

MILNOR K -THEORY OF LOCAL RINGS WITH FINITE RESIDUE FIELDS

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ABSTRACT. We propose a definition of improved Milnor K -groups of local rings with finite residue fields, such that the improved Milnor K -sheaf in the Zariski topology is a universal extension of the naive Milnor K -sheaf with a certain transfer map for étale extensions of local rings. The main theorem states that the improved Milnor K -ring is generated by elements of degree one.

INTRODUCTION

It is well known that Milnor K -theory of fields is a very nice cohomology theory in the sense that it encodes important arithmetic information about the field in question. Or in fancy words it is part of a motivic cohomology theory of smooth varieties [11].

In view of this fact the following question, which is the motivation for this article, seems to be reasonable:

Question: How can we generalize Milnor K -theory from fields to commutative rings?

If we want to generalize Milnor K -theory to more general rings we could simply copy the symbolic definition proposed by Milnor [13] for fields to an arbitrary commutative ring A : Let $K_*^{preM}(A)$ be the quotient of the tensor algebra of A^\times modulo the two-sided ideal generated by elements $a \otimes (1 - a)$ for $a, 1 - a \in A^\times$, see Definition 1. This is what we would like to call naive Milnor K -theory. But there are at least two problems:

- (1) Thomason [17] showed that a good definition of Milnor K -theory of smooth varieties, which generalizes the one for fields, does not exist. Here good means that the theory should satisfy standard properties of a cohomology theory like for example \mathbb{A}^1 -homotopy invariance and there should exist a functorial homomorphism to Quillen K -theory. This means that we can expect a good Milnor K -theory only for local rings.

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- (2) Even if we restrict to local rings the functor K_*^{preM} defined above is not what we would like to call Milnor K -theory. For example it does not satisfy the Gersten conjecture, compare Remark 11.

In spite of (2) the naive Milnor K -ring of a local ring with infinite residue field yields a good cohomology theory as the results proved in [15] and [8] suggest. So these considerations reduce our question posed above to the question how to define Milnor K -theory of local rings with finite residue fields. As the author hopes the reader will find a satisfying answer in Section 1.

There we show the following: Let F be an abelian sheaf on the big Zariski site of all schemes. Assume F is continuous, i.e. that it commutes with filtering inverse limits of affine schemes, see Definition 5, and has some kind a transfer for finite étale extensions of local rings with infinite residue fields – for an explanation see Section 1. In Theorem 7 we prove:

Theorem A. *There exists a universal transformation of continuous sheaves $F \rightarrow \hat{F}$ such that \hat{F} has a transfer for finite étale extensions of local rings. Moreover for a local ring A with infinite residue field we have $F(A) = \hat{F}(A)$.*

In fact by what is proved in [8] we can take F to be the sheafification of K_*^{preM} , which we denote by K_*^M , and get some improved Milnor K -sheaf \hat{K}_*^M . In order to convince the reader that this improved Milnor K -sheaf is in fact the correct one, we have collected some basic results on the latter in Proposition 10. We should remark that the ring \hat{K}_*^M does already appear in unpublished notes of Gabber [2], but without the transfer map of [8] it is quite hard to study it.

The second aim of this article is to show that the improved Milnor K -ring is generated by symbols. In fact this is not at all clear in view of the definition of \hat{K}_*^M via Theorem A. Our main result, Theorem 9, whose proof unfortunately requires a very messy calculation in polynomial rings, says:

Theorem B. *The natural homomorphism of Zariski sheaves*

$$K_*^M \longrightarrow \hat{K}_*^M$$

is surjective.

Via the Milnor Conjecture proved by Voevodsky et al. [14] this result has some interesting application to quadratic forms over local rings. Furthermore let $H_{mot}^n(\text{Spec}(A), \mathbb{Z}(n))$ be the hypercohomology of the complex of Zariski sheaves $\mathbb{Z}(n)$ constructed in [11], see also the remark following the corollary. We can deduce in Corollary 15:

Corollary. *The motivic cohomology ring*

$$(H_{mot}^n(\text{Spec}(A), \mathbb{Z}(n)))_{n \geq 0}$$

for a regular local equicharacteristic ring A is generated by elements of degree one.

This corollary was – at least implicitly – predicted by the Beilinson–Lichtenbaum conjectures on motivic cohomology.

A remark on motivic cohomology.

In this article we will use the motivic cohomology ring of a regular scheme X (of finite Krull dimension). In order to fix notation we shortly explain what this means. Voevodsky constructs the motivic complex $\mathbb{Z}(n)$ for $n \geq 0$ which is a complex of Zariski sheaves on the category of smooth separated schemes over a field [11]. Nevertheless the construction of this complex of sheaves works equally well on the small Zariski site of a regular scheme, so that we can define for $n \geq 0$ and $m \in \mathbb{Z}$

$$H_{mot}^m(X, \mathbb{Z}(n)) = \mathbb{H}_{Zar}^m(X, \mathbb{Z}(n)).$$

The important fact about this motivic cohomology functor is that it is continuous, see Definition 5 where one has to assume that all appearing rings are regular.

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First of all I would like to thank Stephen Lichtenbaum and Burt Totaro who explained to me that Corollary 15 was indeed expected to be true as part of the fantastic Beilinson-Lichtenbaum program on motivic cohomology. This was in fact the initial motivation for the work presented here.

I am indebted to Wilberd van der Kallen for explaining me the different presentations available for K_2 of a local ring. Furthermore I thank Uwe Jannsen for many helpful comments and his encouragement.

1. IMPROVED MILNOR K -THEORY

Let A be a commutative unital ring and let

$$\mathcal{T}_*(A^\times) = \bigoplus A^\times \otimes \cdots \otimes A^\times$$

be the tensor algebra over the units of A .

The usual definition of Milnor K -groups is:

Definition 1. Define the graded ring $K_*^{preM}(A)$ for a commutative ring A by

$$K_*^{preM}(A) = \mathcal{T}_*(A^\times) / (\{a \otimes (1 - a) \mid a, 1 - a \in A^\times\}).$$

Let K_*^M be the sheaf of graded rings associated to K_*^{preM} on the site of all schemes with the Zariski topology.

For a ring A we will denote $K_*^M(\text{Spec } A)$ also by $K_*^M(A)$.

It is clear that $K_1^M(A) = A^\times$ for every ring A . Below we will improve the definition of Milnor K -theory in order to get a sensible theory for local rings with finite residue fields. Therefore K_*^M will usually be called naive K -theory.

The following facts are standard for Milnor K -groups of local rings with infinite residue fields, see [15], [16] and [18].

Proposition 2. *Let A be a local ring with infinite residue field. Then we have*

- (1) *The natural map $K_2^M(A) \rightarrow K_2(A)$ from Milnor K -theory to algebraic K -theory is an isomorphism.*
- (2) *The relation $\{a, -a\} = 0$ holds in $K_2^M(A)$ for $a \in A^\times$.*
- (3) *The ring $K_*^M(A)$ is skew-symmetric.*
- (4) *For $a_1, \dots, a_n \in A^\times$ with $a_1 + \dots + a_n = 1$ the relation*

$$0 = \{a_1, \dots, a_n\} \in K_n^M(A)$$

holds.

Moreover there exists a transfer for Milnor K -groups which is constructed in [8] and whose main properties we recall in the next proposition. The transfer will be essential for the constructions of this paper.

Let $i : A \rightarrow B$ be a finite étale extension of local rings with infinite residue fields. Fix an explicit presentation $B \cong A[t]/(f)$ which exists by EGA IV 18.4.5.

Proposition 3. *For a fixed presentation of B over A there exists a canonical transfer homomorphisms*

$$N_{B/A} : K_n^M(B) \longrightarrow K_n^M(A)$$

satisfying:

- (1) *$N_{B/A} : K_1^M(B) \rightarrow K_1^M(A)$ is the usual norm map on unit groups.
 $N_{B/A} : K_0^M(B) \rightarrow K_0^M(A)$ is multiplication by $\deg(B/A)$.*
- (2) *The projection formula holds, i.e. for $x \in K_n^M(A)$, $y \in K_m^M(B)$ we have*

$$x N_{B/A}(y) = N_{B/A}(i_*(x)y) \in K_{n+m}^M(A).$$

- (3) *If A contains a field the transfer does not depend on the presentation of B over A chosen.*
- (4) *Let $j : A \rightarrow A'$ be a homomorphism of local rings and let $i' : A' \rightarrow B' = B \otimes_A A'$ be the induced inclusion, for which we fix the induced presentation. Assume B' is local. Then the diagram*

$$\begin{array}{ccc} K_n^M(B) & \longrightarrow & K_n^M(B') \\ \downarrow N_{B/A} & & \downarrow N_{B'/A'} \\ K_n^M(A) & \xrightarrow{j_*} & K_n^M(A') \end{array}$$

commutes.

Proof. This is proved in [8]. □

Next we consider general abelian sheaves with a weak form of a transfer. In fact we will construct the improved Milnor K -groups axiomatically such that they have a transfer map.

Let \mathfrak{S} be the category of abelian sheaves on the big Zariski site of all schemes. Let $\mathfrak{S}\mathfrak{T}$ be the full subcategory of \mathfrak{S} such that a sheaf F is in $\mathfrak{S}\mathfrak{T}$ if for every finite étale extension of local rings $i : A \subset B$ there exists a compatible system of norms

$$[N_{B'/A'} : F(B') \rightarrow F(A')]_{A'}$$

where A' runs over all local A -algebras for which $B' = B \otimes_A A'$ is also local. Compatibility means that if $A' \rightarrow A''$ are both local A -algebras such that $B' = B \otimes_A A'$ and $B'' = B \otimes_A A''$ are local the diagram

$$\begin{array}{ccc} F(B') & \longrightarrow & F(B'') \\ N_{B'/A'} \downarrow & & \downarrow N_{B''/A''} \\ F(A') & \longrightarrow & F(A'') \end{array}$$

commutes. Furthermore we assume that if $i' : A' \rightarrow B'$ is the induced inclusion our norm $N_{B'/A'}$ satisfies

$$N_{B'/A'} \circ i'_* = \text{deg}(B/A) \text{id}_{F(A')}.$$

Let $\mathfrak{S}\mathfrak{T}^\infty$ be the full subcategory of sheaves in \mathfrak{S} which have such norms if we restrict to the system of local A -algebras A' with infinite residue fields.

Proposition 4. *The functor K_n^M is an object of $\mathfrak{S}\mathfrak{T}^\infty$ for all $n \geq 0$.*

Proof. Immediate from Proposition 3. □

Actually the Milnor K -functor should have some more global canonical transfer but at the moment we can define it only in the case of equicharacteristic schemes. We shall not be concerned with this problem here.

Let F be a covariant functor from rings to abelian groups.

Definition 5. The functor F is called continuous if for every filtering direct limit of rings

$$A = \varinjlim A_i$$

the natural homomorphism

$$\varinjlim F(A_i) \longrightarrow F(A)$$

is an isomorphism.

A (pre-)sheaf is called continuous if its restriction to affine schemes is continuous in the above sense.

Proposition 6. *The Milnor K -sheaf K_*^M is continuous.*

Proof. It is clear from the definition that $K_*^{\text{pre}M}$ is continuous. Furthermore a simple calculation shows that if a presheaf is continuous the associated sheaf in the Zariski topology is so too. □

Our existence and uniqueness result, which is motivated by a construction of improved Milnor K -theory due to Gabber [2], reads now:

Theorem 7. *For every continuous $F \in \mathfrak{S}\mathfrak{T}^\infty$ there exists a universal continuous $\hat{F} \in \mathfrak{S}\mathfrak{T}$ and a natural transformation $F \rightarrow \hat{F}$. That is for arbitrary continuous $G \in \mathfrak{S}\mathfrak{T}$ and natural transformation $F \rightarrow G$ there exists a unique natural transformation $\hat{F} \rightarrow G$ such that the diagram commutes.*

$$\begin{array}{ccc} F & \xrightarrow{\quad} & \hat{F} \\ & \searrow & \swarrow \text{---} \\ & & G \end{array}$$

Moreover for a local ring A with infinite residue field we have $F(A) = \hat{F}(A)$.

Before we can give the proof we have to recall the construction of the rational function ring $A(t)$ over a ring A and some of its basic properties. For a commutative ring A we let $A(t_1, \dots, t_n)$ be the ring $A[t_1, \dots, t_n]_S$ where S is the multiplicative set consisting of all polynomials $\sum_I a_{(I)} t^I$ such that the ideal in A generated by all the coefficients $a_{(I)} \in A$ is A itself. If A is local with maximal ideal m the ring $A(t_1, \dots, t_n)$ is local too, in fact it is easy to see that for A local $S = A[t_1, \dots, t_n] - m_t$ where m_t is the prime ideal $m A[t_1, \dots, t_n]$. Denote by $i : A \rightarrow A(t)$ the natural ring homomorphism. Denote by i_1 resp. i_2 the natural ring homomorphism $A(t) \rightarrow A(t_1, t_2)$ which sends t to t_1 resp. t_2 .

Lemma 8. *If $A \subset B$ is a finite étale extension of local rings there is a canonical isomorphism $B \otimes_A A(t_1, \dots, t_n) \xrightarrow{\sim} B(t_1, \dots, t_n)$*

Proof. For simplicity we restrict to $n = 1$. Let m be the maximal ideal of A . Consider the finite extension of rings $A[t] \rightarrow B[t]$. Let m_t^A be the prime ideal $m A[t]$ and m_t^B be the prime ideal $m B[t]$. The latter ideal is indeed a prime ideal, because $m B$ is the maximal ideal of B by assumption. This also implies that

$$B(t) = B[t]_{m_t^B}.$$

Moreover the finiteness of $A \subset B$ implies that m_t^B is the only prime ideal over m_t^A . Recall that according to standard facts $B[t] \otimes_{A[t]} A(t)$ is the semi-local ring whose maximal ideals correspond to the finite set of prime ideals in $B[t]$ which lie over m_t^A . But as we saw there is only one prime ideal over m_t^A , namely m_t^B , so

$$B \otimes_A A(t) = B[t] \otimes_{A[t]} A(t) \quad \text{and} \quad B(t) = B[t]_{m_t^B}$$

must be isomorphic. \square

Proof of Theorem 7. For an arbitrary Zariski sheaf G we let \hat{G} be the Zariski sheafification of the following presheaf defined on affine schemes:

$$\text{Spec}(A) \mapsto \ker[G(A(t)) \xrightarrow{i_1^* - i_2^*} G(A(t_1, t_2))]$$

We claim that if G is an object in $\mathfrak{S}\mathfrak{T}^\infty$ then \hat{G} is an object in $\mathfrak{S}\mathfrak{T}$. The continuity of \hat{G} follows because the presheaf on affine schemes just defined

is clearly continuous and the Zariski sheaffication of a continuous presheaf is continuous. For a finite étale extension of local rings $A \subset B$ Lemma 8 and the existence of a compatible system of norms allow us to write down the commutative diagram

$$\begin{array}{ccccc} G(B) & \longrightarrow & G(B(t)) & \xrightarrow{i_{1*} - i_{2*}} & G(B(t_1, t_2)) \\ & & \downarrow N & & \downarrow N \\ G(A) & \longrightarrow & G(A(t)) & \xrightarrow{i_{1*} - i_{2*}} & G(A(t_1, t_2)) \end{array}$$

So there exists a norm map $\hat{G}(B) \rightarrow \hat{G}(A)$ for which one easily verifies the compatibility with base change. This shows that \hat{G} is an object in \mathfrak{ST} .

The next proposition will be essential for the proof of the universal property of \hat{F} .

Proposition 9. *Let $G \in \mathfrak{ST}$ (resp. $G \in \mathfrak{S}$) be continuous. Then for a local ring A (resp. a local ring with infinite residue field) we have $G(A) = \hat{G}(A)$.*

Proof. First we prove the statement in parenthesis. So let A have infinite residue field and let $G \in \mathfrak{S}$ be continuous. We will prove the injectivity of $G(A) \rightarrow \hat{G}(A)$ first. In the following $S' \subset S$ will always be some finitely generated submonoid where $S \subset A[t]$ is defined as above. So by continuity we clearly have

$$\hat{G}(A) \subset G(A[t]_S) = \varinjlim_{S'} G(A[t]_{S'})$$

So it is enough to show that $G(A) \rightarrow G(A[t]_{S'})$ is injective for every S' . For fixed S' we will explain how to choose an element $\alpha \in A$ such that for all $p \in S'$ we have $p(\alpha) \in A^\times$. Let $p_1, \dots, p_r \in S'$ be generators of the finitely generated monoid S' . Because the residue field of A is infinite, it is possible to find $\alpha \in A$ with $p_1(\alpha) \cdots p_r(\alpha) \in A^\times$. This is the element α we were looking for. Let $\pi : A[t]_{S'} \rightarrow A$ be the ring homomorphism such that t maps to α . As

$$G(A) \longrightarrow G(A[t]_{S'}) \xrightarrow{\pi_*} G(A)$$

is the identity the injectivity of the first arrow follows.

For the surjectivity of $G(A) \rightarrow \hat{G}(A)$ we argue similarly. Let $S' \subset A(t)^\times \cap A[t]$ and $S'' \subset A(t_1, t_2)^\times \cap A[t_1, t_2]$ be some finitely generated submonoids and $x \in G(A[t]_{S'})$ such that the arrow

$$G(A[t]_{S'}) \xrightarrow{i_{1*} - i_{2*}} G(A[t_1, t_2]_{S''})$$

is well defined and kills x . Since S' and S'' are finitely generated and the residue field of A is infinite, we can construct an element $\alpha \in A$ such that for all $p \in S'$ we have $p(\alpha) \in A^\times$ and for all $p \in S''$ we have $p(t, \alpha) \in A(t)^\times$. Denote by $\pi : A[t]_{S'} \rightarrow A$ resp. $\pi' : A[t_1, t_2]_{S''} \rightarrow A(t)$ the ring

homomorphisms sending t to α resp. t_1 to t and t_2 to α . Now the sequence of equalities

$$i_* \circ \pi(x) = \pi' \circ i_{2*}(x) = \pi' \circ i_{1*}(x) = im(x) \in G(A(t))$$

proves the surjectivity of $G(A) \rightarrow \hat{G}(A)$.

Next we prove that for $G \in \mathfrak{S}\mathfrak{T}$ continuous and A local $G(A) \rightarrow \hat{G}(A)$ is an isomorphism. We prove injectivity first. Fix an arbitrary prime p . Consider a tower of finite étale extensions of A

$$A \subset A_1 \subset A_2 \subset \cdots \subset A_\infty$$

such that A_m is local, $[A_m : A_{m-1}] = p$ and $\cup_m A_m = A_\infty$. Now $G(A_\infty) = \hat{G}(A_\infty)$ according to the first part of the proof. Consider $x \in \ker[G(A) \rightarrow \hat{G}(A)]$. So by continuity $x \in \ker[G(A) \rightarrow G(A_m)]$ for some $m > 0$. The existence of a transfer homomorphism $N : G(A_m) \rightarrow G(A)$ with

$$G(A) \longrightarrow G(A_m) \xrightarrow{N} G(A)$$

equal to multiplication by p^m shows that $p^m x = 0$. As this holds for all primes p we have proved injectivity.

In order to prove surjectivity of $G(A) \rightarrow \hat{G}(A)$ consider $x \in \hat{G}(A)$ and fix a prime p and a tower of finite étale extensions as in the injectivity proof. Again observe that $G(A_\infty) = \hat{G}(A_\infty)$. Because of the continuity of G there exists $x_m \in G(A_m)$ which has the same image in $\hat{G}(A_m)$ as x . The commutative diagram

$$\begin{array}{ccc} G(A_m) & \longrightarrow & \hat{G}(A_m) \\ N_{A_m/A} \downarrow & & \downarrow N_{A_m/A} \\ G(A) & \xrightarrow{i_*} & \hat{G}(A) \end{array}$$

implies that $i_* \circ N_{A_m/A}(x_m) = p^m x$. So if we choose two different primes p_1, p_2 we see that there exists an integer n with $p_1^n x \in im(i_*)$ and $p_2^n x \in im(i_*)$. Choose $\alpha, \beta \in \mathbb{Z}$ such that $\alpha p_1^n + \beta p_2^n = 1$. Then $x = \alpha p_1^n x + \beta p_2^n x \in im(i_*)$, what we had to show. \square

Now we can finish the proof. Let F and G be as in the statement of the theorem. Define the homomorphism $\hat{F} \rightarrow \hat{G}$ such that the following diagram is commutative

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & \nearrow & \downarrow \alpha \\ \hat{F} & \longrightarrow & \hat{G} \end{array}$$

where α is an isomorphism by Proposition 9. The uniqueness of the homomorphism $\hat{F} \rightarrow G$ follows from the commutative diagram

$$\begin{array}{ccccc} F(A(t)) & \xrightarrow{\beta} & \hat{F}(A(t)) & \longleftarrow & \hat{F}(A) \\ \downarrow & & \downarrow & & \downarrow \\ G(A(t)) & \xlongequal{\quad} & G(A(t)) & \xleftarrow{\gamma} & G(A) \end{array}$$

where A is a local ring, since by Proposition 9 β is an isomorphism and γ is injective. □

The next proposition comprises basic information about our improved Milnor K -theory \hat{K}_*^M . We will only sketch the proofs.

Proposition 10. *Let (A, m) be a local ring. Then:*

- (1) $\hat{K}_1^M(A) = A^\times$.
- (2) $\hat{K}_*^M(A)$ has a natural graded commutative ring structure.
- (3) Proposition 2 and 3 remain true for any any local ring A if we replace K_*^M by \hat{K}_*^M .
- (4) If F is a field we have $K_n^M(F) = \hat{K}_n^M(F)$.
- (5) For every $n \geq 0$ there exists a universal natural number M_n such that if $|A/m| > M_n$ the natural homomorphism

$$K_n^M(A) \longrightarrow \hat{K}_n^M(A)$$

is an isomorphism.

- (6) There exists a homomorphism

$$K_n(A) \longrightarrow \hat{K}_n^M(A)$$

such that the composition

$$\hat{K}_n^M(A) \longrightarrow K_n(A) \longrightarrow \hat{K}_n^M(A)$$

is multiplication by $(n-1)!$ and the composition

$$K_n(A) \longrightarrow \hat{K}_n^M(A) \longrightarrow K_n(A)$$

is the Chern class $c_{n,n}$.

- (7) If (A, I) is a Henselian pair in the sense of [3] and $s \in \mathbb{N}$ is invertible in A/I the map

$$\hat{K}_n^M(A)/s \longrightarrow \hat{K}_n^M(A/I)/s$$

is an isomorphism.

- (8) Let A be regular and equicharacteristic, $F = Q(A)$ and $X = \text{Spec } A$. The Gersten conjecture holds for Milnor K -theory, i.e. the Gersten complex

$$0 \longrightarrow \hat{K}_n^M(A) \longrightarrow K_n^M(F) \longrightarrow \bigoplus_{x \in X(1)} K_{n-1}^M(k(x)) \longrightarrow \cdots$$

is exact.

- (9) Let X be a regular scheme containing a field. There is a natural isomorphism

$$H_{Zar}^n(X, \hat{K}_n^M) \cong CH^n(X).$$

- (10) If A is equicharacteristic of characteristic prime to 2 the map

$$i_A : \hat{K}_3^M(A) \longrightarrow K_3(A)$$

is injective.

- (11) Let A be regular and equicharacteristic. There is a natural isomorphism

$$\hat{K}_n^M(A) \xrightarrow{\sim} H_{mot}^n(\text{Spec}(A), \mathbb{Z}(n))$$

onto motivic cohomology.

Proof. (1): Since K_1^M is an objects in $\mathfrak{S}\mathfrak{X}$, the isomorphism $\hat{K}_1^M(A) = K_1^M(A) = A^\times$ follows from Proposition 9.

(2): This follows immediately from the hat construction in the proof of Theorem 7.

(3): It is well known that the sheaf associated to $X \mapsto K_2(X)$ is an object in $\mathfrak{S}\mathfrak{X}$, so that $\hat{K}_2(A) = K_2(A)$ by Proposition 9. But if A is a local ring with infinite residue field we have $K_2^M(A) = K_2(A)$ according to Proposition 2 (1) and the isomorphism of sheaves $\hat{K}_2^M = K_2$ follows from the definition of the ‘hat’ in the proof of Theorem 7. The rest follows from the injectivity of

$$\hat{K}_n^M(A) \longrightarrow \hat{K}_n^M(A(t)) = K_n^M(A(t)).$$

(4): If F is infinite this follows from Proposition 9, if F is finite then $K_n^M(F) = 0$ and so it suffices to show that $K_n^M(F) \rightarrow \hat{K}_n^M(F)$ is surjective. Let $s_i : K_n^M(F(t_1, \dots, t_i)) \rightarrow K_n^M(F(t_1, \dots, t_{i-1}))$ be the specialization homomorphism which maps $\{t_i^j + a_{j-1}t_i^{j-1} + \dots + a_0\}$ to 0 for $a_k \in F(t_1, \dots, t_{i-1})$ ($0 \leq k < j$) so that

$$K_n^M(F(t_1, \dots, t_{i-1})) \xrightarrow{inc_{i-1}} K_n^M(F(t_1, \dots, t_i)) \xrightarrow{s_i} K_n^M(F(t_1, \dots, t_{i-1}))$$

is the identity. Then with the notation as in the proof of the theorem we have for $x \in \hat{K}_n^M(F) \subset K_n^M(F(t))$

$$\hat{K}_n^M(F) \ni inc_0 \circ s_1(x) = s_2 \circ i_{2*}(x) = s_2 \circ i_{1*}(x) = x.$$

(5): It was shown in [8] that there exists an $M_n \in \mathbb{N}$ such that the statement of Proposition 3 remains true if the local ring A has more than M_n elements in its residue field and if $deg(B/A) \leq 3$. Now an analog of the second part of proof of Proposition 9 with $p_1 = 2$ and $p_2 = 3$ gives (5).

(6): This follows immediately from [15].

(7): An elementary calculation shows that

$$K_n^M(A)/s \longrightarrow K_n^M(A/I)/s$$

is an isomorphism for every local ring A with s invertible in A/I . Now a norm argument shows the analogous result for the improved Milnor K -groups. This is accomplished by choosing an étale local extension $A \subset A'$ of some prime power degree q , coprime to s , such that the residue field of A' has more than M_n elements. Here M_n is the natural number from part (5). Observe that $(A', A'/I)$ is again a Henselian pair. Consider the commutative diagram

$$\begin{array}{ccc} \hat{K}_n^M(A')/s & \longrightarrow & \hat{K}_n^M(A'/I)/s \\ N \downarrow & & \downarrow N \\ \hat{K}_n^M(A)/s & \longrightarrow & \hat{K}_n^M(A/I)/s \end{array}$$

where the upper horizontal arrow is an isomorphism by what has been said so far. A simple diagram chase shows that the kernel and cokernel of $\hat{K}_n^M(A)/s \rightarrow \hat{K}_n^M(A/I)/s$ have exponent q and must therefore vanish.

(8): This complex was constructed in [7]. Again if A has more than M_n elements in its residue field the result was proven in [8]. If not one uses a norm trick as in (7).

(9): Immediate from (8).

(10): If A is a field this was shown by Kahn using Voevodsky's proof of the Milnor conjecture [6]. If A is regular it follows from the field case and (8). If A is not regular we first use the norm trick and can and will assume that A has more than M_3 elements in its residue field. Next we use Hoobler's trick [4] and choose some regular equicharacteristic local ring A' such that there exists an exact sequence

$$0 \longrightarrow I \longrightarrow A' \longrightarrow A \longrightarrow 0$$

such that (A', I) is a Henselian pair. Let x be in $\ker(i_A)$. Then $2x = 0$ according to (6). Next choose $x' \in K_3^M(A') = \hat{K}_3^M(A')$ which maps to x . An elementary argument left to the reader shows that we can choose x' such that $2x' = 0$. Now remember that the two torsion in $K_3(A')$ is isomorphic to the two torsion in $K_3(A)$ by the rigidity of algebraic K -theory, see [3], so that $i_{A'}(x') = 0$ and finally $x' = 0$ by the regular case proved above.

(11): Improved Milnor K -theory fulfills the Gersten conjecture by (8) and motivic cohomology does so by standard facts, so the result follows from an easy diagram chase of Gersten complexes using the fact that (11) is known if A is a field. The details can be found in [8, Section 7]. \square

Remark 11. In general the map

$$K_n^M(A) \longrightarrow \hat{K}_n^M(A)$$

is not an isomorphism. For example for $n = 2$ we have $\hat{K}_2^M(A) = K_2(A)$ according to Proposition 10(3) and it was shown in the appendix to [5] that for $A = \mathbb{F}_2[t]_{(t)}$ the groups $K_2^M(A)$ and $K_2(A)$ are not isomorphic.

2. GENERATION BY SYMBOLS

In this section we propose a conjectural determination of $\hat{K}_*^M(A)$. Set

$$I = \ker[K_2^M(A) \rightarrow K_2(A)] .$$

One can show that $I = 0$ if $|A/m| > 3$, where m is the maximal ideal of A . In fact there are explicit descriptions of I by van der Kallen, Maazen and Stienstra [18], [12] and Kolster [10].

Conjecture 12. *For any local ring A the natural homomorphism of graded rings*

$$K_*^M(A)/(I) \longrightarrow \hat{K}_*^M(A)$$

is an isomorphism.

We will prove the surjectivity as our main theorem, which is Theorem B of the introduction.

Theorem 13. *Let A be a local ring. Then the map*

$$\eta_A : K_*^M(A) \longrightarrow \hat{K}_*^M(A)$$

is surjective.

In the proof we use the statement of the theorem for $n = 2$ which is classical modulo Proposition 10(3), see for example [10]:

Proposition 14 (Dennis–Stein). *The map*

$$K_2^M(A) \longrightarrow K_2(A) = \hat{K}_2^M(A)$$

is surjective.

2.1. Applications.

An interesting application of our theorem concerns motivic cohomology of local rings.

Corollary 15. *Assume A to be regular local and equicharacteristic. The motivic cohomology ring*

$$[H_{mot}^n(A, \mathbb{Z}(n))]_{n \geq 0}$$

is generated by elements of degree one.

Proof. Immediate from Proposition 10(11). □

Finally, all the results in [9, Section 0] remain true for local rings with finite residue fields of characteristic different from 2. For the convenience of the reader we state them without proofs in the next proposition. Let $W(A)$ be the Witt ring of the local ring A and $I_A \subset W(A)$ the fundamental ideal. Assume that the residue field of A has characteristic different from 2.

Proposition 16. *If A is equicharacteristic the following holds:*

- (1) If A is regular and $i : A \rightarrow F$ is the inclusion into the fraction field, $F = Q(A)$, the following conditions are equivalent:
- (i) $q \in I_A^n \subset W(A)$.
 - (ii) $i_*(q) \in I_F^n \subset W(F)$.
- (2) If $A \subset A'$ is a finite étale extension with A' local the transfer

$$N_{A'/A} : W(A') \rightarrow W(A)$$

maps $I_{A'}^n$ to I_A^n .

2.2. Proof of Theorem 13.

The heart of the proof will be to show that for a finite étale extension $A \subset B$ of local rings of degree $p = 2$ or 3 the transfer $N_{B/A} : \hat{K}_n^M(B) \rightarrow \hat{K}_n^M(A)$ maps the image of η_B to the image of η_A . In the proof of this fact an elementary but technical reasoning reduces us to $n = 1$ if $p = 2$ and $n = 2$ if $p = 3$ – this is the reason why we have to restrict to these two special primes. But in both cases we know that η_A is surjective, in fact for $n = 2$ this is the proposition due to Dennis and Stein mentioned above. Finally, using this key result the standard norm trick allows us to reduce the proof of the surjectivity of η_A to the case of infinite residue fields.

We start the proof by fixing $p = 2$ or 3 . Consider a tower of finite étale extensions of A

$$A \subset A_1 \subset A_2 \subset \cdots \subset A_\infty$$

such that A_m is local, $[A_m : A_{m-1}] = p$ and $\cup_m A_m = A_\infty$.

From Proposition 10(3) we know

- (1) $K_n^M(A_\infty) = \hat{K}_n^M(A_\infty)$.
- (2) There exist transfers

$$N_{A_m/A_{m-1}} : \hat{K}_n^M(A_m) \longrightarrow \hat{K}_n^M(A_{m-1})$$

such that the composite

$$\hat{K}_n^M(A_{m-1}) \longrightarrow \hat{K}_n^M(A_m) \xrightarrow{N} \hat{K}_n^M(A_{m-1})$$

is multiplication by p and such that the projection formula holds, compare Proposition 3 and the hat construction in the proof of theorem 7.

Now let x be in $\hat{K}_n^M(A)$ and let x_m be the induced element in $\hat{K}_n^M(A_m)$. By (1) there exists $x'_\infty \in K_n^M(A_\infty)$ with $\eta_{A_\infty}(x'_\infty) = x_\infty$. Because of the continuity of K_n^M and \hat{K}_n^M (Proposition 6) there exists $m \in \mathbb{N}$ and $x'_m \in K_n^M(A_m)$ with $\eta_{A_m}(x'_m) = x_m$. Now the next lemma produces $x' \in K_n^M(A)$ with $\eta_A(x') = p^m x$.

So making this construction for $p = 2$ and $p = 3$ we find $m_2, m_3 \geq 0$ and $x_2^*, x_3^* \in K_n^M(A)$ such that

$$\begin{aligned} \eta_A(x_2^*) &= 2^{m_2} x \\ \eta_A(x_3^*) &= 3^{m_3} x \end{aligned}$$

Choose $\alpha, \beta \in \mathbb{Z}$ satisfying

$$\alpha 2^{m_2} + \beta 3^{m_3} = 1.$$

Then $\eta_A(\alpha x_2^* + \beta x_3^*) = x$. So we deduce that $K_n^M(A) \rightarrow \hat{K}_n^M(A)$ is surjective. In order to complete the proof we have to show:

Lemma 17. *With the notation of the theorem for $p = 2$ or 3 and $A \subset B$ a finite étale extension of local rings of degree p we have*

$$N_{B/A}(\text{im } \eta_B) \subset \text{im } \eta_A.$$

For $p = 2$ resp. $p = 3$ the proof of this lemma is reduced to the case $n = 1$ resp. $n = 2$ by the projection formula (Proposition 3(2)) and the next two sublemmas. But for $n \leq 2$ the lemma is clear as $K_1^M(A) = \hat{K}_1^M(A)$ and as $K_2^M(A) \rightarrow \hat{K}_2^M(A)$ is surjective by Proposition 14.

Sublemma 18. *For $p = 2$ the subgroup $\text{im } \eta_B \subset \hat{K}_n^M(B)$ is generated by symbols*

$$\{a_1, a_2, \dots, a_n\} \in \hat{K}_n^M(B)$$

with $a_1 \in B^\times$ and $a_i \in A^\times$ for $i > 1$.

Sublemma 19. *For $p = 3$ the subgroup $\text{im } \eta_B \subset \hat{K}_n^M(B)$ is generated by symbols*

$$\{a_1, a_2, \dots, a_n\} \in \hat{K}_n^M(B)$$

with $a_1, a_2 \in B^\times$ and $a_i \in A^\times$ for $i > 2$.

By EGA IV 18.4.5 we can write $B = A[t]/(f)$ where $f = t^p + \alpha_{p-1}t^{p-1} + \dots + \alpha_0$ is irreducible modulo the maximal ideal $m \subset A$.

In the proof of the sublemmas we can by induction restrict to $n = 2$ for $p = 2$ and $n = 3$ for $p = 3$. Now the rest of the proof is by brute force.

Proof of Sublemma 18. We start with a symbol $\{a_1t + a_0, b_1t + b_0\} \in \hat{K}_2^M(B)$, $a_1, a_0, b_1, b_0 \in A$ and have to show that it lies in the image of $K_1(A) \otimes_{\mathbb{Z}} K_1(B)$ in $\hat{K}_2^M(B)$.

1st step: Reduce to $a_1 = b_1 = 1$.

If $a_1 \in A^\times$ then we write

$$\{a_1t + a_0\} = \left\{t + \frac{a_0}{a_1}\right\} + \{a_1\}.$$

and multiply from the right with $\{b_1t + b_0\}$. If $a_1 \notin A^\times$ write

$$\{a_1t + a_0\} = -\{t\} + \left\{t - \frac{a_1\alpha_0}{a_0 - a_1\alpha_1}\right\} + \{a_0 - a_1\alpha_1\} \in K_1^M(B).$$

and multiply from the right with $\{b_1t + b_0\}$. Similarly we reduce to $b_1 = 1$.

2nd step: Reduce to $\bar{a}_0 \neq \bar{b}_0 \in A/m$ and $a_1 = b_1 = 1$.

By the first step we can assume $a_1 = b_1 = 1$ and $\bar{a}_0 = \bar{b}_0 \in A/m$.

Case A: $(t + \bar{a}_0)^2 \notin A/m$.

In this case write

$$\{t + a_0, t + b_0\} = -\{t + a_0, t + a_0\} + \left\{t + a_0, t + \frac{a_0 b_0 - \alpha_0}{a_0 + b_0 - \alpha_1}\right\} + \{t + a_0, a_0 + b_0 - \alpha_1\}$$

Remark that $\{t + a_0, t + a_0\} = \{t + a_0, -1\}$, $\bar{a}_0 + \bar{b}_0 - \bar{\alpha}_1 \neq 0$ by assumption and

$$\bar{a}_0 \neq \frac{\bar{a}_0 \bar{b}_0 - \bar{\alpha}_0}{\bar{a}_0 + \bar{b}_0 - \bar{\alpha}_1}.$$

The latter because otherwise \bar{a}_0 would be a zero of

$$(t + \bar{a}_0)(t + \bar{b}_0) - \bar{f}$$

but \bar{f} has no zeros in A/m .

Case B: $(t + \bar{a}_0)^2 \in A/m$.

Choose $c \in A$ with $\bar{c} \neq \bar{a}_0$. Then

$$\{t + a_0, t + b_0\} = -\{t + c, t + b_0\} + \{(a_0 + c - \alpha_1)t + a_0 c_0 - \alpha_0, t + b_0\}.$$

Again as in the previous case $\bar{a}_0 + \bar{c} - \bar{\alpha}_1 \neq 0$ by assumption and

$$\bar{b}_0 \neq \frac{\bar{a}_0 \bar{c} - \bar{\alpha}_0}{\bar{a}_0 + \bar{c}_0 - \bar{\alpha}_1}.$$

3rd step:

We have to show $\{t + a_0, t + b_0\} \in \hat{K}_2^M(B)$, where $\bar{a}_0 \neq \bar{b}_0$, is induced by an element from $K_1^M(A) \otimes K_1^M(B)$. Write

$$\frac{t + a_0}{a_0 - b_0} + \frac{t + b_0}{b_0 - a_0} = 1$$

and correspondingly

$$0 = \left\{ \frac{t + a_0}{a_0 - b_0}, \frac{t + b_0}{b_0 - a_0} \right\} \in \{t + a_0, t + b_0\} + K_1^M(A) \cdot K_1^M(B)$$

□

Proof of Sublemma 19. By a simple linear change of variables $t \mapsto t + c$, $c \in A$, we can and will assume $\alpha_1 \in A^\times$. We start with a symbol

$$\{a_2 t^2 + a_1 t + a_0, b_2 t^2 + b_1 t + b_0, c_2 t^2 + c_1 t + c_0\} \in \hat{K}_3^M(B).$$

1st step: Reduce to $a_2, b_2, c_2 \in A^\times \cup \{0\}$.

Let $a_2 \in m$. Then either $a_1 \in A^\times$ or $a_0 \in A^\times$. Assume for example $a_1 \in A^\times$. Then write

$$\{a_2 t^2 + a_1 t + a_0\} = -\{t\} + \{(a_1 - a_2 \alpha_2)t^2 + (a_0 - a_2 \alpha_1)t - a_2 \alpha_0\} \in K_1^M(B)$$

Similarly for b and c .

2nd step: Reduce to $a_2 = b_2 = c_2 = 0$.

If $a_2 \in A^\times$ write

$$\{a_2 t^2 + a_1 t + a_0\} = -\left\{t + \frac{a_2 \alpha_2 - a_1}{a_2}\right\} + \{\dots\} \in K_1^M(B)$$

where \dots stands for some polynomial in $A[t]$ of degree one.

3rd step: Reduce to $a_1, b_1, c_1 \in A^\times$ and $a_2 = b_2 = c_2 = 0$.

If $a_1 \in m$ let $\beta = a_0 - a_1\alpha_2$ and $c = \frac{\beta\alpha_2 + a_1\alpha_1}{\beta}$ and write

$$\{a_1t + a_0\} = -2\{t\} - \{t + c\} + \{\dots\} \in K_1^M(B)$$

where \dots stands for some polynomial of degree one with an invertible degree one coefficient. Here we use the fact $\alpha_1 \in A^\times$.

In the following we consider without restriction a symbol $\{t + a_0, t + b_0, t + c_0\} \in \hat{K}_3^M(B)$.

4th step: Reduce to $\bar{a}_0 \neq \bar{b}_0$.

We can assume $\bar{a}_0 = \bar{b}_0 = \bar{c}_0$ because otherwise a permutation finishes the step. Now argue as follows: Choose $\bar{c} \in A/m$, $\bar{c} \neq \bar{a}_0$. Then we can find $\bar{d} \in A/m$ such that

$$(t + \bar{a}_0)(t + \bar{c})(t + \bar{d}) \equiv \bar{g} \pmod{\bar{f}}$$

with $\deg \bar{g} < 2$. If $\bar{d} = \bar{a}_0$ set $d = b_0$ and lift \bar{c} to $c \in A$ such that

$$(t + a_0)(t + c)(t + d) \equiv g \pmod{f}$$

with $\deg g < 2$. If $\bar{d} \neq \bar{a}_0$ lift \bar{c} and \bar{d} arbitrarily to elements $c, d \in A$ such that with the notation as above $\deg g < 2$.

Case A: $\deg \bar{g} = 1$.

Observe that g is clearly coprime to $t + b_0$. So it is enough to write

$$\{t + a_0, t + b_0, t + c_0\} = (-\{t + d\} - \{t + c\} + \{g\})\{t + b_0, t + c_0\}.$$

Case B: $\deg \bar{g} = 0$.

Similar to Case A it is clearly enough to show

$$(1) \quad \{g, t + b_0, t + c_0\} \in K_1^M(A) \cdot K_2^M(B) \subset \hat{K}_3^M(B).$$

We have $g = q_1t + q_0$, $q_1 \in m$. Let $\beta = q_0 + (1 - q_1)b_0 - c_0$ and write

$$g/\beta + (1 - q_1)(t + b_0)/\beta - (t + c_0)/\beta = 1$$

But Proposition 10(3) resp. Proposition 2(4) applied to the last equation gives (1).

5th step:

We have to show that $\{t + a_0, t + b_0, t + c_0\} \in \hat{K}_3^M(B)$ with $\bar{a}_0 \neq \bar{b}_0$ is induced by an element from $K_1^M(A) \cdot K_2^M(B)$. Write

$$\frac{t + a_0}{a_0 - b_0} + \frac{t + b_0}{b_0 - a_0} = 1$$

and correspondingly

$$0 = \left\{ \frac{t + a_0}{a_0 - b_0}, \frac{t + b_0}{b_0 - a_0}, t + c_0 \right\} \in \{t + a_0, t + b_0, t + c_0\} + K_1^M(A) \cdot K_2^M(B).$$

□

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