

FREYD'S GENERATING HYPOTHESIS FOR GROUPS WITH PERIODIC COHOMOLOGY

SUNIL K. CHEBOLU, J. DANIEL CHRISTENSEN, AND JÁN MINÁČ

ABSTRACT. Let G be a finite group and let k be a field whose characteristic p divides the order of G . Freyd's generating hypothesis for the stable module category of G is the statement that a map between finite-dimensional kG -modules in the thick subcategory generated by k factors through a projective if the induced map on Tate cohomology is trivial. We show that if G has periodic cohomology then the generating hypothesis holds if and only if the Sylow p -subgroup of G is C_2 or C_3 . We also give some other conditions that are equivalent to the GH for groups with periodic cohomology.

CONTENTS

1. Introduction	2
Acknowledgements:	3
2. Some results from representation theory	3
2.1. Periodic cohomology	3
2.2. The principal block and $\text{thick}_G(k)$	4
3. Examples	6
3.1. Non-trivial identity ghosts	6
3.2. The alternating group A_4 when $p = 2$	6
3.3. The symmetric group S_3 when $p = 3$	7
4. Groups with periodic cohomology for which the GH holds	7
4.1. Direct products	7
4.2. Reduction to the normal case	8
4.3. The Sylow p -subgroup is C_2	9
4.4. The Sylow p -subgroup is C_3	9
5. Groups with periodic cohomology for which the GH fails	10
References	11

Date: October 17, 2007.

2000 Mathematics Subject Classification. Primary 20C20, 20J06; Secondary 55P42.

Key words and phrases. Tate cohomology, generating hypothesis, stable module category, ghost map, principal block, thick subcategory, periodic cohomology.

1. INTRODUCTION

Motivated by the celebrated generating hypothesis (GH) of Peter Freyd in homotopy theory [14] and its analogue in the derived category of a commutative ring [17, 18], we have formulated in [11] the analogue of Freyd's GH in the stable module category $\text{stmod}(kG)$ of a finite p -group G , where k is a field of characteristic p . (The stable module category is the tensor triangulated category obtained from the category of finitely generated left kG -modules by killing the projective modules.) In this setting, the GH is the statement that any map that induces the trivial map in Tate cohomology is trivial in the stable module category $\text{stmod}(kG)$ (i.e., factors through a projective). In [6] we showed that the only non-trivial p -groups for which is this true are C_2 and C_3 . The goal of the current project is to describe the analogue of this hypothesis for arbitrary finite groups and determine for which groups it is true. It turns out that the above formulation of the GH is not appropriate for arbitrary finite groups, for, in general, a finite group G can admit a non-projective kG -module whose Tate cohomology is trivial. Clearly the identity map on such a module will disprove the GH, so it is unreasonable to expect Tate cohomology to detect all non-trivial maps in $\text{stmod}(kG)$. As we justify in Section 3.1, instead one has to restrict to the thick subcategory $\text{thick}_G(k)$ generated by k in $\text{stmod}(kG)$. (This is the smallest full subcategory of $\text{stmod}(kG)$ that contains k and closed under exact triangles and direct summands.) So the modified GH for a group ring kG is the statement that Tate cohomology detects all non-trivial maps in $\text{thick}_G(k)$, i.e. that the Tate cohomology functor

$$\begin{aligned} \text{thick}_G(k) &\longrightarrow \widehat{H}^*(G, k)\text{-modules} \\ M &\longmapsto \widehat{H}^*(G, M) \end{aligned}$$

is faithful. If G is a p -group, there is only one simple kG -module, namely the trivial module k , consequently $\text{thick}_G(k) = \text{stmod}(kG)$. Therefore this modified GH agrees with the aforementioned version of the GH for p -groups. In this paper we determine those finite groups with periodic cohomology for which the modified GH holds. Our results can be summarised in:

Theorem 1.1. *Let G be a non-trivial finite group that has periodic cohomology and let k be a field of characteristic p that divides the order of G . Then the following are equivalent.*

- (1) *The Sylow p -subgroup of G is either C_2 or C_3 .*
- (2) *The Tate cohomology functor detects all non-trivial maps in $\text{thick}_G(k)$. That is, the GH holds for kG .*
- (3) *The thick subcategory $\text{thick}_G(k)$ consists of finite direct sums of suspensions of k .*
- (4) *The Tate cohomology functor detects all non-trivial maps in the localising subcategory $\text{loc}_G(k)$ generated by k .*
- (5) *The localising subcategory $\text{loc}_G(k)$ consists of arbitrary direct sums of suspensions of k .*

If we replace $\text{thick}_G(k)$ by the subcategory of finite-dimensional modules in the principal block, or $\text{loc}_G(k)$ by the subcategory of all modules in the principal block, we obtain additional equivalent statements.

Maps of kG -modules that induce the trivial map in Tate cohomology are called *ghosts*. In this terminology, our main result (the equivalence (1) \iff (2) of the above theorem)

states that there are no non-trivial ghosts in $\text{thick}_G(k)$ if and only if the Sylow p -subgroup is C_2 or C_3 .

It is worth pointing out that the GH for kG depends only on G and the characteristic of k . This is clear from the equivalence (1) \iff (2).

Although we have generalised our result for p -groups from [6], we should stress that our proof in [6] does not directly generalise. Several obstacles and subtle issues that arise in studying the GH for non- p -groups are illustrated in Section 3 where we work out some examples of the GH in detail. One new additional technique used here is block theory. In particular, we make good use of the main theorems of Brauer and the Green correspondence along with some knowledge of the structure of modules in the principal block for groups with a cyclic normal Sylow p -subgroup via Brauer trees. A result [2, 4] concerning the thick subcategory generated by k also plays an important role; see Theorem 2.4.

In work with Carlson [7] the first and third authors have disproved the GH for groups with non-periodic cohomology using techniques from Auslander-Reiten theory and support varieties. Combined with the results of this paper, this gives a complete classification of the group algebras of finite groups for which the GH holds. Some related questions which are motivated by the GH have also been studied in [10].

The paper is organised as follows. The proof of the main theorem (the equivalence of statements (1) and (2) above) occupies Sections 4 and 5. In Section 4 it is shown that the GH holds when the Sylow p -subgroup of G is either C_2 or C_3 . The failure of the GH for all other groups with periodic cohomology is shown in Section 5. In Section 2 we recall several results from representation theory which are used in the later sections. This section also contains the proofs of the other equivalences of Theorem 1.1. Section 3 contains a few important examples which illustrate some issues that arise when studying the GH for non- p -groups. The reader who is only interested in the proof of the main theorem may skip Sections 2 and 3 and can refer to Section 2 when necessary.

All groups in this paper are non-trivial finite groups and the characteristic p of the field always divides the order of G . We work in the stable module category of kG and we freely use standard facts about this category which can be found in [8].

Acknowledgements: The first and third authors carried out some of this work at the PIMS algebra summer school at the University of Alberta. They would like to thank the University of Alberta, the organisers (A. Adem, J. Kuttler and A. Pianzola) of the summer school, and V. Chernousov for their hospitality. We are grateful to J. Carlson for his work in [9] which inspired us to consider almost split sequences in the last section. Calculations using Peter Webb's reps package [22] for GAP [15] were useful in our search for non-trivial ghosts.

2. SOME RESULTS FROM REPRESENTATION THEORY

In this section we collect some known results from representation theory which we will need in the sequel.

2.1. Periodic cohomology. We say that kG , or simply G when there is no confusion, has *periodic cohomology* if there is a positive integer d such that $\Omega^d k$ is stably isomorphic to k . When this is the case, the *period* is the smallest such d . It is a well-known fact due to Artin and Tate that a finite group G has periodic cohomology over a field k of characteristic p if and only if the Sylow p -subgroup of G is cyclic or a generalised quaternion group.

We begin with a proposition which forms the backbone of our analysis.

Proposition 2.1 ([11]). *Let G be a finite group with periodic cohomology. Then the GH holds for kG if and only if every module in $\text{thick}_G(k)$ is a sum of suspensions of k . In particular, the GH holds for kG if and only if every indecomposable non-projective kG -module in $\text{thick}_G(k)$ is stably isomorphic to $\Omega^i k$ for some i .*

Proof. We sketch a proof here; more details can be found in [11]. Let M be in $\text{thick}_G(k)$. Since the trivial representation is periodic, a ghost out of M can be constructed in $\text{thick}_G(k)$ using a triangle of the form

$$\bigoplus_{\text{finite sum}} \Omega^i k \longrightarrow M \xrightarrow{f} U_M.$$

If the GH holds for kG , then f must vanish. Thus the above triangle splits, and so M is a retract of $\bigoplus \Omega^i k$. Since M is finite-dimensional, it follows from the Krull-Schmidt theorem that M is a sum of suspensions of k . The converse is immediate. \square

Thus the GH holds if and only if the number of indecomposable non-projective kG -modules in $\text{thick}_G(k)$ is equal to the period. The next two results give us tools for computing these quantities.

Theorem 2.2 ([20]). *Let G be a finite group with periodic cohomology. When $p = 2$, the period is 1, 2 or 4 when the Sylow 2-subgroup is C_2 , C_{2^r} ($2^r > 2$), or Q_{2^n} , respectively. When p is odd and the Sylow p -subgroup is C_{p^r} , the period is $2\Phi_p$, where Φ_p is the number of automorphisms of C_{p^r} that are given by conjugation by elements in G .*

Theorem 2.3. *Let G be a finite group with cyclic Sylow p -subgroup of order p^r , and let s be the number of simple kG -modules. Then the number of indecomposable non-projective kG -modules is $s(p^r - 1)$. Moreover, if B is a block of kG and e is the number of simple modules lying in B , then the number of indecomposable non-projective kG -modules is $e(p^r - 1)$.*

Proof. The first statement is a simplified version of [12, Prop. 20.11]. The second statement follows from the detailed structure given there, using the fact that a module M lies in the block B if and only if each composition factor of M lies in B . \square

In order to apply Theorem 2.3, we next need to understand the relationship between $\text{thick}_G(k)$ and the principal block.

2.2. The principal block and $\text{thick}_G(k)$. Let B_0 be the principal block of kG , and $\text{stmod}(B_0)$ the stable category of all finitely-generated kG -modules in the principal block. $\text{thick}_G(k)$ will denote the full subcategory of all kG -modules that can be built out of k via cofibre sequences and retractions. $\text{thick}_G(k)$ is clearly a subcategory of $\text{stmod}(B_0)$. The following deep theorem due to Benson, Carlson and Robinson gives a nice group-theoretic criterion for when these subcategories are equal.

Theorem 2.4 ([2, 4]). *If the centraliser of every element of order p in G is p -nilpotent (i.e., has a normal p -complement) then $\text{thick}_G(k) = \text{stmod}(B_0)$.*

We do not need this, but we note that the converse of Theorem 2.4 holds as well.

The condition in the above theorem will be referred to as the BCR condition for G . In the next proposition we will show that this condition is always satisfied when the Sylow p -subgroup of G is cyclic.

Proposition 2.5. *If the Sylow p -subgroup of G is cyclic, then $\text{thick}_G(k) = \text{stmod}(B_0)$.*

Proof. It is enough to verify the BCR condition. To this end, we will use a theorem of Burnside [16, Thm. 4.3] which states that for a finite group A with Sylow p -subgroup P , if $N_A(P) = C_A(P)$ then A is p -nilpotent. So consider an element θ of order p in our group, and choose a Sylow p -subgroup $C_{p^r} = \langle x \rangle$ of G with $\theta = x^{p^{r-1}}$. We have to show that the centraliser $C_G(\theta)$ of θ is p -nilpotent. So we apply Burnside's theorem for $A = C_G(\theta)$. Note that $C_G(\theta)$ contains the Sylow p -subgroup C_{p^r} , so C_{p^r} is also the Sylow p -subgroup of $C_G(\theta)$. Consider an element α in $N_{C_G(\theta)}(C_{p^r})$. We have to show that α centralises x , the generator of C_{p^r} , that is, we have to show that $\alpha x \alpha^{-1} = x$. Since C_{p^r} is the Sylow p -subgroup of G , the index of $C_{C_G(\theta)}(C_{p^r})$ in $N_{C_G(\theta)}(C_{p^r})$ is coprime to p and therefore the inner automorphism $\alpha(-)\alpha^{-1}$ on C_{p^r} must have order coprime to p . Since α belongs to $N_{C_G(\theta)}(C_{p^r})$, we certainly have $\alpha x \alpha^{-1} = x^i$ for some $1 \leq i \leq p^r - 1$ and therefore $\alpha y \alpha^{-1} = y^i$ for any y in C_{p^r} . In particular, $\alpha \theta \alpha^{-1} = \theta^i$. But since α centralises θ , we must have that i is congruent to 1 mod p . Thus $i^{p^{r-1}}$ is congruent to 1 mod p^r , and so the order of the inner automorphism $\alpha(-)\alpha^{-1}$ must be a power of p . Since it is also coprime to p , it must be the identity, so α centralises x as desired. \square

Remark 2.6. The BCR condition is of course not always satisfied. Even in the periodic case there are examples where it fails. Consider the group $G = SL_2(\mathbb{F}_3) \cong Q_8 \rtimes C_3$ with k of characteristic 2. The centraliser of the unique element of order 2 in G is not 2-nilpotent; the centraliser of this element is Q_8 and it does not have a normal 2-complement due to the non-trivial action of C_3 . We learned this example from [5].

Recall that the localising subcategory $\text{loc}_G(k)$ generated by k is the smallest full subcategory of $\text{StMod}(kG)$ that contains k and is closed under exact triangles, direct summands and arbitrary direct sums. It is contained in $\text{StMod}(B_0)$. We now have the following corollary of the above proposition.

Corollary 2.7. *If the Sylow p -subgroup of G is cyclic, then $\text{loc}_G(k) = \text{StMod}(B_0)$.*

Proof. This follows immediately from the above proposition and the following fact due to Ringel and Tachikawa [19]. It states that if G has finite representation type (i.e., the Sylow p -subgroups are cyclic), then every kG -module is a direct sum of finite-dimensional kG -modules. \square

Partial proof of Theorem 1.1. We are now ready to prove all equivalences of Theorem 1.1 except the equivalence (1) \iff (3), which will be proved in the last two sections. For now, we assume (1) \iff (3). In Proposition 2.1 we have seen that (2) \iff (3). The implications (5) \implies (4) \implies (2) are clear. Since (2) \iff (3) \iff (1) we will be done if we can show that (1) and (3) together imply (5). Consider a module M in $\text{loc}_G(k)$. By (1), the Sylow p -subgroup is C_2 or C_3 and so, as mentioned in the proof of Corollary 2.7, M is isomorphic to a direct sum of finite-dimensional modules. Again using (1), Proposition 2.5 implies that the summands of M belong to $\text{thick}_G(k)$. By (3), these summands are direct sums of suspensions of k , and so it follows that M is also a direct sum of (possibly infinitely many) suspensions of k . \square

3. EXAMPLES

In this section we discuss some examples which will help the reader get some insight into the GH. We begin by justifying in the next subsection why we have to restrict to the thick subcategory generated by k when studying the GH for non- p -groups. As pointed out earlier, for p -groups this makes no difference, for the trivial module k is the only simple module over a p -group and therefore $\text{thick}_G(k) = \text{stmod}(kG)$.

3.1. Non-trivial identity ghosts. The key point is that, in general, there can be non-projective modules with trivial Tate cohomology. Clearly the identity map on such a module will be a non-trivial ghost. Examples of such modules abound. For instance, suppose that kG has more than one block. Then any non-projective indecomposable module M that does not belong to the principal block B_0 is an example. Moreover, if $\text{thick}_G(k)$ is a proper subcategory of $\text{stmod}(B_0)$, then there is an indecomposable non-projective module which is in $\text{stmod}(B_0)$ but outside of $\text{thick}_G(k)$ and has trivial Tate cohomology. This happens precisely when the BCR condition fails; see [2, 4]. Thus the interesting question is whether there are any non-trivial ghosts inside $\text{thick}_G(k)$.

In contrast, we show that there are no non-trivial identity ghosts in the thick subcategory generated by k .

Proposition 3.1. *Let M be in $\text{thick}_G(k)$. If the identity map $M \rightarrow M$ is a ghost, then it is trivial in $\text{stmod}(kG)$.*

Proof. This is a standard thick subcategory argument. Consider the full subcategory of all modules X in $\text{stmod}(kG)$ which have the property that $\underline{\text{Hom}}(\Omega^i X, M) = 0$ for all integers i . It is straightforward to verify that this subcategory is closed under retractions and exact triangles. It contains the trivial representation by hypothesis. Thus it contains the thick subcategory generated by k , and hence contains M . In particular the identity map on M is trivial. \square

It is now clear from the above discussion and the proposition that $\text{thick}_G(k)$ is the right category in which to study the GH.

In some favourable cases, even when G is not a p -group, $\text{thick}_G(k)$ can be the whole of $\text{stmod}(kG)$. The GH for such groups can be easily attacked using the restriction-induction technique. We illustrate this in the example of A_4 .

3.2. The alternating group A_4 when $p = 2$. Let k be a field of characteristic 2 and consider the alternating group A_4 . This is a group of order 12 and is generated by x , y and z which satisfy the relations $x^2 = y^2 = (xy)^2 = 1 = z^3$, $zxz^{-1} = y$ and $zyz^{-1} = xy$. Using these relations, one can show that the centraliser of every element of order 2 is 2-nilpotent. Moreover the principal idempotent can be shown to be 1. Therefore, we have $\text{thick}_{A_4}(k) = \text{stmod}(kA_4)$.

Now the subgroup of A_4 generated by x and y is the Klein four group V_4 . So the Sylow 2-subgroup is V_4 . By [6], we know that the GH fails for V_4 . So the induction of a non-trivial ghost over kV_4 will give a non-trivial ghost (see [6, Prop. 2.1]) over kA_4 , thus disproving the GH for kA_4 .

Remark 3.2. The induction functor $\underline{\text{Ind}}: \text{stmod}(kH) \rightarrow \text{stmod}(kG)$ does not in general send $\text{thick}_H(k)$ into $\text{thick}_G(k)$. For example, if \mathbb{F}_3 is the trivial $\mathbb{F}_3 C_3$ module, then it can be shown

that the induced $\mathbb{F}_3(C_2 \times C_3)$ -module $\mathbb{F}_3 \uparrow^{C_2 \times C_3}$ does not belong to the thick subcategory $\text{thick}_{C_2 \times C_3}(\mathbb{F}_3)$. Since the right domain for the GH is $\text{thick}_G(k)$, the above induction strategy does not generalise to arbitrary finite groups.

3.3. The symmetric group S_3 when $p = 3$. In this section we prove that the GH holds in $\text{thick}_{S_3}(k)$ when k has characteristic 3. The argument we give here is a model for the general argument we give in Section 4, and also illustrates Theorem 2.3.

The group S_3 has presentation $\langle x, y \mid x^3 = 1 = y^2, yxy^{-1} = x^{-1} \rangle$. Define elements $e_1 = (1 - y)/2$ and $e_2 = (1 + y)/2$ in $A = kS_3$. Then $e_1 + e_2 = 1$ and it is a straightforward exercise to show that e_1 and e_2 are orthogonal idempotents in A , i.e. $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1e_2 = 0 = e_2e_1$. The principal indecomposable modules Ae_1 and Ae_2 (both 3-dimensional) have composition series of length 3:

$$\begin{aligned} Ae_1 &\supseteq A(x-1)e_1 \supseteq A(x-1)^2e_1 \supseteq 0 \\ Ae_2 &\supseteq A(x-1)e_2 \supseteq A(x-1)^2e_2 \supseteq 0. \end{aligned}$$

These six modules form a complete set of indecomposable kS_3 -modules; see [13, § 64]. Moreover, Ae_1 and Ae_2 are the indecomposable projectives over the simple modules $A(x-1)^2e_1$ and $A(x-1)^2e_2$ respectively. The structure of the simples is as follows: $A(x-1)^2e_2 = k$, the trivial representation, and $A(x-1)^2e_1 = k_{-1}$, on which x acts trivially and y by multiplication by -1 . We now leave it as an amusing exercise for the reader to show that

$$\begin{aligned} k &\cong A(x-1)^2e_2, \\ \Omega k &\cong A(x-1)e_2, \\ \Omega^2 k &\cong A(x-1)^2e_1 (= k_{-1}), \\ \Omega^3 k &\cong A(x-1)e_1, \text{ and} \\ \Omega^4 k &\cong k. \end{aligned}$$

So k has period 4, which agrees with the answer we get from Swan's formula (Thm. 2.2): $2\Phi_3 = 2(2) = 4$. This also shows that every indecomposable non-projective kG -module is isomorphic to $\Omega^i k$ for some i , and so the GH holds for kS_3 .

This example suggests that the GH for non- p -groups is both subtle and interesting.

4. GROUPS WITH PERIODIC COHOMOLOGY FOR WHICH THE GH HOLDS

In this section we show that the GH holds in $\text{thick}_G(k)$ if the Sylow p -subgroup is either C_2 or C_3 . We begin with some results which will be used in the proof.

4.1. Direct products.

Lemma 4.1. *Let G be a finite group that is a product of two groups: $G = A \times B$. Assume that p does not divide the order of B . Then we have*

$$\text{thick}_G(k) \cong \text{thick}_A(k)$$

and

$$\text{loc}_G(k) \cong \text{loc}_A(k)$$

as tensor triangulated categories.

Although this lemma is well-known we could not find a proof in the literature. So we give a proof here for the reader's convenience.

Proof. Consider the inclusion and restriction functors

$$I: \text{thick}_A(k) \hookrightarrow \text{thick}_{A \times B}(k)$$

and

$$R: \text{thick}_{A \times B}(k) \longrightarrow \text{thick}_A(k).$$

The inclusion functor I is well-defined on stable morphisms because p does not divide the order of B . Note that I and R are tensor triangulated functors. We will show that they are inverse to each other. $R \circ I$ is clearly isomorphic to the identity functor on $\text{thick}_A(k)$. Also, I is full. So it is enough to show that the inclusion functor I is essentially surjective.

Let \mathcal{A} be the essential image of I . That is, \mathcal{A} is the class of all M in $\text{thick}_{A \times B}(k)$ such that M is stably isomorphic to IM' for some M' in $\text{thick}_A(k)$. Since \mathcal{A} contains k , it is enough to show that \mathcal{A} is a thick subcategory. By definition, \mathcal{A} is closed under isomorphisms. Suppose that M and N are in \mathcal{A} and there is a triangle $\Omega N \rightarrow L \rightarrow M \rightarrow N$. We know that $M \cong IM'$ and $N \cong IN'$ for some M' and N' in $\text{thick}_A(k)$. Consider the map $IM' \cong M \rightarrow N \cong IN'$. Since I is full, it is induced by a map $M' \rightarrow N'$. Let L' be the fibre of this map, so we have a diagram

$$\begin{array}{ccccccc} \Omega N & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \\ \downarrow \sim & & & & \downarrow \sim & & \downarrow \sim \\ \Omega IN' & \longrightarrow & IL' & \longrightarrow & IM' & \longrightarrow & IN' \end{array}$$

By standard triangulated category arguments, there exists a fill-in $L \rightarrow IL'$ which must be an isomorphism. So \mathcal{A} is closed under taking fibres. Finally, we must show that \mathcal{A} is closed under taking summands. Suppose $Q \cong IQ'$ has a summand M . Let e be the idempotent $Q \rightarrow Q$ with image M . Since I is full and faithful, there is a corresponding idempotent $e': Q' \rightarrow Q'$. Since idempotents split in $\text{stmod}(kG)$, there is a summand M' of Q' with $IM' \cong M$.

The proof of the second isomorphism is identical to that of the first one; one has to just observe that the functors I and R respect arbitrary coproducts. \square

Remark 4.2. This result cannot be generalised to semi-direct products. The example to keep in mind is kS_3 where the characteristic of k is 3. By Swan's formula (Thm. 2.2) or the computations in Section 3.3, the trivial representation k has period 4 in $\text{thick}_{S_3}(k)$ and has period 2 in $\text{thick}_{C_3}(k)$. In particular,

$$\text{thick}_{C_3 \times C_2}(k) \not\cong \text{thick}_{C_3}(k).$$

So while the point of this paper is to show that the GH is determined by the Sylow p -subgroup, it is not because the relevant categories are equivalent.

4.2. Reduction to the normal case. We now use results from block theory to show that when the Sylow p -subgroup D of G is C_p , we can reduce to the case where D is normal. The relevant background material can be found in any standard text book in representation theory which deals with block theory, such as [1, 3].

Theorem 4.3. *Let G be a group which has a cyclic Sylow p -subgroup D , let D_1 be the unique subgroup of D that is isomorphic to C_p and let $N_1 = N_G(D_1)$. Then there is an equivalence of categories*

$$\text{stmod}(B_0) \cong \text{stmod}(b_0),$$

where B_0 is the principal block of kG and b_0 the principal block of kN_1 .

When D is C_p , then $D_1 = D$ and so D is also the Sylow p -subgroup of N_1 and is normal in N_1 .

Proof. Recall that D is the defect group of the principal block. Since $DC_G(D) = C_G(D) \leq N_1$, Brauer's third main theorem says that the block b_0^G corresponding to the principal block b_0 of kN_1 is the principal block B_0 of kG . So by [1, pp. 124–125], there is an equivalence of categories

$$\text{stmod}(B_0) \cong \text{stmod}(b_0).$$

□

By Theorem 4.3, we know that if the Sylow p -subgroup H of G is isomorphic to C_p , then the stable categories of the principal blocks of kG and $kN_G(H)$ are equivalent. So we can assume without loss of generality that H is normal in G .

Also note that if L is an extension of k , then there is a faithful functor

$$L \otimes_k - : \text{thick}_G(k) \longrightarrow \text{thick}_G(L)$$

which sends ghosts to ghosts. So we can assume that k is algebraically closed, and we do so for the remainder of this section. This is convenient because we cite [1] in Section 4.4, and that reference makes the assumption that k is algebraically closed.

4.3. The Sylow p -subgroup is C_2 . If $H = C_2$ is normal in G , then it is actually central in G . By the Schur-Zassenhaus Theorem it follows that $G = C_2 \times L$ for some group L which has odd order. Then by Lemma 4.1 we have that $\text{thick}_G(k)$ is equivalent to $\text{thick}_{C_2}(k)$ as tensor triangulated categories. The GH holds in the latter, so it also holds for kG .

4.4. The Sylow p -subgroup is C_3 . Let $H = C_3$ be normal in G . Now consider the map

$$\begin{aligned} \Xi: G &\longrightarrow \text{Aut}(C_3) \cong C_2 \\ g &\longmapsto g(-)g^{-1} \end{aligned}$$

There are only two possibilities for the image of Ξ :

Case 1: The image of Ξ is trivial. In this case, exactly as before, $G = C_3 \times L$ for some group L whose order is not divisible by 3. So by Lemma 4.1 we have that $\text{thick}_G(k)$ is equivalent to $\text{thick}_{C_3}(k)$ as tensor triangulated categories. The GH holds in the latter, so it also holds for kG .

Case 2: The image of $\Xi = C_2$. Then the centraliser $C_G(C_3)$ has index 2 in G . In this case, Φ_3 , the number of automorphisms of C_3 given by conjugation by elements of G , is equal to 2. By Theorem 2.2 and Proposition 2.1, it is enough to show that the number of indecomposable non-projective kG -modules in the principal block is equal to $2\Phi_3 = 4$. By Theorem 2.3, we know that the number of indecomposable non-projective kG -modules in the principal block is twice the number of simple kG -modules in the principal block. Combining these two, we just have to show that there are only two simple kG -modules in the principal block.

Let P be the indecomposable projective module over k , that is $P/\text{rad}(P) \cong k$. Let W be the module $\text{rad}(P)/\text{rad}^2(P)$. Then the set of all simple kG -modules in the principal block is

$$\{k, W, W \otimes W, W \otimes W \otimes W, \dots\}.$$

This fact can be found in [1, Exercise 13.3], for instance. We will be done if we can show that $k \not\cong W$ and $W \otimes W = k$ because then we will have exactly two simple kG -modules in the principal block, namely k and W . These two facts will become clear once we give the explicit structure of W . Write $G = C_3L$, where L is a complement of C_3 , which exists by the Schur-Zassenhaus theorem, and let x be a generator of C_3 . It can be shown [1, p. 37] that W is a one-dimensional module generated by v such that $x(v) = v$, and for h in L , $h(v) = v$ if h belongs to $C_G(C_3)$ and $-v$ if h does not belong to $C_G(C_3)$. Since $C_G(C_3)$ has index 2, there are elements outside $C_G(C_3)$ which do not fix v , and therefore W is not isomorphic to k . The fact that $W \otimes W \cong k$ is clear since

$$\begin{aligned} x(v \otimes v) &= xv \otimes xv = v \otimes v \\ h(v \otimes v) &= hv \otimes hv = -v \otimes -v = v \otimes v. \end{aligned}$$

This shows that there are exactly two simple kG -modules in the principal block. So we are done.

5. GROUPS WITH PERIODIC COHOMOLOGY FOR WHICH THE GH FAILS

In this section we show that for a group G which has periodic cohomology, the GH fails whenever the Sylow p -subgroup of G is not C_2 or C_3 . In view of Proposition 2.1, in order to disprove the GH for these groups we have to show that there is a module in $\text{thick}_G(k)$ that is not stably isomorphic to a direct sum of suspensions of k . We will show that the middle term of an almost split sequence has this property.

We recall the standard almost split sequence for the reader. Let G be any finite group and let P be the indecomposable projective module over k , that is, $P/\text{rad}P \cong k$. Since kG is a symmetric algebra, we also have $\text{soc}P \cong k$. The quotient $\text{rad}P/\text{soc}P$ is called the heart H_G of G . It occurs as a summand in the middle term of the standard almost split sequence

$$0 \longrightarrow \text{rad}P \longrightarrow H_G \oplus P \longrightarrow P/\text{soc}P \longrightarrow 0.$$

This sequence can also be written as

$$0 \longrightarrow \Omega^1 k \longrightarrow H_G \oplus P \longrightarrow \Omega^{-1} k \longrightarrow 0. \quad (1)$$

It is a non-trivial result of Webb [21, Thm. E] that H_G is an indecomposable kG -module provided the Sylow p -subgroup of G is not a dihedral 2-group.

Theorem 5.1. *Let G be a group which has periodic cohomology for which the Sylow p -subgroup is not C_2 or C_3 . Then the kG -module H_G is an indecomposable non-projective module in $\text{thick}_G(k)$ that is not stably isomorphic to $\Omega^i k$ for any i . In particular, there is a non-trivial ghost out of H_G in $\text{thick}_G(k)$, i.e., the GH fails for kG .*

Proof. It is clear from the short exact sequence (1) that H_G belongs to $\text{thick}_G(k)$. Further, we know from Webb's theorem stated above that H_G is an indecomposable kG -module. So we only have to show that H_G is not projective and that it is not stably isomorphic to $\Omega^i k$ for any i . Both of these statements follow easily by comparing dimensions. The key fact to observe is that the dimension of every projective kG -module is divisible by p^n , the order of the Sylow p -subgroup of G . (One sees this by restricting the projective module to the Sylow p -subgroup P , over which the restriction becomes a free kP -module.) On the other hand, from the definition of H_G , it is clear that $\dim_k H_G \equiv -2 \pmod{p^n}$. So if H_G is projective, then p^n should divide 2, but that would mean that the Sylow p -subgroup is C_2 , which is a contradiction. Therefore H_G has to be non-projective. Using the minimal projective resolution of k and the above fact about dimensions of projective kG -modules, one sees by a straightforward induction on i that $\dim_k \Omega^i k \equiv 1$ or $-1 \pmod{p^n}$. If $H_G \cong \Omega^i k$ for some i , then it follows that the Sylow p -subgroup is either trivial or C_3 . Both cases are ruled out by the hypothesis. Therefore H_G is not stably isomorphic to $\Omega^i k$ for any i . This completes the proof of the theorem. \square

The last two sections together prove our main theorem that if G has periodic cohomology, then the GH holds for kG if and only if the Sylow p -subgroup of G is either C_2 or C_3 .

REFERENCES

- [1] J. L. Alperin. *Local representation theory*, volume 11 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Modular representations as an introduction to the local representation theory of finite groups.
- [2] D. J. Benson. Cohomology of modules in the principal block of a finite group. *New York J. Math.*, 1:196–205, electronic, 1994/95.
- [3] D. J. Benson. *Representations and cohomology. I*, volume 30 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1998.
- [4] D. J. Benson, J. F. Carlson, and G. R. Robinson. On the vanishing of group cohomology. *J. Algebra*, 131(1):40–73, 1990.
- [5] D. J. Benson, Jon F. Carlson, and Jeremy Rickard. Thick subcategories of the stable module category. *Fund. Math.*, 153(1):59–80, 1997.
- [6] David Benson, Sunil K. Chebolu, J. Daniel Christensen, and Ján Mináč. The generating hypothesis for the stable module category of a p -group. *Journal of Algebra*, 310(1):428–433, 2007.
- [7] Jon Carlson, Sunil K. Chebolu, and Ján Mináč. Freyd's generating hypothesis for the stable module category of a finite group. 2007. Preprint.
- [8] Jon F. Carlson. *Modules and group algebras*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1996. Notes by Ruedi Suter.
- [9] Jon F. Carlson, Sunil K. Chebolu, and Ján Mináč. Finite generation of Tate cohomology. 2007. Preprint.
- [10] Sunil K. Chebolu, J. Daniel Christensen, and Ján Mináč. Ghosts in modular representation theory. 2006. Preprint, arXiv:math.RT/0609699.
- [11] Sunil K. Chebolu, J. Daniel Christensen, and Ján Mináč. Groups which do not admit ghosts. *Proc. Amer. Math. Soc.*, to appear. arXiv:math.RT/0610423.
- [12] Charles W. Curtis and Irving Reiner. *Methods of representation theory. Vol. I*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1990. Reprint of the 1981 original, A Wiley-Interscience Publication.

- [13] Charles W. Curtis and Irving Reiner. *Representation theory of finite groups and associative algebras*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. Reprint of the 1962 original, A Wiley-Interscience Publication.
- [14] Peter Freyd. Stable homotopy. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 121–172. Springer, New York, 1966.
- [15] The GAP Group. GAP — Groups, Algorithms, and Programming, version 4.4.9, 2006. <http://www.gap-system.org>
- [16] Daniel Gorenstein. *Finite groups*. Chelsea Publishing Co., New York, second edition, 1980.
- [17] Mark Hovey, Keir Lockridge, and Gena Puninski. The generating hypothesis in the derived category of a ring. *Mathematische Zeitschrift*, 256(4):789–800, 2007.
- [18] Keir Lockridge. The generating hypothesis in the derived category of R -modules. *Journal of Pure and Applied Algebra*, 208(2):485–495, 2007.
- [19] Claus Michael Ringel and Hiroyuki Tachikawa. QF – 3 rings. *J. Reine Angew. Math.*, 272:49–72, 1974.
- [20] Richard G. Swan. Groups with periodic cohomology. *Bull. Amer. Math. Soc.*, 65:368–370, 1959.
- [21] Peter J. Webb. The Auslander-Reiten quiver of a finite group. *Math. Z.*, 179(1):97–121, 1982.
- [22] Peter J. Webb. Reps — a GAP package for modular representation theory, 2007. Available at <http://www.math.umn.edu/~webb/GAPfiles/>

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN ONTARIO, LONDON, ON N6A 5B7, CANADA
E-mail address: `schebolu@uwo.ca`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN ONTARIO, LONDON, ON N6A 5B7, CANADA
E-mail address: `jdc@uwo.ca`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN ONTARIO, LONDON, ON N6A 5B7, CANADA
E-mail address: `minac@uwo.ca`