

ALGEBRAIC K -THEORY AND ABSTRACT HOMOTOPY THEORY

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ABSTRACT. We decompose the K -theory space of a Waldhausen category in terms of its Dwyer-Kan simplicial localization. This leads to a criterion for functors to induce equivalences of K -theory spectra that generalizes and explains many of the criteria appearing in the literature. We show that under mild hypotheses, a weakly exact functor that induces an equivalence of homotopy categories induces an equivalence of K -theory spectra.

1. INTRODUCTION

Quillen's higher algebraic K -theory provides a powerful and subtle invariant of rings and schemes. Waldhausen reformulated the definition and generalized the original input from algebra to homological algebra or homotopy theory; in place of exact categories, which are additive categories with a notion of exact sequence, Waldhausen's construction allows categories equipped with weak equivalences (quasi-isomorphisms) and a notion of cofibration sequence. Although designed to apply homotopy theory and K -theory to geometric topology, the added flexibility of Waldhausen K -theory turns out to be tremendously useful even when studying the original algebraic objects. For instance, the remarkable localization and Mayer-Vietoris theorems of Thomason and Trobaugh [16, 7.4,8.1], which relate the K -theory of a scheme to the K -theories of open covers, depend on the techniques possible only in Waldhausen's framework. One of the most important of these techniques is the ability to change homological models, using different categories of complexes with equivalent K -theory. A central question then becomes how to determine when different models yield the same K -theory [16, 1.9.9].

Waldhausen's approximation theorem [18, 1.6.4] stands as the prototypical example of a K -theory equivalence criterion. Thomason and Trobaugh [16, 1.9.8] specialized Waldhausen's approximation theorem to certain categories of complexes, where for appropriate complicial functors, an equivalence of derived categories implies an equivalence of K -theory. Based on this result and work of the Grothendieck school on K_0 , they articulated the perspective that higher algebraic K -theory "essentially depends only on the derived category" [16, 1.9.9], with a caveat about choice of models. Indeed, Schlichting [15] subsequently constructed examples of Frobenius categories with abstractly equivalent derived categories but different algebraic K -theory groups. On the other hand, for the algebraic K -theory of rings, Dugger and Shipley [2] proved that an abstract equivalence of derived categories

Date: September 7, 2007.

2000 Mathematics Subject Classification. Primary 19D99; Secondary 55U35.

The first author was supported in part by an NSF postdoctoral fellowship, NSF grant DMS-0111298, and a Clay Mathematics Institute Liftoff Fellowship.

The second author was supported in part by NSF grant DMS-0504069.

does imply a K -theory equivalence. Their argument relies on the folk theorem that a Quillen equivalence of model categories induces an equivalence of K -theory of appropriate Waldhausen subcategories. Toën and Vezzosi [17] generalized this from Quillen equivalences to equivalences on Dwyer-Kan simplicial localizations [3]. Other approaches have tried to construct higher algebraic K -theory directly from the derived category [13] (and sequels) or using the Heller-Grothendieck-Keller theory of “derivators” [8, 7, 9, 10] in for example [11].

In this paper, we realize the vision of Thomason and Trobaugh in the context of Waldhausen categories that admit functorial factorization (see Section 2) and whose weak equivalences are closed under retracts and are saturated in the sense of Waldhausen [18, §1.2] (satisfy the two out of three property). We show that under this hypothesis, a weakly exact functor that induces an equivalence of homotopy categories induces an equivalence of K -theory spectra. This hypothesis holds in particular in Waldhausen categories that come from model categories. Functorial factorization generalizes Waldhausen’s notion of “cylinder functor satisfying the cylinder axiom”. Waldhausen categories often have their weak equivalences closed under retracts, but we have included two alternative hypotheses for cases when this does not hold. We prove the following theorem in Section 3.

Theorem 1.1. *Let \mathcal{C} and \mathcal{D} be saturated Waldhausen categories that admit functorial factorization, and let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a weakly exact functor that induces an equivalence on homotopy categories. If one of the following additional hypotheses holds*

- (i) *The weak equivalences of \mathcal{C} and \mathcal{D} are closed under retracts,*
- (ii) *A map f in \mathcal{C} is a weak equivalence if and only the map Ff in \mathcal{D} is a weak equivalence, or*
- (iii) *For any $A, B \in \mathcal{C}$, the image of $\mathrm{Ho}(\mathrm{w}\mathcal{C})(A, B)$ in $\mathrm{Ho}\mathcal{D}(FA, FB)$ coincides with the image of $\mathrm{Ho}(\mathrm{w}\mathcal{D})(FA, FB)$,*

then F induces an equivalence of K -theory spectra.

In the statement, a “weakly exact” functor is a homotopical generalization of an exact functor. An exact functor between Waldhausen categories preserves weak equivalences, cofibrations, and pushouts along cofibrations. A weakly exact functor preserves weak equivalences, but need only preserve cofibrations and pushouts along cofibrations up to weak equivalence (see Definition 2.1 below). The “homotopy category” of a category \mathcal{C} with weak equivalence is the category $\mathrm{Ho}\mathcal{C}$ obtained by formally inverting the weak equivalences. $\mathrm{Ho}\mathcal{C}$ generalizes the derived category to this context; it is typically not a triangulated category without additional hypotheses on \mathcal{C} .

Following Waldhausen’s notation, we have used $\mathrm{w}\mathcal{C}$ and $\mathrm{w}\mathcal{D}$ to denote the subcategories of weak equivalences for \mathcal{C} and \mathcal{D} . The image of $\mathrm{Ho}(\mathrm{w}\mathcal{C})$ in $\mathrm{Ho}\mathcal{C}$ consists of isomorphisms by definition, but might not in general contain all isomorphisms of $\mathrm{Ho}\mathcal{C}$. It does contain all the isomorphisms, however, under the hypotheses of Theorem 1.1 when the weak equivalences of \mathcal{C} are closed under retracts; see Section 6 for a complete discussion. Hypotheses (ii) and (iii) in Theorem 1.1 ensure that the weak equivalences of \mathcal{C} and \mathcal{D} and their formal inverses generate equivalent subcategories of $\mathrm{Ho}\mathcal{D}$ even when they do not necessarily generate all the isomorphisms of $\mathrm{Ho}\mathcal{D}$.

As part of the proof of the Theorem 1.1, we describe a precise relationship between the algebraic K -theory space and the Dwyer-Kan simplicial localization of the Waldhausen category. The Dwyer-Kan simplicial localization associates to any category with weak equivalences simplicial mapping spaces which have the “correct” homotopy type [4] and that characterize the higher homotopy theory of the category [12, 5.7]. We show that algebraic K -theory of a Waldhausen category \mathcal{C} decomposes into pieces that we describe in terms of the Dwyer-Kan “hammock” simplicial mapping spaces [3]. Our description elucidates the nature of the homotopical information encoded by K -theory and provides a conceptual explanation of the equivalence criteria described above.

Theorem 1.2. *Let \mathcal{C} be a saturated Waldhausen category that admits functorial factorization. For $n > 1$, the nerve of wS_n is weakly equivalent to the homotopy coend*

$$\text{hocoend}_{(X_1, \dots, X_n) \in w\mathcal{C}^n} LC(X_{n-1}, X_n) \times \cdots \times LC(X_1, X_2),$$

where LC denotes the Dwyer-Kan hammock localization.

The homotopy coend comes with a map to the classifying space $Bw\mathcal{C}^n$. We can identify this space and the homotopy fiber of the map intrinsically in terms of the Dwyer-Kan simplicial localization. For X an object in \mathcal{C} , let $\text{hAut } X$ denote the subspace of $LC(X, X)$ consisting of the components which have a vertex where all the forward maps are weak equivalences; then $\text{hAut}(X)$ is a grouplike monoid of homotopy automorphisms of X in LC .

Theorem 1.3. *Let \mathcal{C} be a saturated Waldhausen category that admits functorial factorization. For $n \geq 1$, the nerve of wS_n is weakly equivalent to the total space of a fibration where the base is the disjoint union of*

$$B \text{hAut } X_n \times \cdots \times B \text{hAut } X_1$$

over n -tuples of weak equivalences classes of objects of \mathcal{C} , and the fiber is equivalent to

$$LC(X_{n-1}, X_n) \times \cdots \times LC(X_1, X_2)$$

for $n > 1$ and contractible for $n = 1$.

In the proof of Theorem 1.1, we argue that a weakly exact functor that induces an equivalence of homotopy categories comes very near to being a DK-equivalence (a functor that induces a weak equivalence of Dwyer-Kan simplicial localizations); see Corollary 3.7. It remains an interesting question to determine when such a functor is a DK-equivalence. When we drop the weakly exact hypothesis and consider only functors that preserve weak equivalences, we can characterize DK-equivalences in terms of homotopy categories of undercategories. For an object A of \mathcal{C} , let $\mathcal{C} \setminus A$ denote the category of objects in \mathcal{C} under A , i.e., an object consists of a map $A \rightarrow X$ in \mathcal{C} and a map from $A \rightarrow X$ to $A \rightarrow Y$ consists of a map $X \rightarrow Y$ in \mathcal{C} that commutes with the maps from A ; say that such a map is a weak equivalence when its underlying map $X \rightarrow Y$ is a weak equivalence in \mathcal{C} . We can then form the homotopy category $\text{Ho}(\mathcal{C} \setminus A)$ by formally inverting the weak equivalences. We prove in Section 8 the following theorem generalizing the main result of [12].

Theorem 1.4. *Let \mathcal{C} and \mathcal{D} be saturated Waldhausen categories that admit functorial factorization, and let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor that preserves weak equivalences.*

Then F is a DK -equivalence if and only if it induces an equivalence $\mathrm{Ho}(\mathcal{C}) \rightarrow \mathrm{Ho}(\mathcal{D})$ and an equivalence $\mathrm{Ho}(\mathcal{C}\backslash A) \rightarrow \mathrm{Ho}(\mathcal{D}\backslash FA)$ for all objects A of \mathcal{C} .

This interpretation relates to Waldhausen's approximation theorem and provides a conceptual understanding of the role of Waldhausen's approximation property [18, 1.6.4] in the more specialized approximation theorems. Recall that for Waldhausen categories \mathcal{C} and \mathcal{D} , an exact functor $F: \mathcal{C} \rightarrow \mathcal{D}$ satisfies the approximation property if

- (i) A map $f: A \rightarrow B$ is a weak equivalence in \mathcal{C} if and only if the map $F(f): FA \rightarrow FB$ is a weak equivalence in \mathcal{D} .
- (ii) For every map $FA \rightarrow X$ in \mathcal{D} , there exists a cofibration $A \rightarrow B$ in \mathcal{C} and a weak equivalence $FB \rightarrow X$ in \mathcal{D} such that the diagram

$$\begin{array}{ccc} FA & \longrightarrow & X \\ \downarrow & \nearrow \simeq & \\ FB & & \end{array}$$

commutes.

We prove the following theorem in Section 9.

Theorem 1.5. *Let \mathcal{C} be a saturated Waldhausen category where every map factors as a cofibration followed by a weak equivalence. Let \mathcal{D} be a saturated Waldhausen category, and let $F: \mathcal{C} \rightarrow \mathcal{D}$ be an exact functor. If F satisfies Waldhausen's approximation property, then $\mathrm{Ho}(\mathcal{C}) \rightarrow \mathrm{Ho}(\mathcal{D})$ is an equivalence and $\mathrm{Ho}(\mathcal{C}\backslash A) \rightarrow \mathrm{Ho}(\mathcal{D}\backslash FA)$ is an equivalence for all objects A of \mathcal{C} .*

In all of the preceding theorems, we required the hypothesis that the Waldhausen categories be saturated, meaning that the weak equivalences satisfy the two out of three property: For composable maps f and g , if any two of f , g , and $f \circ g$ are weak equivalences then so is the third. This usage differs from the usage of the term in other sources such as [5]. Although much of the work in this paper could be adjusted to avoid this hypothesis, it is a hypothesis so common and pervasive in homotopy theory that to do so would lose more in the awkwardness it would engender than it would gain in the extra abstract generality it might achieve. Rather than continually repeating this hypothesis throughout the rest of the paper, we instead incorporate it by convention in the definition of weak equivalences.

Convention. In this paper we always understand a subcategory of weak equivalences to satisfy the two out of three property. In particular, all Waldhausen categories are assumed to be saturated in the sense of Waldhausen.

The authors would like to thank the Institute for Advanced Study and the University of Chicago for their hospitality while some of this work was being done. The authors would also like to thank H. Miller and L. Hesselholt for asking motivating questions.

2. WEAKLY EXACT FUNCTORS

This section defines weakly exact functor and shows that under mild technical hypotheses a weakly exact functor between Waldhausen categories induces a map between their K -theory spectra. Although we expect that the extra flexibility

provided by stating Theorem 1.1 in terms of weakly exact functors rather than exact functors will increase its applicability, in fact, weakly exact functors play a vital technical role in its proof even in the case when the functor in question is exact. Specifically, the proof requires a version of Theorem 1.2 that is natural in weakly exact functors, which we state as Theorem 2.7 at the end of the section. We begin with the definition of weakly exact functor.

Definition 2.1. Let \mathcal{C} and \mathcal{D} be Waldhausen categories. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is *weakly exact* if the initial map $* \rightarrow F*$ in \mathcal{D} is a weak equivalence and F preserves weak equivalences, weak cofibrations, and homotopy cocartesian squares.

In the definition, a *weak cofibration* is a map that is weakly equivalent (by a zigzag) to a cofibration in the category $\text{Ar}\mathcal{C}$ of arrows in \mathcal{C} , and a *homotopy cocartesian square* is a square diagram that is weakly equivalent (by a zigzag) to a pushout square where one of the parallel sets of arrows consists of cofibrations. It follows that a functor that preserves weak equivalences will preserve weak cofibrations and homotopy cocartesian squares if and only if it takes cofibrations to weak cofibrations and takes pushouts along cofibrations to homotopy cocartesian squares. Clearly the concept of weakly exact functor will only be useful when homotopy cocartesian squares have the usual expected properties. According to [1, §2], these properties hold when the Waldhausen category admits “functorial factorization of weak cofibrations”.

Recall that a Waldhausen category \mathcal{C} admits *functorial factorization* when any map $f: A \rightarrow B$ in \mathcal{C} factors as a cofibration followed by a weak equivalence

$$A \twoheadrightarrow Tf \xrightarrow{\simeq} B,$$

f

functorially in f in the category $\text{Ar}\mathcal{C}$ of arrows in \mathcal{C} . In other words, given the map ϕ of arrows on the left (i.e., commuting diagram),

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ a \downarrow & \phi & \downarrow b \\ A' & \xrightarrow{f'} & B' \end{array} \qquad \begin{array}{ccc} A & \twoheadrightarrow & Tf \xrightarrow{\simeq} B \\ a \downarrow & T\phi \downarrow & \downarrow b \\ A' & \twoheadrightarrow & Tf' \xrightarrow{\simeq} B' \end{array}$$

we have a map $T\phi$ that makes the diagram on the right commute and that satisfies the usual identity and composition relations, $T\text{id}_f = \text{id}_{Tf}$ and $T(\phi' \circ \phi) = T\phi' \circ T\phi$. A cylinder functor satisfying the cylinder axiom in the sense of Waldhausen [18, §1.6] is a factorization functor that in addition satisfies strong exactness properties.

Waldhausen categories that admit functorial factorization have as a consequence the property that every map is weakly equivalent to a cofibration. This isn’t always the case in examples of interest, especially in “Waldhausen subcategories”. To get around this, in [1] we worked in terms of the technical hypothesis that \mathcal{C} admit *functorial factorization of weak cofibrations (FFWC)* [1, 2.2], which means that the weak cofibrations can be factored functorially (in $\text{Ar}\mathcal{C}$) as above. Our interest in FFWC is the following theorem proved in this section.

Theorem 2.2. *Let \mathcal{C} and \mathcal{D} be Waldhausen categories admitting FFWC and let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a weakly exact functor. Then F induces a map of K-theory spectra.*

We prove Theorem 2.2 using the S'_\bullet construction of [1, §2]. To put this in context, we begin by reviewing the S_\bullet construction in detail. Recall that Waldhausen's S_\bullet construction produces a simplicial Waldhausen category $S_\bullet\mathcal{C}$ from a Waldhausen category \mathcal{C} and is defined as follows. Let $\text{Ar}[n]$ denote the category with objects (i, j) for $0 \leq i \leq j \leq n$ and a unique map $(i, j) \rightarrow (i', j')$ for $i \leq i'$ and $j \leq j'$. $S_n\mathcal{C}$ is defined to be the full subcategory of the category of functors $A: \text{Ar}[n] \rightarrow \mathcal{C}$ such that:

- $A_{i,i} = *$ for all i ,
- The map $A_{i,j} \rightarrow A_{i,k}$ is a cofibration for all $i \leq j \leq k$, and
- The diagram

$$\begin{array}{ccc} A_{i,j} & \longrightarrow & A_{i,k} \\ \downarrow & & \downarrow \\ A_{j,j} & \longrightarrow & A_{j,k} \end{array}$$

is a pushout square for all $i \leq j \leq k$,

where we write $A_{i,j}$ for $A(i, j)$. The last two conditions can be simplified to the hypothesis that each map $A_{0,j} \rightarrow A_{0,j+1}$ is a cofibration and the induced maps $A_{0,j}/A_{0,i} \rightarrow A_{i,j}$ are isomorphisms. This becomes a Waldhausen category by defining a map $A \rightarrow B$ to be a weak equivalence when each $A_{i,j} \rightarrow B_{i,j}$ is a weak equivalence in \mathcal{C} , and to be a cofibration when each $A_{i,j} \rightarrow B_{i,j}$ and each induced map $A_{i,k} \cup_{A_{i,j}} B_{i,j} \rightarrow B_{i,k}$ is a cofibration in \mathcal{C} . The following definition gives a homotopical version of this construction for Waldhausen categories that admit FFWC.

Definition 2.3. Let \mathcal{C} be a Waldhausen category that admits FFWC. Define $S'_n\mathcal{C}$ to be the full subcategory of functors $A: \text{Ar}[n] \rightarrow \mathcal{C}$ such that:

- The initial map $* \rightarrow A_{i,i}$ is a weak equivalence for all i ,
- The map $A_{i,j} \rightarrow A_{i,k}$ is a weak cofibration for all $i \leq j \leq k$, and
- The diagram

$$\begin{array}{ccc} A_{i,j} & \longrightarrow & A_{i,k} \\ \downarrow & & \downarrow \\ A_{j,j} & \longrightarrow & A_{j,k} \end{array}$$

is a homotopy cocartesian square for all $i \leq j \leq k$.

We define a map $A \rightarrow B$ to be a weak equivalence when each $A_{i,j} \rightarrow B_{i,j}$ is a weak equivalence in \mathcal{C} , and to be a cofibration when each $A_{i,j} \rightarrow B_{i,j}$ is a cofibration in \mathcal{C} and each induced map $A_{i,k} \cup_{A_{i,j}} B_{i,j} \rightarrow B_{i,k}$ is a weak cofibration in \mathcal{C} .

Clearly S'_\bullet assembles into a simplicial category with the usual face and degeneracy functors. Furthermore, we have the following comparison result [1, 2.8,2.9].

Proposition 2.4. *Let \mathcal{C} be a Waldhausen category admitting FFWC.*

- (i) $S'_\bullet\mathcal{C}$ is a simplicial Waldhausen category admitting FFWC.
- (ii) The inclusion $S_\bullet \rightarrow S'_\bullet$ is a simplicial exact functor.
- (iii) For each n , the inclusion $\text{w}S_n \rightarrow \text{w}S'_n$ induces a weak equivalence on nerves.

The following proposition is now clear from the definition of weakly exact. Theorem 2.2 then follows from this proposition and the previous proposition by iterating the S'_\bullet construction.

Proposition 2.5. *Let \mathcal{C} and \mathcal{D} be Waldhausen categories admitting FFWC and let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a weakly exact functor. Then for each n , $S'_n F: S'_n \mathcal{C} \rightarrow S'_n \mathcal{D}$ is a weakly exact functor.*

Finally, we can use the S'_\bullet construction to express the full naturality of the weak equivalences in Theorem 1.2. Unfortunately, the hypothesis FFWC is not quite strong enough. We need a slight refinement of this hypothesis:

Definition 2.6. Let \mathcal{C} be a Waldhausen category. We say that \mathcal{C} has *functorial mapping cylinders for weak cofibrations (FMCWC)* when \mathcal{C} admits functorial factorization of weak cofibrations by a functor T together with a natural transformation $B \rightarrow Tf$ splitting the natural weak equivalence $Tf \rightarrow B$, for weak cofibrations $f: A \rightarrow B$.

$$\begin{array}{ccc} & B & \\ & \downarrow & \searrow \scriptstyle = \\ A & \xrightarrow{\quad} & Tf \xrightarrow{\simeq} B, \\ & \searrow \scriptstyle f & \nearrow \scriptstyle \simeq \end{array}$$

Functorial factorization of all maps implies functorial mapping cylinders: For a map $f: A \rightarrow B$, the factorization of the map from the coproduct, $A \vee B \rightarrow B$, provides the functorial mapping cylinder. Thus, functorial factorization of all maps is equivalent to the conjunction of functorial mapping cylinders for weak cofibrations and all maps being weak cofibrations.

For a Waldhausen category \mathcal{C} that has functorial mapping cylinders for weak cofibrations, and A, B objects in \mathcal{C} , we use $LC_{\text{co}}(A, B)$ to denote the components of the Dwyer-Kan hammock function complex $LC(A, B)$ that correspond to the weak cofibrations; precisely, $LC_{\text{co}}(A, B)$ consists of those components that contain as a vertex a zigzag where all the forward arrows are weak cofibrations. Likewise, we use $LC_{\text{w}}(A, B)$ to denote the components of the Dwyer-Kan hammock function complex $LC(A, B)$ that contain a zigzag where all the forward arrows are weak equivalences. Then LC_{co} and LC_{w} are simplicial subcategories of the Dwyer-Kan simplicial localization LC . We prove the following generalization of Theorems 1.2 and 1.3 in Section 7.

Theorem 2.7. *Let \mathcal{C} be a saturated Waldhausen category that has FMCWC.*

- (i) *For $n > 1$, the nerve of $\text{w}S'_n$ is weakly equivalent to the homotopy coend*

$$\text{hocoend}_{(X_1, \dots, X_n) \in \text{w}\mathcal{C}^n} LC_{\text{co}}(X_{n-1}, X_n) \times \cdots \times LC_{\text{co}}(X_1, X_2),$$

naturally in weakly exact functors.

- (ii) *The nerve of $\text{w}\mathcal{C}$ is weakly equivalent to the disjoint union of $B \text{hAut } X$ over the weak equivalence classes of objects of \mathcal{C} .*
 (iii) *For $n \geq 1$, the nerve of $\text{w}S'_n$ is weakly equivalent to the total space of a fibration where the base is the disjoint union of*

$$B \text{hAut } X_n \times \cdots \times B \text{hAut } X_1$$

over n -tuples of weak equivalence classes of objects of \mathcal{C} , and the fiber is equivalent to

$$LC_{\text{co}}(X_{n-1}, X_n) \times \cdots \times LC_{\text{co}}(X_1, X_2).$$

for $n > 1$ and contractible for $n = 1$.

3. OUTLINE OF THE PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1 from Theorem 2.7 above, Theorem 3.5 below, and Propositions 3.8 and 3.9 below, all of which are proved in later sections. Throughout this section (and this section only), we fix \mathcal{C} , \mathcal{D} , and $F: \mathcal{C} \rightarrow \mathcal{D}$, satisfying the hypotheses of Theorem 1.1. Moreover, we fix factorization functors T on \mathcal{C} and \mathcal{D} ; we use the following terminology and notation.

Definition 3.1. For X in \mathcal{C} (resp. \mathcal{D}), the *cone on X* , CX , is $T(X \rightarrow *)$ and the *suspension of X* , ΣX , is CX/X . Let EX denote the cofiber sequence

$$X \rightarrow CX \rightarrow \Sigma X$$

viewed as an object of $S_2\mathcal{C}$ (resp. $S_2\mathcal{D}$), with $A_{0,1} = X$, $A_{0,2} = CX$, and $A_{1,2} = \Sigma X$.

It follows from the functoriality of T that CX , ΣX , and EX assemble to functors in X . A straightforward application of factorization and [1, 2.5] (or the gluing axiom) shows that these functors preserve weak equivalences and homotopy co-cartesian squares. This gives the following proposition; the corresponding result holds for \mathcal{D} .

Proposition 3.2. *The functors C and Σ are weakly exact functors $\mathcal{C} \rightarrow \mathcal{C}$, and E is a weakly exact functor $\mathcal{C} \rightarrow S_2\mathcal{C}$.*

The factorization functor for \mathcal{C} induces a factorization functor on $S_2\mathcal{C}$, and so E induces a map of K -theory spectra $K\mathcal{C} \rightarrow KS_2\mathcal{C}$. Applying the Additivity Theorem [18, 1.4.2, 1.3.2.(3)], we see that on K -theory, the sum in the stable category of the maps induced by the identity and suspension is the map induced by the cone. Since the cone induces the trivial map, it follows that Σ induces on $K\mathcal{C}$ the map -1 in the stable category. In particular, we obtain the following corollary; the corresponding result holds for \mathcal{D} .

Corollary 3.3. *Σ induces a weak equivalence on K -theory spectra $K\mathcal{C} \rightarrow K\mathcal{C}$.*

Although we do not assume any relationship between the factorization functors on \mathcal{C} and \mathcal{D} , nevertheless, we can relate the suspensions.

Proposition 3.4. *There is a functor $\Xi: \mathcal{C} \rightarrow \mathcal{D}$ and natural weak equivalences*

$$F\Sigma X \xleftarrow{\simeq} \Xi X \xrightarrow{\simeq} \Sigma FX.$$

For example, ΞX can be defined as the pushout

$$\begin{array}{ccc} FX \rightarrow T(F(X \rightarrow CX)) & & \\ \downarrow & & \downarrow \\ F* & \longrightarrow & \Xi X. \end{array}$$

The factorization weak equivalence $T(F(X \rightarrow CX)) \rightarrow FCX$ and the universal property of the pushout induces the map $\Xi X \rightarrow F\Sigma X$, which is a weak equivalence by [1, 2.5]. Functoriality of T in $\text{Ar } \mathcal{D}$ then gives a map under FX ,

$$T(F(X \rightarrow CX)) \longrightarrow T(FX \rightarrow *) = CFX,$$

which is a weak equivalence since the initial map to each is a weak equivalence. This map and the final map $F* \rightarrow *$ induce the map $\Xi X \rightarrow \Sigma FX$, which is a weak equivalence by the gluing axiom.

To take advantage of the suspension functor, we need to relate it to the Dwyer-Kan function complexes. For this we use the following application of Theorem 6.2 from Section 6. Again, the corresponding theorem also holds for \mathcal{D} .

Theorem 3.5. *If the diagram on the left below is homotopy cocartesian in \mathcal{C} ,*

$$\begin{array}{ccc} A \longrightarrow B & & LC(D, Y) \longrightarrow LC(C, Y) \\ \downarrow & & \downarrow \\ C \longrightarrow D & & LC(B, Y) \longrightarrow LC(A, Y) \end{array}$$

then for any object Y in \mathcal{C} , the diagram on the right is homotopy cartesian in the category of simplicial sets.

Applying this theorem to the homotopy cocartesian square defining the suspension, we obtain the following corollary.

Corollary 3.6. *For any X, Y in \mathcal{C} , $LC(\Sigma X, Y)$ is weakly equivalent to the based loop space of $LC(X, Y)$, based at the trivial map $X \rightarrow Y$.*

Applying F to the homotopy cocartesian square defining the suspension, we see that likewise $LD(F\Sigma X, FY)$ is weakly equivalent to the based loop space of $LD(FX, FY)$, based at F of the trivial map $X \rightarrow Y$, or equivalently, based at the trivial map $FX \rightarrow FY$ since it is in the same component.

Iterating the suspension in the previous proposition, we see that $\pi_n LC(X, Y)$ based at the trivial map is

$$\pi_0 LC(\Sigma^n X, Y) = \text{Ho } \mathcal{C}(\Sigma^n X, Y).$$

Since F induces an equivalence $\text{Ho } \mathcal{C} \rightarrow \text{Ho } \mathcal{D}$, we obtain the following corollary.

Corollary 3.7. *For any X, Y in \mathcal{C} , the restriction of*

$$LF: LC(X, Y) \longrightarrow LD(FX, FY)$$

to the component of the trivial map is weak equivalence; the map $LF: LC(\Sigma X, Y) \rightarrow LD(F\Sigma X, FY)$ is a weak equivalence.

To apply the previous corollary, we also need to know that the components of $LC(\Sigma X, \Sigma X)$ in $\text{hAut } \Sigma X$ correspond to the same components of $LD(F\Sigma X, F\Sigma X)$ that are in $\text{hAut } F\Sigma X$. Since these components consist of exactly the components representing the image of $\text{Ho } \text{w}\mathcal{C}(\Sigma X, \Sigma X)$ and $\text{Ho } \text{w}\mathcal{D}(F\Sigma X, F\Sigma X)$, respectively, this is clear when hypothesis (iii) in the statement of Theorem 1.1 holds. The other two cases are handled by the following propositions. The first is a special case of Theorem 6.4.

Proposition 3.8. *If hypothesis (i) in the statement of Theorem 1.1 holds, then a map in \mathcal{C} is a weak equivalence if and only if it represents an isomorphism in $\text{Ho } \mathcal{C}$, and likewise for \mathcal{D} .*

The second proposition is proved in Section 5 as Corollary 5.8.

Proposition 3.9. *If hypothesis (ii) in the statement of Theorem 1.1 holds, then so does hypothesis (iii).*

We have now assembled all we need to prove Theorem 1.1. By Proposition 2.4, we can use $N(\text{w}S'_\bullet \mathcal{C})$, the diagonal of the nerve of the simplicial category $\text{w}S'_\bullet \mathcal{C}$, as

a model for the (delooped) K -theory space of \mathcal{C} . Since suspension induces a weak equivalence on K -theory, the telescope under suspension

$$\mathrm{Tel}_\Sigma N(\mathrm{w}S'_\bullet \mathcal{C}) = \mathrm{Tel}(N(\mathrm{w}S'_\bullet \mathcal{C}) \xrightarrow{\Sigma} N(\mathrm{w}S'_\bullet \mathcal{C}) \xrightarrow{\Sigma} N(\mathrm{w}S'_\bullet \mathcal{C}) \xrightarrow{\Sigma} \cdots)$$

is equivalent to $N(\mathrm{w}S'_\bullet \mathcal{C})$ via the inclusion. The same observations apply to \mathcal{D} , and Proposition 3.4 provides homotopies to construct a map of telescopes

$$(3.10) \quad \mathrm{Tel}_\Sigma N(\mathrm{w}S'_n \mathcal{C}) \longrightarrow \mathrm{Tel}_\Sigma N(\mathrm{w}S'_n \mathcal{D})$$

for all n . Now we use the models from Theorem 2.7. We write

$$LC(X_1, \dots, X_n) = LC(X_{n-1}, X_n) \times \cdots \times LC(X_1, X_2)$$

and similarly for \mathcal{D} to save space. Then by Theorem 2.7.(i), the square in the homotopy category formed by the S'_n constructions

$$\begin{array}{ccc} N(\mathrm{w}S'_n \mathcal{C}) & \xrightarrow{F_*} & N(\mathrm{w}S'_n \mathcal{D}) \\ \Sigma_* \downarrow & & \downarrow \Sigma_* \\ N(\mathrm{w}S'_n \mathcal{C}) & \xrightarrow{F_*} & N(\mathrm{w}S'_n \mathcal{D}) \end{array}$$

is isomorphic in the homotopy category to the square in the homotopy category formed by the homotopy coends

$$\begin{array}{ccc} \mathrm{hocoend} LC(X_1, \dots, X_n) & \xrightarrow{LF_*} & \mathrm{hocoend} LD(Y_1, \dots, Y_n) \\ L\Sigma_* \downarrow & & \downarrow L\Sigma_* \\ \mathrm{hocoend} LC(X_1, \dots, X_n) & \xrightarrow{LF_*} & \mathrm{hocoend} LD(Y_1, \dots, Y_n), \end{array}$$

where the homotopy coends are over $(X_1, \dots, X_n) \in \mathrm{w}\mathcal{C}^n$ and $(Y_1, \dots, Y_n) \in \mathrm{w}\mathcal{D}^n$. Writing \mathcal{C}_Σ and \mathcal{D}_Σ for the full subcategories of \mathcal{C} and \mathcal{D} consisting of objects that are weakly equivalent to suspensions, it is clear that the vertical map of arrows factors through the arrow

$$\mathrm{hocoend}_{(X_1, \dots, X_n) \in \mathrm{w}\mathcal{C}_\Sigma^n} LC(X_1, \dots, X_n) \xrightarrow{LF_*} \mathrm{hocoend}_{(Y_1, \dots, Y_n) \in \mathrm{w}\mathcal{D}_\Sigma^n} LD(Y_1, \dots, Y_n).$$

Corollary 3.7 and Theorem 2.7 imply this latter map is a weak equivalence, and we conclude that the map of telescopes (3.10) is a weak equivalence. This completes the proof of Theorem 1.1.

4. UNIVERSAL SIMPLICIAL QUASIFIBRATIONS

In this section, we introduce the first of two techniques which provide the foundation for our subsequent work. The theorems of this paper depend on machinery for solving two related problems: The identification of certain squares of simplicial sets as homotopy cartesian squares and the identification of the homotopy fiber of certain maps of simplicial sets. Quillen's Theorem B [14] and its simplicial variant [18, 1.4.B] provide a flexible tool for these purposes. We rely on a particular formulation of Theorem B in terms of a notion of "universal simplicial quasifibration". Our exposition and viewpoint on the subject is heavily influenced by postings of Tom Goodwillie on Don Davis' algebraic topology mailing list [6].

Recall that a map $X \rightarrow Y$ of spaces is a quasifibration when for every point x of X , the map from the fiber to the homotopy fiber is a weak equivalence. We say that a map of simplicial sets $X_\bullet \rightarrow Y_\bullet$ is a quasifibration when its geometric realization is a quasifibration of spaces.

Definition 4.1. A map of simplicial sets $X_\bullet \rightarrow Y_\bullet$ is a *universal simplicial quasifibration* when for every map of simplicial sets $Z_\bullet \rightarrow Y_\bullet$, the induced map of the pullback $X_\bullet \times_{Y_\bullet} Z_\bullet \rightarrow Z_\bullet$ is a quasifibration.

The definition specifies a class of maps for which it is easy to identify pullbacks as homotopy pullbacks. The following proposition implies that to verify that a map is a universal simplicial quasifibration, it suffices to check the condition on the simplexes of X_\bullet ; for these, checking that the pullback map to the simplex is a quasifibration then amounts to checking that the fiber over a vertex includes as a weak equivalence. The proof in one direction is the restriction of the universal simplicial quasifibration property to the standard simplices; the proof in the other direction Waldhausen's version of Theorem B [18, 1.4.B].

Proposition 4.2. *A map of simplicial sets $X_\bullet \rightarrow Y_\bullet$ is a universal simplicial quasifibration if and only if for every n and every map $\Delta[n] \rightarrow Y_\bullet$, the induced map of the pullback $X_\bullet \times_{Y_\bullet} \Delta[n] \rightarrow \Delta[n]$ is a quasifibration.*

In general the simplicial sets we use arise as homotopy colimits. Thus, it is convenient to state the following proposition. For a small category \mathcal{C} , we write NC to denote the simplicial nerve.

Proposition 4.3. *Let \mathcal{C} be a small category and F a functor from \mathcal{C} to simplicial sets. Suppose that for every map $f: x \rightarrow y$ in \mathcal{C} , the induced map $F(f): F(x) \rightarrow F(y)$ is a weak equivalence. Then the map $\text{hocolim}_{\mathcal{C}} F \rightarrow NC$ is a universal simplicial quasifibration.*

Proof. We can identify an n -simplex of NC as a functor $\sigma: \Delta[n] \rightarrow \mathcal{C}$, where $\Delta[n]$ is the poset of $0, \dots, n$ under \leq . The pullback of $\text{hocolim}_{\mathcal{C}} F$ over this simplex is $\text{hocolim}_{\Delta[n]} F \circ \sigma$. For any vertex i , the inclusion of the fiber, $F(\sigma(i))$, in $\text{hocolim}_{\Delta[n]} F \circ \sigma$ is a homotopy equivalence. \square

The same proof also gives the following proposition, which we apply directly in the proof of Theorem 2.7.

Proposition 4.4. *Let \mathcal{C} be a small category and F a functor from $\mathcal{C}^{\text{op}} \times \mathcal{C}$ to simplicial sets. Suppose that for every z in \mathcal{C} and every map $f: x \rightarrow y$ in \mathcal{C} , the induced maps $F(f, z): F(y, z) \rightarrow F(x, z)$ and $F(z, f): F(z, x) \rightarrow F(z, y)$ are weak equivalences. Then the map $\text{hocoend}_{\mathcal{C}} F \rightarrow NC$ is a universal simplicial quasifibration.*

We also repeatedly use the following refinement of Quillen's Theorem B [14]. Recall that for a functor $\phi: \mathcal{D} \rightarrow \mathcal{C}$ and a fixed object Z of \mathcal{C} , the comma category $Z \downarrow \phi$ has as objects the pairs $(Y, Z \rightarrow \phi Y)$ consisting of an object Y of \mathcal{D} and a map in \mathcal{C} from Z to ϕY . A morphism

$$(Y, Z \rightarrow \phi Y) \longrightarrow (Y', Z \rightarrow \phi Y')$$

consists of a map $Y \rightarrow Y'$ in \mathcal{D} such that the diagram

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ \phi Y & \longrightarrow & \phi Y' \end{array}$$

in \mathcal{D} commutes. The following theorem gives a useful sufficient condition for a commuting square of functors to induce a homotopy cartesian square of nerves.

Theorem 4.5. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be small categories, and let*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\delta} & \mathcal{C} \\ \downarrow & & \downarrow \gamma \\ \mathcal{B} & \xrightarrow{\beta} & \mathcal{A} \end{array}$$

be a (strictly) commuting diagram of functors. If the following two conditions hold, then the induced square of nerves is homotopy cartesian:

- (i) For every map $X \rightarrow X'$ in \mathcal{A} , the induced functor on comma categories $X' \downarrow \beta \rightarrow X \downarrow \beta$ induces a weak equivalence of nerves, and
- (ii) For every object Z in \mathcal{C} , the functor $Z \downarrow \delta \rightarrow \gamma Z \downarrow \beta$ induces a weak equivalence of nerves.

Proof. As in Quillen's proof of Theorem B, we have natural weak equivalences

$$\mathrm{hocolim}_{X \in \mathcal{A}} N(X \downarrow \beta) \longrightarrow N\mathcal{B} \quad \text{and} \quad \mathrm{hocolim}_{Z \in \mathcal{C}} N(Z \downarrow \delta) \longrightarrow N\mathcal{D}.$$

These then fit into the commutative diagram on the left

$$\begin{array}{ccc} \mathrm{hocolim}_{Z \in \mathcal{C}} N(Z \downarrow \delta) & \longrightarrow & N\mathcal{C} \\ \downarrow & & \downarrow \\ \mathrm{hocolim}_{X \in \mathcal{A}} N(X \downarrow \beta) & \longrightarrow & N\mathcal{A} \end{array} \quad \begin{array}{ccc} \mathrm{hocolim}_{Z \in \mathcal{C}} N(\gamma Z \downarrow \beta) & \longrightarrow & N\mathcal{C} \\ \downarrow & & \downarrow \\ \mathrm{hocolim}_{X \in \mathcal{A}} N(X \downarrow \beta) & \longrightarrow & N\mathcal{A} \end{array}$$

weakly equivalent to the diagram of nerves in question. By (ii), we have that the canonical map

$$\mathrm{hocolim}_{Z \in \mathcal{C}} N(Z \downarrow \delta) \longrightarrow \mathrm{hocolim}_{Z \in \mathcal{C}} N(\gamma Z \downarrow \beta)$$

is a weak equivalence, and it follows that the square on the right above is weakly equivalent to the square on the left. The square on the right is a pullback square of simplicial sets, and by Proposition 4.3, (i) implies that the bottom horizontal map is a universal simplicial quasifibration. We conclude that the square is homotopy cartesian. \square

5. HOMOTOPY CALCULI OF FRACTIONS AND MAPPING CYLINDERS

In this section, we describe the second technical device essential to the proof of the main theorems, the homotopy calculi of fraction introduced in [3]. When a category with weak equivalences admits a homotopy calculus of fractions, the Dwyer-Kan function complexes $LC(A, B)$ admit significantly smaller models that are nerves of categories of words of a specified type. We begin the section with a concise review of this theory. We then prove that a Waldhausen category with FMCWC admits a homotopy calculus of left fractions.

We begin with the notation for the categories of words of specified types. Let \mathcal{C} be a category with a subcategory $w\mathcal{C}$ of weak equivalences. We consider the words on letters \mathbf{C}, \mathbf{W} , and \mathbf{W}^{-1} : To every such word \mathbf{Y} and pair of objects A, B in \mathcal{C} , we associate a category $\mathbf{Y}(A, B)$, where the objects are roughly speaking words in \mathcal{C} of the type specified by the letters in \mathbf{Y} and the morphisms are the weak equivalences. The precise definition is as follows.

Definition 5.1. Let Υ be a word of length n on letters \mathbf{C} , \mathbf{W} , and \mathbf{W}^{-1} , and let A, B be objects of \mathcal{C} . We define the $\Upsilon(A, B)$ to be the following category. An object in $\Upsilon(A, B)$ consists of:

- A collection of objects X_1, \dots, X_{n-1} of \mathcal{C} ,
- A map $f_i: X_{i-1} \rightarrow X_i$ in \mathcal{C} whenever the i -th letter of Υ is \mathbf{C} for some $1 \leq i \leq n$,
- A map $f_i: X_{i-1} \rightarrow X_i$ in $w\mathcal{C}$ whenever the i -th letter of Υ is \mathbf{W} for some $1 \leq i \leq n$, and
- A map $f_i: X_i \rightarrow X_{i-1}$ in $w\mathcal{C}$ whenever the i -th letter of Υ is \mathbf{W}^{-1} for some $1 \leq i \leq n$,

where we interpret X_n as A and X_0 as B in the conditions above. A morphism in $\Upsilon(A, B)$ from $\{X_i, f_i\}$ to $\{X'_i, f'_i\}$ is a collection of maps $g_i: X_i \rightarrow X'_i$ in $w\mathcal{C}$ that are the identity on A and B and make the evident diagram commute.

$$\begin{array}{ccccccc}
 & & X_{n-1} & \xrightarrow{f_{n-1}} & \cdots & \xrightarrow{f_2} & X_1 \\
 & \nearrow f_n & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & \searrow f_1 \\
 A & & & & & & & B \\
 & \searrow f'_n & \downarrow g_{n-1} & & \downarrow g_1 & & \downarrow g_1 & \nearrow f'_1 \\
 & & X'_{n-1} & \xrightarrow{f'_{n-1}} & \cdots & \xrightarrow{f'_2} & X'_1
 \end{array}$$

The numbering, which may seem unusual in the diagrams, is forced by the convention that the letters in the word follow composition order, where we think of the object $\{X_i, f_i\}$ as representing a formal composition $f_1 \circ \cdots \circ f_n$ with the i -th map corresponding to the i -th letter (with f_i^{-1} in place of f_i in the formal composition when the i -th letter is \mathbf{W}^{-1}). Our words indicate the same categories and diagrams as those in [3], which are numbered slightly differently.

In our work below, the three most important words are $\mathbf{W}^{-1}\mathbf{C}$, $\mathbf{W}^{-1}\mathbf{W}$, and $\mathbf{W}^{-1}\mathbf{CW}^{-1}$. For convenience, we spell out these categories explicitly.

Example 5.2. The categories of words $\mathbf{W}^{-1}\mathbf{C}$, $\mathbf{W}^{-1}\mathbf{W}$, and $\mathbf{W}^{-1}\mathbf{CW}^{-1}$.

- (i) $\mathbf{W}^{-1}\mathbf{C}$: An object $\{X, f_1, f_2\}$ is pictured on the left and a map from $\{X, f_1, f_2\}$ to $\{X', f'_1, f'_2\}$ is pictured on the right.

$$\begin{array}{ccc}
 A & \xrightarrow{f_2} & X \xleftarrow{f_1} B \\
 & & \simeq \\
 & & X'
 \end{array}
 \qquad
 \begin{array}{ccc}
 & & X \\
 & \nearrow f_2 & \downarrow \simeq & \searrow f_1 \\
 A & & & B \\
 & \searrow f'_2 & \downarrow \simeq & \nearrow f'_1 \\
 & & X' &
 \end{array}$$

- (ii) $\mathbf{W}^{-1}\mathbf{W}$: Objects and maps look the same, but the map $A \rightarrow X$ is required to be in $w\mathcal{C}$.
- (iii) $\mathbf{W}^{-1}\mathbf{CW}^{-1}$: An object $\{X_1, X_2, f_1, f_2, f_3\}$ is pictured on the left and a map from $\{X_1, X_2, f_1, f_2, f_3\}$ to $\{X'_1, X'_2, f'_1, f'_2, f'_3\}$ is pictured on the

right.

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & X_2 & \xrightarrow{f_2} & X_1 \\
 & f_3 \swarrow & \downarrow & \downarrow & \swarrow f_1 \\
 A & \xleftarrow[\simeq]{f_3} & X_2 & \xrightarrow{f_2} & X_1 & \xleftarrow[\simeq]{f_1} & B \\
 & \nwarrow f'_3 & \downarrow & \downarrow & \nwarrow f'_1 \\
 & & X'_2 & \xrightarrow{f'_2} & X'_1
 \end{array} \\
 \end{array}$$

As described in [3, 5.5], the function complex $LC(A, B)$ is a colimit of the nerves of these categories. The hypothesis of a homotopy calculi of fractions is a homotopical requirement on how these nerves fit together. The following definition is [3, 6.1]. Although we use only homotopy calculus of left fractions and homotopy calculus of two-sided fractions in our work below, we include the definition of homotopy calculus of right fractions for completeness.

Definition 5.3 (Homotopy calculi of fractions). Let \mathcal{C} be a category with a subcategory of weak equivalences $w\mathcal{C}$.

- (i) \mathcal{C} admits a *homotopy calculus of two-sided fractions (HC2F)* means that for every pair of integers $i, j \geq 0$, the functors given by inserting an identity morphism in the $(i+1)$ -st spot,

$$\begin{aligned}
 \mathbf{W}^{-1}\mathbf{C}^{i+j}\mathbf{W}^{-1} &\longrightarrow \mathbf{W}^{-1}\mathbf{C}^i\mathbf{W}^{-1}\mathbf{C}^j\mathbf{W}^{-1} \\
 \mathbf{W}^{-1}\mathbf{W}^{i+j}\mathbf{W}^{-1} &\longrightarrow \mathbf{W}^{-1}\mathbf{W}^i\mathbf{W}^{-1}\mathbf{W}^j\mathbf{W}^{-1},
 \end{aligned}$$

induce weak equivalences on nerves for every pair of objects of \mathcal{C} .

- (ii) \mathcal{C} admits a *homotopy calculus of left fractions (HCLF)* means that for every pair of integers $i, j \geq 0$, the functors given by inserting an identity morphism in the $(i+1)$ -st spot,

$$\begin{aligned}
 \mathbf{W}^{-1}\mathbf{C}^{i+j} &\longrightarrow \mathbf{W}^{-1}\mathbf{C}^i\mathbf{W}^{-1}\mathbf{C}^j \\
 \mathbf{W}^{-1}\mathbf{W}^{i+j} &\longrightarrow \mathbf{W}^{-1}\mathbf{W}^i\mathbf{W}^{-1}\mathbf{W}^j,
 \end{aligned}$$

induce weak equivalences on nerves for every pair of objects of \mathcal{C} .

- (iii) \mathcal{C} admits a *homotopy calculus of right fractions (HCRF)* means that for every pair of integers $i, j \geq 0$, the functors given by inserting an identity morphism in the i -th spot,

$$\begin{aligned}
 \mathbf{C}^{i+j}\mathbf{W}^{-1} &\longrightarrow \mathbf{C}^i\mathbf{W}^{-1}\mathbf{C}^j\mathbf{W}^{-1} \\
 \mathbf{W}^{i+j}\mathbf{W}^{-1} &\longrightarrow \mathbf{W}^i\mathbf{W}^{-1}\mathbf{W}^j\mathbf{W}^{-1},
 \end{aligned}$$

induce weak equivalences on nerves for every pair of objects of \mathcal{C} .

Dwyer and Kan observe [3, 6.1, §9] that if \mathcal{C} has a homotopy calculus of left or right fractions, then \mathcal{C} has a homotopy calculus of two-sided fractions. The following proposition [3, 6.2] explains the utility and terminology of the definition.

Proposition 5.4. *Let \mathcal{C} be a category with a subcategory of weak equivalences $w\mathcal{C}$.*

- (i) *If \mathcal{C} admits a homotopy calculus of two-sided fractions, then the maps*

$$\begin{aligned}
 N\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(A, B) &\longrightarrow LC(A, B) \\
 N\mathbf{W}^{-1}\mathbf{W}\mathbf{W}^{-1}(A, B) &\longrightarrow L(w\mathcal{C})(A, B)
 \end{aligned}$$

are weak equivalences.

(ii) If \mathcal{C} admits a homotopy calculus of left fractions, then the maps

$$\begin{aligned} \mathbf{N}\mathbf{W}^{-1}\mathbf{C}(A, B) &\longrightarrow LC(A, B) \\ \mathbf{N}\mathbf{W}^{-1}\mathbf{W}(A, B) &\longrightarrow L(\mathbf{w}\mathcal{C})(A, B) \end{aligned}$$

are weak equivalences.

(iii) If \mathcal{C} admits a homotopy calculus of right fractions, then the maps

$$\begin{aligned} \mathbf{N}\mathbf{C}\mathbf{W}^{-1}(A, B) &\longrightarrow LC(A, B) \\ \mathbf{N}\mathbf{W}\mathbf{W}^{-1}(A, B) &\longrightarrow L(\mathbf{w}\mathcal{C})(A, B) \end{aligned}$$

are weak equivalences.

The following is the main theorem of this section. In it, $\tilde{\mathbf{co}}\mathcal{C}$ denotes the category of weak cofibrations (q.v. [1, 2.4]), which by definition contains the weak equivalences $\mathbf{w}\mathcal{C}$ as a subcategory.

Theorem 5.5. *Let \mathcal{C} be a Waldhausen category with FMCWC. Then \mathcal{C} , $\tilde{\mathbf{co}}\mathcal{C}$, and $\mathbf{w}\mathcal{C}$ admit homotopy calculi of left fractions.*

Under the hypothesis of FMCWC, the previous theorem in particular allows us to model the function complex $LC(A, B)$ in terms of the categories of words $\mathbf{W}^{-1}\mathbf{C}(A, B)$ and $\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(A, B)$. The following theorem identifies the subsets $LC_{\mathbf{co}}(A, B)$ and $LC_{\mathbf{w}}(A, B)$ in these terms. In it, for $\mathbf{Y} = \mathbf{W}^{-1}\mathbf{C}$ and $\mathbf{Y} = \mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}$, we write $\mathbf{Y}(A, B)_{\tilde{\mathbf{co}}}$ and $\mathbf{Y}(A, B)_{\mathbf{w}}$ for the full subcategory of $\mathbf{Y}(A, B)$ of diagrams whose forward map is a weak cofibration and weak equivalence, respectively.

Theorem 5.6. *Let \mathcal{C} be a Waldhausen category with FMCWC. Then the maps*

$$\begin{aligned} \mathbf{W}^{-1}\mathbf{C}(A, B)_{\tilde{\mathbf{co}}} &\longrightarrow LC_{\mathbf{co}}(A, B), & \mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(A, B)_{\tilde{\mathbf{co}}} &\longrightarrow LC_{\mathbf{co}}(A, B), \\ \mathbf{W}^{-1}\mathbf{C}(A, B)_{\mathbf{w}} &\longrightarrow LC_{\mathbf{w}}(A, B), & \text{and} & \mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(A, B)_{\mathbf{w}} &\longrightarrow LC_{\mathbf{w}}(A, B) \end{aligned}$$

are weak equivalences

Proof. We do the case for the weak cofibrations; the case for the weak equivalences is identical. Since $\mathbf{W}^{-1}\mathbf{C}(A, B)_{\tilde{\mathbf{co}}}$ and $\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(A, B)_{\tilde{\mathbf{co}}}$ are collections of components of the categories $\mathbf{W}^{-1}\mathbf{C}(A, B)$ and $\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(A, B)$, and $LC_{\mathbf{co}}(A, B)$ is a collection of components of the simplicial set $LC(A, B)$, we just need to check that these components coincide. For this, it suffices to show that every component of $LC_{\mathbf{co}}(A, B)$ contains as a vertex the image of an object of $\mathbf{W}^{-1}\mathbf{C}(A, B)_{\tilde{\mathbf{co}}}$. The key observation is that the category $\mathbf{W}^{-1}\mathbf{C}(A, B)_{\tilde{\mathbf{co}}}$ is precisely the category of words $\mathbf{W}^{-1}\mathbf{C}(A, B)$ in the category $\tilde{\mathbf{co}}\mathcal{C}$. By definition, a component of $LC_{\mathbf{co}}(A, B)$ contains a vertex represented by a zigzag in which all the forward arrows are weak cofibrations. Interpreting this zigzag as a vertex in $L(\tilde{\mathbf{co}}\mathcal{C})(A, B)$, we get a path in $L(\tilde{\mathbf{co}}\mathcal{C})(A, B)$ to a vertex in the image of $\mathbf{W}^{-1}\mathbf{C}(A, B)_{\tilde{\mathbf{co}}}$ since $\tilde{\mathbf{co}}\mathcal{C}$ admits a homotopy calculus of left fractions. The inclusion of $\tilde{\mathbf{co}}\mathcal{C}$ induces a map $L(\tilde{\mathbf{co}}\mathcal{C})(A, B) \rightarrow LC_{\mathbf{co}}(A, B)$ that gives us a path in $LC_{\mathbf{co}}(A, B)$ to a vertex in the image of $\mathbf{W}^{-1}\mathbf{C}(A, B)_{\tilde{\mathbf{co}}}$. \square

Using the previous theorem and the observation in its proof that $\mathbf{W}^{-1}\mathbf{C}(A, B)_{\tilde{\mathbf{co}}}$ and $\mathbf{W}^{-1}\mathbf{C}(A, B)$ coincide with the categories of words $\mathbf{W}^{-1}\mathbf{C}(A, B)_{\mathbf{w}}$ for $\tilde{\mathbf{co}}\mathcal{C}$ and $\mathbf{w}\mathcal{C}$ (respectively), we obtain the following corollary.

Theorem 6.1. *Let \mathcal{C} be a Waldhausen category that has FMCWC and whose weak equivalences are closed under retracts. Let $A \rightarrow B$ be a weak cofibration and $A \rightarrow C$, $B \rightarrow D$, and $C \rightarrow D$ maps that make the square on the left commute.*

$$\begin{array}{ccc} A \xrightarrow{wc} B & & LC(D, E) \rightarrow LC(C, E) \\ \downarrow & & \downarrow \\ C \rightarrow D & & LC(B, E) \rightarrow LC(A, E) \end{array}$$

Then the square on the left is homotopy cocartesian in \mathcal{C} if and only if for every E the square of simplicial sets on the right is homotopy cartesian.

In fact, the forward direction holds under significantly weaker hypotheses, and we state this as the following theorem, which implies Theorem 3.5 in Section 3. A similar result is the main theorem of [19].

Theorem 6.2. *Let \mathcal{C} be a Waldhausen category that admits a HCLF. For a cofibration $A \rightarrow B$, a map $A \rightarrow C$, $D = B \cup_A C$, and any object E , the following square is homotopy cartesian:*

$$\begin{array}{ccc} LC(D, E) & \rightarrow & LC(C, E) \\ \downarrow & & \downarrow \\ LC(B, E) & \rightarrow & LC(A, E) \end{array}$$

The previous theorems make it easier to check in certain cases that functors are weakly exact. Applying Theorem 6.2 in \mathcal{C} and Theorem 6.1 in \mathcal{D} gives the following corollary.

Corollary 6.3. *Let \mathcal{C} be a Waldhausen category that admits a HCLF, and let \mathcal{D} be a Waldhausen category that has FMCWC and whose weak equivalences are closed under retracts. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor that preserves weak equivalences and weak cofibrations. If F is a DK-equivalence, then F preserves homotopy cocartesian squares and so is weakly exact.*

The hypothesis that weak equivalences are closed under retracts is familiar from the theory of model categories. Two other properties of weak equivalences in model categories are currently somewhat less well-known but explored in [5]. The more subtle of these is the “two out of six” property [5, 7.3], which we abbreviate to *DKHS-2/6*. The subcategory $w\mathcal{C}$ satisfies DKHS-2/6 when for any three composable maps

$$f: A \longrightarrow B, \quad g: B \longrightarrow C, \quad h: C \longrightarrow D,$$

if the composites $g \circ f$ and $h \circ g$ are in $w\mathcal{C}$, then so are the original maps f , g , and h . The proof of Theorem 6.1 depends more directly on the other property, which [5, 8.4] calls “saturated” and we call *DKHS-saturated* (to avoid confusion with Waldhausen’s terminology). By definition, the localization functor $\mathcal{C} \rightarrow \text{Ho}\mathcal{C}$ sends weak equivalences to isomorphisms. We say that the weak equivalences of \mathcal{C} are DKHS-saturated when a map is a weak equivalence if and only if its image in $\text{Ho}\mathcal{C}$ is an isomorphism. Note that when \mathcal{C} admits a homotopy calculus of two-sided fractions, the weak equivalences of \mathcal{C} are DKHS-saturated if and only if the subcategory of $\text{Ho}\mathcal{C}$ generated by the weak equivalences and their inverses consists of all the isomorphisms of $\text{Ho}\mathcal{C}$. We prove the following theorem at the end of the section.

Theorem 6.4. *Let \mathcal{C} be a Waldhausen category that has FMCWC. The following are equivalent:*

- (i) *The weak equivalences are closed under retracts.*
- (ii) *The weak equivalences satisfy the DKHS-2/6 property.*
- (iii) *The weak equivalences are DKHS-saturated.*

We can now prove Theorem 6.1, assuming Theorems 6.2 and 6.4.

Proof of Theorem 6.1 from Theorems 6.2 and 6.4. The “only if” direction follows from Theorem 6.2 and [1, 2.5]. For the “if” direction, by factoring the weak cofibration $A \rightarrow B$ as a cofibration followed by a weak equivalence, it suffices to consider the case when $A \rightarrow B$ is a cofibration. Consider the induced map $B \cup_A C \rightarrow D$; in the commutative diagram of simplicial sets

$$\begin{array}{ccccc} LC(D, E) & \rightarrow & LC(B \cup_A C, E) & \rightarrow & LC(C, E) \\ & \searrow & \downarrow & & \downarrow \\ & & LC(B, E) & \longrightarrow & LC(A, E) \end{array}$$

both the outer “square” and the inner square are homotopy cartesian. It follows that the map $LC(D, E) \rightarrow LC(B \cup_A C, E)$ is a weak equivalence for all objects E , and so in particular, $B \cup_A C \rightarrow D$ is an isomorphism in $\text{Ho}\mathcal{C}$. Because \mathcal{C} is DKHS-saturated by Theorem 6.4, $B \cup_A C \rightarrow D$ is a weak equivalence. \square

The proof of Theorem 6.2 is slightly more complicated. We apply Quillen Theorem B as formulated in Theorem 4.5 to the short hammock version of $LC(B, E)$, the nerve of the category $\mathbf{W}^{-1}\mathbf{C}(A, E)$. Recall that $\mathbf{W}^{-1}\mathbf{C}(A, E)$ is the category whose objects are the zigzags \overleftrightarrow{X}

$$A \rightarrow X \overleftarrow{\simeq} E$$

and whose maps are the maps $X \rightarrow X'$ under A and E , as in Example 5.2.(i). Composition with $f: A \rightarrow B$ induces a functor $f^*: \mathbf{W}^{-1}\mathbf{C}(B, E) \rightarrow \mathbf{W}^{-1}\mathbf{C}(A, E)$. Theorem 6.2 is an immediate consequence of Theorem 4.5 and the following lemma.

Lemma 6.5. *Assume \mathcal{C} admits a HCLF and $f: A \rightarrow B$ is a cofibration. For any map $\overleftrightarrow{X} \rightarrow \overleftrightarrow{X}'$ in $\mathbf{W}^{-1}\mathbf{C}(A, E)$ the induced map of comma categories from $\overleftrightarrow{X}' \downarrow f^*$ to $\overleftrightarrow{X} \downarrow f^*$ induces a weak equivalence of nerves.*

Proof. The argument is another application of Theorem 4.5. Let $\text{w}\mathcal{C}_E$ denote the full subcategory of $\text{w}\mathcal{C}$ consisting of those objects that are weakly equivalent to E . We have a functor $G_{\overleftrightarrow{X}}$ from $\overleftrightarrow{X} \downarrow f^*$ to $\text{w}\mathcal{C}_E$ that sends the object \overleftrightarrow{Y}

$$\begin{array}{ccc} A \rightarrow X \overleftarrow{\simeq} E & & \\ \downarrow & & \downarrow \simeq \\ B \rightarrow Y & & \end{array}$$

to Y . For a map $\overleftarrow{X} \rightarrow \overleftarrow{X}'$ in $\mathbf{W}^{-1}\mathbf{C}(A, E)$, consider the following strictly commuting diagram of functors:

$$(6.6) \quad \begin{array}{ccc} \overleftarrow{X}' \downarrow f^* & \xrightarrow{G_{\overleftarrow{X}'}} & \mathbf{wC}_E \\ \downarrow & & \downarrow = \\ \overleftarrow{X} \downarrow f^* & \xrightarrow{G_{\overleftarrow{X}}} & \mathbf{wC}_E. \end{array}$$

We verify that this diagram satisfies the hypotheses of Theorem 4.5. For any object H in \mathbf{wC}_E , the comma category $H \downarrow G_{\overleftarrow{X}}$ has as objects the diagrams of the form

$$\begin{array}{ccc} A \rightarrow X & \xleftarrow{\simeq} & E \\ \downarrow & & \downarrow \simeq \\ B \rightarrow Y & \xleftarrow{\simeq} & H. \end{array}$$

Using the universal property of the pushout, we see that this category is equivalent to the full subcategory of $\mathbf{W}^{-1}\mathbf{C}(B \cup_A X, H)$ of those zigzags

$$B \cup_A X \rightarrow Y \xleftarrow{\simeq} H$$

for which the composite map $E \rightarrow B \cup_A X \rightarrow Y$ is a weak equivalence. Thus, $H \downarrow G_{\overleftarrow{X}}$ is equivalent to a disjoint union of certain components of $\mathbf{W}^{-1}\mathbf{C}(B \cup_A X, H)$. Hypotheses (i) and (ii) of Theorem 4.5 follow from the fact that $N(\mathbf{W}^{-1}\mathbf{C}(B \cup_A X, H))$ preserves weak equivalences in H and X . We conclude that diagram 6.6 is a homotopy cartesian square. Since the vertical on the right is a weak equivalence, so is the vertical map on the left. \square

It remains to prove Theorem 6.4. Obviously (iii) implies both (i) and (ii); we show that (i) implies (ii) and (ii) implies (iii).

Proof that (i) implies (ii). Let $A \rightarrow B \rightarrow C \rightarrow D$ be a sequence of composable maps, with $A \rightarrow C$ and $B \rightarrow D$ weak equivalences. Since $A \rightarrow C$ is a weak equivalence, it is a weak cofibration, and so we can factor it as a cofibration $A \rightarrow C'$ followed by a split weak equivalence $C' \rightarrow C$. Let $B' = C' \cup_A B$.

$$\begin{array}{ccc} A \xrightarrow{\simeq} C & & A \xrightarrow{\simeq} C' \\ \simeq \downarrow & \nearrow \simeq & \downarrow \\ C' \longleftarrow C & & B \xrightarrow{\simeq} B' \end{array}$$

We have a composite map $f: C \rightarrow C' \rightarrow B'$, and the compatible maps $C' \rightarrow C$ and $B \rightarrow C$ induce a map $g: B' \rightarrow C$ such that $g \circ f$ is the identity on C . Since the composite map $B \rightarrow C \rightarrow D$ is a weak equivalence, the composite of g with $C \rightarrow D$ is a weak equivalence. We therefore obtain a commutative diagram

$$\begin{array}{ccccc} C & \xrightarrow{f} & B' & \xrightarrow{g} & C \\ \downarrow & & \downarrow \simeq & & \downarrow \\ D & \xrightarrow{=} & D & \xrightarrow{=} & D, \end{array}$$

where both horizontal composites are the identity and the middle vertical map is a weak equivalence. We conclude from (ii) that the map $C \rightarrow D$ is a weak equivalence, and it follows that $B \rightarrow C$ and $A \rightarrow B$ are weak equivalences. \square

Proof that (ii) implies (iii). (cf. [5, 36.4]) Let $a: A \rightarrow B$ be a map in \mathcal{C} that becomes an isomorphism in $\text{Ho}\mathcal{C}$. Since \mathcal{C} admits a HCLF, the inverse isomorphism from $\text{Ho}\mathcal{C}$ is represented by a zigzag (in \mathcal{C}) of the form

$$(6.7) \quad B \xrightarrow{b} C \xleftarrow[\simeq]{c} A$$

for some C . Moreover, applying factorization, we can assume without loss of generality that $c: A \rightarrow C$ is a cofibration as well as a weak equivalence. The composite zigzag

$$A \xrightarrow{b \circ a} C \xleftarrow[\simeq]{c} A$$

is in the component of the identity on A , and so $b \circ a$ is a weak equivalence. Let $B' = B \cup_A C$, and let $C' = C \cup_B B'$.

$$(6.8) \quad \begin{array}{ccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C \\ \downarrow c \simeq & & \downarrow \simeq & & \downarrow \simeq \\ B & \xrightarrow{b} & C & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

The composite $C \rightarrow C'$ is a weak equivalence because it is the pushout of the weak equivalence $b \circ a$ over the cofibration c . The zigzag

$$B \xrightarrow{b} C \longrightarrow B' \xleftarrow[\simeq]{c} B$$

is in the component representing the composite of the zigzag (6.7) with a , i.e., the component containing the identity of B . It follows that in the diagram (6.8), the horizontal composite map $B \rightarrow B'$ is a weak equivalence. Applying DKHS-2/6 to the bottom horizontal sequence of maps in (6.8), we conclude that b is a weak equivalence, and hence that a is a weak equivalence. \square

7. PROOF OF THEOREMS 1.2, 1.3, AND 2.7

We begin with the proof of part (i) of Theorem 2.7. The first reduction is to replace wS'_n with a simpler category. Let $F'_{n-1}\mathcal{C}$ denote the Waldhausen category whose objects are the sequences of $n-1$ composable weak cofibrations in \mathcal{C} . We have an exact forgetful functor $S'_n\mathcal{C} \rightarrow F'_{n-1}\mathcal{C}$ that sends an object $\{A_{i,j}\}$ of $S'_n\mathcal{C}$ to the sequence

$$A_{0,1} \longrightarrow A_{0,2} \longrightarrow \cdots \longrightarrow A_{0,n}.$$

The Waldhausen category $F'_{n-1}\mathcal{C}$ is the analogue for $S'_n\mathcal{C}$ of the Waldhausen category $F_{n-1}\mathcal{C}$, whose objects are the sequences of $n-1$ composable cofibrations in \mathcal{C} . The forgetful functor $S_n \rightarrow F_{n-1}\mathcal{C}$ is exact and an equivalence of categories, whose inverse equivalence is also exact. Proposition 2.4.(iii) and the analogous fact for F_{n-1} and F'_{n-1} then implies the following proposition.

Proposition 7.1. *If \mathcal{C} admits FFWC, the forgetful functor $wS'_n\mathcal{C} \rightarrow wF'_{n-1}\mathcal{C}$ induces a weak equivalence of nerves.*

We use proposition 7.1 to simplify one side of the equivalence in part (i) of Theorem 2.7, and we use homotopy calculus of left fractions to simplify the other side. Although the categories $\mathbf{W}^{-1}\mathbf{C}$ produce much more manageable simplicial sets than the hammock function complexes, they are contravariant in weak equivalences of each variable and so do not have the right functoriality to fit into a homotopy coend. The categories $\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}$ are covariant in weak equivalences of the source variable and contravariant in weak equivalences of the target variable, which is the opposite variance expected of a function complex. We do likewise have such an opposite variance on the hammock function complexes $LC(X, Y)$ since the category LC contains “backward” copies of the weak equivalences. The following lemma compares the homotopy coend in Theorem 2.7 with the homotopy coend for the opposite variance.

Lemma 7.2. *When \mathcal{C} has FMCWC,*

$$\text{hocoend}_{(X_1, \dots, X_n) \in \mathbf{wC}^n} LC_{\text{co}}(X_{n-1}, X_n) \times \cdots \times LC_{\text{co}}(X_1, X_2)$$

and

$$\text{hocoend}_{(X'_1, \dots, X'_n) \in (\mathbf{wC}^{\text{op}})^n} LC_{\text{co}}(X'_{n-1}, X'_n) \times \cdots \times LC_{\text{co}}(X'_1, X'_2)$$

are weakly equivalent.

Proof. Write B and B' for these homotopy coends, and let D be the homotopy coend

$$\begin{aligned} D = \text{hocoend}_{(X_1, X'_1, \dots, X_n, X'_n) \in (\mathbf{wC} \times \mathbf{wC}^{\text{op}})^n} & \\ \mathbf{W}^{-1}\mathbf{W}(X'_n, X_n) \times LC_{\text{co}}(X'_{n-1}, X_n) \times \mathbf{W}^{-1}\mathbf{W}(X_{n-1}, X'_{n-1}) \times \cdots & \\ \cdots \times \mathbf{W}^{-1}\mathbf{W}(X_2, X'_2) \times LC_{\text{co}}(X'_1, X_2) \times \mathbf{W}^{-1}\mathbf{W}(X_1, X'_1). & \end{aligned}$$

Composition then induces maps $D \rightarrow B$ and $D \rightarrow B'$. Let C be the homotopy colimit

$$\begin{aligned} C = \text{hocolim}_{(X_1, X'_1, \dots, X_n, X'_n) \in (\mathbf{wC} \times \mathbf{wC}^{\text{op}})^n} & \\ \mathbf{W}^{-1}\mathbf{W}(X'_n, X_n) \times \mathbf{W}^{-1}\mathbf{W}(X_{n-1}, X'_{n-1}) \times \cdots \times \mathbf{W}^{-1}\mathbf{W}(X_1, X'_1). & \end{aligned}$$

We have an evident map $D \rightarrow C$ obtained by dropping the LC_{co} factors, and we have maps $C \rightarrow N\mathbf{wC}^n$ and $C \rightarrow N(\mathbf{wC}^{\text{op}})^n$ obtained from the canonical map $C \rightarrow N((\mathbf{wC} \times \mathbf{wC}^{\text{op}})^n)$ by dropping the X'_i or the X_i respectively. We then have the following commuting diagrams

$$\begin{array}{ccc} D & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & N\mathbf{wC}^n \end{array} \qquad \begin{array}{ccc} D & \longrightarrow & C \\ \downarrow & & \downarrow \\ B' & \longrightarrow & N(\mathbf{wC}^{\text{op}})^n \end{array}$$

that are easily seen to be pullback squares. By Proposition 4.3, the right vertical maps are universal simplicial quasifibrations. Since for each vertex of $N\mathbf{wC}^n$ and of $N(\mathbf{wC}^{\text{op}})^n$, the fiber is contractible, the right vertical maps are weak equivalences. It follows that the left vertical maps are weak equivalences, and this gives a zigzag of weak equivalences relating B and B' . \square

We now prove part (i) of Theorem 2.7 by comparing $N(\mathbf{w}F'_{n-1})$ with the homotopy coend in Lemma 7.2. Thus, let \mathcal{C} be a Waldhausen category that has FMCWC and fix $n \geq 2$. We prove the comparison in a sequence of reductions $\mathbb{A}, \mathbb{B}, \mathbb{C}$ obtained from applying simplicial homotopy theory.

Lemma 7.3. *$N\mathbf{w}F'_{n-1}\mathcal{C}$ is equivalent to the diagonal simplicial set of the bisimplicial set \mathbb{A} that has as its (q, r) simplices the commutative diagrams of the following form*

$$\begin{array}{ccccccc}
 X_{0,1} & \xrightarrow{wc} & \cdots & \xrightarrow{wc} & X_{0,n} & \xleftarrow{\simeq} & B_0 \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} B_q \\
 \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \\
 \vdots & \xrightarrow{wc} & \vdots & \xrightarrow{wc} & \vdots & & \\
 \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \\
 A_q \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} A_0 & \xleftarrow{\simeq} & X_{r,1} & \xrightarrow{wc} & \cdots & \xrightarrow{wc} & X_{r,n},
 \end{array}$$

where the maps labelled “ \simeq ” are weak equivalences and the maps labelled “ wc ” are weak cofibrations.

Proof. Each (q, r) simplex is specified by an r -simplex of $N\mathbf{w}F'_{n-1}\mathcal{C}$, a q -simplex of $N\mathbf{w}(\mathcal{C} \setminus X_{r,1})$ and a q -simplex of $N\mathbf{w}(\mathcal{C}/X_{0,n})$. Regarding $N\mathbf{w}F'_{n-1}\mathcal{C}$ as a bisimplicial set constant in the q direction, we get a bisimplicial map from \mathbb{A} to $N\mathbf{w}F'_{n-1}\mathcal{C}$. Since for each fixed r -simplex of $N\mathbf{w}F'_{n-1}\mathcal{C}$, the simplicial sets $N\mathbf{w}(\mathcal{C} \setminus X_{r,1})$ and $N\mathbf{w}(\mathcal{C}/X_{0,n})$ are contractible, the bisimplicial map induces a weak equivalence on diagonals. \square

In the case $n = 2$, the diagonal of the bisimplicial set \mathbb{A} is

$$\text{hocoend}_{(A,B) \in (\mathbf{wC}^{\text{op}})^2} N(\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(A, B)_{\tilde{\text{co}}}),$$

where, as in Theorem 5.6, $\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(A, B)_{\tilde{\text{co}}}$ denotes the full subcategory of $\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(A, B)$ of diagrams where the forward arrow is a weak cofibration. Lemma 7.2 then finishes the argument for the case $n = 2$. Now assume $n \geq 3$.

Lemma 7.4. *The diagonal of the bisimplicial set \mathbb{A} is weakly equivalent to the diagonal of the bisimplicial set \mathbb{B} that has as its (q, r) simplices the commutative diagrams of the following form*

$$\begin{array}{ccccccccccc}
 Y_{0,1} & \xrightarrow{wc} & Z_{0,1} & \xleftarrow{\simeq} & Y_{0,2} & \xrightarrow{wc} & \cdots & \xleftarrow{\simeq} & Y_{0,n-1} & \xrightarrow{wc} & Z_{0,n-1} & \xleftarrow{\simeq} & B_0 \\
 \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \\
 \vdots & \xrightarrow{wc} & \vdots & \xleftarrow{\simeq} & \vdots & \xrightarrow{wc} & \vdots & \xleftarrow{\simeq} & \vdots & \xrightarrow{wc} & \vdots & \xleftarrow{\simeq} & \\
 \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \\
 A_0 \xleftarrow{\simeq} Y_{r,1} & \xrightarrow{wc} & Z_{r,1} & \xleftarrow{\simeq} & Y_{r,2} & \xrightarrow{wc} & \cdots & \xleftarrow{\simeq} & Y_{r,n-1} & \xrightarrow{wc} & Z_{r,n-1} & &
 \end{array}$$

together with sequences of weak equivalences

$$A_q \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} A_0 \qquad B_0 \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} B_q.$$

Proof. We have an inclusion of \mathbb{A} in \mathbb{B} by inserting identity maps in the appropriate columns. The lemma now follows from Theorem 5.5 and homotopy calculus of two-sided fractions [3, 9.4,9.5]. \square

Lemma 7.5. *The diagonal of the bisimplicial set \mathbb{B} is weakly equivalent to the diagonal of the bisimplicial set \mathbb{C} that has as its (q, r) simplices the commutative diagrams of the following form*

$$\begin{array}{ccccccc}
 Y_{0,1} & \xrightarrow{wc} & Z_{0,1} & & Y_{0,2} & \xrightarrow{wc} & Z_{0,2} & \cdots & Y_{0,n-1} & \xrightarrow{wc} & Z_{0,n-1} & \xleftarrow{\simeq} & B_0 \\
 \downarrow \simeq & & \downarrow \simeq & \nearrow \simeq & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\
 \vdots & \xrightarrow{wc} & \vdots & & \vdots & \xrightarrow{wc} & \vdots & \cdots & \vdots & \xrightarrow{wc} & \vdots & & \\
 \downarrow \simeq & & \downarrow \simeq & \nearrow \simeq & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\
 A_0 & \xleftarrow{\simeq} & Y_{r,1} & \xrightarrow{wc} & Z_{r,1} & & Y_{r,2} & \xrightarrow{wc} & Z_{r,2} & \cdots & Y_{r,n-1} & \xrightarrow{wc} & Z_{r,n-1}
 \end{array}$$

together with sequences of weak equivalences

$$A_q \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} A_0 \quad B_0 \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} B_q.$$

Proof. Fix A_0, B_0 and consider the simplicial set \mathbb{C}_k that has its r -simplices the pairs of commutative diagrams

$$\begin{array}{ccccccc}
 Y_{0,1} & \xrightarrow{wc} & Z_{0,1} & \xleftarrow{\simeq} & Y_{0,2} & \xrightarrow{wc} & \cdots & \xleftarrow{\simeq} & Y_{0,k} & \xrightarrow{wc} & Z_{0,k} \\
 \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
 \vdots & \xrightarrow{wc} & \vdots & \xleftarrow{\simeq} & \vdots & \xrightarrow{wc} & \cdots & \xleftarrow{\simeq} & \vdots & \xrightarrow{wc} & \vdots \\
 \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
 A_0 & \xleftarrow{\simeq} & Y_{r,1} & \xrightarrow{wc} & Z_{r,1} & \xleftarrow{\simeq} & Y_{r,2} & \xrightarrow{wc} & \cdots & \xleftarrow{\simeq} & Y_{r,k} & \xrightarrow{wc} & Z_{r,k}
 \end{array}$$

and

$$\begin{array}{ccccccc}
 & & Z_{0,k} & & Y_{0,k+1} & \xrightarrow{wc} & Z_{0,k+1} & \cdots & Y_{0,n-1} & \xrightarrow{wc} & Z_{0,n-1} & \xleftarrow{\simeq} & B_0 \\
 & & \downarrow \simeq & \nearrow \simeq & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\
 & & \vdots & \xrightarrow{wc} & \vdots & & \vdots & \xrightarrow{wc} & \vdots & & \vdots & & \\
 & & \downarrow \simeq & \nearrow \simeq & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\
 & & Y_{r,k+1} & \xrightarrow{wc} & Z_{r,k+1} & \cdots & Y_{r,n-1} & \xrightarrow{wc} & Z_{r,n-1} & & & &
 \end{array}$$

Then \mathbb{C}_1 is \mathbb{C} and \mathbb{C}_{n-1} is \mathbb{B} . We construct a zigzag of weak equivalences between \mathbb{C}_{k+1} and \mathbb{C}_k with the diagonal of a bisimplicial set in the middle. Let \mathbb{D}_k be the bisimplicial set that has as its (r, s) simplices the commutative diagrams of the

Composing induces a bisimplicial map from the homotopy coend to \mathbb{C} that is easily seen to be a weak equivalence.

Part (iii) of Theorem 1.3 follows from Proposition 4.4 and part (ii). Thus, it remains to prove part (ii), namely, that $Nw\mathcal{C}$ is weakly equivalent to the disjoint union of $BhAut(X)$. Fixing X in \mathcal{C} , the undercategory of X in $w\mathcal{C}$ has contractible nerve. Then the (cartesian) commutative diagram of categories

$$\begin{array}{ccc} \mathbf{W}^{-1}\mathbf{W}(X, X) & \longrightarrow & w\mathcal{C}\backslash X \\ \downarrow & & \downarrow \\ w\mathcal{C}\backslash X & \longrightarrow & w\mathcal{C} \end{array}$$

satisfies the hypotheses of Theorem 4.5. Thus, $hAut(X) \simeq \mathbf{W}^{-1}\mathbf{W}(X, X)$ is equivalent to the loop space of $Nw\mathcal{C}$ based at X .

This completes the proof of Theorem 2.7, which implies Theorems 1.2 and 1.3. We close the section with a remark on Theorem 1.2.

Remark 7.6. The decomposition of Theorem 1.2 does not fit into a simplicial structure to give a “construction” of the algebraic K -theory spectrum. An indirect construction of the algebraic K -theory spectrum for certain categories enriched in simplicial sets via the category of simplicial functors can be found in [17, §4]. The Dwyer-Kan simplicial localization of a category that admits functorial factorization satisfies the hypotheses there, by Theorem 6.2.

A direct construction in terms of the Dwyer-Kan simplicial localization would include a description of face and degeneracy maps fitting the pieces together compatibly with the simplicial structure on the S_\bullet construction. Although we do not produce such a construction, we can reinterpret some of the simplicial structure maps of S_\bullet in terms of the spaces described above. The degeneracy maps, as in S_\bullet , are induced simply by repeating an object X_i and using the identity map in $LC(X_i, X_i)$, where we understand X_0 as the distinguished zero object of \mathcal{C} . The face maps d_2, \dots, d_{n-1} are induced by composition

$$LC(X_i, X_{i+1}) \times LC(X_{i-1}, X_i) \longrightarrow LC(X_{i-1}, X_{i+1}).$$

The face maps d_1 and d_n essentially drop the first and n -th objects, respectively. The face map d_0 , which in S_n corresponds to replacing the sequence of cofibrations $X_1 \rightarrow \dots \rightarrow X_n$ with the quotient $X_2/X_1 \rightarrow \dots \rightarrow X_n/X_1$, is the impediment to making the spaces above into a simplicial object. Under the hypotheses of the Theorem 2.7, Dwyer-Kan mapping complexes take pushouts along cofibrations to homotopy pullbacks, i.e., take homotopy cocartesian squares to homotopy cartesian squares. Roughly speaking, the face map d_0 on the spaces above would involve composition and taking homotopy fibers, but appears difficult to describe directly.

8. PROOF OF THEOREM 1.4

In this section we prove Theorem 1.4 following ideas in [12]. The first key step is relating the homotopy categories of undercategories to the higher homotopy data implicit in the Dwyer-Kan simplicial localization.

Theorem 8.1. *Let \mathcal{C} be a Waldhausen category that admits a HCLF and let A be an object of \mathcal{C} . Let $\vec{B} = A \rightarrow B$ be a cofibration viewed as an object of $\mathcal{C}\backslash A$, let $\vec{C} = A \rightarrow C$ be an object of $\mathcal{C}\backslash A$, and write \vec{A} for $\text{id}: A \rightarrow A$, viewed as an object*

of $\mathcal{C}\backslash A$. The following square is homotopy cartesian:

$$\begin{array}{ccc} L(\mathcal{C}\backslash A)(\vec{B}, \vec{C}) & \rightarrow & L(\mathcal{C}\backslash A)(\vec{A}, \vec{C}) \\ \downarrow & & \downarrow \\ LC(B, C) & \longrightarrow & LC(A, C) \end{array}$$

Proof. Since \mathcal{C} admits a HCLF, it suffices to show that the square

$$\begin{array}{ccc} \mathbf{W}^{-1}\mathbf{C}_A(\vec{B}, \vec{C}) & \rightarrow & \mathbf{W}^{-1}\mathbf{C}_A(\vec{A}, \vec{C}) \\ \downarrow & & \downarrow \\ \mathbf{W}^{-1}\mathbf{C}(B, C) & \longrightarrow & \mathbf{W}^{-1}\mathbf{C}(A, C) \end{array}$$

is homotopy cartesian, where we have written $\mathbf{W}^{-1}\mathbf{C}_A$ for the categories of words $\mathbf{W}^{-1}\mathbf{C}$ in $\mathcal{C}\backslash A$ to avoid confusion. We apply Theorem 4.5. An easy check of the definitions shows that this square satisfies the hypothesis (ii), and it satisfies hypothesis (i) by Lemma 6.5 (with $E = C$). \square

The previous theorem identifies the Dwyer-Kan function complexes in $\mathcal{C}\backslash A$ for cofibrations. In the context of Theorem 1.4, functorial factorization allows us to extend this to compute the Dwyer-Kan function complexes for arbitrary objects of $\mathcal{C}\backslash A$.

Proposition 8.2. *Let $\mathcal{C}\backslash_{\text{co}}A$ denote the full subcategory of $\mathcal{C}\backslash A$ consisting of the cofibrations. If \mathcal{C} has functorial factorization, then the inclusion $\mathcal{C}\backslash_{\text{co}}A \rightarrow \mathcal{C}\backslash A$ is a DK-equivalence.*

For a map $f: B \rightarrow C$ in \mathcal{C} , let $\Omega_f LC(B, C)$ denote the space of based loops in the geometric realization of $LC(B, C)$, based at the vertex f . Thinking of $\Omega_f LC(B, C)$ as the homotopy pullback of the diagram

$$\begin{array}{ccc} & LC(B, C) & \\ & \downarrow \Delta & \\ \{(f, f)\} & \longrightarrow & LC(B, C) \times LC(B, C), \end{array}$$

we can identify the lower right term as $LC(B \vee B, C)$ by Theorem 6.2; we then get the following result as a corollary of the previous proposition and theorem.

Corollary 8.3. *Let \mathcal{C} be a Waldhausen category that satisfies functorial factorization. Let $f: B \rightarrow C$ be a map in \mathcal{C} . Viewing B as an object under $B \vee B$ via the codiagonal map, and C as such via the composite with f , then the loop space $\Omega_f LC(B, C)$ is weakly equivalent to $L(\mathcal{C}\backslash B \vee B)(B, C)$.*

In comparing function complexes for \mathcal{C} to function complexes for \mathcal{D} , we need the following proposition, which is an easy consequence of Theorem 8.1 and Proposition 8.2.

Proposition 8.4. *Let $A' \rightarrow A$ be a map in \mathcal{C} that is an isomorphism in $\text{Ho}\mathcal{C}$. If \mathcal{C} has functorial factorization, then the induced functor $\mathcal{C}\backslash A \rightarrow \mathcal{C}\backslash A'$ is a DK-equivalence.*

Finally, we prove Theorem 1.4. Clearly, by Theorem 8.1 and Proposition 8.2, a DK-equivalence implies an equivalence of homotopy categories and homotopy

categories of all under categories. For the converse, note that $\mathcal{C} \setminus_{\text{co}} B \vee B$ inherits from \mathcal{C} the property of being a Waldhausen category with functorial factorization. Likewise, note that for an arbitrary component of $\mathcal{L}\mathcal{C}(B, C)$, we can find a vertex ϕ of the form

$$B \xrightarrow{f} X \xleftarrow{\simeq} C$$

by Theorem 5.5. The loop space based at ϕ , $\Omega_\phi \mathcal{L}\mathcal{C}(B, C)$, is then homotopy equivalent to $\Omega_f \mathcal{L}\mathcal{C}(B, X)$, and so weakly equivalent to $L(\mathcal{C} \setminus B \vee B)(B, X)$, as per Corollary 8.3. Thus, iterating Corollary 8.3 identifies the homotopy groups of $\mathcal{L}\mathcal{C}(B, C)$ at arbitrary basepoints in terms of sets of maps in the homotopy categories of undercategories, as in [12, 5.4].

Specifically, we can identify $\pi_n(\mathcal{L}\mathcal{C}(B, C))$ based at ϕ as $\text{Ho}(\mathcal{C} \setminus S^{n-1})(B, X)$, for certain objects S^{n-1} formed inductively as follows: Starting with $S^{-1} = *$ and $B^0 = B$, S^n is formed as the coproduct $B^n \cup_{S^{n-1}} B^n$ in $\mathcal{C} \setminus_{\text{co}} S^{n-1}$ where B^n is an object of $\mathcal{C} \setminus_{\text{co}} S^{n-1}$ with a weak equivalence $B^n \rightarrow B$ in $\mathcal{C} \setminus S^{n-1}$. Now in \mathcal{D} , we can perform the analogous construction starting with $S_{\mathcal{D}}^{-1} = *$ and $B_{\mathcal{D}}^0 = FB$ to form $S_{\mathcal{D}}^{n-1}$ and $B_{\mathcal{D}}^n$. When inductively we choose the weak equivalence $B_{\mathcal{D}}^n \rightarrow FB$ to factor through FB^n in $\mathcal{D} \setminus S_{\mathcal{D}}^{n-1}$, then $S_{\mathcal{D}}^n \rightarrow FB$ factors through FS^n . Since we have not assumed that F is weakly exact, we cannot conclude that the map $S_{\mathcal{D}}^n \rightarrow FS^n$ is a weak equivalence; however, we do have the following lemma, which then completes the proof of Theorem 1.4.

Lemma 8.5. *For all n , the restriction of F to a functor $\mathcal{C} \setminus S^n \rightarrow \mathcal{D} \setminus S_{\mathcal{D}}^n$ induces an equivalence of homotopy categories $\text{Ho}(\mathcal{C} \setminus S^n) \rightarrow \text{Ho}(\mathcal{D} \setminus S_{\mathcal{D}}^n)$.*

Proof. For $n = -1$, $S^{-1} = *$ and $S_{\mathcal{D}}^{-1} = *$, the initial object in each category and the equivalence $\text{Ho}\mathcal{C} \rightarrow \text{Ho}\mathcal{D}$ is part of the hypothesis on F . Now by induction, assume that $\text{Ho}(\mathcal{C} \setminus S^{n-1}) \rightarrow \text{Ho}(\mathcal{D} \setminus S_{\mathcal{D}}^{n-1})$ is an equivalence. We have that $S^n = B^n \cup_{S^{n-1}} B^n$ is the coproduct of two copies of B in $\text{Ho}(\mathcal{C} \setminus S^{n-1})$ and $S_{\mathcal{D}}^n = B_{\mathcal{D}}^n \cup_{S_{\mathcal{D}}^{n-1}} B_{\mathcal{D}}^n$ is the coproduct of two copies of FB in $\text{Ho}(\mathcal{D} \setminus S_{\mathcal{D}}^{n-1})$. It follows that the map $S_{\mathcal{D}}^n \rightarrow FS^n$ is an isomorphism in $\text{Ho}(\mathcal{D} \setminus S_{\mathcal{D}}^{n-1})$ and hence in $\text{Ho}\mathcal{D}$. By Proposition 8.4, the map $S_{\mathcal{D}}^n \rightarrow FS^n$ induces an equivalence $\text{Ho}(\mathcal{D} \setminus FS^n) \rightarrow \text{Ho}(\mathcal{D} \setminus S_{\mathcal{D}}^n)$, and by hypothesis on F , the functor $\text{Ho}(\mathcal{C} \setminus S^n) \rightarrow \text{Ho}(\mathcal{D} \setminus FS^n)$ is an equivalence. The functor $\text{Ho}(\mathcal{C} \setminus S^n) \rightarrow \text{Ho}(\mathcal{D} \setminus S_{\mathcal{D}}^n)$ in question is the composite of these two equivalences. \square

9. PROOF OF THEOREM 1.5

This section proves Theorem 1.5, which relates the approximation property to the homotopy categories of the undercategories. It is convenient to prove the theorem in the following form.

Theorem 9.1. *Let \mathcal{C} be a Waldhausen category where every map factors as a cofibration followed by a weak equivalence. Let \mathcal{D} be a category with weak equivalences and let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor that satisfies the approximation property, preserves finite coproducts, and preserves pushouts where one leg in \mathcal{C} is a cofibration. Then F induces an equivalence $\text{Ho}\mathcal{C} \rightarrow \text{Ho}\mathcal{D}$.*

Note that we do not assume that \mathcal{D} has all finite coproducts or pushouts; only the finite coproducts and pushouts required by the hypotheses are assumed to exist.

Theorem 9.1 implies Theorem 1.5: When \mathcal{C} has the property that every map can be factored as a cofibration followed by a weak equivalence, then for any object A , the inclusion of $\mathcal{C}\backslash_{\text{co}}A$ in $\mathcal{C}\backslash A$ (with notation as in Proposition 8.2 above) satisfies the hypothesis of Theorem 9.1 and so induces an equivalence of homotopy categories. Likewise, F regarded as a functor $\mathcal{C}\backslash_{\text{co}}A \rightarrow \mathcal{D}\backslash FA$ satisfies the hypothesis of Theorem 9.1 and so induces an equivalence of homotopy categories. It follows that F induces an equivalence of homotopy categories $\text{Ho}(\mathcal{C}\backslash A) \rightarrow \text{Ho}(\mathcal{D}\backslash FA)$.

The remainder of the section is devoted to the proof of Theorem 9.1. We begin by constructing a functor $R: \mathcal{D} \rightarrow \text{Ho}\mathcal{C}$. For each object X in \mathcal{D} , apply the approximation property to the initial map $F* \rightarrow X$ to choose an object RX in \mathcal{C} and a weak equivalence $\epsilon_X: FRX \rightarrow X$ in \mathcal{D} . For each map in \mathcal{D} , $f: X \rightarrow Y$, apply the approximation property to the map

$$F(RX \vee RY) \cong FRX \vee FRY \xrightarrow{f \circ \epsilon_X + \epsilon_Y} Y$$

to obtain an object Qf of \mathcal{C} , cofibrations $RX \rightarrow Qf$ and $RY \rightarrow Qf$, and a commuting diagram in \mathcal{D} .

$$(9.2) \quad \begin{array}{ccccc} FRX & \longrightarrow & FQf & \longleftarrow & FRY \\ \simeq \downarrow \epsilon_X & & \searrow \simeq & & \simeq \downarrow \epsilon_Y \\ X & \xrightarrow{\quad f \quad} & & & Y \end{array}$$

It follows from part (i) of the approximation property then that the map $RY \rightarrow Qf$ is a weak equivalence, and so we obtain a zigzag in \mathcal{C} ,

$$RX \rightarrow Qf \xleftarrow{\simeq} RY.$$

Let Rf be the map $RX \rightarrow RY$ in $\text{Ho}\mathcal{C}$ represented by this zigzag. The following lemma implies that Rf is independent of the choice of Qf .

Lemma 9.3. *Let $f: X \rightarrow Y$ be a map in \mathcal{D} . Let*

$$RX \xrightarrow{\alpha} B \xleftarrow{\beta} RY$$

be any zigzag in \mathcal{C} . If there exists a map $\gamma: FB \rightarrow Y$ that makes the diagram

$$\begin{array}{ccccc} FRX & \longrightarrow & FB & \xleftarrow{\simeq} & FRY \\ \simeq \downarrow \epsilon_X & & \searrow \gamma & & \simeq \downarrow \epsilon_Y \\ X & \xrightarrow{\quad f \quad} & & & Y \end{array}$$

commute in \mathcal{D} , then $Rf = \beta^{-1} \circ \alpha$ in $\text{Ho}\mathcal{C}$.

Proof. By construction, the map $RX \vee RY \rightarrow Qf$ is a cofibration, and by factorization, we can assume without loss of generality that the map $\alpha + \beta: RX \vee RY \rightarrow B$ is a cofibration. Since the maps $FB \rightarrow Y$ and $FQf \rightarrow Y$ both compose to the same maps $FRX \rightarrow Y$ and the same maps $FRY \rightarrow Y$, we obtain a map

$$F(B \cup_{RX \vee RY} Qf) \cong FB \cup_{FRX \vee FRY} FQf \longrightarrow Y.$$

Applying the approximation property to this map, we obtain an object C in \mathcal{C} and a map

$$B \cup_{RX \vee RY} Qf \longrightarrow C$$

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