THE K-THEORY OF TORIC VARIETIES

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Abstract. Recent advances in computational techniques for K-theory allow us to describe the K-theory of toric varieties in terms of the K-theory of fields and simple cohomological data.

1. Introduction

In this paper, we revisit the K-theory of toric varieties, using the new perspective afforded by the recent papers [18], [2], [3]. These papers provide a new technique for computations of the K-theory of a singular algebraic variety X over a field of characteristic 0, in terms of the homotopy K-theory of X and cohomological data: the cyclic homology of X and the *cdh*-cohomology of the sheaves Ω^p of Kähler differentials.

The homotopy K-theory $KH_*(X)$ of an affine toric variety is just the algebraic K-theory of a Laurent polynomial ring, and is well understood. Even when X is a non-affine toric variety, $KH_*(X)$ is tractable; we show in Proposition 5.6 that it is a summand of $K_*(X)$. This allows us to give a short proof in Proposition 5.7 of Gubeladze's classical theorem (in [11]) that $K_0(X) = \mathbb{Z}$ for affine X.

This reduces the problem of understanding $K_*(X)$ to that of understanding the cyclic homology of X and its *cdh*-cohomology. Because toric varieties admit resolutions of singularities that are formed in a purely combinatorial manner, it turns out this is indeed an accessible problem.

The main goal of this paper is to use these new techniques to give a streamlined approach to two of Gubeladze's recent results concerning the K-theory of toric varieties: examples of toric varieties with "huge" Grothendieck groups [14] and his "Dilation Theorem" (verifying the "nilpotence conjecture") [15]. Our proof of this theorem is considerable shorter than the original. On the other hand, our approach and Gubeladze's are cousins in the sense that they have a common ancestor: Cortiñas' verification of the KABI conjecture [1].

Since varieties are locally smooth in the *cdh*-topology, it is not surprising that the *cdh*-fibrant version of cyclic homology is strongly related to the *cdh* cohomology of the sheaf Ω^p of Kähler differentials. Theorem 4.1 below shows that, for a toric variety X, the *cdh* cohomology of Ω^p is computed by the Zariski cohomology of Danilov's sheaf of differentials $\tilde{\Omega}^q_X$. Since the global sections of Ω^p_X and $\tilde{\Omega}^p_X$ can be

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computed explicitly for toric varieties, we are able to find easily examples of toric varieties with huge Grothendieck groups; see Example 5.10.

Gubeladze's Dilation Theorem (stated and proven in Theorem 6.9 below) asserts, roughly speaking, that after inverting the action of "dilations," the K-theory of a toric variety becomes homotopy invariant. Our Theorem 6.6 shows that, after inverting the action of dilations, the global sections of $\tilde{\Omega}^q_X$ agree with the Hochschild homology groups $HH_q(X)$. By the technique of [2], this quickly leads to our new proof of Gubeladze's theorem.

Notation. Throughout this paper, we will adhere to the following notation. Let N be a free abelian group of rank $n < \infty$ and let $M = N^* = \mathrm{Hom}(N, \mathbb{Z})$. Define $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} = \mathrm{Hom}_\mathbb{R}(N_\mathbb{R}, \mathbb{R}) \cong M \otimes_\mathbb{Z} \mathbb{R}$. For $m \in M_\mathbb{R}, n \in N_\mathbb{R}$, let $\langle m, n\rangle$ denote the value of m at n. Finally, let k denote a field of characteristic 0.

2. Review of toric varieties

The material in this section may be found in standard texts, such as [9] or [5].

A *strongly convex rational cone* in $N_\mathbb{R}$ is a subset $\sigma \subset N_\mathbb{R}$ that is a cone spanned by finitely many vectors in N and that contains no lines. That is, $\sigma = \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_k$ for some $v_1, \dots, v_k \in N \subset N_\mathbb{R}$ and whenever both u and $-u$ belong to σ, we must have $u = 0$. Given such a cone σ, let $\sigma^\vee \subset M_\mathbb{R}$ denote the dual cone, defined to consist of those $m \in M_\mathbb{R}$ such that $\langle m, -\rangle \geq 0$ on σ. Note that $\sigma^\vee \cap M$ is the abelian monoid (under addition of functions) of linear functions with integer coefficients on $N_\mathbb{R}$ whose restrictions to σ are nowhere negative. A *face* of σ is a subset τ of the form

$$\sigma(m) = \{n \in \sigma \mid \langle m, n\rangle = 0\} \tag{2.1}$$

for some $m \in \sigma^\vee$. Observe that a face of a strongly convex rational cone is again a strongly convex rational cone. We write $\tau \prec \sigma$ to indicate that τ is a face of σ.

Recall that k denotes a field of characteristic zero.

The affine toric k-variety associated to a strongly convex rational cone σ is $U_\sigma = \mathrm{Spec}\, k[\sigma^\vee \cap M]$. We write elements of the monoid ring $k[\sigma^\vee \cap M]$ as k-linear combinations of the set of formal symbols $\{\chi^m \mid m \in \sigma^\vee \cap M\}$, so that multiplication in this ring is given on this k-basis by $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$.

A *fan* Δ in $N_\mathbb{R}$ is a finite collection of strongly convex rational cones in $N_\mathbb{R}$ such that (1) any face of a cone in Δ is again in Δ and (2) the intersection of any two cones in Δ is a face of each. If τ is a face of σ, then $U_\tau \to U_\sigma$ is an open immersion, because the evident map $k[\sigma^\vee \cap M] \to k[\tau^\vee \cap M]$ is given by inverting a finite number of the χ^m. It follows that for any fan Δ, we may form a scheme $X(\Delta)$ by patching together the affine schemes U_σ corresponding to

cones σ along the open subschemes associated to their intersections.

We call $X(\Delta)$ the *toric variety* associated to Δ.

Orbits. We write $T_N = \mathrm{Spec}\, k[M]$ for the n-dimensional torus associated to N. Observe that T_N acts on each U_σ — equivalently, the ring $k[\sigma^\vee \cap M]$ is naturally M-graded with weight m part being $k \cdot \chi^m$, if $m \in \sigma^\vee$, and 0 if $m \notin \sigma^\vee$. Since these actions are compatible, the torus T_N acts on $X(\Delta)$.

The orbits of this action are tori, and are in 1–1 correspondence with the cones of Δ; thus $X(\Delta)$ is the disjoint union of the orbits $\mathrm{orb}(\tau)$ corresponding to the $\tau \in \Delta$. To describe the orbit for τ, let $\mathbb{Z}(\tau \cap N)$ denote the subgroup of N generated by $\tau \cap N$, and let $\overline{N}$ be the free abelian group $N/\mathbb{Z}(\tau \cap N)$. Then $\mathrm{orb}(\tau) \cong T_{\overline{N}}$. Note that the orbit corresponding to the minimal cone $\{0\}$ is the dense open $\mathrm{orb}(0) = U_0$, and is naturally isomorphic to T_N.

We write $V_\Delta(\sigma)$ for the closure of $\mathrm{orb}(\sigma)$ in $X(\Delta)$. The orbits in $V_\Delta(\sigma)$ are indexed by the *star* of σ, $\mathrm{Star}_\Delta(\sigma)$, defined as the set of cones in Δ containing σ:

$$V_\Delta(\sigma) = \coprod_{\sigma \prec \tau} \mathrm{orb}(\tau).$$

Each orbit-closure $V_\Delta(\sigma)$ has the structure of a toric variety. To see this, let $\overline{N} = N/\mathbb{Z}(\sigma \cap N)$. Then $\{\overline{\epsilon} \,|\, \sigma \prec \epsilon\}$ forms a fan in $\overline{N}_\mathbb{R}$, and the corresponding toric variety is $V_\Delta(\sigma)$. The torus $T_{\overline{N}}$ is a quotient of T_N and the inclusion $V_\Delta(\sigma) \subset X(\Delta)$ is T_N-equivariant, the action of T_N on $V_\Delta(\sigma)$ being induced by the quotient map $T_N \twoheadrightarrow T_{\overline{N}}$. In the case where Δ has a single maximal cone ϵ, so that σ is a face of ϵ, we have

$$V_\Delta(\sigma) = \mathrm{Spec}\, k[\epsilon^\vee \cap M \cap \sigma^\perp],$$

and the closed immersion $V_\Delta(\sigma) \hookrightarrow U_\epsilon$ is given by the ring surjection

$$\mathrm{Spec}\, k[\epsilon^\vee \cap M] \twoheadrightarrow \mathrm{Spec}\, k[\epsilon^\vee \cap M \cap \sigma^\perp]$$

sending χ^m to 0, if $m \notin \sigma^\perp$, and to χ^m, if $m \in \sigma^\perp$.

It is useful to regard the open complement of $V_\Delta(\sigma)$ in $X(\Delta)$ as the toric variety corresponding to the largest sub-fan of Δ in $N_\mathbb{R}$ that does not contain τ.

Every toric variety is normal, but need not be smooth. A toric variety $X(\Delta)$ is smooth if and only if, for every cone σ in the fan Δ, the minimal lattice points along the 1-dimensional faces (rays) of σ form part of a $\mathbb{Z}$-basis of N. In particular, in order for $X(\Delta)$ to be smooth, the set of rays of each cone must be $\mathbb{R}$-linearly independent (such a cone is said to be *simplicial*).

Resolution of Singularities. We will need a detailed description of resolutions of singularities for toric varieties, which we now recall from [9]. If $v \in N$ is contained in one (or more) of the cones of Δ, one may subdivide Δ by the ray $\rho = \mathbb{R}_{\geq 0} v$ through v to form a new fan Δ' in $N_\mathbb{R}$ as follows: If $\tau \in \Delta$ does not contain ρ, then τ is also a cone of Δ'. For each cone $\tau \in \Delta$ containing ρ and for each face ν of τ not containing ρ, Δ' contains the cone spanned by ρ and ν:

$$\tilde{\nu} := \nu + \mathbb{R}_{\geq 0}\rho.$$

Finally, ρ itself belongs to Δ'. Thus if $\sigma \in \Delta$ is the minimal cone of Δ containing ρ, then Δ' is the disjoint union of $\Delta \setminus \mathrm{Star}_\Delta(\sigma)$ and $\mathrm{Star}_{\Delta'}(\rho)$.

There is a map of toric varieties $X' = X(\Delta') \to X = X(\Delta)$ and it is proper, birational, and equivariant with respect to the action of the torus T_N. Starting with any toric variety $X(\Delta)$, one can arrive at a desingularization of $X(\Delta)$ by performing a finite number of subdivisions of this type.

Suppose Δ' is the fan obtained by subdividing Δ by inserting a ray ρ, and let $\sigma \in \Delta$ be the minimal cone in Δ containing ρ. Then the description of the orbit-closures given above makes it clear that

$$\begin{array}{ccc} V' = V_{\Delta'}(\rho) & \xrightarrow{i'} & X(\Delta') = X' \\ \downarrow & & \downarrow \pi \\ V = V_\Delta(\sigma) & \xrightarrow{i} & X(\Delta) = X \end{array} \tag{2.2}$$

is an abstract blow-up square. That is, this a pull-back square in which the horizontal arrows are closed immersions and the map on open complements is an isomorphism:

$$X(\Delta') \setminus V_{\Delta'}(\rho) \xrightarrow{\cong} X(\Delta) \setminus V_\Delta(\sigma).$$

As with any abstract blow-up, the maps $\{X(\Delta') \to X(\Delta), V_\Delta(\sigma) \to X(\Delta)\}$ form a covering for the *cdh*-topology. Recall that the torus T_N acts on each variety in the above square and each map in this square is T_N-equivariant.

3. Danilov's sheaves $\tilde\Omega^p$

In this section, we introduce the coherent sheaves $\tilde\Omega^p_X$, first defined by Danilov [5, 4.2]. We will see in the next section that their Zariski cohomology groups turn out to give the *cdh*-cohomology groups of Ω^p.

Given a fan Δ, let $\Delta(1)$ denote the collection of rays in Δ; the 1-skeleton of Δ is the fan $\Delta(1) \cup \{0\}$ and its toric variety $X^{(1)}$ lies in the smooth locus of $X(\Delta)$.

Definition 3.1. For a toric k-variety $X = X(\Delta)$ defined by a fan Δ in $N_\mathbb{R}$, we define $\tilde\Omega^p_X$ to be the coherent sheaf on X fitting into the exact sequence

$$0 \to \tilde\Omega^p_X \to \mathcal{O}_X \otimes_\mathbb{Z} \wedge^p(M) \xrightarrow{\delta} \bigoplus_{\rho\in\Delta(1)} \mathcal{O}_{V_\Delta(\rho)} \otimes_\mathbb{Z} \wedge^{p-1}(M \cap \rho^\perp).$$

The component of the map δ indexed by ρ sends $f\otimes(m_1\wedge\cdots\wedge m_p)$ in $\mathcal{O}_X\otimes\wedge^p_\mathbb{Z}(M)$ to

$$i^*(f) \otimes \left(\sum_i (-1)^i \langle m_i, n_\rho\rangle m_1 \wedge \cdots \wedge \hat{m}_i \wedge \cdots \wedge m_p\right)$$

where $i : V_\Delta(\rho) \hookrightarrow X$ is the canonical closed immersion, and $n_\rho \in N$ is the minimal lattice point on ρ. By convention, $\tilde\Omega^0_X = \mathcal{O}_X$.

On the affine U_σ, the ring $\mathcal{O}(U_\sigma)$ is M-graded, so the sections of $\mathcal{O}_X \otimes \wedge^p M$ are M-graded with $\wedge^p M$ in weight 0; the weight m summand is $k \cdot \chi^m \otimes \wedge^p M$ if $m \in \sigma^\vee$. Since δ is graded, it follows that each $\tilde\Omega^p_X(U_\sigma)$ is M-graded.

Remark 3.2. Sections of $\tilde\Omega^1_X$ may be considered as differential forms on X, with $1 \otimes m$ corresponding to the form $d\log(\chi^m) = d\chi^m/\chi^m$. On a nonsingular cone σ, we may identify $\mathcal{O}\otimes\wedge^p M$ with the locally free sheaf $\Omega^p(\log D)$ of differentials with logarithmic poles along $D = \cup V(\rho)$. This identifies the map δ with the residue map, so we have $\Omega^p|_{U_\sigma} \cong \tilde\Omega^p|_{U_\sigma}$.

As shown by Danilov [5, 4.3], the sheaf $\tilde\Omega^p_X$ is naturally isomorphic to $j_*(\Omega^p_U)$, where $j : U \hookrightarrow X$ is the immersion of the open subscheme U of smooth points of X. Applying Remark 3.2 to $X^{(1)} \hookrightarrow U$, we see that $\tilde\Omega^p_X = j^{(1)}_*(\Omega^p_{X^{(1)}})$ where $j^{(1)} : X^{(1)} \hookrightarrow X$ is the evident open immersion.

We will need an explicit description of the M-grading on $\tilde\Omega^1$, or rather on the module of sections $\tilde\Omega^1(U_\sigma)$ over an affine toric variety U_σ. (See [5, 4.2.3].) When $m \in \sigma^\vee\cap M$, its weight m summand is the subspace $\tilde\Omega^1(U_\sigma)_m = k\cdot\chi^m\otimes(M\cap\sigma(m)^\perp)$ of the weight m summand $k\cdot\chi^m\otimes M$ of $\mathcal{O}(U_\sigma)\otimes M$. Here $\sigma(m)^\perp$ is the orthogonal complement of the face $\sigma(m)$ of σ defined in (2.1) by the vanishing of m: For

$m \notin \sigma^\vee$, $\tilde\Omega^1(U_\sigma)_m = 0$ because $\mathcal{O}(U_\sigma)_m = 0$. More generally, we have for $m \in M$ and $p \geq 0$

$$\tilde\Omega^p_X(U_\sigma)_m = \begin{cases} k\cdot\chi^m \otimes \wedge^p(M\cap\sigma(m)^\perp) & \text{if } m\in\sigma^\vee \\ 0 & \text{if } m\notin\sigma^\vee. \end{cases} \tag{3.3}$$

It is instructive to compare (3.3) to the analogous formula for $\Omega^p(U_\sigma)$ and $HH_p(U_\sigma)$, which are graded by the submonoid $\sigma^\vee \cap M$ of M. There is a natural map from the module Ω^p_X of Kähler differentials to $\tilde\Omega^p_X$. On U_σ it is the M-graded map induced by the M-graded map $\Omega^p(U_\sigma) \to \mathcal{O}(U_\sigma)\otimes\wedge^p(M)$ defined by:

$$\chi^{m_0}\, d\chi^{m_1}\wedge\cdots\wedge d\chi^{m_p} \mapsto (1/p!)\chi^m \otimes \big(m_1\wedge\cdots\wedge m_p\big), \qquad m=\sum m_i. \tag{3.4}$$

Recall that the orbit-closure $V(\tau)$ for the face τ is $\mathrm{Spec}(k[\sigma^\vee\cap M\cap\tau^\perp])$.

Lemma 3.5. *For each $m\in\sigma^\vee\cap M$, let $V=V(\sigma(m))$ denote the orbit-closure for the face $\sigma(m)$ of σ. Then the closed immersion $V\subset U_\sigma$ induces an isomorphism $HH_*(U_\sigma)_m \cong HH_*(V)_m$. In particular, for all p:*

$$\Omega^p_X(U_\sigma)_m = \Omega^p(V)_m$$

Proof. For convenience, let us set $A=\sigma^\vee\cap M$ and $B=A\cap\sigma(m)^\perp$, so that $U_\sigma=\mathrm{Spec}(k[A])$ and $V(\sigma(m))=\mathrm{Spec}(k[B])$. The immersion $V\subset U_\sigma$ corresponds to a surjection $k[A]\to k[B]$, which is split by the evident inclusion $\iota: k[B]\to k[A]$. Hence $HH_*(k[B])$ is a summand of $HH_*(k[A])$, and it suffices to show that ι induces a surjection on the weight m summand of the complex for Hochschild homology.

Now the degree p part of the Hochschild complex for $k[A]$ is $k[A]^{\otimes p+1}$, so the weight m summand has a basis consisting of the $\chi^{u_0}\otimes\chi^{u_1}\cdots\otimes\chi^{u_p}$ where $u_i\in A$ and $\sum u_i = m$. If $n\in\sigma(m)$, then $\langle u_i,n\rangle\geq 0$ and $\sum_i\langle u_i,n\rangle = \langle m,n\rangle = 0$. This forces each $\langle u_i,n\rangle=0$, *i.e.,* $u_i\in B$. Hence $k[B]^{\otimes p+1}_m = k[A]^{\otimes p+1}_m$, as claimed. □

Lemma 3.6. *Every orbit blow-up square* (2.2) *determines a distinguished triangle on X_{Zar} of the form*

$$\tilde\Omega^p_X \to \mathbb{R}\pi_*\tilde\Omega^p_{X'}\oplus i_*\tilde\Omega^p_V \to \mathbb{R}\pi_* i'_*\tilde\Omega^p_{V'} \to \tilde\Omega^p_X[1],$$

and hence a long exact sequence of Zariski cohomology groups:

$$\cdots\to H^q(X,\tilde\Omega^p)\to H^q(X',\tilde\Omega^p)\oplus H^q(V,\tilde\Omega^p)\to H^q(V',\tilde\Omega^p)\to H^{q+1}(X,\tilde\Omega^p)\to\cdots.$$

Proof. We have short exact sequences of coherent sheaves

$$0\to\tilde\Omega^p_{(X,V)}\to\tilde\Omega^p_X\to i_*\tilde\Omega^p_V\to 0$$

on X, and $0\to\tilde\Omega^p_{(X',V')}\to\tilde\Omega^p_{X'}\to i_*\tilde\Omega^p_{V'}\to 0$ on X'. Applying $\mathbb{R}\pi_*$ to the latter yields a morphism of distinguished triangles

$$\begin{array}{ccccc} \tilde\Omega^p_{(X,V)} & \longrightarrow & \tilde\Omega^p_X & \longrightarrow & i_*\tilde\Omega^p_V \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{R}\pi_*\tilde\Omega^p_{(X',V')} & \longrightarrow & \mathbb{R}\pi_*\tilde\Omega^p_{X'} & \longrightarrow & \mathbb{R}\pi_* i_*\tilde\Omega^p_{V'} \end{array}$$

Danilov proved in [6, Prop 1.8] that the left vertical map is a quasi-isomorphism, *i.e.,* that $\mathbb{R}^j\pi_*\tilde\Omega^p_{(X',V')}=0$ for $j>0$, and $\tilde\Omega^p_{(X,V)}\xrightarrow{\simeq}\pi_*\tilde\Omega^p_{(X',V')}$. The distinguished triangle follows from this in a standard way. □

Remark 3.7. Danilov [5, 8.5.1] proved that if $\pi : X' \to X$ is a morphism of toric varieties resulting from a subdivision of the fan, then $\mathcal{O}_X \xrightarrow{\simeq} \mathbb{R}\pi_*\mathcal{O}_{X'}$, *i.e.*, $\pi_*\mathcal{O}_{X'} = \mathcal{O}_X$ and $R^i\pi_*\mathcal{O}_{X'} = 0$ for $i > 0$. This proves that toric varieties have (at most) rational singularities.

4. The *cdh*-cohomology of Ω^p for toric varieties

In this short section, we prove Theorem 4.1, that Danilov's sheaves compute the *cdh*-cohomology groups $H^*_{\mathrm{cdh}}(X, \Omega^p)$ for toric varieties.

Theorem 4.1. *Let X be an arbitrary toric k-variety. There is an isomorphism*

$$H^*_{Zar}(X, \tilde{\Omega}^p_X) \cong H^*_{cdh}(X, \Omega^p)$$

for all p, natural for morphisms of toric varieties and for the closed embedding of an orbit-closure of X into X.

Example 4.2. The case $* = 0$ of Theorem 4.1 is that $\tilde{\Omega}^p(X) \cong H^0_{\mathrm{cdh}}(X, \Omega^p)$. This is equivalent to Danilov's calculation [6, 1.5] that in (2.2), $\tilde{\Omega}^p_X \xrightarrow{\simeq} \pi_*\tilde{\Omega}^p_{X'}$ for all p.

For the proof, we recall that $H^*_{\mathrm{cdh}}(X, \Omega^p)$ is just the Zariski hypercohomology of the complex $\mathbb{R}a_*a^*\Omega^p|_X$, where $a : (Sch/k)_{\mathrm{cdh}} \to (Sch/k)_{\mathrm{Zar}}$ is the morphism of sites and $|_X$ denotes the restriction from the big Zariski site $(Sch/k)_{\mathrm{Zar}}$ to X_{Zar}.

Recall that we can resolve the singularities of a toric variety via equivariant blow-up squares of the form (2.2). Iterating the orbit blow-up operations described in (2.2), as in [7, 6.2.5] we can find a smooth toric *cdh*-hypercover $\pi : Y_\bullet \to X$. The following Mayer-Vietoris lemma is an immediate consequence of [21, 12.1].

Lemma 4.3. *For every cdh sheaf $\mathcal{F}$, $\mathbb{R}a_*\mathcal{F}|_X \cong \mathbb{R}\pi_*(\mathbb{R}a_*\mathcal{F}|_{Y_\bullet})$.*

Proof of Theorem 4.1. As in [7, 5.2.6], Lemma 3.6 implies that the maps $\tilde{\Omega}^p_X \to \mathbb{R}\pi_*\tilde{\Omega}^p_{Y_\bullet}$ are quasi-isomorphisms. By Remark 3.2, the maps $\Omega^p_{Y_\bullet} \to \tilde{\Omega}^p_{Y_\bullet}$ are isomorphisms. Hence we have quasi-isomorphisms of complexes of Zariski sheaves on X:

$$\mathbb{R}\pi_*\Omega^p_{Y_\bullet} \xrightarrow{\simeq} \mathbb{R}\pi_*\tilde{\Omega}^p_{Y_\bullet} \xleftarrow{\simeq} \tilde{\Omega}^p_X.$$

Now by [3, 2.5], we have $\Omega^p_{Y_n} \cong \mathbb{R}a_*a^*\Omega^p|_{Y_n}$. Applying Lemma 4.3 to $\mathcal{F} = a^*\Omega^p$ yields:

$$\mathbb{R}a_*a^*\Omega^p|_X \xrightarrow{\simeq} \mathbb{R}\pi_*(\mathbb{R}a_*a^*\Omega^p|_{Y_\bullet}) \cong \mathbb{R}\pi_*\Omega^p_{Y_\bullet}.$$

Applying $H^*_{\mathrm{Zar}}(X, -)$ yields $H^*_{\mathrm{cdh}}(X, \Omega^p) \xrightarrow{\simeq} H^*_{\mathrm{Zar}}(Y_\bullet, \Omega^p) \cong H^*_{\mathrm{Zar}}(X, \tilde{\Omega}^p)$, an isomorphism which is natural in the pair $Y_\bullet \to X$. As any two smooth toric hypercovers have a common refinement, the isomorphism $\tilde{\Omega}^p_X \simeq \mathbb{R}a_*a^*\Omega^p|_X$ in the derived category is independent of $Y_\bullet$. The asserted naturality follows. □

Now recall that every variety is locally smooth for the *cdh* topology. Hence the Hochschild-Kostant-Rosenberg theorem implies that the Hochschild homology sheaf HH_n has $a^*HH_n \cong a^*\Omega^n$. We write $\mathbb{H}_{\mathrm{cdh}}(X, HH)$ for $\mathbb{R}a_*a^*$ applied to the Hochschild complex, and $\mathbb{H}_{\mathrm{cdh}}(X, HH^{(t)})$ for its summand in Hodge weight t. We write the Zariski hypercohomology of these complexes as $\mathbb{H}^*_{\mathrm{cdh}}(X, HH)$ and $\mathbb{H}^*_{\mathrm{cdh}}(X, HH^{(t)})$, respectively. By [3, 2.2], $\mathbb{H}_{\mathrm{cdh}}(X, HH^{(t)}) \cong \mathbb{R}a_*a^*\Omega^t[t]$. Hence Theorem 4.1 translates into the following language:

Corollary 4.4. *For every toric variety X, $\mathbb{H}^n_{cdh}(X, HH^{(t)}) \cong H^{t+n}_{Zar}(X, \tilde{\Omega}^t_X)$, and*

$$\mathbb{H}^n_{cdh}(X, HH) \cong \bigoplus\nolimits_{t\geq 0} H^{t+n}_{Zar}(X, \tilde{\Omega}^t_X).$$

The Hochschild homology in 4.4 is taken over any field k of characteristic zero. Since every toric variety $X = X_k$ over k is obtained by base-change from a toric variety $X_{\mathbb{Q}}$ over the ground field $\mathbb{Q}$, flat base-change yields $\Omega^*_{X/k} \cong \Omega^*_{X_{\mathbb{Q}}/\mathbb{Q}} \otimes_{\mathbb{Q}} k$, and the Künneth formula yields $\Omega^*_{X/\mathbb{Q}} = \Omega^*_{X_{\mathbb{Q}}/k} \otimes_{\mathbb{Q}} \Omega^*_{k/\mathbb{Q}} = \Omega^*_{X/k} \otimes_k \Omega^*_{k/\mathbb{Q}}$. Similar formulas hold for $HH_*(X/\mathbb{Q})$ and hence for $\mathbb{H}^*_{\mathrm{cdh}}(X, HH(-/\mathbb{Q}))$.

We define $\tilde{\Omega}^t_{X/\mathbb{Q}}$ to be $j_*\Omega^t_{X/\mathbb{Q}}$. The above remarks imply that $\tilde{\Omega}^t_X \cong \tilde{\Omega}^t_{X_{\mathbb{Q}}/\mathbb{Q}} \otimes_{\mathbb{Q}} k$, and that there is also a Künneth formula $\tilde{\Omega}^*_{X/\mathbb{Q}} \cong \tilde{\Omega}^*_X \otimes_k \Omega^*_{k/\mathbb{Q}}$.

Hence we have have the following variant of the previous corollary.

Corollary 4.5. *For every toric k-variety X,*

$$\mathbb{H}^n_{cdh}(X, HH^{(t)}(-/\mathbb{Q})) \cong H^{t+n}_{Zar}(X, \tilde{\Omega}^t_{X/\mathbb{Q}}) \cong \bigoplus_{i+j=t} H^{t+n}_{Zar}(X, \tilde{\Omega}^i_X) \otimes_k \Omega^j_{k/\mathbb{Q}},$$

and

$$\mathbb{H}^n_{cdh}(X, HH(-/\mathbb{Q})) \cong \bigoplus_{t\geq 0} H^{t+n}_{Zar}(X, \tilde{\Omega}^t_{X/\mathbb{Q}}).$$

5. K-theory and cyclic homology of toric varieties

Recall from section 3 that $\tilde{\Omega}^p_X$ has both a combinatorial definition, and an interpretation as $j_*\Omega^p_U$ where $j : U \hookrightarrow X$ is the inclusion of the smooth locus. In this section, we study the exterior differentiation map $d : \tilde{\Omega}^p_X \to \tilde{\Omega}^{p+1}_X$ which arises as the pushforward of the de Rham differential $d : \Omega^p_U \to \Omega^{p+1}_U$. The following combinatorial description of this map is useful.

Lemma 5.1. ([5, 4.4]) *The map $d : \tilde{\Omega}^p_X \to \tilde{\Omega}^{p+1}_X$ induced by exterior differentiation $d : \Omega^p_U \to \Omega^{p+1}_U$ is the M-graded map which in weight m is $k\chi^m \otimes (m_1 \wedge \cdots) \mapsto k\chi^m \otimes (m \wedge m_1 \wedge \cdots)$. That is, it is induced by:*

$$(\mathcal{O}_X(U_\sigma)_m \otimes_{\mathbb{Z}} \wedge^p M) \cong \wedge^p M \xrightarrow{m\wedge -} \wedge^{p+1} M \cong (\mathcal{O}_X(U_\sigma)_m \otimes_{\mathbb{Z}} \wedge^{p+1} M).$$

Pushing forward the de Rham complex Ω^*_U, we see that the $\tilde{\Omega}^p_X$'s fit together to form a "log de Rham" complex $\tilde{\Omega}^*_X$ on X. There is a natural map $\Omega^*_X \to \tilde{\Omega}^*_X$ of complexes, which is an isomorphism on the smooth locus of X. Similarly, pushing forward the de Rham complex $\Omega^*_{U/\mathbb{Q}}$ from the smooth locus to X, we obtain a log de Rham complex $\tilde{\Omega}^*_{X/\mathbb{Q}}$.

As in [2] and [3], $\mathbb{H}_{\mathrm{cdh}}(X, HC)$ denotes $\mathbb{R}a_*a^*$ applied to the cyclic homology cochain complex, and $\mathbb{H}_{\mathrm{cdh}}(X, HC^{(t)})$ is its summand in Hodge weight t. The Zariski hypercohomology of these complexes is written as $\mathbb{H}^*_{\mathrm{cdh}}(X, HC)$ and $\mathbb{H}^*_{\mathrm{cdh}}(X, HC^{(t)})$, respectively, and is called the *cdh-fibrant cyclic homology* of X.

By [3, 2.2], $\mathbb{H}_{\mathrm{cdh}}(X, HC^{(t)}) \cong \mathbb{R}a_*a^*\Omega^{\leq t}[2t]$, where $\Omega^{\leq t}$ denotes the brutal truncation of the de Rham complex. Similarly, we write $\tilde{\Omega}^{\leq t}_X$ for the brutal truncation of the Danilov complex $\tilde{\Omega}^*_X$. By Theorem 4.1, $\mathbb{H}_{\mathrm{cdh}}(X, HC^{(t)}) \cong \tilde{\Omega}^{\leq t}_X[2t]$.

As with Hochschild homology, the cyclic homology in the above paragraph is taken over k. As in the previous section, we may also consider cyclic homology taken over the ground field $\mathbb{Q}$, and we also have $\mathbb{H}_{\rm cdh}(X, HC^{(t)}(-/\mathbb{Q})) \cong \mathbb{R}a_*a^*\Omega^{\le t}_{/\mathbb{Q}}[2t]$, again by [3, 2.2].

Again by Theorem 4.1, we have an isomorphism in the derived category:

$$\mathbb{R}a_*a^*\Omega^{\le t}_{/\mathbb{Q}} \simeq \tilde\Omega^{\le t}_{X/\mathbb{Q}}.$$

Concatenating these identifications, we have:

Proposition 5.2. *If X is a toric k-variety, the cdh-fibrant cyclic homology is given by the formula:*

$$\mathbb{H}^{-n}_{cdh}(X, HC) \cong \bigoplus\nolimits_{t\ge 0} H^{2t-n}_{Zar}(X, \tilde\Omega^{\le t}_X).$$

and

$$\mathbb{H}^{-n}_{cdh}(X, HC(-/\mathbb{Q})) \cong \bigoplus\nolimits_{t\ge 0} H^{2t-n}_{Zar}(X, \tilde\Omega^{\le t}_{X/\mathbb{Q}}).$$

Example 5.3. The case $t = 0$ of 5.2 yields the formula

$$HC^{(0)}_n(X) = H^{-n}_{\rm Zar}(X, \mathcal{O}) \xrightarrow{\simeq} H^{-n}_{\rm cdh}(X, \mathcal{O}) = \mathbb{H}^{-n}_{\rm cdh}(X, HC^{(0)}).$$

This illustrates the interconnections between the case $p = 0$ of Theorem 4.1, Danilov's calculation in Remark 3.7, and the convention that $\tilde\Omega^0_X = \mathcal{O}_X$.

These calculations tell us about the algebraic K-theory of toric varieties, via the following translation of [3, 1.6] into the present language.

Definition 5.4. Let $\mathcal{F}_{HC}[1]$ denote the mapping cone complex of $HC(-/\mathbb{Q}) \to \mathbb{R}a_*a^*HC(-/\mathbb{Q})$; the indexing we use is such that there is a long exact sequence:

$$\cdots \to H^{-n}(X, \mathcal{F}_{HC}) \to HC_n(X/\mathbb{Q}) \to \mathbb{H}^{-n}_{\rm cdh}(X, HC(-/\mathbb{Q})) \to \cdots.$$

Theorem 5.5. *(*[3, 1.6]*) For every X in Sch/k, there is a long exact sequence*

$$\cdots \to KH_{n+1}(X) \to H^{-n}_{Zar}(X, \mathcal{F}_{HC}[1]) \to K_n(X) \to KH_n(X) \to \cdots.$$

For toric varieties, the sequence (5.5) splits:

Proposition 5.6. *For every toric variety X, $K_*(X) \to KH_*(X)$ is a split surjection. Hence*

$$K_n(X) \cong KH_n(X) \oplus H^{-n}_{Zar}(X, \mathcal{F}_{HC}[1]).$$

Proof. For each affine cone σ, $M(\sigma) := M \cap \sigma^\perp$ is a free abelian monoid, so $T_\sigma = \mathrm{Spec}(k[M(\sigma)])$ is a torus. We first claim that the inclusion $i_\sigma : k[M(\sigma)] \rightarrowtail k[M \cap \sigma^\vee]$, or surjection $U_\sigma \twoheadrightarrow T_\sigma$, induces an isomorphism on KH-theory, *i.e.,*

$$K(T_\sigma) \xrightarrow{\simeq} KH(T_\sigma) \xrightarrow{\simeq} KH(U_\sigma). \tag{5.6a}$$

Since (5.6a) factors $K(T_\sigma) \to K(U_\sigma) \to KH(U_\sigma)$, this proves the lemma for U_σ.

Because T_σ is regular, the first map is an isomorphism. For a suitable rational $n \in \sigma$, evaluation at n is a monoid map from $M \cap \sigma^\vee$ to $\mathbb{N}$ with kernel $M(\sigma)$.

This gives $k[M \cap \sigma^\vee]$ the structure of an $\mathbb{N}$-graded algebra with $k[M(\sigma)]$ in degree zero. Hence i_σ induces an isomorphism $KH(k[M(\sigma)]) \cong KH(k[M \cap \sigma^\vee])$, as claimed.

If τ is a face of σ, we have a commutative diagram

$$\begin{array}{ccc} k[M(\sigma)] & \longrightarrow & k[M \cap \sigma^\vee] \\ {\scriptstyle \rm into}\downarrow & & \downarrow{\scriptstyle \rm into} \\ k[M(\tau)] & \longrightarrow & k[M \cap \tau^\vee]. \end{array}$$

Thus the isomorphism in (5.6a) is natural in σ, for σ a face of a fan Δ, and so is the splitting of $K(U_\sigma) \to KH(U_\sigma)$. Since $K(X)$ is the homotopy limit over Δ of the $K(U_\sigma)$, and similarly for $KH(X)$, the homotopy limit of the splittings provides a splitting of the map $K(X) \to KH(X)$. □

Remark 5.6.1. The proof amounts to the observation that there is an algebraic homotopy from U_σ onto its smallest orbit $orb(\sigma)$, and that this homotopy is natural with respect to face inclusions.

The sequence (5.5) is compatible with the decomposition arising from the Adams operations because the Chern character is, by [4]. Thus $K_*^{(i)}(X)$ and $KH_*^{(i)}(X)$ fit into a long exact sequence with $\mathcal{F}_{HC}^{(i-1)}$. For example, it is immediate from Example 5.3 that $\mathcal{F}_{HC}^{(0)}(X)$ is acyclic, proving that $K_*^{(1)}(X) \cong KH_*^{(1)}(X)$ for toric varieties. The case $* = 0$, which is a well known assertion about the Picard group of normal varieties, has the following extension:

Proposition 5.7. *If $X = U_\sigma$ is an affine toric k-variety, then $K_0(X) = \mathbb{Z}$.*

Proof. Note that the coordinate ring of U_σ is graded, so $KH_0(X) = \mathbb{Z}$.

By 5.5, we need to show that $\mathbb{H}^0(X, \mathcal{F}_{HC}) = 0$. Since $HC_{-1}(X) = 0$, we are reduced to proving that the map

$$HC_0(X) \to \mathbb{H}^0_{\text{cdh}}(X, HC)$$

is onto. By 5.2, the target of this map is $\bigoplus_{t\geq 0} H^{2t}_{\text{Zar}}(X, \tilde{\Omega}^{\leq t}_{/\mathbb{Q}})$. Since X is affine, we have $H^{2t}_{\text{Zar}}(X, \tilde{\Omega}^{\leq t}_{/\mathbb{Q}}) = 0$ for all $t > 0$. Finally, when $t = 0$ we have

$$H^0_{\text{Zar}}(X, \tilde{\Omega}^{\leq 0}_{/\mathbb{Q}}) = H^0_{\text{Zar}}(X, \mathcal{O}_X) = HC_0(X). \qquad \square$$

Remark 5.7.1. A much better version of this Corollary was proven years ago by Gubeladze [11]: For a PID R, every finitely projective module over $R[A]$, where A is a semi-normal, abelian, cancellative monoid without non-trivial units, is free. This was extended to the case where R is regular by Swan [22].

Of course, the dictionary coming from [3] via 5.5 also allows us to say something about the higher K-theory of toric varieties. Let $K_n^{(i)}(X)$ denote the weight i part of $K_n(X) \otimes \mathbb{Q}$ with respect to the Adams operations, *i.e.,* the eigenspace where $\psi^k = k^i$ for all k. We adopt the parallel notation $KH_n^{(i)}(X)$ for the weight i part of $KH_n(X)$.

The absolute cotangent sheaf $\mathbb{L}_X$ of $X/\mathbb{Q}$ has $\mathbb{L}_X^{\geq 0} = \Omega^1_{X/\mathbb{Q}}$ and $H^{1-n}(X, \mathbb{L}_X) = HH_n^{(1)}(X/\mathbb{Q})$; see [27, 8.8.9]. There is a natural map $\mathbb{L}_X \to \Omega^1_{X/\mathbb{Q}} \to \tilde{\Omega}^1_{X/\mathbb{Q}}$.

Corollary 5.8. *For any toric k-variety X, we have a distinguished triangle*

$$\mathcal{F}_{HC}^{(1)} \to \mathbb{L}_X \to \tilde{\Omega}^1_{X/\mathbb{Q}} \to \mathcal{F}_{HC}^{(1)}[1],$$

and hence an isomorphism $K_q^{(2)}(X) \cong KH_q^{(2)}(X) \oplus H^{2-q}_{Zar}(X, \mathbb{L}_X \to \tilde{\Omega}^1_{X/\mathbb{Q}})$.

Proof. The Zariski sheaf $HC^{(1)}$ is the mapping cone of $\mathcal{O} \to \mathbb{L}_X$; see [27, 9.8.18]. Since $\mathbb{R}a_*(a^*\mathcal{O})|X = \mathcal{O}_X$ by Remark 3.7, and $\mathbb{H}_{\text{cdh}}(X, HC^{(1)}) \simeq (\mathcal{O} \to \tilde{\Omega}^1_X)[2]$ by 5.2, it follows that the mapping cone $\mathcal{F}_{HC}^{(1)}$ of $HC^{(1)} \to \mathbb{H}_{\text{cdh}}(X, HC^{(1)})$ is also the mapping cone of $\mathbb{L}_X \to \tilde{\Omega}^1_X$. This proves the first assertion; the second assertion follows from this, Proposition 5.6 and [3, 1.6]. □

The techniques of [3] allow us to find examples of toric varieties with "huge" K_0 and K_1 groups, in the spirit of [25], [12] and [14]. Our toric varieties will have quotient singularities because all the cones will be simplices; see [9].

Example 5.9. Let $N = \mathbb{Z}^3$, and let us to agree to write elements of N as column vectors and elements of $M \cong \mathbb{Z}^3$ as row vectors. Define τ to be the cone in the xy-plane of $N_{\mathbb{R}} = \mathbb{R}^3$ spanned by the vectors $e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $e_1 + 2e_2 = \begin{bmatrix}1\\2\\0\end{bmatrix}$. Then U_τ is a singular, affine toric k-variety.

In fact, $U_\tau = \operatorname{Spec}\big(k[X,Y,Z]/(YZ - X^2)[T,T^{-1}]\big)$, where $X = \chi^{(1,0,0)}$, $Y = \chi^{(0,1,0)}$, $Z = \chi^{(2,-1,0)}$ and $T^{\pm 1} = \chi^{(0,0,\pm 1)}$. This is because $\tau^\vee \cap M$ is generated by the vectors $(1,0,0), (0,1,0), (2,-1,0)$ and $(0,0,\pm 1)$.

Let $m \in M$ be the vector $(1,0,0)$. Its face is $\tau(m) = \{0\}$, so $\tau(m)^\perp = M$. We see from (3.3) that $\tilde{\Omega}^1(U_\tau)_m = k \cdot X \otimes M \cong k^3$. The forms dX, $X dY/Y$ and $X dT/T$ form a basis. On the other hand, $\Omega^1(U_\tau)_m$ is the k-vector space spanned by $\chi^u d(\chi^v)$ with $u, v \in \tau^\vee \cap M$ satisfying $u + v = m$. It is easy to see that the only $u, v \in \tau^\vee \cap M$ satisfying $u + v = (1,0,0)$ are when u, v is $\{(0,0,-j), (1,0,j)\}$. Thus $\Omega^1(U_\tau)_m$ is the 2-dimensional vector space spanned by dX and $X dT/T$. It follows that $\Omega^1(U_\tau) \to \tilde{\Omega}^1(U_\tau)$ is not onto in weight m.

Similar reasoning shows that for $m = (1,0,c)$ we also have $\tilde{\Omega}^1(U_\tau)_m \cong k^3$ on $T^c\, dX$, $T^c X\, dY/Y$ and $T^{c-1} X\, dT$, and that $\tilde{\Omega}^1(U_\tau)_m = \Omega^1(U_\tau)$ for all other m. (It is useful to use the fact that $\Omega^1(U_\tau)$ is a submodule of $\tilde{\Omega}^1(U_\tau)$ by [23].) Thus $\tilde{\Omega}^1(U_\tau)/\Omega^1(U_\tau) \cong k[T,T^{-1}]$. By the Künneth formula,

$$\operatorname{coker}\big\{\Omega^1(U_\tau/\mathbb{Q}) \to \tilde{\Omega}^1(U_\tau/\mathbb{Q})\big\} \cong \tilde{\Omega}^1(U_\tau)/\Omega^1(U_\tau).$$

As in Proposition 5.6, it is easy to see that $KH_*(U_\tau) \cong K_*(k[T,T^{-1}])$. Hence 5.8 implies that $K_1^{(2)}(U_\tau)$ is isomorphic to a nonzero k-vector space:

$$K_1^{(2)}(U_\tau) \cong H^1_{\mathrm{Zar}}(U_\tau, \Omega^1 \to \tilde{\Omega}^1) \cong \tilde{\Omega}^1(U_\tau)/\Omega^1(U_\tau) \cong k[T,T^{-1}].$$

Example 5.10. We now extend the τ of Example 5.9 to form a fan Δ consisting of two 3-dimensional cones σ_1, σ_2 (together with all of their faces) such that $\sigma_1 \cap \sigma_2 = \tau$. Specifically, let σ_1 and σ_2 be spanned by the two edges of τ together with

$$v_1 = \begin{bmatrix}-1\\0\\+1\end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix}-1\\0\\-1\end{bmatrix},$$

respectively. Let $X = X(\Delta)$, so $X = U_{\sigma_1} \cup U_{\sigma_2}$ and $U_\tau = U_{\sigma_1} \cap U_{\sigma_2}$. It follows from 5.6 that $KH_0(X) = \mathbb{Z} \oplus \mathbb{Z}$ and that

$$K_0(X) \cong \mathbb{Z}^2 \oplus H^1_{\mathrm{Zar}}(X, \mathcal{F}_{HC}).$$

We will show that the right-hand term is nonzero; since it is a k-vector space, it will follow that $K_0(X)$ contains the additive group underlying a non-zero k-vector space. Taking k to be uncountable, for example $k = \mathbb{C}$, we see $K_0(X)$ is uncountable.

Because the singular locus of X is 1-dimensional, $H^n(X, \mathbb{L}_X) = H^n(X, \Omega^1_X)$ for $n > 0$. By Corollary 5.8,

$$K_0^{(2)}(X) = H^1_{\mathrm{Zar}}(X, \mathcal{F}_{HC}) = H^2_{\mathrm{Zar}}(X, \Omega^1 \to \tilde{\Omega}^1).$$

From the Mayer-Vietoris sequence for the given cover of X, and Proposition 5.7, we see that there is an exact sequence

$$\tilde{\Omega}^1(U_{\sigma_1})/\Omega^1(U_{\sigma_1}) \oplus \tilde{\Omega}^1(U_{\sigma_2})/\Omega^1(U_{\sigma_2}) \to \tilde{\Omega}^1(U_\tau)/\Omega^1(U_\tau) \to K_0^{(2)}(X) \to 0.$$

By Example 5.9, $\tilde{\Omega}^1(U_\tau)/\Omega^1(U_\tau)$ is zero except in weights $m = (1,0,c)$, $c \in \mathbb{Z}$, where it is spanned by the forms $T^c X dY/Y$. For such m, $\tau(m) = \{0\}$. If $c > 0$ then $m \in \sigma_1^\vee$ and the element $\chi^m dY/Y \in \tilde{\Omega}^1(U_{\sigma_1})$ maps to $T^c X\, dY/Y \in \tilde{\Omega}(U_\tau)$. If $c < 0$ then $m \in \sigma_2^\vee$ and the element $\chi^m dY/Y \in \tilde{\Omega}^1(U_{\sigma_1})$ maps to $T^c X\, dY/Y \in \tilde{\Omega}(U_\tau)$.

We are left with the form $X\, dY/Y$ in weight $m = (1,0,0)$. Since $m \notin \sigma_i^\vee$ for $i = 1,2$, we have $\tilde{\Omega}^1(U_{\sigma_1})_m = \tilde{\Omega}^1(U_{\sigma_2})_m = 0$. This proves that

$$K_0^{(2)}(X) \cong \tilde{\Omega}^1(U_\tau)/\Omega^1(U_\tau)_{(1,0,0)} \cong k.$$

As in Gubeladze's example of toric varieties with "huge" Grothendieck groups in [14], we can further extend Δ to obtain a complete fan consisting of simplicial cones $\overline{\Delta}$, so that $\overline{X} = X(\overline{\Delta})$ is a projective closure of X and such that $Y = X(\overline{\Delta} - \Delta)$ is smooth. Since Y and X form an open cover of $\overline{X}$, we see that $K_0(\overline{X})$ also contains the additive group underlying a non-zero k-vector space.

6. Gubeladze's Dilation Theorem

The main goal of this section is to give a new proof of Gubeladze's Dilation Theorem [15] for the K-theory of monoid rings, which we obtain in 6.10 as a corollary of a version of this result valid for all toric varieties (Theorem 6.9).

For a toric variety $X = X(\Delta)$ with Δ a fan in $N_\mathbb{R}$ and integer $c \in \mathbb{N}$, define $\theta_c : X(\Delta) \to X(\Delta)$ to be the endomorphism of toric varieties induced by the endomorphism of the lattice N given by multiplication by c. If $\sigma \subset N_\mathbb{R}$ is a cone, the map $\theta_c : U_\sigma \to U_\sigma$ of affine toric k-varieties is induced by the ring endomorphism of $k[\sigma^\vee \cap M]$ that sends χ^m to χ^{cm}. That is, this is the map that raises all monomials to the c-th power. Observe that if $k = \mathbb{F}_p$ and $c = p$, this is precisely the Frobenius endomorphism, and it useful to think of θ_c as a generalization of Frobenius that exists in the category of toric varieties.

Fix a sequence $\mathfrak{c} = (c_1, c_2, \dots)$ of integers with $c_i \geq 2$ for all i. If F is a contravariant functor from toric varieties to abelian groups, we define $F^\mathfrak{c}$ by

$$F(X)^\mathfrak{c} = \varinjlim \left(F(X) \xrightarrow{\theta_{c_1}^*} F(X) \xrightarrow{\theta_{c_2}^*} \cdots \right).$$

Gubeladze's Dilation Theorem asserts that the natural map $K_*(X) \to KH_*(X)$ induces an isomorphism $K_*(X)^\mathfrak{c} \to KH_*(X)^\mathfrak{c}$ for any toric variety X. Our proof of this theorem involves computing $HH_q(X)^\mathfrak{c}$ where HH_* denotes Hochschild homology.

Fix a cone σ. As in the proof of Lemma 3.5, the chain complex defining the Hochschild homology of $k[\sigma^\vee \cap M]$ is $\sigma^\vee \cap M$-graded with the weight of $\chi^{m_0} \otimes \cdots \otimes \chi^{m_p}$ defined to be $m_0 + \cdots + m_p$, and the Hochschild homology groups of U_σ are $\sigma^\vee \cap M$-graded $k[\sigma^\vee \cap M]$-modules. A fortiori, they are M-graded, with zero in weight m if $m \notin \sigma^\vee$. Since $\theta_c(\chi^{m_0} \otimes \cdots) = \chi^{cm_0} \otimes \cdots$, θ_c sends the weight m summand to the weight cm summand.

The Hochschild homology of a non-affine variety is defined by taking Zariski hypercohomology of the sheafification of the complex defined just as in the definition of $HH_*(R)$, but with $\mathcal{O}_X \otimes_k \cdots \otimes_k \mathcal{O}_X$ in place of $R \otimes_k \cdots \otimes_k R$ (see [24, 4.1]).

For a toric variety $X = X(\Delta)$, we may compute $HH_*(X)$ as follows: Let $\sigma_1, \dots, \sigma_m$ denote the maximal cones in the fan Δ. For each $1 \le i_0 \le \cdots \le i_p \le m$, we may form the complex defining the Hochschild homology of the affine toric variety $U_{\sigma_{i_0} \cap \cdots \cap \sigma_{i_p}}$. We then assemble these into a bicomplex in the usual Čech manner and take the homology of the associated total complex.

Lemma 6.1. *For any toric variety $X = X(\Delta)$, the groups $HH_*(X)$ have a natural M-grading, and the endomorphism θ_c maps the weight m summand to the weight cm summand.*

Proof. We have seen that the Hochschild complexes forming the columns of the bicomplex are M-graded. Since the ring maps are all M-graded, the Čech differentials are also M-graded. Since $HH_*(X)$ is the homology of an M-graded bicomplex, it is M-graded. Since the map θ_c sends the weight m subcomplex to the weight cm subcomplex, it has the same effect on homology. □

Remark 6.1.1. This construction implies that the Čech spectral spectral sequence is M-graded:

$$E^1_{pq} = \bigoplus_{i_0 < \cdots < i_p} HH_q(U_{\sigma_{i_0} \cap \cdots \cap \sigma_{i_p}}) \Rightarrow HH_{q-p}(X).$$

Lemma 6.2. *Set $A = \sigma^\vee \cap M$. If $m \in A$ lies on no proper face of $\sigma^\vee$, then $A + \langle -m \rangle = M$, and $k[A][\chi^{-m}] = k[M]$.*

Proof. Since $k[A][\chi^{-m}] = k[A + \langle -m \rangle]$, it suffices to prove the first assertion, *i.e.,* that every $t \in M$ is of the form $a - im$ for some positive integer i. Fix a nonzero $n \in N$. The assumption that m lies on no proper face of $\sigma^\vee$ implies that $\langle m, n \rangle > 0$. Hence $\langle t + im, n \rangle > 0$ for $i \gg 0$. Since $\sigma \cap N$ is finitely generated, it follows that $t + im \in A$ for $i \gg 0$, as claimed. □

Lemma 6.3. *The map $\theta_c : \Omega^q(U_\sigma)_m \to \Omega^q(U_\sigma)_{cm}$ is multiplication by $c^q \chi^{(c-1)m}$.*

Proof. When $\sum u_i = m$, θ_c takes $\omega = \chi^{u_0} d\chi^{u_1} \wedge \cdots d\chi^{u_q}$ to $c^q \chi^m \omega$. □

Remark 6.3.1. The same proof shows that the map $\theta_c : \tilde{\Omega}^q(U_\sigma)_m \to \tilde{\Omega}^q(U_\sigma)_{cm}$ is multiplication by $c^q \chi^{(c-1)m}$. By (3.3), this is an isomorphism for all $c \neq 0$.

Proposition 6.4. *For any toric k-variety X, the natural maps* (3.4) *induce isomorphisms, for all q:*

$$\Omega^q(X)^{\mathfrak{c}} \to \tilde{\Omega}^q(X)^{\mathfrak{c}}$$

Proof. We may assume $X = U_\sigma$, so that $\Omega^q(X) = \Omega^q_{k[A]}$ for $A = \sigma^\vee \cap M$. It suffices to check that the map is an isomorphism in each weight $m \in M_{\mathfrak{c}}$; without loss of generality, one may assume $m \in M$. By Lemma 3.5, $(\Omega^q_{k[A]})_m \cong (\Omega^q_{k[B]})_m$, where $B = A \cap \sigma(m)^\perp$. By Lemma 6.3, θ_c coincides with multiplication by $c^q \chi^{(c-1)m}$ both as a map $(\Omega^q_{k[A]})_m \to (\Omega^q_{k[A]})_{cm}$ and as a map $(\Omega^q_{k[B]})_m \to (\Omega^q_{k[B]})_{cm}$. Hence the group

$$\Omega^q(X)^{\mathfrak{c}}_m = \varinjlim \left((\Omega^q_{k[A]})_m \xrightarrow{\theta_{c_1}} (\Omega^q_{k[A]})_{c_1 m} \xrightarrow{\theta_{c_2}} \cdots \right)$$

is the weight m part of the localization of $\Omega^q_{k[B]}$ at χ^m, *i.e.,* of $\Omega^q(k[B][\chi^{-m}])$. By construction, m is not on any proper face of $\sigma(m)^\vee \cap \sigma(m)^\perp$. By Lemma 6.2,

$$\Omega^q(k[B][\chi^{-m}])_m \cong \Omega^q(k[B + \langle -m \rangle])_m = \Omega^q(k[T])_m, \qquad T = M \cap \sigma(m)^\perp.$$

Since T is a free abelian group, $(\Omega^q_{k[T]})_m \cong \wedge^q(T)\otimes k$. Now recall that by Remark 6.3.1 and (3.3) we also have

$$(\tilde\Omega^q_{k[T]})^{\mathfrak{c}}_m \cong \tilde\Omega^q(U_\sigma)^{\mathfrak{c}}_m = \tilde\Omega^q(U_\sigma)_m \cong k\cdot\chi^m\otimes\wedge^q(T),$$

The map $(\Omega^q_{k[T)})_m \to (\tilde\Omega^q_{k[T)})_m$ is given by (3.4), and it is an isomorphism by inspection. □

In order to prove an analogous result for Hochschild homology, we need to briefly review the decomposition of Hochschild homology into summands given by the (higher) André-Quillen homology groups. For more details, we refer the reader to [20, 3.5] or [27, 8.8].

For a commutative k-algebra R, one forms a simplicial polynomial k-algebra $R_\bullet$ and a simplicial ring map $R_\bullet \to R$ which is a homotopy equivalence on underlying simplicial sets. The (higher) *cotangent complex* $\mathbb{L}^{(q)}_{X/k}$ is defined to be the simplicial R-module $R\otimes_{R_\bullet}\Omega^q_{R_\bullet}$, and the André-Quillen homology groups of R are defined to be $D^{(q)}_p(R) = H_p(\mathbb{L}^{(q)}_{X/k})$. The R-modules $D^{(q)}_p(R)$ are independent up to isomorphism of the choices made.

In general, there is a natural spectral sequence of R-modules

$$D^{(q)}_p(R) \Longrightarrow HH_{p+q}(R)$$

and a natural R-module isomorphism $D^{(q)}_0(R) \cong \Omega^q_{R/k}$. Since we are assuming $\mathrm{char}(k)=0$, this spectral sequence degenerates to give a natural decomposition of R-modules

$$HH_n(R) \cong \bigoplus_{p+q=n} D^{(q)}_p(R) = \Omega^q_{R/k}\oplus \bigoplus_{p+q=n,p>0} D^{(q)}_p(R).$$

Since the André-Quillen homology groups are functorial for ring maps, the endomorphisms θ_{c_i} preserve this decomposition.

Lemma 6.5. *Let U_σ be an affine toric variety. Then the $D^{(q)}_p(U_\sigma)$ are M-graded modules and, for every $m\in\sigma^\vee\cap M$, the map $\theta_c : D^{(q)}_p(U_\sigma)_m \to D^{(q)}_p(U_\sigma)_{cm}$ is multiplication by $c^q\chi^{(c-1)m}$.*

Proof. Let $A=\sigma^\vee\cap M$ and form a simplicial resolution of A by free abelian monoids $A_\bullet\to A$. That is, $A_\bullet$ is a simplicial abelian monoid which in each degree is free abelian and the map of simplicial abelian monoids $A_\bullet\to A$ is a homotopy equivalence. This is possible by the same basic cotriple resolution used to form simplicial free resolutions of k-algebras (see [27, 8.6]). For functorial reasons, $k[A_\bullet]\to k[A]$ is a free simplicial resolution of $k[A]$. We therefore have

$$D^{(q)}_p(k[A]) = H_p(k[A]\otimes_{k[A_\bullet]}\Omega^q_{k[A_\bullet]}).$$

For each n, the ring $k[A_n]$ is M-graded by the maps $\delta_n : A_n\to A\subset M$. Thus the simplicial ring $k[A_\bullet]$ is also M-graded and the map $k[A_\bullet]\to k[A]$ of simplicial rings preserves this grading. It follows that $k[A]\otimes_{k[A_\bullet]}\Omega^q_{k[A_\bullet]}$ is naturally M-graded, where the weight of $\chi^{u_0}\otimes d(\chi^{u_1})\wedge\cdots\wedge d(\chi^{u_q})$ is $u_0+\delta_n(u_1)+\cdots+\delta_n(u_q)$, for any $u_0\in A$ and $u_1,\dots,u_q\in A_n$. Hence $D^{(q)}_p(k[A])$ is an M-graded $k[A]$-module, and it is clear that, for any positive integer c, the endomorphism θ_c of $D_p^{(q)}(k[A])$ maps the weight m summand to the weight cm summand. To prove that the map

$$\theta_c : D_p^{(q)}(k[A])_m \to D_p^{(q)}(k[A])_{cm}$$

coincides with multiplication by $c^q\chi^{(c-1)m}$, it suffices to prove the analogous assertion for the M-graded $k[A]$-modules $k[A] \otimes_{k[A_n]} \Omega^q_{k[A_n]}$. The proof of this is exactly like the proof of Lemma 6.3, using $\omega = \chi^{u_0} \otimes d\chi^{u_1} \wedge \cdots \wedge d\chi^{u_q}$. □

Theorem 6.6. *For any toric k-variety X, the natural maps*

$$\Omega^q(X)^{\mathfrak{c}} \to HH_q(X)^{\mathfrak{c}}$$

are isomorphisms, for all q.

Proof. By the spectral sequence in 6.1.1, we may assume that X is affine, say of the form $X = U_\sigma$ for some cone σ. Setting $A = \sigma^\vee \cap M$, the coordinate ring of X is $k[A]$. To establish the isomorphism $\Omega^p(U_\sigma)^{\mathfrak{c}} \cong HH_p(U_\sigma)^{\mathfrak{c}}$ it suffices to prove that

$$D_p^{(q)}(k[\sigma^\vee \cap M])^{\mathfrak{c}} = 0$$

for all $p > 0$. As in the proof of Proposition 6.4, it suffices to fix an arbitrary $m \in M$ and show that the weight m part vanishes. By Lemma 3.5, $D_p^{(q)}(k[A])_m \cong D_p^{(q)}(k[B])_m$, where $B = A \cap \sigma(m)^\perp$. By Lemma 6.5, θ_c coincides with multiplication by $c^q\chi^{(c-1)m}$ both as a map $D_p^{(q)}(k[A])_m \to D_p^{(q)}(k[A])_{cm}$ and as a map $D_p^{(q)}(k[B])_m \to D_p^{(q)}(k[B])_{cm}$. Hence the weight m summand

$$D_p^{(q)}(X)^{\mathfrak{c}}_m = \varinjlim \left(D_p^{(q)}(k[A])_m \xrightarrow{\theta_{c_1}} D_p^{(q)}(k[A])_{c_1 m} \xrightarrow{\theta_{c_2}} \cdots \right)$$

is the weight m part of the localization of $D_p^{(q)}(k[B])$ at χ^m, *i.e.*, of $D_p^{(q)}(k[B][\chi^{-m}])$.

Recall that $\sigma(m) \subset \sigma$ denotes the face of σ (possibly just the origin) on which $m = 0$. By Lemma 6.2,

$$D_p^{(q)}(k[B][\chi^{-m}])_m \cong D_p^{(q)}(k[B + \langle -m \rangle])_m = D_p^{(q)}(k[T])_m, \quad T = M \cap \sigma(m)^\perp.$$

Since $T = M \cap \sigma(m)^\perp$ is a free abelian group, we have

$$D_p^{(q)}(k[B][\frac{1}{\chi^m}]) = D_p^{(q)}(k[T]) = 0$$

for all $p > 0$. This proves that $D_p^{(q)}(k[A])^{\mathfrak{c}} = 0$ for all $p > 0$, proving the theorem. □

Corollary 6.7. *For any field k of characteristic 0 and any toric k-variety X, we have a natural isomorphism for all n:*

$$HH_n(X/\mathbb{Q})^{\mathfrak{c}} \xrightarrow{\simeq} \mathbb{H}^{-n}_{cdh}(X, HH(-/\mathbb{Q}))^{\mathfrak{c}}.$$

The right hand side in 6.7 denotes Hochschild homology with *cdh* descent imposed (and localized by $\mathfrak{c}$). (On both sides, we take Hochschild homology over $\mathbb{Q}$.)

Proof. Let us write $X_{\mathbb{Q}}$ for the model of X defined over the rationals and $X_k = X$ for the model over k. We have $X_k = X_{\mathbb{Q}} \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec} k$.

The natural map

$$HH_n(X_{k/k})^{\mathfrak{c}} \longrightarrow \mathbb{H}^{-n}_{\mathrm{cdh}}(X, HH)^{\mathfrak{c}}$$

is an isomorphism. Since both sides satisfy Zariski descent, this is an immediate consequence Theorem 4.1 and Theorem 6.6. The Künneth formula for Hochschild homology, described before Corollary 4.5, gives

$$HH_*(X/\mathbb{Q})^{\mathfrak{c}} \cong HH_*(X_{\mathbb{Q}}/\mathbb{Q})^{\mathfrak{c}} \otimes_{\mathbb{Q}} \Omega^*_{k/\mathbb{Q}}.$$

In particular, one gets long exact sequences for $HH_*(-/\mathbb{Q})^{\mathfrak{c}}$ associated to abstract blow-ups of toric k-varieties. Since the map

$$HH_n(X_k/\mathbb{Q})^{\mathfrak{c}} \cong \mathbb{H}^{-n}_{\mathrm{cdh}}(X, HH(-/\mathbb{Q}))^{\mathfrak{c}}$$

is an isomorphism whenever X is smooth by [3, 2.4], the result holds by induction and the five-lemma. □

Corollary 6.8. *For any field k of characteristic* 0 *and any toric k-variety X, and all n, we have*

$$HC_n(X/\mathbb{Q})^{\mathfrak{c}} \cong \mathbb{H}^{-n}_{cdh}(X, HC(-/\mathbb{Q}))^{\mathfrak{c}}.$$

Proof. There is a map from the SBI sequence for HH and HC to the SBI sequence for its *cdh*-fibrant variant. Applying the exact functor $(-)^{\mathfrak{c}}$ yields a similar map of long exact sequences, every third term of which is the isomorphism of Corollary 6.7. The result now follows by induction on n, since all complexes are cohomologically bounded above. □

Theorem 6.9. *For any field k of characteristic* 0 *and any toric k-variety X, we have*

$$K_*(X)^{\mathfrak{c}} \cong KH_*(X)^{\mathfrak{c}}.$$

Proof. Since $(-)^{\mathfrak{c}}$ is exact, it suffices by Theorem 5.5 to show that $H^*_{\mathrm{Zar}}(X, \mathcal{F}_{HC})^{\mathfrak{c}}$ vanishes. Again because $(-)^{\mathfrak{c}}$ is exact, we have a long exact sequence

$$\cdots \to H^n_{\mathrm{Zar}}(X, \mathcal{F}_{HC})^{\mathfrak{c}} \to HC_{-n}(X/\mathbb{Q})^{\mathfrak{c}} \to \mathbb{H}^n_{\mathrm{cdh}}(X, HC(-/\mathbb{Q}))^{\mathfrak{c}}$$

The desired vanishing follows from the previous corollary. □

Corollary 6.10. *(Gubeladze's Dilation Theorem) Let Γ be an arbitrary commutative, cancellative, torsionfree monoid without nontrivial units. Then for every sequence $\mathfrak{c}$ and every p, $(K_p(k[\Gamma])/K_p(k))^{\mathfrak{c}} = 0$.*

Proof. To prove the Dilation Theorem, it suffices to prove it for all "affine positive normal" monoids, *i.e.,* for monoids of the form $\Gamma = \sigma^\vee \cap M$ such that $\sigma^\perp = 0$. This is a reformulation of [12, 3.4], and is stated explicitly in [15, Proposition 2.1] (up to the typo that $K_p(R[M])$ should be $K_p(R[M])/K_p(R)$).

For such Γ, $X = \mathrm{Spec}(k[\Gamma])$ is a toric variety, and the proof of Proposition 5.6 above shows that $k[\Gamma]$ is $\mathbb{N}$-graded with k in weight 0. Hence $KH(X) \simeq K(\mathrm{Spec}\, k)$. The result now follows from Theorem 6.9. □

Remark 6.11. In [16], Gubeladze proves an unstable version of his Dilation Theorem for the groups K_1 and K_2, which is valid for any regular coefficient ring in place of the field k. In [17], he proves that his Dilation Theorem remains valid if one replaces the field k by any regular coefficient ring that contains a copy of $\mathbb{Q}$.

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