Zero cycles of degree one on projective varieties of type E_7 and the norm principle

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Abstract

Let G be an anisotropic simple algebraic group of type E_7 over a field k. Using the Gille-Merkurjev norm principle we prove that the G-variety of parabolic subgroups of type 7 (the enumeration of simple roots follows Bourbaki) does not have a zero-cycle of degree 1.

1 Introduction

Let G be a simple algebraic group of inner type over a field k. In the present note we study the following problem: Assume we are given an anisotropic projective G-homogeneous variety over k. Does it have a zero-cycles of degree 1?

It is well known that this question has a negative answer for central simple algebras, i.e., when G is a group of type A_n. For quadratic forms the problem above is equivalent to the Springer theorem. Moreover, this problem is closely related to Serre's question, whether the map $H^1(k, G) \to \prod H^1(K_i, G)$ is injective when K_i are finite field extensions of k such that $\gcd[K_i : k] = 1$. (Serre's question has a positive answer for classical groups due to Bayer-Fluckiger and Lenstra).

Thus, the anisotropic G-varieties of classical types A_n, B_n and partially C_n and D_n do not have zero-cycles of degree 1. (Nevertheless, one should be careful here, since there are examples (due to Parimala) of classical projective homogeneous varieties of *outer type* which have a zero-cycle of degree 1).

The anisotropic varieties of complete flags of type F_4 and E_6 do not have zero-cycles of degree 1 either. This result follows from the existence of the

*Supported partially by DFG, project GI706/1-1.

Rost invariant. The same is true for a variety of complete flags of type E_7. This is proved by Ph. Gille in [Gi97], where the norm principle of algebraic groups was used. The norm principle can be applied to all algebraic groups and gives an alternative proof for the classical groups as well, e.g., of the Springer theorem. In other words, the norm principle provides a uniform approach to all known cases.

In the present note we apply the Gille-Merkurjev norm principle [Gi97], [Me96] to the groups of type ${}^1E_6$ and E_7 and prove that the anisotropic varieties of parabolic subgroups of type 1 (for E_6) and 7 (for E_7) don't have zero-cycles of degree 1. Apart from this, we prove that the anisotropic groups of type E_7 remain anisotropic after odd degree field extensions.

2 Norm principle

Let k be a perfect field with $\operatorname{char} k \neq 2, 3$, G_1 a simple simply connected algebraic group over k of inner type, Δ its Dynkin diagram, and Δ_0 its Tits index (see [Ti65]).

Let G be a connected reductive group over k and

$$1 \to G_1 \to G \xrightarrow{f} T = \mathbb{G}_m \to 1$$

an exact sequence. The cocharacter group T_* can be canonically identified with the group $\mathbb{Z}$. A cocharacter $\varphi \in T_*$ is called f-special, if there is a k-homomorphism $g\colon \mathbb{G}_m \to G$ such that $f \circ g = \varphi$.

Denote as $Z' = G/[G, G]$, C the center of G_1, Z the center of G, and μ the center of $[G, G]$.

We can represent the homomorphism f as a composition $G \to Z' \to T$. In particular, there is the induced homomorphism $\alpha\colon Z'_* \to T_*$ between the cocharacter groups. The exact sequence $1 \to \mu \to Z \to Z' \to 1$ induces a homomorphism $\beta\colon Z'_* \to \mu(-1)$, and the canonical epimorphism $C \to \mu$ induces a map $\gamma\colon C_* \to \mu(-1)$, where $\mu(-1)$ is the Tate twist, i.e., $\mu(-1) = \operatorname{Hom}(\mu_n, \mu)$ for any n with $\mu^n = 1$. For a cocharacter $\varphi \in T_*$ we define a subset $X(\varphi) \subset C_*$ as the set $\gamma^{-1}(\beta(\alpha^{-1}(\{\varphi\})))$.

Assume now that the Dynkin diagram Δ has no multiple edges. Following [Me96, (5.8)] we identify C_* and the character group C^* and consider $X(\varphi)$ as a subset of C^*. Define now $\Omega(\varphi)$ as the set of all subsets $\Theta \subset \Delta$ such that the elements $\{\bar{\omega}_i|_C, i \in \Delta \setminus \Theta\}$, where $\bar{\omega}_i$ denotes the i-th fundamental weight of G_1, generate a subgroup of C^* that intersects $X(\varphi)$.

To motivate our approach we recall first some classical results from the theory of quadratic forms. Let A be a central simple k-algebra of degree ≥ 4 with an orthogonal involution σ of the first kind. There is the following exact sequence of groups:

$$1 \to G_1 = \mathrm{Spin}(A, \sigma) \to G = \Gamma(A, \sigma) \xrightarrow{f} \mathbb{G}_m \to 1,$$

where f is the spinor norm homomorphism.

By [Me96, Lemma 6.2] an odd cocharacter $\varphi \in \mathbb{Z} = (\mathbb{G}_m)_*$ is f-special iff G_1 has a maximal parabolic subgroup of type 1 defined over k. This observation together with [Me96, Lemma 3.4] immediately implies that the group G_1 has a parabolic subgroup of type 1 defined over k iff G_1 has such a parabolic subgroup over an odd degree field extension K/k. This result is known as *the Springer theorem* for quadratic forms.

In the same way [Me96, (6.3)] shows that an orthogonal involution σ as above is hyperbolic over k iff it is hyperbolic over an odd degree field extension K/k. The sequence

$$1 \to G_1 = \mathrm{O}^+(A, \sigma) \to G = \mathrm{GO}^+(A, \sigma) \xrightarrow{f} \mathbb{G}_m \to 1,$$

where f is the multiplier map, plays here a crucial role.

Assume now that G_1 is a simply connected group of type E_7. Let (A, σ, π), where A is a central simple k-algebra with a symplectic involution σ and $\pi\colon A \to A$ a linear map, be *a gift* associated with G_1 (see [Fe72], [Ga01a] and [Ga01b]). An invertible element $h \in A$ is called a similarity if there exists $\alpha_h \in k^*$ (the multiplier of h) such that $\sigma(h)h = \alpha_h \cdot 1$ and $\pi(hah^{-1}) = \alpha_h h\pi(a)h^{-1}$ for all $a \in A$. Then G_1 coincides with the similarities of this gift with multiplier 1. Let G be the group of all similarities. Then G is a connected reductive group and there is the following exact sequence of algebraic groups:

$$1 \to G_1 \to G \xrightarrow{f} T = \mathbb{G}_m \to 1,$$

where the map f is defined on k-points as $h \mapsto \alpha_h$.

In this situation we can prove the following proposition.

2.1 Proposition. *An odd cocharacter $\varphi \in T_* = \mathbb{Z}$ is f-special iff the maximal parabolic subgroup of G_1 of type 7 is defined over k.*

Proof. First we compute $X(\varphi)$. In our situation $[G, G] = G_1$, $Z' = T = \mathbb{G}_m$, $\mu = \mu_2$, $C = \mu_2$, and $Z = \mathbb{G}_m$. Thus, the maps α and γ are the identity maps.

The map $\beta\colon Z'_* = \mathbb{Z} \to \mu_2(-1) = \mathbb{Z}/2$ is the usual factor-map. Therefore, $X(\varphi) = \{\chi\}$, where as χ we denote a unique non-trivial element of $C_* \simeq C^*$.

Assume that an odd cocharacter φ is f-special. By [Me96, Proposition 5.8] the image of $X(\varphi)$ with respect to the Tits homomorphism (see [Ti71]) $C^* \to \mathrm{Br}(k)$ is trivial. This means that the Tits algebra A of G_1 is split.

By [Me96, Lemma 5.4] the group G_1 is isotropic. If the parabolic subgroup of G_1 of type 7 is not defined, then by Tits's classification [Ti65, p. 59] the Tits index of G_1 equals $\Delta_0 = \Delta \setminus \{\alpha_1\}$. But the restriction to the center of the 1-st fundamental weight is trivial (see [Ti90, p. 653]). So, we come to a contradiction with [Me96, Theorem 5.6].

Conversely, since the restriction to the center C of the 7-th fundamental weight equals χ (see [Ti90, p. 653]), it follows that $\Delta \setminus \{\alpha_7\} \in \Omega(\varphi)$. The cocharacter φ is now f-special by [Me96, Theorem 5.6]. □

2.2 Corollary. *Let G_1 be an anisotropic algebraic group of type E_7 over a perfect field k with $\operatorname{char} k \neq 2, 3$. Then the variety X of its parabolic subgroups of type 7 does not have a zero-cycle of degree 1.*

Proof. Let K/k be an odd degree field extension such that X_K is isotropic. We may assume $K \subset \bar{k}$. Then all odd cocharacters $\varphi \in T_* = \mathbb{Z}$ are f_K-special by the previous proposition. By [Me96, Lemma 3.4] the cocharacter $[K:k]\cdot\varphi$ is f-special. Since $[K:k]$ is odd, the previous proposition implies that the variety X is isotropic over k. A contradiction. Therefore the image of the degree homomorphism $\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$ is contained in $2\mathbb{Z}$. In particular, the degree of any closed point of X is even. □

2.3 Remark. If the Tits algebras of G_1 are trivial, then the image of the degree homomophism $\mathrm{CH}_0(X) \to \mathbb{Z}$ equals $2\mathbb{Z}$.

2.4 Corollary. *A group G_1 as in the statement of the previous corollary does not split over an odd degree field extension.*

Using similar arguments one can show the following well-known statement. Opposite to the traditional approach our proof does not use cohomological invariants of Albert algebras.

2.5 Proposition. *Let G_1 be an anisotropic algebraic group of type ${}^1\mathrm{E}_6$ over a perfect field k with $\operatorname{char} k \neq 2, 3$. Then the variety X of its parabolic subgroups of type 1 (resp. 6) does not have a zero-cycle of degree 1.*

Proof. If G_1 has a non-trivial Tits algebra, then the statement is obvious.

Assume that G_1 has trivial Tits algebras. Let J be an Albert algebra associated with G_1 (see [Ga01a]). A k-linear map $h\colon J \to J$ is called a similarity if there exists $\alpha_h \in k^*$ (the multiplier of h) such that $N(h(j)) = \alpha_h N(j)$ for all $j \in J$, where N stands for the cubic norm on J. Then G_1 coincides with the similarities of this Jordan algebra with multiplier 1. Let G be the group of all similarities. Then G is a reductive group and there is the following exact sequence of algebraic groups:

$$1 \to G_1 \to G \xrightarrow{f} T = \mathbb{G}_m \to 1,$$

where the map f is defined on k-points as $h \mapsto \alpha_h$.

Let $\varphi \in T_* = \mathbb{Z}$ be a cocharacter coprime to 3. We claim that φ is f-special iff the maximal parabolic subgroup of G_1 of type 1 (or equivalently of type 6) is defined over k.

First we compute $X(\varphi)$. In our situation $[G,G] = G_1$, $Z' = T = \mathbb{G}_m$, $\mu = \mu_3$, $C = \mu_3$, $Z = \mathbb{G}_m$, and the group $C_* \simeq C^* = \mathbb{Z}/3 = \{0, \bar{\omega}_1|_C, \bar{\omega}_6|_C = -\bar{\omega}_1|_C\}$, where $\bar{\omega}_i|_C$ denotes the restriction of the i-th fundamental weight of G_1 to the center, $i = 1, 6$. Therefore, $X(\varphi) = \{\bar{\omega}_1|_C\}$ or $X(\varphi) = \{\bar{\omega}_6|_C\}$ (it depends on φ).

Assume that a prime to 3 cocharacter φ is f-special. By [Me96, Lemma 5.4] the group G_1 is isotropic. If the parabolic subgroup of G_1 of type 1 is not defined, then by Tits's classification [Ti65, p. 58] the Tits index of G_1 equals $\Delta_0 = \Delta \setminus \{\alpha_2, \alpha_4\}$ (the enumeration of roots follows Bourbaki). But the restrictions to the center of the 2-nd and of the 4-th fundamental weights are trivial (see [Ti90, p. 653]). So, we come to a contradiction with [Me96, Theorem 5.6].

Conversely, assume that the parabolic subgroup of G_1 of type 1 is defined over k. Since $\Delta \setminus \{\alpha_1, \alpha_6\} \in \Omega(\varphi)$, the cocharacter φ is now f-special by [Me96, Theorem 5.6].

Let now K/k be a field extension of degree coprime to 3 such that X_K is isotropic. We may assume $K \subset \bar{k}$. Then all coprime to 3 cocharacters $\varphi \in T_* = \mathbb{Z}$ are f_K-special by the previous arguments. By [Me96, Lemma 3.4] the cocharacter $[K : k] \cdot \varphi$ is f-special. Since $[K : k]$ is coprime to 3, the previous arguments imply that the variety X is isotropic over k. A contradiction. Therefore the image of the degree homomorphism $\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$ is contained in $3\mathbb{Z}$. In particular, the degree of any closed point of X is divisible by 3. $\square$

2.6 Remark. If the Tits algebras of G_1 are trivial, then the image of the degree homomophism $\mathrm{CH}_0(X) \to \mathbb{Z}$ equals $3\mathbb{Z}$.

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