HOMOLOGICAL SYMBOLS AND THE QUILLEN CONJECTURE

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Abstract. We formulate a "correct" version of the Quillen conjecture on the cohomology of linear groups by defining an unstable form of Milnor K-theory and show that this version can be solved by a finite process.

1. Introduction

Let R be a commutative ring with unit and $GL_1(R) = U(R)$ the group of invertible elements in R. Recall that the Milnor K-theory $MK(R)$ of R is the quadratic algebra generated by the group of units $U(R)$ with the relations $u \otimes v = 0$ for $u + v = 1$. Mimicking this construction we define a new algebra $MH(R)$ called the *algebra of homological symbols* of R as a quadratic algebra generated by the homology $HU(R)$ of the group of units with relations to be determined.

The role of this new algebra is to approximate the homology algebra $H(R)$ of all the general linear groups $GL_n(R)$ of finite rank n over R i.e. the direct sum of all the modules $HGL_n(R)$ taken over all nonnegative integers n with the product induced by the matrix block multiplication. The diagonal embeddings of the n-fold products of $U(R)$ with itself in $GL_n(R)$ induce a canonical algebra homomorphism from the tensor algebra on $HU(R)$ to $H(R)$. If the relations in $MH(R)$ vanish in $H(R)$ under this homomorphism, then we get a canonical homomorphism of algebras from $MH(R)$ to $H(R)$.

Definition 1.1. The *algebra of homological symbols* $MH(R)$ is the quadratic algebra generated by $HU(R)$ with relations those elements of the 2-fold tensor product of $HU(R)$ with itself which vanish in $HGL_2(R)$ under the canonical homomorphism.

One challenging problem is to calculate the quadratic algebra $MH(R)$ for any ring R and any homology theory H. To hint to the depth of this problem let $H(R, \infty)$ be the colimit of $HGL_n(R)$ under the maps induced by the tail inclusions $GL_n(R)$ into $GL_{n+1}(R)$ and similarly

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$MH(R,\infty)$ be the corresponding colimit of the modules $MH(R,n)$ of the "rank n elements" in $MH(R)$. Then the induced canonical homomorphism from $MH(R,\infty)$ to $H(R,\infty)$ is an isomorphism for suitable H and R as a consequence of the Bloch-Kato conjecture [12].

In terms of this algebra some of the previous results and conjectures in the unstable range can also be reformulated. For instance, if we replace in Definition 1.1 the homology $HGL_2(R)$ by the homology of an etale model for the classifying space $BGL_2(R)$ for certain rings R and homology theories H then we obtain a new algebra say the algebra of *etale homological symbols* $H^M(R)$ (see also Definition 3.5) which comes with a natural comparison map $MH(R) \to H^M(R)$. Then our results in [2, 3] can be reformulated by saying that for suitable choices for R and H we conjecture that this comparison map is an isomorphism and we prove that if our conjecture holds then Quillen's conjecture fails in high ranks. Recall that Quillen's conjecture [11] states that the mod ℓ cohomology of $GL_n(R)$ is a free module over a certain ring of Chern classes for any regular prime ℓ and suitable rings R containing $1/\ell$ and the ℓ-roots of unity. Our conjecture applied to the case where R is the ring from Quillen's conjecture and H is the mod ℓ homology can be verified for $\ell = 2, 3$ [1, 3, 10].

Based on this evidence, the Conjecture 4.2 should be the "correct" conjecture and the aim of this paper is to offer a group theoretical approach to this conjecture modulo some computer implementation to be done in a follow up paper. This would settle Quillen's conjecture one prime at a time extending the results to $\ell = 5, 7, \ldots$. More precisely, our main result is:

Theorem 1.2. *Let* $R = \mathbb{Z}[1/\ell, \zeta]$ *where* ℓ *is an odd regular prime number,* ζ *a primitive* ℓ*-root of unity and* H *the mod* ℓ *homology theory. Then there is an effective finite process to calculate the relations of* $MH(R)$ *in homological degree two.*

An "effective finite process" means that there are finitely many explicit words we need to check in a finitely presented explicit group. A more precise formulation is given in Theorem 6.1. We will exemplify this process for $\ell = 3$ reproving our previous result but the method used in this paper is general and new. We hope that the algebra $MH(R)$ which looks like some unstable version of Milnor K-theory would be an interesting object to study for other problems as well. For instance, H can be taken to be a generalized homology theory at the classifying space level and there are no restrictions on R.

The paper is organized as follows. After reviewing some group homology in §2, we formulate our conjecture and observe that for a given prime ℓ there are only finitely many relations to calculate §§3,4. Then we replace $GL_2(R)$ by a group SE_2 and prove in §5 that SE_2 is finitely presented, a key result formulated explicitly as Theorem 5.2. In §6 we give an explicit finite method based on Hopf's formula to verify these relations in homology degree two, exemplifying by the case $\ell = 3$. In the final section §7 we make some remarks on the relations in the higher homological degrees.

2. Group homology preliminaries

Let G be a group and $\mathcal{B}_*G$ the (normalized) bar complex with $\mathcal{B}_iG$ the free abelian group generated by the set of symbols $[x_1|...|x_i]$ where x_1,..., x_i are non-identity elements of G. The boundary operator is given by the formula

$$\partial[x_1|...|x_s] = [x_2|...|x_s]+\sum_{j=1}^{s-1}(-1)^j[x_1|...|x_jx_{j+1}|...|x_s]+(-1)^s[x_1|...|x_{s-1}]$$

where $[x_1|...|x_jx_{j+1}|...|x_s]$ equals zero if x_jx_{j+1} is the identity of G. By definition, the group homology of G with $\mathbb{Z}$- respectively $\mathbb{Z}/\ell$-coefficients is the homology of the complex $\mathcal{B}_*G$ respectively $\mathcal{B}_*G\otimes\mathbb{Z}/\ell$ and is denoted by $H_*(G,\mathbb{Z})$ respectively $H_*(G,\mathbb{Z}/\ell)$ where ℓ is a prime number. There are various ways to produce cycles representing elements in these homology groups.

First of all, the complex $\mathcal{B}_*G$ is endowed with a shuffle product

$$[x_1|...|x_i]\wedge[x_{i+1}|...|x_{i+s}] = \sum(-1)^\sigma[x_{\sigma(1)}|...|x_{\sigma(i+s)}]$$

where the sum is over all the permutations σ of $i+s$ letters that shuffle 1,..., i with $i+1$,..., $i+s$ (i.e. $\sigma^{-1}(1) < ... < \sigma^{-1}(i)$ and $\sigma^{-1}(i+1) < ... < \sigma^{-1}(i+s)$) and $(-1)^\sigma$ is the signature of σ. With respect to the shuffle product, $\mathcal{B}_*G$ is an (anti-)commutative, associative, and unital graded algebra. We remark that

Lemma 2.1. *The Leibniz formula*

$$\begin{aligned}\partial([x_1|...|x_i]\wedge[x_{i+1}|...|x_{i+s}]) &= (\partial[x_1|...|x_i])\wedge[x_{i+1}|...|x_{i+s}]\\ &\quad+(-1)^i[x_1|...|x_i]\wedge(\partial[x_{i+1}|...|x_{i+s}])\end{aligned}$$

holds if only if $x_jx_k = x_kx_j$ for all $j \leq i < k$.

The proof is immediate. In particular, we have

Corollary 2.2. *The element of $\mathcal{B}_iG$ defined by*

$$\langle x_1, x_2, ..., x_i\rangle = [x_1]\wedge[x_2]\wedge...\wedge[x_i]$$

for x_1,x_2,..., x_i elements of G commuting with one another is a cycle that modulo the boundaries is i-linear and skew-symmetric in x_1,..., x_i.

One way to find cycles mod ℓ is to define for each non-negative integer k and each ℓ-torsion element v of G an element in $\mathcal{B}_{2k}G$ by the formula

$$[v]^{(k)} = \sum [v^{i_1}|v|v^{i_2}|v|...|v^{i_k}|v]$$

where the sum is taken over all integers i_1,i_2,..., i_k from 1 to $\ell - 1$ with the convention that the sum is the identity element [ ] if $k = 0$. An easy calculation shows that $[v]^{(1)} \wedge [v]^{(1)} = 2[v]^{(2)}$ where $v^\ell = 1$. By induction we establish that

Lemma 2.3. *For any ℓ-torsion element v of G and any non-negative integers k and s the following equation*

$$[v]^{(k)} \wedge [v]^{(s)} = \binom{r+s}{s} [v]^{(k+s)}$$

holds in $\mathcal{B}_*G$.

Again an easy calculation shows that $\partial[v]^{(1)} = \ell[v]$ where $v^\ell = 1$ and by using Lemma 2.3 and Lemma 2.1 we can establish by induction the following:

Proposition 2.4. *For each ℓ-torsion element v of G and positive integer k the following equation*

$$\partial[v]^{(k)} = \ell[v]^{(k-1)} \wedge [v]$$

holds in $\mathcal{B}_*G$. In particular, $[v]^{(k)}$ is a cycle mod ℓ.

Remark 2.5. For a given group G, the short exact sequence of complexes

$$0 \to \mathcal{B}_*G \otimes \mathbb{Z}/\ell \xrightarrow{1\otimes\ell} \mathcal{B}_*G \otimes \mathbb{Z}/\ell^2 \to B_*G \otimes \mathbb{Z}/\ell \to 0$$

where the homomorphism $1 \otimes \ell$ is the multiplication by ℓ on the second factor, induces a long exact sequence of homology groups

$$\to H_i(G, \mathbb{Z}/\ell) \to H_i(G, \mathbb{Z}/\ell^2) \to H_i(G, \mathbb{Z}/\ell) \xrightarrow{\beta} H_{i-1}(G, \mathbb{Z}/\ell) \to ...$$

where β is a Bockstein homomorphism. By a diagram chasing, Proposition 2.4 implies that modulo the boundaries

$$\beta[v]^{(k)} = [v]^{(k-1)} \wedge [v]$$

for any ℓ-torsion element v of G and positive integer k.

By Lemma 2.1, if A is an abelian group, then $\mathcal{B}_*A$ is a differential graded algebra inducing the same structure on $\mathcal{B}_*A \otimes \mathbb{Z}/\ell$ and hence, inducing a graded algebra structure on $H_*(A, \mathbb{Z}/\ell)$. Moreover let ${}_\ell A$ denote the ℓ-torsion subgroup of A and $\Gamma({}_\ell A)$ the algebra of divided powers generated in degree two by ${}_\ell A$ over the field with ℓ elements $\mathbb{F}_\ell$. Then Lemma 2.3 and Proposition 2.4 shows that the map ${}_\ell A \to H_2(A, \mathbb{Z}/\ell)$ induced by $v \mapsto [v]^{(1)}$ can be extended to a graded algebra homomorphism

$$\Gamma({}_\ell A) \to H_*(A, \mathbb{Z}/\ell). \tag{2.1}$$

Similarly, let $\Lambda(A \otimes \mathbb{Z}/\ell)$ denote the exterior algebra generated in degree one by $A \otimes \mathbb{Z}/\ell$ over $\mathbb{F}_\ell$. Then Corollary 2.2 shows that the map $A \otimes \mathbb{Z}/\ell \to H_1(A, \mathbb{Z}/\ell)$ induced by $a \otimes 1 \mapsto [a]$ extends to a graded algebra homomorphism

$$\Lambda(A \otimes \mathbb{Z}/\ell) \to H_*(A, \mathbb{Z}/\ell). \tag{2.2}$$

Theorem 2.6 ([4] p. 126)**.** *If ℓ is an odd prime and A is an abelian group, then the maps* (2.1) *and* (2.2) *induce a natural isomorphism of graded algebras*

$$\Gamma({}_\ell A) \otimes \Lambda(A \otimes \mathbb{Z}/\ell) \approx H_*(A, \mathbb{Z}/\ell).$$

For later use, let G and K be two groups and recall the Künneth isomorphism [5], p. 218,

$$H_*(G, \mathbb{Z}/\ell) \otimes H_*(K, \mathbb{Z}/\ell) \approx H_*(G \times K, \mathbb{Z}/\ell) \tag{2.3}$$

induced by the natural map

$$[x_1|...|x_i] \otimes [x_{i+1}|...|x_{i+s}] \mapsto [x_1 \times 1|...|x_i \times 1] \wedge [1 \times x_{i+1}|...|1 \times x_{i+s}]$$

where x_j is an element of G for $j \leq i$ and an element of K for $j > i$.

If both G and K are abelian, the left hand side of (2.3) is a graded algebra under the rule

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(a \wedge c) \otimes (b \wedge d)$$

where a, c are (homogenous) elements of $H_*(G, \mathbb{Z}/\ell)$ with $|c|$ the degree of c and similarly for b, d in $H_*(K, \mathbb{Z}/\ell)$. In this setting, (2.3) is a graded algebra isomorphism.

3. An algebra of homological symbols

For the rest of this paper we make the following:

Notation 3.1. Let $R = \mathbb{Z}[\zeta, 1/\ell]$ be a ring of S-integers where ℓ is a regular odd prime number and ζ is a primitive ℓ-root of unity.

The motivation for introducing the concept of "homological symbol" comes from the following construction. Denote by GL_n the discrete group of invertible $n \times n$-matrices over R and by BGL_n its (natural) classifying space. Recall that the group homology of GL_n is the same as the singular homology of BGL_n. This classifying space fits into a diagram

$$BGL_1^{\times n} \xrightarrow{\iota_n} BGL_n \xrightarrow{f_n} BGL_n^{et}$$

where ι_n is the classifying space map induced by the canonical inclusion $GL_1^{\times n} \subset GL_n$ and f_n is the natural etale approximation map at the prime ℓ as constructed by [7]. The above diagram induces a diagram in homology

$$(3.1) \quad H_i(GL_1^{\times n}, \mathbb{Z}/\ell) \xrightarrow{\iota_{n*}} H_i(GL_n, \mathbb{Z}/\ell) \xrightarrow{f_{n*}} H_i(BGL_n^{et}, \mathbb{Z}/\ell)$$

whose terms will be denoted from the left to the right by T_{in}, H_{in}, and H_{in}^{et}. As i and n run over all non-negative integers, these terms form three double graded algebras denoted respectively by T_{**}, H_{**}, and H_{**}^{et}, where the product is induced by the matrix-block multiplication $GL_j \times GL_k \to GL_{j+k}$.

It is immediate that T_{**} can be identified via the Künneth isomorphism (2.3) applied to $G = K = GL_1$ with a tensor algebra generated in rank one by $H_*(GL_1, \mathbb{Z}/\ell)$. Moreover, by the Dirichlet Unit Theorem, GL_1 is the product of a cyclic group containing ζ and a free abelian group of rank $r = (\ell - 1)/2$ generated by the fundamental units say e_1, e_2, ..., e_r. Hence, by the Theorem 2.6 we have that

Proposition 3.2. *Each element of $H_*(GL_1, \mathbb{Z}/\ell)$ is represented by a formal sum of cycles of the form*

$$[\zeta]^{(k)} \wedge \langle u_1, ..., u_i \rangle$$

with $\mathbb{Z}/\ell$-coefficients, where $\{u_1, ..., u_i\}$ runs over all the subsets of $\{\zeta, e_1, ..., e_r\}$ and k over all non-negative integers.

Notation 3.3. Via the composition of the maps (3.1) we have a graded algebra homomorphism denoted by g from T_{**} to H_{**}^{et}.

To describe the kernel of the map g let $\alpha : GL_1^{\times 2} \to GL_1^{\times 2}$ send each pair of units $u \times v$ to $u^{-1}v \times uv$ and α_* denote by abuse the composed map

$$H_*(GL_1, \mathbb{Z}/\ell)^{\otimes 2} \approx H_*(GL_1^{\times 2}, \mathbb{Z}/\ell) \xrightarrow{\alpha_*} H_*(GL_1^{\times 2}, \mathbb{Z}/\ell) \subset T_{**}$$

where the first isomorphism is a Künneth isomorphism. If z is an element of $H_*(GL_1, \mathbb{Z}/\ell)$ represented by $[\zeta]^{(j)} \wedge \langle v_1, ..., v_s \rangle$ as in Proposition

3.2, a short calculation shows that

$$\begin{aligned}(3.2)\qquad \alpha_*([\zeta]^{(k)}\wedge\langle u_1,...,u_i\rangle\otimes z) &= [\zeta^{-1}\times\zeta]^{(k)}\wedge[\zeta\times\zeta]^{(j)}\\ &\quad \wedge\langle u_1^{-1}\times u_1,...,u_i^{-1}\times u_i\rangle\\ &\quad \wedge\langle v_1\times v_1,...,v_s\times v_s\rangle.\end{aligned}$$

where homology classes are eventually denoted by their representative cycles.

Theorem 3.4. *[3, 8] The graded algebra homomorphism $g: T_{**}\to H_{**}^{et}$ is surjective and its kernel is generated as a two-sided ideal by rank two elements of the form* (3.2) *where i, k run over all non-negative integers such that $i-k$ is positive and even, $\{u_1,...,u_i\}$ runs over all the subsets of $\{\zeta,e_1,...,e_r\}$, and z runs over a set of generators for $H_*(GL_1,\mathbb{Z}/\ell)$.*

Based on this theorem, the purpose of this paper is to study the following algebra:

Definition 3.5. Let H_{**}^M be the *algebra of etale homological symbols at ℓ* defined as the bi-graded algebra generated in rank one by $H_*(GL_1,\mathbb{Z}/\ell)$ with relations in rank two given by (3.2) subject to the conditions in the Theorem 3.4.

Remark 3.6. Theorem 3.4 can be reformulated by saying that $g: T_{**}\to H_{**}^{et}$ induces a natural isomorphism of bi-graded algebras $H_{**}^M\approx H_{**}^{et}$. The point of the definition is that H_{**}^M is a quadratic algebra (with respect to the rank) that mimics a similar construction for the Milnor K-theory but starting with $H_*(GL_1,\mathbb{Z}/\ell)$ instead of GL_1. To emphasis this point of view, (3.2) correspond to some "homological Steinberg relations at ℓ" via the Künneth isomorphism

$$H_*(GL_1^{\times 2},\mathbb{Z}/\ell)\approx H_*(GL_1,\mathbb{Z}/\ell)\otimes H_*(GL_1,\mathbb{Z}/\ell)$$

induced by the map sending

$$[u\times v]^{(k)}\mapsto\sum_{h=0}^{k}[u]^{(h)}\otimes[v]^{(k-h)}\text{ and }[a\times b]\mapsto[a]\otimes 1+1\otimes[b]$$

where a, b, u, v are elements of GL_1 with $u^\ell=v^\ell=1$.

4. A VANISHING CONJECTURE

From the diagram (3.1), the homomorphism $g: T_{**}\to H_{**}^{et}$ factorizes through a graded algebra homomorphism $\iota: T_{**}\to H_{**}$ induced by the canonical inclusions. A natural question is whether ι induces a natural graded algebra homomorphism $H_{**}^M\to H_{**}$, i.e. whether ι maps the relations in H_{**}^M to zero in H_{**}. Equivalently, we ask whether the algebra of etale homological symbols is the same as the algebra of general homological symbols as defined in Definition 1.1. To answer this question for a given ℓ, only finitely many relations need to be verified in H_{**}.

To see this let SL_n be the subgroup of GL_n consisting of matrices with determinant 1 and construct a commutative diagram

$$\begin{tikzcd}
GL_1^{\times 2} \arrow[r, "\tau\times 1"] \arrow[d, "\alpha"'] & SL_2 \times GL_1 \arrow[d, "\mu"'] \\
GL_1^{\times 2} \arrow[r, "\iota"] & GL_2
\end{tikzcd}$$

where $\tau : GL_1 \to SL_2$ sends a unit u to the diagonal matrix $\begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}$ and μ sends a matrix A and a unit u to their matrix by scalar product. By passing to homology we see that $\tau_*(y) = 0$ implies

$$\iota_*\alpha_*(y \otimes z) = \mu_*(\tau_*(y) \otimes z) = 0$$

for all z in $H_*(GL_1, \mathbb{Z}/\ell)$ and conversely, by a spectral sequence argument [3] if the above equation holds for all z it follows that $\tau_*(y) = 0$ in H_*SL_2.

Proposition 4.1. *[3] The relations in H^M_{**} vanish in H_{**} under the map ι induced by the canonical inclusions, if and only if for each subset $\{u_1, ..., u_i\}$ of $\{\zeta, e_1, ..., e_r\}$ and each integer k with $i - k$ positive and even there are two chains y and z in $\mathcal{B}_*SL_2$ such that*

$$[\tau(\zeta)]^{(k)} \wedge \langle \tau(u_1), ..., \tau(u_i)\rangle = \partial y + \ell z.$$

To analyze the last equation, let SE_2 be the group generated by the disjoint union of GL_1 and R subject to the relations [6]

$$\lceil x \rceil \lceil 0 \rceil \lceil y \rceil = -\lceil x + y \rceil \tag{4.1}$$

$$u^{-1}\lceil x \rceil = \lceil xu^2 \rceil u \tag{4.2}$$

$$\lceil u \rceil \lceil u^{-1} \rceil \lceil u \rceil = -u^{-1} \tag{4.3}$$

for $u, -1 \in GL_1$ and $x, y, 0 \in R$ together with the relations in GL_1, where the elements of R are written between $\lceil$ and $\rceil$ to distinguish them from those of GL_1. It is immediate that by sending $\lceil x \rceil$ to $\begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}$ and u to $\tau(u)$ for $x \in R$ and $u \in GL_1$ we get a well-defined group homomorphism $SE_2 \to SL_2$ which is an isomorphism if R is Euclidean [6]. Moreover the map $\tau : GL_1 \to SL_2$ factorizes through the canonical inclusion $GL_1 \to SE_2$. In this slightly more general context, we formulate a vanishing conjecture:

Conjecture 4.2. *There are chains y and z in $\mathcal{B}_*SE_2$ such that the equations in Proposition 4.1 (same restrictions) hold with each $\tau(u)$ replaced by the image of $u \in GL_1$ in SE_2 under the canonical inclusion.*

In other words, $[\zeta]^{(k)} \wedge \langle u_1, ..., u_i\rangle$'s vanish in $H_*(SE_2, \mathbb{Z}/\ell)$ under the stated conditions. If this conjecture is true for a given ℓ, it is obvious that only finitely many verifications are needed.

Remark 4.3. SE_2 is generated by $\lceil 0 \rceil$, $\lceil \frac{1}{\ell} \rceil$, and ζ. A proof will be given in the next section.

Remark 4.4. One way to find evidence for the Conjecture 4.2 in the homology degree two is via Hopf's formula [4]:

$$H_2(SE_2, \mathbb{Z}) \approx \frac{K \cap [F, F]}{[F, K]}, \tag{4.4}$$

where according to Remark 4.3, F is the free group generated by $\lceil 0 \rceil$, $\lceil \frac{1}{\ell} \rceil$, and ζ and K is the kernel of the canonical map $F \to SE_2$. For A, B subsets of F, the notation $[A, B]$ means the subgroup of F generated by the commutators $[a, b] = aba^{-1}b^{-1}$ with $a \in A$ and $b \in B$. The isomorphism (4.4) is induced by the map $\mathcal{B}_2SE_2 \to F$ sending $[g|h]$ to $\phi(g)\phi(h)\phi(gh)^{-1}$ where g, h are in SE_2 and ϕ is a right inverse of the canonical map $F \to SE_2$. In particular, if $gh = hg$, then $\langle g, h\rangle$ maps to the commutator $[\phi(g), \phi(h)]$ modulo $[F, K]$.

5. A FINITE PRESENTATION FOR SE_2

To begin the analysis of the conjecture as suggested by Remark 4.4 we simplify the notations by setting $a = \lceil 0 \rceil$ and $b = \lceil \frac{1}{\ell} \rceil$ and we will refer to (4.1) as *type I relations*, to (4.2) as *type II relations* and to (4.3) as *type III relations*. Also, we introduce further notations:

Notation 5.1. For each $i \mod \ell$ define in SE_2

$$-1 = a^2, \ b_i = \zeta^{ri} b \zeta^{ri}, \ (b_0 = b)$$

and for each $i \in \{1, ..., r\}$ define $u_i = a^{-1}w_i^{-1}$ where

$$w_i \ = \ (-1)^{r-1}(bb_i^{-1})^{\ell} a \prod_{j=0}^{\ell-2} (b_{ji}a)^{\ell-j} b_{(\ell-1)i}(bb_i^{-1})^{\ell}.$$

Finally, we define in SE_2

$$\ell = (-1)^r \zeta^{r^2+\frac{1}{2}(r^2+r)}(u_1u_2...u_r)^2.$$

With these notations and terminology, the goal of this section is to prove that SE_2 has a finite presentation:

Theorem 5.2. *Under the assumption that* -1*,* ζ*,* $1-\zeta^i$ *for* i *integer from* 1 *to* r *generate the group of units* GL_1*, the group* SE_2 *is generated by* ζ*,* a*, and* b *subject to the following* $\frac{1}{8}(9\ell^2+4\ell+59)$ *relations:*

$$-b = b(-1),\ -\zeta = \zeta(-1) \tag{5.1}$$

$$(a(ba)^{\ell})^3 = 1,\ ((ba)^{\ell}a^{-1})^3 = -1 \tag{5.2}$$

$$(-1)^2 = 1,\ \zeta^{\ell} = 1,\ u_iu_j = u_ju_i,\ u_i\zeta = \zeta u_i \tag{5.3}$$

$$w_i^2 = -1,\ \zeta a\zeta = a,\ b_su_i^{-1}(ab_{s+i})^2 = -u_ib_sab_{s+2i} \tag{5.4}$$

$$b_sab_t = b_tab_s,\ b_sa = \ell(b_sa)^{\ell^2}\ell^{-1},\ b\prod_{\nu=1}^{\ell-1}(ab_{\nu}) = a \tag{5.5}$$

where i,j *run over integers from* 1 *to* r*, and* s,t *over integers* $\bmod \ell$*.*

Remark 5.3. The assumption that GL_1 is generated by the cyclotomic units as in the Theorem 5.2 is equivalent with the assumption that the class number of $\mathbb{Q}(\zeta+\zeta^{-1})$ is $h^+=1$, by [14], p. 145. By the same source, $h^+=1$ for all $\ell \leq 67$ for instance.

In particular, this theorem implies Remark 4.3. The proof of the theorem is given by a sequence of lemmas. First we prove that the relations occurring in Theorem 5.2 hold in SE_2. In what follows we tacitly assume the relations in GL_1.

Lemma 5.4. *In* SE_2*, the element* -1 *is central while the elements* b_ia *for* $i \bmod \ell$ *commute with one another. In particular,* (5.1) *and the first relation in* (5.5) *hold.*

Proof. By type I relations $-1=a^2$ as in Notation 5.1. Furthermore, by type II relations the unit -1 is central in SE_2 and this fact combined with the previous statement is equivalent to (5.1). Also, the type II relations together with the fact that $\zeta^{2r}=\zeta^{\ell-1}=\zeta^{-1}$ imply that

$$\lceil \frac{\zeta^i}{\ell} \rceil = \zeta^{ri}\lceil \frac{1}{\ell} \rceil \zeta^{ri} = b_i \tag{5.6}$$

for each $i \bmod \ell$. The fact that the elements b_ia commute with one another follows now from the type I relations. This property is the first relation in (5.5). □

Lemma 5.5. *The unit* $1-\zeta^i$ *in* GL_1 *is represented in* SE_2 *by the word* u_i *as defined in Notation 5.1 for all* i *from* 1 *to* r*. In particular, the relations* (5.3) *hold.*

Proof. In the ring R we have the following equation:

$$\frac{1}{1-\zeta^i} = \frac{1}{\ell}\sum_{j=0}^{\ell-1}(\ell-j)\zeta^{ij}$$

which by type I relations, (5.6), and the fact that -1 is central implies

$$\lceil \frac{1}{1-\zeta^i} \rceil = \prod_{j=0}^{\ell-2} (b_{ij}a)^{\ell-j} b_{(\ell-1)i}(-1)^r. \tag{5.7}$$

Similarly, by type I relations we have

$$\lceil \zeta^i \rceil = \lceil \frac{\ell\zeta^i}{\ell} \rceil = (b_i a)^{\ell-1} b_i \tag{5.8}$$

which plugged in the equation

$$\lceil 1-\zeta^i \rceil \lceil 0 \rceil \lceil \zeta^i \rceil = -\lceil 1 \rceil$$

and after multiplying the equation on the right by a gives:

$$\lceil 1-\zeta^i \rceil a (b_i a)^{\ell} = -(ba)^{\ell}.$$

Solving this equation by using Lemma 5.4 and $-1 = a^2$ gives

$$\lceil 1-\zeta^i \rceil = (bb_i^{-1})^{\ell} a. \tag{5.9}$$

Plugging (5.7) and (5.9) in the type III relation

$$(1-\zeta^i)^{-1} = -\lceil 1-\zeta^i \rceil \lceil (1-\zeta^i)^{-1} \rceil \lceil 1-\zeta^i \rceil$$

we deduce that

$$(1-\zeta^i)^{-1} = (-1)^{r-1} (bb_i^{-1})^{\ell} a \prod_{j=0}^{\ell-2} (b_{ji}a)^{\ell-j} b_{(\ell-1)i} (bb_i^{-1})^{\ell} a.$$

Using now Notation 5.1, we conclude the proof of the lemma. □

Corollary 5.6. *The unit element ℓ in GL_1 is represented in SE_2 by the word given in Notation 5.1.*

Proof. It follows from Lemma 5.5 and the equation

$$\ell = \prod_{i=1}^{\ell-1} (1-\zeta^i) = \prod_{i=1}^{r} (-\zeta^{r+i})(1-\zeta^i)^2$$

which holds in the group GL_1. □

Remark 5.7. If a ring element x is written as

$$x = \frac{1}{\ell^{2k+1}} \sum_{\nu=1}^{N} \zeta^{i_\nu} \tag{5.10}$$

where k, i_1,..., i_N are nonnegative integers, then by type II and type I relations we have

$$\lceil x \rceil = \ell^k b_{i_1} a b_{i_2} a \ldots b_{i_{N-1}} a b_{i_N} \ell^k (-1)^{N-1}. \tag{5.11}$$

In particular, combining this equation with Lemma 5.5, we conclude that a, b, ζ generate SE_2 under the assumption of Theorem 5.2.

Proposition 5.8. *In SE_2 all the relations* (5.1) - (5.5) *hold.*

Proof. The relations (5.2) are obtained by combining the type I relation $\lceil -1\rceil\lceil 0\rceil\lceil 1\rceil = -\lceil 0\rceil$ with (5.8) for $i = 0$ and the type III relations applied to the units 1 and -1.

The relations $\zeta a\zeta = a$ and $w_i^2 = -1$ follow from the type II relations applied to ζ, a, and u_i, the later in view of Lemma 5.5, and $a^2 = -1$. This proves the first two relations in (5.4). Applying type I and II relations to the identity

$$\frac{1}{\ell}\zeta^j(1-\zeta^i)^2 + \frac{2}{\ell}\zeta^{j+i} = \frac{1}{\ell}(\zeta^j + \zeta^{j+2i}),$$

we obtain in view of Lemma 5.5, the last relation in (5.4).

The second relation in (5.5) comes from type I and II relations:

$$b_i = \lceil\frac{\zeta^i}{\ell}\rceil = \lceil\frac{\zeta^i\ell^2}{\ell^3}\rceil = \ell(b_i a)^{\ell^2-1} b_i \ell$$

combined with Corollary 5.6. The last relation in (5.5) follows in a similar manner from the cyclotomic equation in R. This completes the proof in view of Lemma 5.5 and 5.4. □

Now we show that all the relations in SE_2 are a consequence of those in Theorem 5.2. To this end we take the equation (5.11) as a definition for its left hand side. In what follows we assume that GL_1 is generated by the cyclotomic units as in Remark 5.3. In agreement with Lemma 5.5 we define $1-\zeta^i$ to be the word u_i and thus GL_1 is generated by -1, ζ, and u_i for i from 1 to r. Also *we assume that the relations in Theorem 5.2 hold* for the rest of the section. In particular, by (5.3) we can assume that all the relations in GL_1 hold in SE_2.

Lemma 5.9. *For all i we have $u_i a u_i = a$ and in particular, $\ell a\ell = a$.*

Proof. By Notation 5.1, $u_i^{-1} = w_i a$, so that

$$u_i^{-1} a u_i^{-1} = w_i a^2 w_i a = -w_i^2 a = a$$

since $a^2 = w_i^2 = -1$ is a central element of order two by (5.1), (5.3), and (5.4). The second relation $\ell a\ell = a$ follows now from the definition of ℓ as in Notation 5.1, the first relation, and (5.4). □

Corollary 5.10. *For every unit u in GL_1 we have $uau = a$.*

Lemma 5.11. *The definition* (5.11) *is invariant under the following three elementary operations applied to* (5.10)*: (1) permuting $i_1,\ldots,i_N$,* *(2) multiplying the numerator and denominator by ℓ^2, and (3) adding $1+\zeta+...+\zeta^{\ell-1}$ to the numerator.*

Proof. When permuting i_1,...,i_N we use (5.5) to show that the right hand side of (5.11) is invariant. Under the operation (2), we show that the same formula is invariant by induction on N. The case $N=1$:

$$\ell^k b_{i_1} \ell^k = \ell^{k+1}(b_{i_1}a)^{\ell^2} a^{-1}\ell^{k+1}$$

can be reduced to $\ell^{-1}a^{-1} = a^{-1}\ell$ and this last relation follows from Lemma 5.9. The induction step follows by (5.5). Under the operation (3), the formula (5.11) becomes

$$\ell^k b_{i_1}a...b_{i_N}aba...b_{\ell-1}\ell^k(-1)^{N+\ell-1}.$$

By (5.5), $ba...b_{\ell-1}$ can be replaced by a^2 and then we can use $a^2=-1$ to prove the invariance. ☐

Lemma 5.12. *Any two representations of a ring element x as a sum of the form* (5.10) *are related by a sequence of elementary operations as defined in Lemma 5.11.*

This lemma is a consequence of the irreducibility of the cyclotomic polynomial $1+t+...+t^{\ell-1}$ over the integers. The proof is left as an exercise in elementary algebra.

Proposition 5.13. *The type I relations in SE_2 are a consequence of Theorem 5.2 relations.*

Proof. Let two ring elements be represented as

$$x = \frac{1}{\ell^{2k+1}}\sum_{\nu=1}^{N}\zeta^{i_\nu}, \quad y = \frac{1}{\ell^{2t+1}}\sum_{\mu=1}^{M}\zeta^{j_\mu}.$$

Then their sum $x+y$ can be represented as

$$x+y = \frac{1}{\ell^{2k+1}}(\sum_{\nu=1}^{N}\zeta^{i_\nu} + \sum_{\mu=1}^{M}\ell^{2i}\zeta^{j_\mu})$$

where we assume that $i=k-t$ is a nonnegative integer. By Lemma 5.12 any other similar representation of $x+y$ is related to the above one by a sequence of elementary transformations. By Lemma 5.11 we can use the above representation to verify the type I relation:

$$\lceil x+y\rceil = \ell^k b_{i_1}a...b_{i_N}a(b_{j_1}a)^{\ell^{2i}}...(b_{j_M}a)^{\ell^{2i}}a^{-1}\ell^k(-1)^{N+\ell^2M-1}.$$

By (5.5) and Lemma 5.11 applied to the ℓ^{2i}-power factors, the above formula equals

$$\ell^k b_{i_1}a...b_{i_N}a\ell^{-i}b_{j_1}a...ab_{j_M}\ell^{k-i}(-1)^{N+M-1}.$$

By Lemma 5.9, we have $a\ell^{-i} = \ell^k a \ell^t$ and thus, the formula above is $-\lceil x\rceil\lceil 0\rceil\lceil y\rceil$ proving the proposition. ☐

Lemma 5.14. *If the type II relations are satisfied by a unit u in GL_1 and two ring elements x and y in R, then the type II relation will be satisfied by u and the ring element $x+y$.*

Proof. By Proposition 5.13 and Corollary 5.10 we have the following calculation:

$$\begin{aligned}\lceil x+y\rceil &= -\lceil x\rceil a\lceil y\rceil = -u\lceil xu^2\rceil uau\lceil yu^2\rceil u\\ &= u\lceil xu^2+yu^2\rceil u = u\lceil (x+y)u^2\rceil u.\end{aligned}$$

Observe that the xu^2+xu^2 and $(x+y)u^2$ are related by elementary transformations and Lemma 5.11 applies. ☐

Lemma 5.15. *If the type II relations are satisfied by two units u and v in GL_1 and for all ring elements x in R, then the type II relations are satisfied by the product uv and the inverse u^{-1} and all ring elements.*

Proof. Just plug-in $y = xu^2$ in the equation $v\lceil xv^2\rceil v = \lceil x\rceil$, multiply on the left and right by u and use $uv = vu$ which holds in GL_1. So that uv satisfies type II relations for any y and a similar argument works for u^{-1}. ☐

Lemma 5.16. *The units ζ and u_i for i from 1 to r satisfy the type II relations for any ring element x.*

Proof. In view of Lemma 5.14 it is enough to check the case when $x = \zeta^j\ell^{-2k-1}$. For ζ this case follows directly by definitions and the relations in GL_1:

$$\zeta\lceil\frac{\zeta^j}{\ell^{2k+1}}\rceil\zeta = \ell^k\zeta b_j\zeta\ell^k = \ell^k b_{j-2}\ell^k = \lceil\frac{\zeta^{j-2}}{\ell^{2k+1}}\rceil.$$

For u_i we have to prove that:

$$u_i\lceil\frac{\zeta^j u_i^2}{\ell^{2k+1}}\rceil u_i = u_i\ell^k\lceil\frac{\zeta^j-2\zeta^{j+i}+\zeta^{j+2i}}{\ell}\rceil\ell^k u_i = \lceil\frac{\zeta^j}{\ell^{2k+1}}\rceil.$$

By using the definition (5.11) and Proposition 5.13 we reduce the identity above in a similar manner as in the proof of Proposition 5.8 to

$$-u_i b_j a b_{j+2i}(b_{j+i}ab_{j+i})^{-1}a^{-1}u_i = b_j.$$

But this is a rearrangement of the last relation in (5.4).

☐

Proposition 5.17. *In SE_2 the type II relations follow from the relations in Theorem 5.2.*

Proof. It follows from Lemma 5.15, Lemma 5.16 and the the way GL_1 is generated. □

Lemma 5.18. *The relation* (5.9) *is a consequence of the Theorem 5.2 relations.*

Proof. We represent $1-\zeta^i$ by the following sum:

$$1-\zeta^i = \frac{1}{\ell}(2\ell + \ell\zeta + ... + \ell\zeta^{i-1} + \ell\zeta^{i+1} + ... + \ell\zeta^{\ell-1}).$$

Applying the defining formula (5.11), we get

$$\lceil 1-\zeta^i \rceil = (ba)^{2\ell}(b_1a)^{\ell}...(b_{i-1}a)^{\ell}(b_{i+1}a)^{\ell}...(b_{\ell-1}a)^{\ell}a^{-1}(-1)^{\ell^2-1}.$$

By using (5.5), the last formula becomes

$$(ba)^{\ell}(bab_1a...b_{\ell-1}a)^{\ell}(b_ia)^{-\ell}a^{-1} = (ba)^{\ell}a^{2\ell}(b_ia)^{-\ell}a^{-1}$$

which in view of $a^2=-1$ and (5.5) again, implies (5.9). □

Lemma 5.19. *The units* -1, 1, *and* u_i *for* i *from* 1 *to* r *satisfy the type III relations.*

Proof. In view of Proposition 5.13, the first two equations to verify are:

$$-1 = \lceil 1 \rceil^3 = \lceil \frac{\ell}{\ell} \rceil^3 = ((ba)^{\ell}a^{-1})^3 \text{ and } 1 = \lceil -1 \rceil^3 = (-a\lceil 1 \rceil^{-1}a^{-1})^3$$

and these follow from (5.2). Again by Proposition 5.13 the relation (5.7) holds and in view of Notation 5.1 can be rewritten as

$$\lceil \frac{1}{1-\zeta^i} \rceil = -a^{-1}(bb_i^{-1})^{-\ell}w_i(bb_i^{-1})^{-\ell}$$

Combining this equation with Lemma 5.18, the type III equation

$$-u_i^{-1} = \lceil u_i \rceil \lceil u_i^{-1} \rceil \lceil u_i \rceil$$

can be rewritten as $u_i^{-1} = w_ia$ which is true by definition. □

Proposition 5.20. *In* SE_2 *the type III relations follow from Theorem 5.2 relations.*

Proof. The main observation is the fact that by using Proposition 5.17, if a unit u satisfies the type III relation then so does uv^2 for any unit v. Indeed,

$$\lceil uv^2 \rceil \lceil u^{-1}v^{-2} \rceil \lceil uv^2 \rceil = v^{-1}\lceil u \rceil \lceil u^{-1} \rceil \lceil u \rceil v^{-1} = -u^{-1}v^{-2}.$$

Since 1, -1 and u_i generate GL_1 up to squares, the conclusion now follows from Lemma 5.19. □

This finishes the proof of Theorem 5.2.

6. Evidence for the conjecture 4.2

Let $1 \to K \to F \to SE_2 \to 1$ be the finite presentation as given for example by Theorem 5.2. In order to verify that a two dimensional cycle σ for SE_2 as in Proposition 4.1 is a boundary mod ℓ we first write this cycle as a word w in F via the explicit map given in Remark 4.4. Now observe that $F \mod [F, F]$ is torsion free, so that if $w = x^\ell$ has a solution x in $F \mod [F, F]$ knowing that w is an element in $[F, F] \cap K$ and thus $w = 1 \mod [F, F]$, then $x \in [F, F]$. We need only to verify if w is an element in $[F, K]K^\ell$ by checking if w is trivial in $F/[F, K]K^\ell$ where K^ℓ is the normal subgroup with relators the ℓ-powers of the relators in K. This means that there is an element x in K such that $w = x^\ell \mod [F, K]$. By the previous observation, x must be in $[F, F]$ so that w is trivial in $([F, F] \cap K/[F, K]) \otimes \mathbb{Z}/\ell$ and conclude by the universal coefficients that this is equivalent to the Conjecture 4.2. To summarize, we have with Notation 5.1,

Theorem 6.1. *Conjecture 4.2 holds in homological degree two if and only if the following* $\frac{1}{2}(r^2 + r)$ *commutators* $[\zeta, u_s]$ *and* $[u_i, u_j]$ *vanish in* $F/[F, K]K^\ell$ *for all* $1 \le s \le r$ *and* $1 \le i < j \le r$.

Remark 6.2. The fundamental units e_i in the Conjecture 4.2 can always be replaced by the the cyclotomic units u_i as in Remark 5.3 because even if h^+ is not 1 it is relatively prime to ℓ due to the fact that ℓ was assumed to be regular.

Remark 6.3. To exemplify the process in Theorem 6.1, we implement the algorithm using GAP [9] as follows. Start with the free group ”f” generated by ”z”,”a”,”b”,”b1”,...,”b(l-1)”,”-1”,”w1”,...,”wr”,”u1”,...,”ur”, ”l” and a list ”k” consisting of all the relators defined by Notation 5.1 and Theorem 5.2. Then form the list ”c” of all the commutators ”[x,y]” where ”x” is ”z”, ”a”, or ”b” and ”y” runs through the list ”k”. Finally, we concatenate the list ”c” and the list ”pk” of all the ℓ-powers in ”k” and obtain the list ”ck”. To check whether a commutator as in Theorem 6.1 is trivial in the factor group ”h:=f/ck” we use a rewriting system for ”h”.

Definition 6.4. To speed up the verifications for $\ell = 3$ we define a group G given by the generators s, r, t, u, v and the relators:

$$s^3r^{-2}, (sr)^2r^{-2}, t^3r^{-2}, r^2t^3, (t^{-1}sr)^3r^{-2}, (tr^{-1}s)^3r^{-2}, s^3u^{-2}, (su)^2u^{-2},$$
$$(su)^2u^{-2}, v^3u^{-2}, u^2v^3, (v^{-1}su)^3u^{-2}, (vu^{-1}s)^3u^{-2}, t^{-1}stsr^2u^{-1}s^{-3}v,$$
$$t^{-1}stsrs^2ru^{-1}s^{-2}vs^{-3}, rt^{-1}s^3u^{-1}s^{-1}v^{-1}s^{-1}vu^{-1},$$
$$rsr^{-2}s^{-1}t^{-1}tr^{-1}s^{-2}v(u^{-1}s^{-3}v)^2u^{-1}.$$

Lemma 6.5. *Let $\ell = 3$ and SE_2 be presented as in Theorem 5.2. The map sending s, r, t, u, v respectively to $-\zeta^2$, a, $-a(ba)^3$, $-w_1$, $w_1(ab^{-1}b_1a)^{-3}$ defines a group homomorphism from G to SE_2.*

Proof. It is enough to use the presentation of SE_2 for $\ell = 3$, add the new generators $s = -\zeta^2$, $r = a$, $t = -a(ba)^3$, $u = -w_1$, $v = w_1(ab^{-1}b_1a)^{-3}$ and check each relation defining G in the new presentation of SE_2. The routine is the following:

```
f:=FreeGroup("z","a","b","b1","b2","-1","w1","u1","3",
"s","t","v");;
k:=[f.6^-1*f.2^2,
f.4^-1*f.1*f.3*f.1,
f.5^-1*f.1^2*f.3*f.1^2,
f.7^-1*(f.3*f.4^-1)^3*f.2*(f.3*f.2)^3*(f.4*f.2)^2*f.5
*(f.3*f.4^-1)^3,
f.8*f.7*f.2,
f.9^-1*f.6*f.1^2*f.8^2,
f.6*f.3*f.6^-1*f.3^-1,
f.6*f.1*f.6^-1*f.1^-1,
(f.2*(f.3*f.2)^3)^3,
((f.3*f.2)^3*f.2^-1)^3*f.6^-1,
f.6^2,
f.1^3,
f.8*f.1*f.8^-1*f.1^-1,
f.7^2*f.6^-1,
f.1*f.2*f.1*f.2^-1,
f.3*f.8^-1*(f.2*f.4)^2*f.5^-1*f.2^-1*f.3^-1*f.8^-1*f.6^-1,
f.4*f.8^-1*(f.2*f.5)^2*f.3^-1*f.2^-1*f.4^-1*f.8^-1*f.6^-1,
f.5*f.8^-1*(f.2*f.3)^2*f.4^-1*f.2^-1*f.5^-1*f.8^-1*f.6^-1,
f.3*f.2*f.4*f.3^-1*f.2^-1*f.4^-1,
f.3*f.2*f.5*f.3^-1*f.2^-1*f.5^-1,
f.4*f.2*f.5*f.4^-1*f.2^-1*f.5^-1,
f.2^-1*f.3^-1*f.9*(f.3*f.2)^9*f.9^-1,
f.2^-1*f.4^-1*f.9*(f.4*f.2)^9*f.9^-1,
f.2^-1*f.5^-1*f.9*(f.5*f.2)^9*f.9^-1,
f.3*f.2*f.4*f.2*f.5*f.2^-1,
#additional generators
f.6*f.1^2*f.10^-1,
f.6*f.2*(f.3*f.2)^3*f.11^-1,
f.7*(f.2*f.3^-1*f.4*f.2)^-3*f.12^-1];
h:=f/k;
RequirePackage("kbmag");
```

```
R:=KBMAGRewritingSystem(h);;
MakeConfluent(R);
F:=FreeStructureOfRewritingSystem(R);
#checking relations in G
r1:=ReducedWord(R,F.10^3*F.2^-2);
r2:=ReducedWord(R,(F.10*F.2)^2*F.2^-2);
r3:=ReducedWord(R,F.11^3*F.2^-2);
r4:=ReducedWord(R,F.2^2*F.11^3);
r5:=ReducedWord(R,(F.11^-1*F.10*F.2)^3*F.2^-2);
r6:=ReducedWord(R,(F.11*F.2^-1*F.10)^3*F.2^-2);
r7:=ReducedWord(R,F.10^3*F.7^-2);
r8:=ReducedWord(R,(F.10*F.7)^2*F.7^-2);
r9:=ReducedWord(R,F.12^3*F.7^-2);
r10:=ReducedWord(R,F.7^2*F.12^3);
r11:=ReducedWord(R,(F.12^-1*F.10*F.7)^3*F.7^-2*F.6);
r12:=ReducedWord(R,(F.12*F.7^-1*F.10)^3*F.7^-2*F.6);
r13:=ReducedWord(R,F.11^-1*F.10*F.11*F.10*F.2^2*F.7^-1
*F.10^-3*F.12*F.6);
r14:=ReducedWord(R,F.11^-1*F.10*F.11*F.10*F.2*F.10^2
*F.2*F.7^-1*F.10^-2*F.12*F.10^-3*F.6);
r15:=ReducedWord(R,F.2*F.11^-1*F.10^3*F.7^-1*F.10^-1
*F.12^-1*F.10^-1*F.12*F.7^-1);
r16:=ReducedWord(R,F.2*F.10*F.2^-2*F.10^-1*F.11^-1
*F.10^-1*F.11*F.2^-1*F.10^-2*F.12*(F.7^-1
*F.10^-3*F.12)^2*F.7^-1*F.6);
```

□

Theorem 6.6. *Conjecture 4.2 is true for $\ell = 3$.*

Proof. According to the above lemma it is enough to apply Remark 6.3 to G as defined in Definition 6.4 rather than SE_2. The word $w = urs^{-1}r^{-1}u^{-1}s$ of G maps to $[u_1^{-1}, \zeta]$ in SE_2 and it will be verified to be the identity.

```
f:=FreeGroup("s","r","t","u","v");;
k:=[f.1^3*f.2^-2,
(f.1*f.2)^2*f.2^-2,
f.3^3*f.2^-2,
f.2^2*f.3^3,
(f.3^-1*f.1*f.2)^3*f.2^-2,
(f.3*f.2^-1*f.1)^3*f.2^-2,
f.1^3*f.4^-2,
(f.1*f.4)^2*f.4^-2,
```

```
f.5^3*f.4^-2,
f.4^2*f.5^3,
(f.5^-1*f.1*f.4)^3*f.4^-2,
(f.5*f.4^-1*f.1)^3*f.4^-2,
f.3^-1*f.1*f.3*f.1*f.2^2*f.4^-1*f.1^-3*f.5,
f.3^-1*f.1*f.3*f.1*f.2*f.1^2*f.2*f.4^-1*f.1^-2*f.5*f.1^-3,
f.2*f.3^-1*f.1^3*f.4^-1*f.1^-1*f.5^-1*f.1^-1*f.5*f.4^-1,
f.2*f.1*f.2^-2*f.1^-1*f.3^-1*f.1^-1*f.3*f.2^-1*f.1^-2
*f.5*(f.4^-1*f.1^-3*f.5)^2*f.4^-1];
i:=x->x^-1;;
ik:=List(k,i);
c1:=ListN(f.1*k*f.1^-1,ik,\*);
c2:=ListN(f.2*k*f.2^-1,ik,\*);
c3:=ListN(f.3*k*f.3^-1,ik,\*);
c4:=ListN(f.4*k*f.4^-1,ik,\*);
c5:=ListN(f.5*k*f.5^-1,ik,\*);
c:=Concatenation(c1,c2,c3,c4,c5);
p:=x->x^3;;
pk:=List(k,p);
ck:=Concatenation(c,pk);
h:=f/ck;
RequirePackage("kbmag");
R:=KBMAGRewritingSystem(h);;
MakeConfluent(R);
F:=FreeStructureOfRewritingSystem(R);
w:=ReducedWord(R,F.4*F.2*F.1^-1*F.2^-1*F.4^-1*F.1);
gap> w;
<identity ...>
```

Since for $\ell = 3$ there is only one cycle $\sigma = \langle \zeta, 1-\zeta \rangle$ to verify, the Conjecture 4.2 is true for $\ell = 3$. $\square$

7. The general case of Theorem 1.2

To state and prove the Theorem 1.2 in general we need only to find a finite process to verify the Conjecture 4.2 in all homological degrees. This was done in the previous section for all cycles of homological dimension two. To extend this process to higher homological dimensions we can follow [13]. For the rest of this section denote $G = SE_2$. For each $c \geq 1$, let us start with c finite presentations

$$1 \to K_i \to F_i \to G \to 1$$

for $i = 1, ..., c$ that could be as in Theorem 5.2. Let $F = F_1 \star ... \star F_c$ be the free product of all $F_i's$ and identify each F_i and K_i with its canonical image in F for $i = 1, ..., c$. Define inductively

$$[a_1, ..., a_c] = [[a_1, ..., a_{c-1}], a_c]$$

for a_i group elements and $[S_1, ..., S_c]$ the group generated by $[a_1, ..., a_c]$ with $a_i \in S_i$ for S_i subgroups of a group S. In particular, define

$$\gamma_c(S) = [S, ..., S] \quad (c \text{ times, } c \geq 1).$$

Let K be the kernel of the canonical projection $F \to G$ induced by the canonical projections $F_i \to G$ for $i = 1, ..., c$. By the main result in [13] we can identify the homology group $H_{2c}(G, \mathbb{Z})$ with a subgroup

$$H_{2c}(G, \mathbb{Z}) \approx \frac{([K_1, ..., K_c] \cap N)\gamma_{c+1}(K)}{[K_1, ..., K_c, F]\gamma_{c+1}(K)} \subset \frac{F}{[K_1, ..., K_c, F]\gamma_{c+1}(K)}.$$

for some suitable normal subgroup $N \subset F$. With little effort, we can show that the last factor group is finitely presented. In this way $2c$-cycles in $H_{2c}(G, \mathbb{Z}/\ell)$ can be represented by words in finitely presented groups. To check $(2c+1)$-cycles in $H_{2c+1}(G, \mathbb{Z}/\ell)$ we can shift the dimensions via the short exact sequence

$$IG \to \mathbb{Z}G \to \mathbb{Z}$$

where $\mathbb{Z}G \to \mathbb{Z}$ is the augmentation map and IG the augmentation ideal. Since the calculations are involved it will be postponed to a future paper.

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