Lambda^l_{\mathbf{F}_l}(X)$ is $\overline{SDT}$. Then the natural homomorphisms*

$$H^{p,*}(X, \mathbf{Z}/l) \otimes_1 \mathbf{F}_l \to H^p(t_{\mathbf{C}}(X), \mathbf{F}_l) \tag{68}$$

are isomorphisms.

Proof: We may replace the right hand side of (68) by $Hom(t^{tr}_{\mathbf{C}}\mathbf{L}\Lambda^l_{\mathbf{F}_l}(X), \mathbf{F}_l[*])$ which implies that it is sufficient to verify that (68) is an isomorphism for X such that $\mathbf{L}\Lambda^l(X) = \Sigma^i l_q$. In that case $H^{*,*}(X, \mathbf{Z}/l)$ is a free module of rank one over $H^{*,*}(Spec(k))$ and the statement is obvious.

The motivic Adem relations demonstrated in [25] imply that there is a homomorphism

$$\mathcal{A}^{*,*}(k) \to \mathcal{A}^* \tag{69}$$

which sends P^i to P^i and β to β and which is of the form

$$H^{*,*}(k) \to H^{*,*}(\bar{k}) = \mathbf{F}_l[\tau] \stackrel{\tau\mapsto 1}{\to} \mathbf{F}_l$$

on $H^{*,*}$. The same relations imply immediately the following result.

Lemma 4.40 *Let k be an algebraically closed field of characteristic $\neq l$ then (69) defines an isomorphism*

$$\mathcal{A}^{*,*}(k) \otimes_1 \mathbf{F}_l \to \mathcal{A}^* \tag{70}$$

Lemma 4.41 *The square*

$$\begin{array}{ccc} \mathcal{A}^{*,*} & \stackrel{\iota_n}{\longrightarrow} & H^{*,*}(K_n) \\ \downarrow & & \downarrow \\ \mathcal{A}^* & \stackrel{\iota_{2n}}{\longrightarrow} & H^*(K^{top}_{2n}) \end{array} \tag{71}$$

commutes.

Proof: We will only give a sketch of the argument. From general functoriality it is sufficient to verify that the image $P^{i,n}$ (resp. $\beta P^{i,n}$) of the motivic class $P^i\iota_n$ (resp. $\beta P^i\iota_n$) in $H^*(K^{top}_{2n})$ is $P^i\iota_{2n}$ (resp. $\beta P^i\iota_{2n}$). Knowing the Cartan formula and Adem relations for the motivic reduced powers we can deduce that the family of operations defined by the classes $P^{i,n}$ and $\beta P^{i,n}$ satisfy the list of properties which uniquely characterize the reduced power operations (see e.g. [9]).

Proposition 4.42 *Let k be as above. Then the homomorphism*

$$u \otimes_1 Id : \mathcal{A}^{*,*} \otimes_1 \mathbf{F}_l \to M^{*,*} \otimes_1 \mathbf{F}_l \tag{72}$$

is an isomorphism.

Proof: Since nothing changes when we pass from an algebraically closed field to its algebraically closed extension we may assume that $k = \mathbf{C}$.

The vertical arrows of (71) factor as

$$\mathcal{A}^{*,*} \to \mathcal{A}^{*,*} \otimes_1 \mathbf{F}_l \to \mathcal{A}^*$$

$$H^{*,*}(K_n) \to H^{*,*}(K_n) \otimes_1 \mathbf{F}_l \to H^*(K_{2n}^{top})$$

where the second arrows of both factorizations are isomorphisms - the first one by Lemma 4.40 and the second one by Lemma 4.39. We conclude that the maps

$$\mathcal{A}^{*,*} \otimes_1 \mathbf{F}_l \to H^{*,*}(K_n) \otimes_1 \mathbf{F}_l$$

are isomorphic to the maps

$$\mathcal{A}^* \to H^*(K_{2n}^{top})$$

defined by the action of the topological Steenrod algebra on ι_{2n} and one verifies easily that these isomorphisms identify the maps defined by (67) with the similar maps defined by the topological suspension morphisms

$$\Sigma^2 K_{2n}^{top} \to K_{2n+2}^{top}.$$

We conclude that (72) is isomorphic to the map

$$\mathcal{A}^* \to \lim_{\leftarrow n} H^{*+2n}(K_{2n}^{top}) \tag{73}$$

defined by the action of the topological Steenrod algebra on the canonical cohomology classes of K_{2n}^{top} which is an isomorphism by [9].

Proposition 4.43 *The homomorphism*

$$u \otimes_0 Id : \mathcal{A}^{*,*} \otimes_0 \mathbf{F}_l \to M^{*,*} \otimes_0 \mathbf{F}_l$$

is a monomorphism.

Proof: Following the notations of [6] let P^I denote the element of $\mathcal{A}^{*,*}$ corresponding to an admissible sequence $I = (\epsilon_0, s_1, \ldots, s_k, \epsilon_k)$. We need to show that for any non-trivial linear combination $P = \sum a_I P^I$ there exists n such that $P(\iota_n) \neq 0$ in $H^{*,*}(K_n) \otimes_0 \mathbf{F}_l$. Bi-stability of operations together with the universal property of ι_n imply that it is sufficient to find any $X \in \Delta^{op}(Sm/k)_+^{\#}$ and a class $w \in H^{*,*}(X)$ such that $P(w) \neq 0$ in $H^{*,*}(X) \otimes_0 \mathbf{F}_l$. Using the computation of the action of motivic Steenrod algebra on the cohomology of $B\mu_l$ and the proof of [9, Proposition VI.2.4] one can easily see that it can be done by taking $X = (B\mu_l)^N$ for some N (cf. [6, Proposition 11.4]).

Lemma 4.44 *Let $R^* \to S^*$ be a homomorphism of non-negatively graded commutative rings and $u : M^* \to N^*$ a homomorphism of free non-negatively graded free modules over R^*. Assume that $R^0 \to S^0$ and $M^* \otimes_R S \to N^* \otimes_R S$ are isomorphisms. Then u is an isomorphism.*

Proof: See [].

Lemma 4.45 *$u : M^* \to N^*$ a homomorphism of free non-negatively graded free modules over $F[\tau]$ where F is a field and $gr(\tau) = 1$. Assume that $u_1 = u \otimes_1 Id$ is surjective and $u_0 = u \otimes_0 Id$ injective. Then u is an isomorphism.*

Proof: Let us show that u is surjective. Then u_0 is surjective and our result will follow from Lemma 4.44. The condition that u_0 is a monomorphism shows that if $u(x) = \tau y$ then $x = \tau x'$ and, since N^* is a free module $u(x') = y$. Let $x \in N^n$. Since u_1 is surjective there exists $z \in M^*$ and $a \in N^*$ such that $u(z) = x + (1 - \tau)a$. Let $a = \sum_{i=0}^m a_i$ and $z = \sum_{i=0}^r z_i$ be the decompositions of a and z into homogenous elements. For $i \neq n$ we have

$$u(z_i) = a_{i+1} - \tau a_i$$

Since $a_{m+1} = 0$ we may use the previous reasoning to show that a_m is in the image of u. Therefore so is a_{m-1} etc. down to a_{n+1}. Modifying z accordingly we may assume that $m = n - 1$ i.e. $a_i = 0$ for $i \geq n$. Then $u(z_n) = x - \tau a_{n-1}$ and we conclude that x is in the image of u by obvious induction on n.

Theorem 4.46 *Let k be a field of characteristic zero. Then u is an isomorphism.*

Proof: By Lemma 4.44 it is enough to show that $u \otimes_0 Id$ is an isomorphism. Since $u \otimes_0 Id$ does not change when we pass to an algebraic closure of the base field we may assume that k is algebraically closed. Then the result follows from Propositions 4.42, 4.43 and Lemma 4.45.

Remark 4.47 It should be possible to prove with our methods an analog of Theorem 4.46 for fields of positive characteristic $\neq l$. There are two issues which have to be addressed. One is to prove results for the etale realization similar to the ones we have proved for the topological one. Another one is to extend reduced power operations to motivic cohomology of non-smooth schemes in order to be able to use Theorem 4.20 instead of Theorem 4.25.

Remark 4.48 An unstable analog of Theorem 4.46 appears to be false i.e. the motivic cohomology of individual spaces K_n are not generated as algebras by elements of the form $P^I(\iota_n)$. In particular, [24, Lemma 2.2] is probably false[5].

Let the base field be **C**. In view of the unstable analog of Proposition 4.42 for any class $a \in H^{*,*}(K_n)$ there exist m such that $\tau^m a$ can be represented as a polynomial of $P^I(\iota_n)$. We claim that there are classes a for which the smallest m satisfying this condition is > 0.

[5] A patch to [24] which replaces Lemma 2.2 can be found in [28].

If we consider all the motivic Eilenberg-MacLane spaces then it is obvious. Indeed, the motivic cohomology of weight zero of $K(\mathbf{Z}/l, n, 0)$ are the same as the topological cohomology of $K(\mathbf{Z}/l, n)$. On the other hand all the motivic power operations shift the weight so applying P^I to the canonical element in $H^{n,0}$ we get elements in $H^{p,q}$ with $q > 0$.

Existence of such classes for the spaces $K_n = K(\mathbf{Z}/l, 2n, n)$ is less obvious. They do exist if the unstable motivic cohomology operations

$$P^i_{KM} : H^{p,q} \to H^{p+2i(l-1),lq}$$

constructed in the context of the higher Chow groups by Kriz and May can be extended to the motivic cohomology of objects such as K_n. In that case the first example I know would be $w = Sq^{16}_{KM} Sq^7 Sq^3 Sq^1(\iota_3)$ for which one would expect

$$\tau w = Sq^{16} Sq^7 Sq^3 Sq^1(\iota_3)$$

but which can not be obtained as a polynomial of $P^I(\iota_3)$ itself. The complexity of the example is due to the fact that one needs to find an admissible sequence with low excess and high weight shift.

It is possible that the motivic cohomology classes of $K(\mathbf{Z}/l, p, q)$ for $p \geq 2q$ can be represented as polynomials of classes obtained from the canonical one using both stable and unstable power operations.

5 Appendices

1 Admissible categories

Definition 5.1 *A full subcategory C of Sch/k is called admissible if*

1. *$Spec(k)$ and $\mathbf{A}^1$ are in C*
2. *for X and Y in C the product $X \times Y$ is in C*
3. *if X is in C and $U \to X$ is etale then U is in C*
4. *for X and Y in C the coproduct $X \amalg Y$ is in C*

If in addition C is closed under the formation of quotients with respect to actions of finite groups it will be called f-admissible.

Lemma 5.2 *The categories of all schemes of finite type, of smooth quasi-projective schemes and of smooth quasi-affine schemes over any field are admissible.*

Lemma 5.3 *The categories of quasi-projective and quasi-affine schemes over any field are f-admissible.*

Lemma 5.4 *The categories of normal quasi-projective and normal quasi-affine schemes over a perfect field are f-admissible.*

Proof: The only non-trivial point is to check that the product of two normal quasi-projective schemes over a perfect field is normal. This follows from [3, 6.8.5] and [3, 17.15.14.2].

Lemma 5.5 *The categories of quasi-projective and quasi-affine semi-normal schemes over a perfect field are f-admissible.*

Proof: Let us consider for example the quasi-projective case. The product of two semi-normal schemes over a perfect field is semi-normal by [2, Corollary 5.9]. The fact that a scheme etale over a semi-normal one is semi-normal follows from the results of [2] as well. Let us show that if X is semi-normal then X/G is semi-normal. Let $p : U \to X/G$ be the semi-normalization of X/G. Then the projection $X \to X/G$ factors through p by the universal property of semi-normalizations and since p is a universal homeomorphism and X is reduced we conclude that $X \to U$ is invariant under G-action. Hence we get an inverse $X/G \to U$.

Remark 5.6 The categories of (semi-)normal quasi-projective and quasi-affine schemes over a non-perfect field are not admissible since the product of the spectra of two inseparable extensions need not be normal.

Remark 5.7 Using the references provided above it is easy to see that the smallest admissible category over any field consists of (smooth) schemes X such that for some n there exists an etale morphism $X \to \mathbf{A}^n$. The smallest f-admissible category over a perfect field consists of finite group quotients of X as above. I do not know whether the same category is f-admissible over any field.

2 Finite group quotients in additive categories

We will need some computations which apply to categorical quotients for finite group actions in any additive category A. In our case the category will be $Cor(C, R)$. Let X be an object with an action of a finite group G. For an element $g \in G$ we let $[g]$ denote the corresponding automorphism $X \to X$. For a subgroup H of G we let $p_H : X \to X/H$ denote the projection. Note that $p_H[hg] = p_H[g]$ for any $h \in H$ and $g \in G$.

If L and M are two subgroups of G and $g \in G$ is such that $gLg^{-1} \in M$ then there is a unique morphism

$$p_{L,M,g} : X/L \to X/M$$

such that

$$p_{L,M,g} p_L = p_M[g] \tag{74}$$

We will write $p_{L,M}$ instead of $p_{L,M,1}$. Also for any $L \subset M$ in G there exists a unique morphism

$$p^{M,L} : X/M \to X/L$$

such that

$$p^{M,L} p_M = (1/|L|) \sum_{g \in M} p_L[g]. \tag{75}$$

These observations are obvious from the definition of categorical quotients. We will need the following result.

Proposition 5.8 *For any X, G and H as above one has:*

1. $p_{H,G} p^{G,H} = |G|/|H| \, Id_{X/G}$
2. *Let $D(H) \subset G$ be a set of representatives for double congugacy classes in G with respect to H. Then one has*

$$p^{G,H} p_{H,G} = \sum_{x \in D(H)} p_{H \cap x^{-1}Hx, H, x} p^{H, H \cap x^{-1}Hx} \tag{76}$$

Proof: The first equality is between morphisms from X/G to itself. Therefore it is sufficient to check that the compositions of these morphisms with p_G coincide. We have:

$$p_{H,G,1} p^{G,H} p_G = p_{H,G,1} (1/|H|) \sum_{g \in G} p_H[g] = (1/|H|) \sum_{g \in G} p_G[g] =$$

$$= |G|/|H| p_G$$

where the first equality holds by (75), the second by (74) and the third because $p_G[g] = p_G$ for all g.

To prove the second statement choose for each $x \in G$ a set $U(H,x) \subset H$ of representatives for $H/(H \cap xHx^{-1})$. Then any element g in G can be written uniquely as a product $g = uxh$ with $h \in H$, $x \in D(H)$ and $u \in U(H,x)$. Since (76) is an equality between two morphisms from X/H to itself it is sufficient to check that their compositions with p_H coincide i.e. that

$$p^{G,H} p_{H,G,1} p_H = \sum_{x \in D(H)} p_{H \cap x^{-1}Hx, H, x} p^{H, H \cap x^{-1}Hx} p_H \tag{77}$$

We have

$$\sum_{x\in D(H)} p_{H\cap x^{-1}Hx,H,x}p^{H,H\cap x^{-1}Hx}p_H = \sum_{x\in D(H)} p_{H\cap x^{-1}Hx,H,x}(1/|H\cap x^{-1}Hx|)\sum_{h\in H} p_{H\cap x^{-1}Hx}[h] =$$

$$= \sum_{x\in D(H)}(1/|H\cap x^{-1}Hx|)\sum_{h\in H} p_H[x][h] = \sum_{x\in D(H)}(1/|H\cap xHx^{-1}|)\sum_{h\in H} p_H[xh]$$

where the first equality holds by (75), the second by (74) and the third because $[x][h]=[xh]$ and the subgroups $H\cap x^{-1}Hx$ and $H\cap xHx^{-1}$ are adjoint. On the other hand

$$p^{G,H}p_{H,G,1}p_H = p^{G,H}p_G = (1/|H|)\sum_{g\in G} p_H[g] =$$

$$= (1/|H|)\sum_{x\in D(H)}\sum_{h\in H}\sum_{u\in U(H,x)} p_H[uxh] = (1/|H|)\sum_{x\in D(H)}\sum_{h\in H}(|H|/|H\cap xHx^{-1}|)p_H[xh] =$$

$$= \sum_{x\in D(H)}\sum_{h\in H}(1/|H\cap xHx^{-1}|)p_H[xh]$$

where the first equality holds by (74), the second one by (75), the third one because $u\in H$ and therefore $p_H[uxh]=p_H[xh]$ and the last one is obvious.

Corollary 5.9 *If H is a normal subgroup in G and $S(H)$ is a set of representatives for the left congugacy classes of H in G then*

$$p^{G,H}p_{H,G} = \sum_{g\in S(H)} p_{H,H,g} \tag{78}$$

Proposition 5.10 *Let X, G, H be as in Proposition 5.8 and let F be a radditive (=additive) functor on A such that $d=|G|/|H|$ is invertible in $F(X/G)$. Then there is a split equalizer sequence*

$$0\to F(X/G)\stackrel{F(p_{H,G})}{\longrightarrow} F(X/H)\rightrightarrows \oplus_{g\in G}F(X/(H\cap g^{-1}Hg))$$

where the first of the two arrows in the pair is $\oplus_g F(p_{H\cap g^{-1}Hg,H,1})$ and the second one is $\oplus_g F(p_{H\cap g^{-1}Hg,H,g})$.

Proof: Set

$$p^* = F(p_{H,G}) : F(X/G)\to F(X/H)$$
$$p_* = F(p^{G,H}) : F(X/H)\to F(X/G)$$

Then $d^{-1}p^*p_*$ is a projector by Proposition 5.8(1) and its image is $F(X/G)$. It remains to check that an element $a\in F(X/H)$ belongs to $Im(p)$ if for all $g\in G$ one has

$$F(p_{H\cap g^{-1}Hg,H,1})(a) = F(p_{H\cap g^{-1}Hg,H,g})(a). \tag{79}$$

By Proposition 5.8(2) we have

$$d^{-1}p^*p_* = d^{-1}\sum_{x\in D(H)} F(p_{H\cap x^{-1}Hx,H,x}p^{H,H\cap x^{-1}Hx})$$

If (79) is satisfied we have

$$d^{-1}\sum_{x\in D(H)} F(p_{H\cap x^{-1}Hx,H,x}p^{H,H\cap x^{-1}Hx})(a) = d^{-1}\sum_{x\in D(H)} F(p_{H\cap x^{-1}Hx,H}p^{H,H\cap x^{-1}Hx})(a) =$$

$$= d^{-1}\sum_{x\in D(H)} |H|/|H\cap x^{-1}Hx|(a)$$

and it remains to notice that

$$\sum_{x\in D(H)} |H|/|H\cap x^{-1}Hx| = |G/H| = d$$

because it is the sum of orders of orbits of the action of H on G/H.

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