

# Operations in Milnor $K$ -theory

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March 31, 2008

## Abstract

We show that operations in Milnor  $K$ -theory mod  $p$  of a field are spanned by divided power operations. After giving an explicit formula for divided power operations and extending them to some new cases, we determine for all fields  $k_0$  and all prime numbers  $p$ , all the operations  $K_i^M/p \rightarrow K_j^M/p$  commuting with field extensions over the base field  $k_0$ . Moreover, the integral case is discussed and we determine the operations  $K_i^M/p \rightarrow K_j^M/p$  for smooth schemes over a field.

## Introduction

Let  $k_0$  be a field and  $p$  be a prime number different from the characteristic of  $k_0$ . In [26], Voevodsky constructs Steenrod operations on the motivic cohomology  $H^{*,*}(X, \mathbf{Z}/p)$  of a general scheme over  $k_0$ . However, when  $p$  is odd or when  $p = 2$  and  $-1$  is a square in  $k_0^\times$ , such operations vanish on the motivic cohomology groups  $H^{i,i}(\text{Spec } k, \mathbf{Z}/p)$  for  $i > 0$  of the spectrum of a field extension  $k$  of  $k_0$ . Here, we define new operations on  $H^{i,i}(\text{Spec } k, \mathbf{Z}/p)$  that only exist for fields.

The same phenomenon happens in étale cohomology, where Steenrod operations, as defined by Epstein in [8], vanish on the étale cohomology  $H_{\text{ét}}^i(\text{Spec } k, \mathbf{Z}/p)$  of a field. Under the assumption of the Bloch-Kato conjecture, our operations give secondary operations relatively to Steenrod operations on the étale cohomology of fields.

Given a base field  $k_0$  and a prime number  $p$ , an operation on  $K_i^M/p$  is a function  $K_i^M(k)/p \rightarrow K_*^M(k)/p$  defined for all fields  $k/k_0$ , compatible with extension of fields. In other words, it is a natural transformation from the functor  $K_i^M/p : \text{Fields}/_{k_0} \rightarrow \text{Sets}$  to the functor  $K_*^M/p : \text{Fields}/_{k_0} \rightarrow \mathbf{F}_p - \text{Algebras}$ . It is important for our purpose that our operations should be functions and not only additive functions. The reason being that additive operations will appear to be trivial in some sense. In these notes, we determine all operations  $K_i^M/p \rightarrow K_*^M/p$  over any field  $k_0$ , no matter if  $p \neq \text{char } k_0$  or not. This is striking, specially in the case when  $i = 1$ .

Let  $n$  be a non-negative integer and  $k$  any field. Let  $x = \sum_{r=1}^l s_r$  be a sum of  $l$  symbols in  $K_i^M(k)/p$ , the mod  $p$  Milnor  $K$ -group of  $k$  of degree  $i$ . We define the  $n^{\text{th}}$  divided power of  $x$ , given as a sum of symbols, by

$$\gamma_n(x) = \sum_{1 \leq l_1 < \dots < l_n \leq l} s_{l_1} \cdot \dots \cdot s_{l_n} \in K_{ni}^M(k)/p.$$

Such a divided power may depend on the way  $x$  has been written as a sum of symbols and thus a well-defined map  $\gamma_n : K_i^M(k)/p \rightarrow K_{ni}^M(k)/p$  may not exist. However,  $\gamma_0(x) = 1$  and  $\gamma_1(x) = x$  and as such,  $\gamma_0$  and  $\gamma_1$  are always well-defined. In his paper [13], Kahn shows that this formula gives well-defined divided powers  $\gamma_n : K_{2i}^M(k)/p \rightarrow K_{2ni}^M(k)/p$  for  $p$  odd and  $\gamma_n : K_i^M(k)/2 \rightarrow K_{ni}^M(k)/2$  for  $k$  containing a square root of  $-1$ . Divided powers are already mentioned in a letter of Rost to Serre, [9]. In this paper, we show that in these cases, divided powers define operations in the above sense and form a basis for all possible operations on mod  $p$  Milnor  $K$ -theory.

In the remaining case, when  $-1$  is not a square in the base field  $k_0$ , divided powers as defined above are not well-defined on mod 2 Milnor  $K$ -theory. However, we will define some new, weaker operations, and show that these new operations are all the possible operations on mod 2 Milnor  $K$ -theory.

Precisely, we will prove :

**Theorem 1** ( $p$  odd). *Let  $k_0$  be any field,  $p$  an odd prime number. The algebra of operations  $K_i^M(k)/p \rightarrow K_*^M(k)/p$  commuting with field extensions over  $k_0$  is*

- If  $i = 0$ , the free  $K_*^M(k_0)/p$ -module of rank  $p$  of functions  $\mathbf{F}_p \rightarrow K_*^M(k_0)/p$ .
- If  $i \geq 1$  is odd, the free  $K_*^M(k_0)/p$ -module

$$K_*^M(k_0)/p \cdot \gamma_0 \oplus K_*^M(k_0)/p \cdot \gamma_1.$$

- If  $i \geq 2$  is even, the free  $K_*^M(k_0)/p$ -module

$$\bigoplus_{n \geq 0} K_*^M(k_0)/p \cdot \gamma_n.$$

**Theorem 2** ( $p = 2$ ). *Let  $k_0$  be any field, and consider the map  $\tau_i : K_*^M(k_0)/2 \rightarrow K_*^M(k_0)/2$ ,  $x \mapsto \{-1\}^{i-1} \cdot x$ . The algebra of operations  $K_i^M(k)/2 \rightarrow K_*^M(k)/2$  commuting with field extensions over  $k_0$  is*

- If  $i = 0$ , the free  $K_*^M(k_0)/2$ -module of rank 2 of functions  $\mathbf{F}_2 \rightarrow K_*^M(k_0)/2$ .
- If  $i = 1$ , the free  $K_*^M(k_0)/2$ -module of rank 2, generated by  $\gamma_0$  and  $\gamma_1$ .
- If  $i \geq 2$ , the  $K_*^M(k_0)/2$ -module

$$K_*^M(k_0)/2 \cdot \gamma_0 \oplus K_*^M(k_0)/2 \cdot \gamma_1 \oplus \bigoplus_{n \geq 2} \text{Ker}(\tau_i) \cdot \gamma_n.$$

Actually, the divided powers  $\gamma_n$  are not always defined with the assumptions of Theorem 2. However, if  $y_n \in \text{Ker}(\tau_i)$ , the map  $y_n \cdot \gamma_n$  will be shown to be well-defined. Notice that when  $-1$  is a square in  $k_0$ , the map  $\tau_i$  is the zero map, and hence  $\text{Ker}(\tau_i) = K_*^M(k_0)/2$ .

Also, the divided powers satisfy the relation  $\gamma_m \cdot \gamma_n = \binom{m+n}{n} \gamma_{m+n}$ . Together with the algebra structure on  $K_*^M(k_0)/p$ , this gives the algebra structure of the algebra of operations  $K_i^M/p \rightarrow K_*^M/p$  over  $k_0$ .

As Nesterenko-Suslin [21] and Totaro [23] have shown, there is an isomorphism  $H^{n,n}(\text{Spec } k, \mathbf{Z}) \xrightarrow{\cong} K_n^M(k)$  between some motivic cohomology groups of  $k$  and the Milnor  $K$ -theory groups of  $k$ . This isomorphism, together with Theorem 1, provides operations on the motivic cohomology groups  $H^{n,n}(\text{Spec } k, \mathbf{Z}/p)$ . Also, since the Bloch-Kato conjecture seems to have been proven by Rost and Voevodsky (see [24] and Weibel's paper [27] that patches lemma 2.2 which is false as stated), this gives the operations in Galois cohomology of fields, with suitable coefficients.

We also describe some new operations in integral Milnor  $K$ -theory over any base field  $k_0$ . Under some reasonable hypothesis on an operation  $\varphi : K_i^M \rightarrow K_*^M$  defined over  $k_0$ , we are able to show that  $\varphi$  is in the  $K_*^M(k_0)$ -span of our weak divided power operations. See sections 2.5 and 3.4.

Finally, we are able to determine operations  $K_i^M/p \rightarrow K_j^M/p$  in the more general set-up of smooth schemes over a field  $k$ . The Milnor  $K$ -theory ring of a smooth scheme  $X$  over  $k$  is defined to be the subring of the Milnor  $K$ -theory of the function field  $k(X)$  whose elements are unramified along all divisors on  $X$ , i.e. who vanish under all residue maps corresponding to codimension-1 points in  $X$ . An operation  $K_i^M/p \rightarrow K_j^M/p$  over the smooth  $k$ -scheme  $X$  is a function that is functorial with respect to morphisms of  $X$ -schemes (see section 4). Once again, if  $p$  is odd and  $i \geq 2$  is even, or if  $p = 2$  and  $k$  contains a square-root of  $-1$ , we have

**Theorem 3.** *Operations  $K_i^M/p \rightarrow K_*^M/p$  over the smooth  $k$ -scheme  $X$  are spanned as a  $K_*^M(X)/p$ -module by the divided power operations.*

Assuming the Bloch-Kato conjecture, we obtain this way all the operations for the unramified cohomology of smooth schemes over the field  $k$ .

The paper is organised as follows. In the first section, we start by recalling some general facts about Milnor  $K$ -theory, particularly the existence of residue and specialization maps. In the second section, we give a detailed account on divided power operations and extend the results mentioned in [13] to the case  $p = 2$ ,  $\sqrt{-1} \notin k^\times$ . We also describe some weak divided power operations for integral Milnor  $K$ -theory. Some applications to cohomological invariants are discussed. In section 3, Theorem 1 and Theorem 2 are proven (see Propositions 3.7, 3.11 and 2.9), and the integral case is discussed (Proposition 3.18). Finally, in the last section, we extend our results to the case of the Milnor  $K$ -theory of smooth schemes over a field  $k$  and prove Theorem 3.

*Acknowledgement.* I am deeply indebted to Burt Totaro for this work. He suggested the problem to me and gave me some helpful advice through his detailed remarks.

# 1 General Facts

All the results in this section can be found in [10], Chapter 7.

Let  $k$  be a field. The  $n^{\text{th}}$  Milnor  $K$ -group  $K_n^{\text{M}}(k)$  is the quotient of the  $n$ -fold tensor power  $(k^\times)^{\otimes n}$  of the multiplicative group  $k^\times$  of the field  $k$  by the relations  $a_1 \otimes \dots \otimes a_n = 0$  as soon as  $a_i + a_j = 1$  for some  $1 \leq i < j \leq n$ . We write  $\{a_1, \dots, a_n\}$  for the image of  $a_1 \otimes \dots \otimes a_n$  in  $K_n^{\text{M}}(k)$ , such elements are called symbols. Thus, elements in  $K_n^{\text{M}}(k)$  are sums of symbols. The relation  $\{x, 1-x\} = 0$  in  $K_2^{\text{M}}(k)$  is often referred to as the Steinberg relation. In particular  $K_0^{\text{M}}(k) = \mathbf{Z}$  and  $K_1^{\text{M}}(k) = k^\times$ . This construction is functorial with respect to field extension. There is a cup-product operation  $K_n^{\text{M}}(k) \times K_m^{\text{M}}(k) \rightarrow K_{n+m}^{\text{M}}(k)$  induced by the tensor product pairing  $(k^\times)^{\otimes n} \times (k^\times)^{\otimes m} \rightarrow (k^\times)^{\otimes (m+n)}$ . We write  $K_*^{\text{M}}(k)$  for the direct sum  $\bigoplus_{n \geq 0} K_n^{\text{M}}(k)$ . As a general fact, for any elements  $x$  and  $y$  in  $k^\times$ , we have the relation  $\{x, y\} = -\{y, x\}$ . Thus, cup product turns  $K_*^{\text{M}}(k)$  into a graded commutative algebra. We now state the easy but important

**Remark 1.1** (see e.g. [10]). It is a general fact that  $\{x, x\} = \{x, -1\}$ . The equality  $\{x, x\} = \{x, -1\} = 0$  happens

- in  $K_2^{\text{M}}(k)/p$  if  $p \neq 2$ .
- in  $K_2^{\text{M}}(k)/2$  if  $-1 \in (k^\times)^2$ .
- in  $K_2^{\text{M}}(k)$  if  $k$  has characteristic 2.

Let  $K$  be a field equipped with a discrete valuation  $v : K^\times \rightarrow \mathbf{Z}$ , and  $A$  its associated discrete valuation ring. Fix a local parameter  $\pi$  and let  $\kappa$  be its residue field. Then, for each  $n \geq 1$ , there exists a unique set of homomorphisms  $\partial_v : K_n^{\text{M}}(K) \rightarrow K_{n-1}^{\text{M}}(\kappa)$  and  $s_\pi : K_n^{\text{M}}(K) \rightarrow K_n^{\text{M}}(\kappa)$  satisfying  $\partial_v(\{\pi, u_2, \dots, u_n\}) = \{\bar{u}_2, \dots, \bar{u}_n\}$  and  $s_\pi(\{\pi^{i_1} u_1, \dots, \pi^{i_n} u_n\}) = \{\bar{u}_1, \dots, \bar{u}_n\}$  for all units  $u_1, \dots, u_n$  in  $A$ , where  $\bar{u}_i$  denotes the image of  $u_i$  in  $\kappa$ . The maps  $\partial_v$  are called the residue maps and the maps  $s_\pi$  are called the specialization maps. The specialization maps depend on the choice of a local parameter, whereas the residue maps don't. It is easy to see that these maps induce well-defined maps on the quotients  $K_*^{\text{M}}/p$ . Moreover, for any  $x \in K_*^{\text{M}}(K)$ , they are related by the formula :

$$s_\pi(x) = \partial_v(\{-\pi\} \cdot x).$$

Let  $k(t)$  be the function field over  $k$  in one variable. A closed point  $P$  in the projective line  $\mathbf{P}_k^1$  over  $k$  determines a discrete valuation on  $k(t)$  and can be viewed as an irreducible polynomial in  $k[t]$  (for  $P = \infty$ , take  $P = t^{-1}$ ). Let  $\partial_P$  (resp.  $s_P$ ) be the corresponding residue map (resp. specialization map) and  $\kappa_P$  the residue field corresponding to the valuation induced by  $P$ . Then, we have Milnor's exact sequence

$$0 \rightarrow K_n^{\text{M}}(k) \rightarrow K_n^{\text{M}}(k(t)) \xrightarrow{\oplus \partial_P} \bigoplus_{P \in \mathbf{P}_k^1 \setminus \{\infty\}} K_{n-1}^{\text{M}}(\kappa_P) \rightarrow 0$$

where the injective arrow is induced by the inclusion of fields  $k \subseteq k(t)$ . Moreover, this sequence is split by  $s_\infty$ . This yields an exact Milnor sequence mod  $p$  for any prime number  $p$ .

Kummer theory defines a map in Galois cohomology  $\partial : k^\times \rightarrow H^1(k, \mu_m)$  where  $\mu_m$  is the group of  $m^{\text{th}}$  roots of unity in a fixed separable closure of  $k$ . Consider the cup-product  $H^1(k, \mu_m) \otimes \dots \otimes H^1(k, \mu_m) \rightarrow H^n(k, \mu_m^{\otimes n})$ . We get a map  $\partial^n : k^\times \otimes \dots \otimes k^\times \rightarrow H^n(k, \mu_m^{\otimes n})$ . Bass and Tate prove in [1] that this map factors through  $K_n^{\text{M}}(k)$  and yields a map  $h_{k,m}^n : K_n^{\text{M}}(k) \rightarrow H^n(k, \mu_m^{\otimes n})$ . The map  $h_{k,m}^n$  is called the Galois symbol. The Bloch-Kato conjecture asserts that the Galois symbol induces an isomorphism  $K_n^{\text{M}}(k)/m \rightarrow H^n(k, \mu_m^{\otimes n})$  for all  $n \geq 0$ , all fields  $k$  and all integer  $m$  prime to the characteristic of  $k$ . The case when  $m$  is a power of 2 is known as Milnor's conjecture and has been proven by Voevodsky in [25]. The case  $n = 0$  is trivial, the case  $n = 1$  is just Kummer theory and Hilbert 90, and the case  $n = 2$  is known as the Merkurjev-Suslin theorem (cf. [19]). Rost and Voevodsky have announced a proof of the Bloch-Kato conjecture in the general case, see [24] and [27]. The proof relies on the existence of reduced power operations.

In the sequel,  $p$  will always denote a prime number, cup product in Milnor  $K$ -theory will be denoted by  $\cdot$  and by definition the group  $K_i^{\text{M}}(k)$  will be 0 as soon as  $i < 0$ . By ring, we mean commutative ring with unit.

## 2 Divided powers

### 2.1 Existence of divided powers in Milnor $K$ -theory

In [13], Kahn mentions the existence of divided powers in all cases of the following proposition. However, we recall the construction of divided powers, as it will prove to be useful for the determination of our operations.

**Definition 2.1.** Let  $n$  be a non-negative integer and  $F$  any field. Let  $x = \sum_{r=1}^l s_r$  be a sum of  $l$  symbols of degree  $i$  in some Milnor  $K$ -group. A divided power of  $x$  is

$$\gamma_n(x) = \sum_{1 \leq l_1 < \dots < l_n \leq l} s_{l_1} \cdot \dots \cdot s_{l_n}.$$

Of course, it is not clear that  $\gamma_n$  should give a well-defined map  $K_i^{\text{M}}(k)/p \rightarrow K_{ni}^{\text{M}}(k)/p$ . However, we have

**Proposition 2.2.**  $\gamma_0 = 1$  and  $\gamma_1 = \text{id}$  are always well-defined.

1. If  $i$  is even  $\geq 2$  and  $p$  is an odd prime number, then there exists a divided power

$$\gamma_n : K_i^{\text{M}}(F)/p \rightarrow K_{ni}^{\text{M}}(F)/p.$$

2. If  $p = 2$  and  $-1 \in (F^\times)^2$ , then for all  $i \geq 2$ , there exists a divided power

$$\gamma_n : K_i^{\text{M}}(F)/2 \rightarrow K_{ni}^{\text{M}}(F)/2.$$

3. If  $i$  is even  $\geq 2$  and  $\text{char } F = 2$ , then there exists a divided power

$$\gamma_n : K_i^M(F) \rightarrow K_{ni}^M(F).$$

**Proof.** We are going to prove that  $\gamma_n$  as defined explicitly doesn't depend on a particular writing of  $x$ . We will give the proof only in the first case. The two remaining cases can be proven exactly the same way, once one remarks that the conditions ( $i$  even and  $p$  odd), ( $p = 2$  and  $-1 \in (F^\times)^2$ ) and ( $i$  even and  $\text{char } F = 2$ ) are here to force :

- $\{a, a\} = 0$
- The algebras  $\bigoplus_{n \geq 0} K_{ni}^M(F)/p$ ,  $\bigoplus_{n \geq 0} K_{ni}^M(F)/2$  and  $\bigoplus_{n \geq 0} K_{ni}^M(F)$  are commutative.

Let  $M_{i,F}$  be the free  $\mathbf{Z}$ -module generated by elements in  $(F^\times)^i$ , and define

$$\begin{aligned} \tilde{\Gamma}_n : M_{i,F} &\rightarrow K_{ni}^M(F)/p \\ \sum_{r=1}^l n_r s_r &\mapsto \sum_{1 \leq l_1 < \dots < l_n \leq l} n_{l_1} s_{l_1} \cdot \dots \cdot n_{l_n} s_{l_n} \end{aligned}$$

The map  $\tilde{\Gamma}_n$  is well-defined because of the commutativity of  $\bigoplus_{n \geq 0} K_{ni}^M(F)/p$  and we want to show that  $\tilde{\Gamma}_n$  factors through  $(F^\times)^{\otimes i}$ . For this purpose, it is enough to show that  $\tilde{\Gamma}_n$  takes the same value on each equivalence class for the quotient map  $M_{i,F} \rightarrow (F^\times)^{\otimes i}$ . We notice that, given  $x$  and  $y$  in  $M_{i,F}$ , the sum formula  $\tilde{\Gamma}_n(x+y) = \sum_{j=0}^n \tilde{\Gamma}_j(x) \tilde{\Gamma}_{n-j}(y)$  holds. Thus, if  $x$  and  $x+y$  have same image in  $(F^\times)^{\otimes i}$ , we want to show that  $\tilde{\Gamma}_n(x+y) = \sum_{j=0}^n \tilde{\Gamma}_j(x) \tilde{\Gamma}_{n-j}(y) = \tilde{\Gamma}_n(x)$ . For this, it is enough to prove that  $\tilde{\Gamma}_j(y) = 0$  for all  $j \geq 1$  and  $y$  mapping to 0 in  $(F^\times)^{\otimes i}$ . Still using the sum formula, it is enough to prove that for any  $1 \leq j \leq i$ , elements of the form  $(a_1, \dots, a_{j-1}, b, a_{j+1}, \dots, a_i) + (a_1, \dots, a_{j-1}, c, a_{j+1}, \dots, a_i) - (a_1, \dots, a_{j-1}, bc, a_{j+1}, \dots, a_i)$  map to 0 under  $\tilde{\Gamma}_n$ . This happens, when  $i \geq 2$ , because  $\{a, a\} = 0$  in  $K_2^M(F)/p$  for all  $a \in F^\times$ .

Therefore, we get a map  $\Gamma_n : (F^\times)^{\otimes i} \rightarrow K_{ni}^M(F)/p$  that satisfies the sum formula.

Now,  $\Gamma_n$  factors through  $\tilde{\gamma}_n : K_i^M(F) \rightarrow K_{ni}^M(F)/p$ . Let  $x = x_1 + x_2$  with  $x_2$  a pure tensor satisfying Steinberg's relation. Then  $\Gamma_n(z) = \Gamma_n(z_1) + \Gamma_{n-1}(z_1) \Gamma_1(z_2) + \dots$ , but  $\Gamma_n(z_2)$  is clearly 0 for  $n \geq 1$ . Hence a map  $\tilde{\gamma}_n : K_i^M(F) \rightarrow K_{ni}^M(F)/p$ .

Finally,  $\tilde{\gamma}_n$  factors through  $\gamma_n : K_i^M(F)/p \rightarrow K_{ni}^M(F)/p$  as easily seen.  $\square$

In all cases of Proposition 2.2, the divided powers satisfy the following properties. Moreover, they are the only set of maps satisfying such properties. See e.g. [13].

**Properties 2.3.** 1.  $\gamma_0(x) = 1$ ,  $\gamma_1(x) = x$ .

2.  $\gamma_n(xy) = x^n \gamma_n(y)$ .

3.  $\gamma_m(x) \gamma_n(x) = \binom{m+n}{n} \gamma_{m+n}(x)$ .

4.  $\gamma_n(x+y) = \sum_{i=0}^n \gamma_i(x) \gamma_{n-i}(y)$ .

5.  $\gamma_m(\gamma_n(x)) = \frac{(nm)!}{m!n!^m} \gamma_{nm}(x)$ .

6.  $\gamma_n(s) = 0$  if  $n \geq 2$  and  $s$  is a symbol.

## 2.2 Divided powers and length

**Definition 2.4.** The *length* of an element  $x \in K_i^M(F)/p$  (resp.  $K_i^M(F)$ ) is the minimum number of symbols appearing in any decomposition of  $x$  as a sum of symbols.

**Remark 2.5.** If  $x \in K_i^M(F)/p$  has length  $l$  and if the divided power  $\gamma_n$  is well-defined on  $K_i^M(F)/p$ , then  $n > l$  implies  $\gamma_n(x) = 0$ . That is,  $\gamma_n$  vanishes on elements of length strictly less than  $n$ .

As was noted by Kahn in [13], the existence of divided powers implies

**Proposition 2.6.** *If there exists an integer  $l$  such that the length of any element in  $K_2^M(F)/p$  (resp.  $K_2^M(F)/2$ ,  $K_2^M(F)$ ) is  $\leq l$ , then for all  $n \geq 2l + 2$ , we have*

- $K_n^M(F)/p = 0$ , if  $p$  is odd.
- $K_n^M(F)/2 = 0$ , if  $-1$  is a square in  $F^\times$ .
- $K_n^M(F) = 0$ , if  $\text{char } F = 2$ .

**Proof.** Let  $x = \{x_1, \dots, x_n\}$  be a symbol in  $K_n^M(F)/p$  with  $p$  odd and  $n \geq 2l + 2$ . Then  $x = \gamma_{l+1}(\{x_1, x_2\} + \dots + \{x_{2l+1}, x_{2l+2}\}) \cdot \{x_{2l+3}, \dots, x_n\}$ . By assumption  $\{x_1, x_2\} + \dots + \{x_{2l+1}, x_{2l+2}\}$  has length at most  $l$  and maps therefore to 0 under  $\gamma_{l+1}$ . The two other cases are similar.  $\square$

**Examples 2.7.** In [3], Becher shows that if  $K_1^M(F)/p = F^\times / (F^\times)^p$  is finite of order  $p^m$ , then the length of elements in  $K_2^M(F)/p$  is always less or equal than  $\frac{m}{2}$  if  $p$  is odd, and is always less or equal than  $\frac{m+1}{2}$  if  $p = 2$ . Hence, if  $K_1^M(F)/p = F^\times / (F^\times)^p$  is finite of order  $p^m$ , the higher Milnor  $K$ -groups  $K_n^M(F)/p$  are zero whenever  $n > 2\lceil \frac{m}{2} \rceil + 1$  if  $p$  is odd and  $n > 2\lfloor \frac{m+1}{2} \rfloor + 1$  if  $p = 2$ . It is worth saying that Becher also shows that these upperbounds are sharp.

## 2.3 Stiefel-Whitney classes of a quadratic form

Let  $q$  be a quadratic form of rank  $r$  over a field  $F$  of characteristic  $\neq 2$ . Then  $q$  admits a diagonal form  $\langle a_1, \dots, a_r \rangle$ . The total Stiefel-Whitney class of  $q$  is defined to be  $w(q) = (1 + \{a_1\}) \dots (1 + \{a_n\}) \in K_*^M(F)/2$ . In [20], Milnor shows that  $w(q)$  is well-defined and doesn't depend on a particular choice of a diagonal form for  $q$ . The  $k^{\text{th}}$  Stiefel-Whitney class  $w_k$  is defined to be the degree  $k$  part of  $w$ .

**Proposition 2.8.** *We have  $w_1 w_2 = w_3$  and more generally if  $n = \sum \varepsilon_i 2^i$  is the binary decomposition of  $n$ , then  $w_n = \prod_{i, \varepsilon_i=1} w_{2^i}$ . Also, when  $-1 \in (F^\times)^2$ ,  $w_{2n} = \gamma_n(w_2)$  and so  $w_{2n+1} = w_1 \cdot \gamma_n(w_2)$ . We also have*

$$w_n = w_1^{\varepsilon_0} \cdot \prod_{i \geq 1, \varepsilon_i=1} \gamma_{2^{i-1}}(w_2).$$

**Proof.** The first point is proved in [20]. The last point was pointed out by Becher in [2, §9] and is a direct consequence of the existence of divided powers mod 2 when  $-1$  is a square in  $F^\times$ , and of the explicit formula defining both the Stiefel-Whitney classes and the divided powers.  $\square$

This result confirms that the invariants  $w_1$  and  $w_2$  of a quadratic form are important. In the literature,  $w_1$  is often referred to as the determinant, and  $w_2$  as the Hasse invariant. Also, the Witt invariant can be expressed in terms of the determinant and the Hasse invariant (see e.g. [17], Proposition V.3.20). A natural question is to ask whether or not a non-degenerate quadratic form is determined, up to isometry, by its total Stiefel-Whitney class. This has been answered by Elman and Lam in [6] : let  $F$  be a field of characteristic not equal to 2, with  $W(F)$  its Witt ring of anisotropic quadratic forms, and  $IF$  its ideal of even-dimensional forms. Let  $I^n F$  denote the  $n$ th power  $(IF)^n$ . Then, the equivalence class of a non-degenerate quadratic form over  $F$  is determined by its dimension and Stiefel-Whitney invariant if and only if  $I^3 F$  is torsion free (as an additive abelian group). This is for example the case of the field of real numbers. However, a real non-degenerate quadratic form of given rank is not solely determined by  $w_1$  and  $w_2$  and as such, the higher Stiefel-Whitney classes do carry a little information beyond what  $w_1$  and  $w_2$  give. Proposition 2.8 shows that when  $-1$  is a square in the base field  $F$ , the higher Stiefel-Whitney classes are completely determined by  $w_1$  and  $w_2$ . This helps to explain why the classes  $w_i$  for  $i \geq 3$  have played very little role in quadratic form theory. Also, in general, it is known that two non-degenerate quadratic forms  $q$  and  $q'$  of same dimension  $\leq 3$  are isometric if and only if they have same  $w_1$  and  $w_2$  (see [17], Proposition V.3.21.). Finally, Elman and Lam gave a description of fields for which non-degenerate quadratic forms of given dimension are totally determined, up to isometry, by their determinant  $w_1$  and Hasse invariant  $w_2$ . This happens if and only if  $I^3 F = 0$  (cf. [7]).

By Milnor's conjecture (proven by Voevodsky in [25]), the above gives divided powers in Galois cohomology. Let  $Et_n$  be the functor that associates to any field  $F$  over  $k_0$  the set of étale algebras of rank  $n$  over  $F$ . In [9], it is proven that the  $H^*(k_0, \mathbf{Z}/2)$ -module  $\text{Inv}_{k_0}(Et_n, \mathbf{Z}/2)$  of natural transformations from the functor  $Et_n$  over  $k_0$  to the functor  $H^*(-, \mathbf{Z}/2)$  over  $k_0$  is free with basis the Galois-Stiefel-Whitney classes  $1, w_1^{gal}, \dots, w_m^{gal}$ ,  $m = \lfloor \frac{n}{2} \rfloor$ . Moreover,  $w_i^{gal} = 0$  for  $i > m$ . Given an étale algebra  $E$  over  $k_0$ , we can consider the non-degenerate quadratic form  $q_E$  on  $E$  viewed as a  $k_0$ -vector space, defined as  $q_E(x) = \text{Tr}_{E/k_0}(x^2)$ . Therefore, we have some invariants, coming from the Stiefel-Whitney classes of quadratic forms. In [12], Kahn proves that for  $E \in Et_n(k_0)$ ,  $w_i^{gal}(E) = w_i(q_E)$  if  $i$  is odd and  $w_i^{gal}(E) = w_i(q_E) + (2) \cdot w_{i-1}(q_E)$  if  $i$  is even. When  $-1$  is a square in  $k_0$ , we get that the higher Galois-Stiefel-Whitney invariants are determined by  $w_1^{gal}$  and  $w_2^{gal}$  in the following way :  $w_{2n+1}^{gal} = w_1^{gal} \cdot \gamma_n(w_2^{gal} - \{2\} \cdot w_1^{gal})$  and  $w_{2n}^{gal} = \gamma_n(w_2^{gal} - \{2\} \cdot w_1^{gal}) + \{2\} \cdot w_1^{gal} \cdot \gamma_{n-1}(w_2^{gal} - \{2\} \cdot w_1^{gal})$ .

However, there are some examples where the divided powers act trivially. For instance, MacDonald computes, for  $n$  odd  $\geq 3$ , the mod 2 cohomological invariants for the groups  $SO(n)$ ,  $\mathbf{Z}/2 \times PGL(n)$ ,  $PSp(2n)$ , and  $F_4$ . These correspond, for  $r = 0, 1, 2$ , and

3 respectively, to  $\text{Inv}_{k_0}(J_n^r, \mathbf{Z}/2)$ , the group of mod 2 invariants for odd degree  $n \geq 3$  Jordan algebras with associated composition algebra of dimension  $2^r$  ( $0 \leq r \leq 3$ ) over the base field  $k_0$  of characteristic supposed to be different from 2. Such algebras are known to be of the form  $H(C, q) = \{x \in M_n(C), B_q^{-1}x^t B_q = x\}$  for  $q$  an  $n$ -dimensional quadratic form of determinant 1 with associated bilinear form  $B_q$  and  $C$  a composition algebra over  $k_0$  of dimension  $2^r$ . The composition algebra comes with a norm form  $\varphi$ , which turns out to be a Pfister form. The group of invariants for  $r$ -Pfister forms ( $r > 0$ ) is the free  $H^*(k_0, \mathbf{Z}/2)$ -module generated by 1 and  $e_r$ , where  $e_r(\langle\langle a_1, \dots, a_r \rangle\rangle) = (a_1) \cdot \dots \cdot (a_r)$ . Write  $J = \varphi \otimes q$  for a Jordan algebra  $J$ , then we have invariants  $v_i = e_r \otimes w_{2i}$  and it is shown in [18] that  $\text{Inv}_{k_0}(J_n^r, \mathbf{Z}/2)$  is the free  $H^*(k_0, \mathbf{Z}/2)$ -module generated by 1,  $v_0, \dots, v_m$ , with  $m$  satisfying  $n = 2m + 1$ . For  $r > 0$ , when  $-1$  is a square in  $k_0$  and because  $e_r \cdot e_r$  is zero, we see that the divided powers vanish on the  $v_i$ 's.

## 2.4 Divided powers in Milnor $K$ -theory mod 2

In this section, we no longer assume  $-1 \in (F^\times)^2$ . We define the map

$$\tau_i : K_*^M(F)/2 \rightarrow K_*^M(F)/2, \quad x \mapsto \{-1\}^{i-1} \cdot x.$$

Let's say a few words about this map. If  $F$  has characteristic  $p > 0$ , then  $\{-1\}$  is zero in  $K_1^M(F)/2$  if  $p \equiv 1 \pmod{4}$ , and in any case  $\{-1, -1\} = 0$  since the groups  $K_n^M(\mathbf{F}_q)$  vanish for finite fields  $\mathbf{F}_q$  and  $n \geq 2$ . So, considering a function field over a finite field, we see that the maps  $\tau_i$  for  $i \geq 2$  are neither injective nor surjective, even when  $-1$  is not a square in  $F$ . If  $F$  is a number field (or a global field), let  $r_1$  be the number of real places of  $F$  and denote them by  $\sigma_i : F \rightarrow \mathbf{R}$ . Bass and Tate show in [1] that for  $n \geq 3$ , the embeddings  $\sigma_i : F \rightarrow \mathbf{R}$  corresponding to the real places of  $F$  induce an isomorphism  $K_n^M(F) \xrightarrow{\oplus \sigma_i} \bigoplus_{i=1}^{r_1} K_n^M(\mathbf{R})/2 \cong (\mathbf{Z}/2)^{r_1}$ . Then, clearly  $K_n^M(F)/2 \cong (\mathbf{Z}/2)^{r_1}$  for  $n \geq 3$ . Also,  $K_1^M(F)/2$  is countably infinite. This shows that  $\tau_i$  cannot be injective.

Hence,  $\text{Ker}(\tau_i)$  is non-trivial in general and the weak divided power operations that we construct below are non-trivial in general.

**Proposition 2.9.** *Let  $n$  be an integer  $\geq 2$  and  $F$  any field. Let  $y_n$  be in the kernel of  $\tau_i$ . Then, if  $s_1, \dots, s_l$  are symbols in  $K_i^M(F)/2$ ,*

$$(y_n \cdot \gamma_n)(s_1 + \dots + s_l) = y_n \cdot \sum_{1 \leq s_{l_1} < \dots < s_{l_n} \leq l} s_{l_1} \cdot \dots \cdot s_{l_n}$$

*is a well-defined map over  $K_i^M(F)/2$ .*

**Proof.** We proceed exactly the same way as in Proposition 2.2, from which we take up the notations. The map  $y_n \cdot \tilde{\Gamma}_n : M_{i,F} \rightarrow K_*^M(F)/2$ ,  $\sum_{r=1}^l n_r s_r \mapsto y_n \cdot \sum_{1 \leq l_1 < \dots < l_n \leq l} n_{l_1} s_{l_1} \cdot \dots \cdot n_{l_n} s_{l_n}$  is well-defined due to the commutativity of the  $\mathbf{F}_2$ -algebra  $K_*^M(F)/2$ . As before,  $\tilde{\Gamma}_n$  satisfies a sum formula which we write  $y_n \cdot \tilde{\Gamma}_n(x+y) = \sum_{j=0}^n \tilde{\Gamma}_j(x) \cdot y_n \cdot \tilde{\Gamma}_{n-j}(y)$  for all  $x$  and  $y$  in  $M_{i,F}$ . To prove that  $\tilde{\Gamma}_n$  factors through  $(F^\times)^{\otimes i}$ , it is enough to show that elements of the form  $y = (a_1, \dots, a_{i-1}, b) + (a_1, \dots, a_{i-1}, c) - (a_1, \dots, a_{i-1}, bc)$  map

to zero under  $y_n \cdot \tilde{\Gamma}_j$  for all  $j \geq 1$ . This is clear for  $j = 1$  and  $j > 3$ . In the case  $j = 2$ , we have  $y_n \cdot \tilde{\Gamma}_2(y) = y_n \cdot (\{a_1, \dots, a_{i-1}, b, a_1, \dots, a_{i-1}, c\} + \{a_1, \dots, a_{i-1}, b, a_1, \dots, a_{i-1}, bc\} + \{a_1, \dots, a_{i-1}, c, a_1, \dots, a_{i-1}, bc\})$ . Notice that  $\{a_1, \dots, a_{i-1}, a_1, \dots, a_{i-1}\} = \{-1\}^{i-1} \{a_1, \dots, a_{i-1}\}$  to conclude  $y_n \cdot \tilde{\Gamma}_2(y) = 0$ . The case  $j = 3$  is similar.

Thus, we have a well-defined map  $y_n \cdot \Gamma_n : (F^\times)^{\otimes i} \rightarrow K_*^M(F)/2$ , and as in the proof of Proposition 2.2, it factors through a well-defined map  $y_n \cdot \gamma_n : K_i^M(F)/2 \rightarrow K_*^M(F)/2$ .  $\square$

**Example 2.10.** Consider again quadratic forms over  $F$ , but this time without assuming  $-1$  is a square in  $F$ , and their Stiefel-Whitney invariants. If  $y \in K_*^M(F)/2$  is such that  $\{-1\} \cdot y = 0$ , then

$$y \cdot w_{2n} = (y \cdot \gamma_n)(w_2).$$

If we write  $w_2$  as an ordered sum of symbols  $\sum_i s_i$ , we can consider  $\gamma_n(w_2)$  to be  $\sum_{i < j} s_i \cdot s_j$ . Of course, this is not well-defined. However, we have the equality for all  $y \in \ker \tau_2$ ,  $y \cdot (\gamma_n(w_2) - w_{2n}) = 0$ . Hence,  $\gamma_n(w_2) - w_{2n}$  must be in the subgroup  $G$  of  $K_{2n}^M(F)/2$  consisting of elements  $z$  such that  $z \cdot y = 0$  for all  $y \in \ker \tau_2$ . In particular,  $G$  contains  $\{-1\} \cdot K_{2n-1}^M(F)/2$ . This means that knowing  $w_1$  and  $w_2$  gives some restriction on the possible values of the higher Stiefel-Whitney classes even when  $-1$  is not a square.

## 2.5 Divided powers in integral Milnor $K$ -theory

In this section,  $\tau_i$  is the map on integral Milnor  $K$ -theory  $K_*^M(F) \rightarrow K_*^M(F)$ ,  $x \mapsto \{-1\}^{i-1} \cdot x$ . The same examples as in the previous section show that this map is not necessarily injective nor surjective.

**Proposition 2.11.** *Let  $n$  and  $i$  be integers  $\geq 2$  with  $i$  even, and  $F$  any field. Let  $y_n$  be an element in the kernel of  $\tau_i$ . Then, if  $s_1, \dots, s_l$  are symbols in  $K_i^M(F)$ ,*

$$(y_n \cdot \gamma_n)(s_1 + \dots + s_l) = y_n \cdot \sum_{1 \leq s_{l_1} < \dots < s_{l_n} \leq l} s_{l_1} \cdot \dots \cdot s_{l_n}$$

*is a well-defined map over  $K_i^M(F)$ .*

**Proof.** Same as for Proposition 2.9 since the algebra  $\bigoplus_{r \geq 0} K_{2r}^M(F)$  is commutative.  $\square$

**Proposition 2.12.** *Let  $n$  and  $i$  be integers  $\geq 2$  with  $i$  odd, and  $F$  any field. Let  $y_n$  be an element in the kernel of  $\tau_i$ , which is of 2-torsion. Then, if  $s_1, \dots, s_l$  are symbols in  $K_i^M(F)$ ,*

$$(y_n \cdot \gamma_n)(s_1 + \dots + s_l) = y_n \cdot \sum_{1 \leq s_{l_1} < \dots < s_{l_n} \leq l} s_{l_1} \cdot \dots \cdot s_{l_n}$$

*is a well-defined map  $K_i^M(F)$  to  $K_{ni}^M(F)$ .*

**Proof.** Notice that the map  $y_n \cdot \tilde{\Gamma}_n : M_{i,F} \rightarrow K_*^M(F)$ ,  $\sum_{l=1}^r n_l s_l \mapsto y_n \cdot \sum_{1 \leq l_1 < \dots < l_n \leq r} n_{l_1} s_{l_1} \cdot \dots \cdot n_{l_n} s_{l_n}$  is well-defined because  $y_n$  is of 2-torsion. Now, the proof is the same as for Proposition 2.9.  $\square$

### 3 Operations in Milnor $K$ -theory of a field

We start this section with a result that will be of constant use.

**Proposition 3.1.** *Let  $a$  be in  $K_n^M(k_0)$  and suppose that for all extension  $k/k_0$  and for all  $x \in K_*^M(k)$  we have  $a \cdot x = 0$ , then  $a$  is necessarily 0 in  $K_n^M(k_0)$ . Moreover the same result holds mod  $p$ .*

**Proof.** Let  $a$  be as in the proposition. Consider the map  $K_n^M(k_0) \rightarrow K_{n+1}^M(k_0(t))$ ,  $a \mapsto \{t\} \cdot a$ . This map is injective since it admits a left inverse, namely the residue map  $\partial_0^M$  at the point  $0 \in \mathbf{P}_{k_0}^1$ . Indeed, we have the formula  $\partial_0^M(\{t\} \cdot a) = a$ . The residue map, as defined in [10, Chapter 7], is a homomorphism and hence induces a well-defined residue map mod  $p$ ,  $K_{n+1}^M(k_0(t))/p \rightarrow K_n^M(k_0)/p$ . Hence, the same arguments apply in the mod  $p$  case.  $\square$

**Definition 3.2.** An operation  $\varphi : K_i^M/p \rightarrow K_*^M/p$  over a field  $k_0$  is a natural transformation from the functor  $K_i^M/p : \mathbf{Fields}/_{k_0} \rightarrow \mathbf{Sets}$  to the functor  $K_*^M/p : \mathbf{Fields}/_{k_0} \rightarrow \mathbf{F}_p - \mathbf{Algebras}$ . In other words, it is a set of functions  $\varphi : K_i^M(k)/p \rightarrow K_*^M(k)/p$  defined for all extensions  $k$  of  $k_0$  such that for any extension  $l$  of  $k$ , the following diagram commutes:

$$\begin{array}{ccc} K_i^M(l)/p & \xrightarrow{\varphi} & K_*^M(l)/p \\ \uparrow & & \uparrow \\ K_i^M(k)/p & \xrightarrow{\varphi} & K_*^M(k)/p \end{array}$$

**Example.** Divided powers are indeed operations in the above sense (when they are well-defined). So, if for instance  $p$  is odd and  $i$  is even, any sum of divided power operations with coefficients in  $K_*^M(k_0)/p$  gives an operation  $K_i^M/p \rightarrow K_*^M/p$  over  $k_0$ . Our main theorems say that this gives all the possible operations.

**Example.** Suppose  $p$  is odd,  $i \geq 2$  is even and  $k$  is an extension of  $k_0$ . The map  $K_i^M(k)/p \rightarrow K_{2i}^M(k)/p$ ,  $x \mapsto x^2$  defines an operation over  $k_0$ . It is easy to check that this operation corresponds to  $2 \cdot \gamma_2$ . More generally, it is straightforward to check that any map of the form  $x \mapsto x^q$  defines an operation  $K_i^M(k)/p \rightarrow K_{qi}^M(k)/p$  and that it is a sum of divided powers. Of course, this is a particular case of Theorem 1.

**Definition 3.3.** Let  $k_0$  be any field and  $K$  an extension of  $k_0$  endowed with a discrete valuation  $v$  such that its valuation ring  $R = \{x \in K, v(x) \geq 0\}$  contains  $k_0$ , so that the residue field  $\kappa$  is an extension of  $k_0$ . We say that specialization maps commute with an operation  $\varphi : K_i^M \rightarrow K_*^M$  over  $k_0$  if for any extension  $K/k_0$  as above, we have a commutative diagram

$$\begin{array}{ccc} K_i^M(K)/p & \xrightarrow{\varphi} & K_*^M(K)/p \\ s_\pi \downarrow & & s_\pi \downarrow \\ K_i^M(\kappa)/p & \xrightarrow{\varphi} & K_*^M(\kappa)/p \end{array}$$

where  $\pi$  is any uniformizer for the valuation  $v$ .

**Example.** Divided power operations over  $k_0$  do commute with specialization maps. This is clear from the definition of specialization maps.

### 3.1 Operations $K_1^M/p \times \dots \times K_1^M/p \rightarrow K_*^M/p$

The following theorem is essential in the determination of operations  $K_i^M/p \rightarrow K_*^M/p$  for  $i \geq 2$ .

**Theorem 3.4.** *Let  $k_0$  be any field and let  $p$  be a prime number. The algebra of operations*

$$\underbrace{K_1^M(k)/p \times \dots \times K_1^M(k)/p}_{r \text{ times}} \longrightarrow K_*^M(k)/p$$

for fields  $k \supseteq k_0$  and commuting with field extension over  $k_0$  is the free module over  $K_*^M(k_0)/p$  with basis the operations  $(\{a_1\}, \dots, \{a_r\}) \mapsto \{a_{i_1}, \dots, a_{i_s}\}$  for all subsets  $1 \leq i_1 < \dots < i_s \leq r$ .

For example, given an operation  $\psi : K_1^M(k)/p \times K_1^M(k)/p \rightarrow K_5^M(k)/p$ , there exist  $a \in K_5^M(k_0)/p$ ,  $b_1$  and  $b_2 \in K_4^M(k_0)/p$  and  $c \in K_3^M(k_0)/p$ , such that for any  $(\{x\}, \{y\}) \in K_1^M(k)/p \times K_1^M(k)/p$ , we have  $\psi(\{x\}, \{y\}) = a + b_1 \cdot \{x\} + b_2 \cdot \{y\} + c \cdot \{x, y\}$ .

This result is proven for Galois cohomology in [9, Theorem 16.4] in the particular case when the characteristic of  $k_0$  is different from  $p$ . Using the Galois symbol and the fact that the Bloch-Kato conjecture is true, we could conclude directly in that case. However, we prefer to give a direct proof that does not rely on this very difficult result obtained by Rost and Voevodsky. Moreover, in the case  $p = \text{char } k_0$  the differential symbols  $\psi_k^n : K_n^M(k)/p \rightarrow \nu(n)_k$ ,  $\{x_1, \dots, x_n\} \mapsto (dx_1/x_1) \wedge \dots \wedge (dx_n/x_n)$  are isomorphisms for all  $n \geq 0$  (Bloch-Gabber-Kato Theorem) and hence give us the operations for logarithmic differentials. For more details on the differential symbols, we refer to [10, Ch. 9].

**Proof of Theorem 3.4.** The proof goes in three steps. In the first step, we show that an operation  $K_1^M/p \rightarrow K_*^M/p$  over  $k_0$  is determined by the image of  $\{t\} \in K_1^M(k_0(t))/p$  where  $t$  is a transcendental element over  $k_0$ . In the second step, we determine the image of  $\{t\}$ . Finally, in the last step we conclude by induction on the number  $r$  of factors. Let  $\varphi : K_1^M/p \rightarrow K_*^M/p$  be an operation over  $k_0$ .

*Step 1.* The operation  $\varphi : K_1^M/p \rightarrow K_*^M/p$  over  $k_0$  is determined by the image  $\varphi(\{t\})$  of  $\{t\} \in K_1^M(k_0(t))/p$  in  $K_*^M(k_0(t))/p$ , for  $t$  transcendental over  $k_0$ . Indeed, consider a field extension  $k/k_0$  and an element  $e \in k$ . If  $e$  is not algebraic over  $k_0$ , then  $\varphi(\{e\})$  is the image in  $K_*^M(k)/p$  of the element  $\{e\} \in K_1^M(k_0(e))/p$ . If  $e$  is algebraic and if  $k$  possesses a transcendental element  $t$  over  $k_0$ , then  $et^p$  is transcendental over  $k_0$ . Also, in  $K_1^M(k)/p$ ,  $\{et^p\} = \{e\}$ , and so  $\varphi(\{e\})$  is determined by  $\varphi(\{et^p\})$ . Finally, if  $e \in k$  is algebraic over  $k_0$ , consider the function field  $k(t)$  and the commutative diagram

$$\begin{array}{ccc} K_1^M(k)/p & \xrightarrow{i} & K_1^M(k(t))/p \\ \varphi \downarrow & & \varphi \downarrow \\ K_*^M(k)/p & \xrightarrow{i} & K_*^M(k(t))/p \end{array}$$

where  $i$  is the map induced by the inclusion of fields  $k \subset k(t)$ . We can write  $\varphi(i(\{e\})) = i(\varphi(\{e\}))$ . The element  $\varphi(\{e\})$  is determined by the previous case. By Milnor's exact sequence,  $i$  is injective. Therefore,  $\varphi(\{e\})$  is uniquely determined.

*Step 2.* The element  $\varphi(\{t\}) \in K_*^M(k_0(t))/p$  has residue 0 for all residue maps corresponding to closed points in  $\mathbf{P}_{k_0}^1 \setminus \{0, \infty\}$ . Indeed, let  $P$  be a closed point in  $\mathbf{P}_{k_0}^1 \setminus \{\infty\}$  and view it as a monic non-constant irreducible polynomial in  $k_0[t]$ . We have a commutative diagram with injective vertical arrows induced by the obvious inclusion of fields (they are injective by Milnor's exact sequence).

$$\begin{array}{ccccc} K_1^M(k_0(t, X))/p & \xrightarrow{\varphi} & K_*^M(k_0(t, X))/p & \xrightarrow{\partial_{P \otimes k_0(X)}} & K_*^M(\kappa_P(X))/p \\ \uparrow i & & \uparrow i & & \uparrow i \\ K_1^M(k_0(t))/p & \xrightarrow{\varphi} & K_*^M(k_0(t))/p & \xrightarrow{\partial_P} & K_*^M(\kappa_P)/p \end{array}$$

where  $\partial_{P \otimes k_0(X)}$  is the residue map at the polynomial  $P \in k_0[t]$  seen as a polynomial in  $k_0(X)[t]$  via the obvious inclusion of fields  $k_0 \subset k_0(X)$  (See Lemma 3.15). Consider the element  $\{tX^p\} \in K_1^M(k_0(t, X))/p$ . Then by definition,  $\{tX^p\} = \{t\}$  and thus this element comes from an element in  $K_1^M(k_0(t))/p$ . The diagram then tells us that  $\varphi(\{tX^p\})$  can only have non-zero residues at polynomials  $P$  in  $k_0(X)[t]$  with coefficients in  $k_0$ .

Given a polynomial  $Q$  in  $k_0[t]$ , let's write  $Q_X$  for the polynomial in  $k_0(X)[t]$  defined by  $Q_X(t) = Q(tX^p)$ . Then, via the isomorphism  $k_0(t) \xrightarrow{\cong} k_0(tX^p)$ ,  $t \mapsto tX^p$ , we get that  $\varphi(\{tX^p\})$  has non-zero residues only at polynomials of the form  $Q_X$ .

Exploiting the fact that  $\{t\} = \{tX^p\}$  in  $K_1^M(k_0(t, X))/p$ , we deduce that if  $\varphi(\{t\}) = \varphi(\{tX^p\})$  has non-zero residue at some polynomial  $Q_X$  as above, then  $Q_X$  must come from a polynomial  $P \in k_0[t]$ . Concretely, we must have

$$Q(tX^p) = \alpha(X)P(t), \quad \text{for some } \alpha(X) \in k_0(X), P \in k_0[t].$$

Having in mind that  $P$  is monic irreducible, this implies  $P = t$  as easily seen.

So, we have proven that  $\varphi(\{t\})$  is unramified outside  $\{0, \infty\}$ . Writing  $a$  for  $\partial_0 \varphi(\{t\})$ ,  $\varphi(\{t\}) - a \cdot \{t\}$  is then unramified on  $\mathbf{P}^1 \setminus \{\infty\}$ . By Milnor's exact sequence, this implies that this element comes from an element  $b \in K_*^M(k_0)/p$ . Therefore, we have  $\varphi(\{t\}) = a \cdot \{t\} + b$ . Combining with step 1, this tells us that there exist  $a$  and  $b$  in  $K_*^M(k_0)/p$ , such that for any field extension  $k/k_0$  and for any  $x \in K_1^M(k)/p$ ,  $\varphi(x) = a \cdot x + b$ . This defines an operation as one can easily check.

*Step 3.* We conclude by induction on  $r$ . The case  $r = 1$  has been treated in steps 1 and 2. Now, write  $(K_1^M(k)/p)^r = (K_1^M(k)/p)^{r-1} \times K_1^M(k)/p$  and write  $(x, a_r)$  for the element  $(a_1, \dots, a_r) \in (K_1^M(k)/p)^r$ . Fix a field  $k/k_0$  and  $x \in (K_1^M(k)/p)^{r-1}$ , then  $\varphi(x, a_r)$  defines an operation  $K_1^M(k)/p \rightarrow K_*^M/p$  over  $k$ . Steps 1 and 2 yield the existence of elements  $c_x$  and  $d_x$  in  $K_*^M(k)/p$  such that  $\varphi(x, a_r) = c_x \cdot a_r + d_x$ . The maps  $x \mapsto c_x$  and  $x \mapsto d_x$  define operations  $(K_1^M(k)/p)^{r-1} \rightarrow K_*^M/p$  over  $k_0$ . By the induction hypothesis, they are of the form stated in the theorem. It is now easy to see that  $\varphi(x, a_r) = c_x \cdot a_r + d_x$  is of the required form. It remains to prove that the operations  $(a_1, \dots, a_r) \mapsto \{a_{i_1}, \dots, a_{i_s}\}$  for

subsets  $1 \leq i_1 < \dots < i_s \leq r$  form a free basis. Let  $\varphi$  be an operation  $(K_1^M/p)^r \rightarrow K_*^M/p$  over  $k_0$ . By the above, we know that there exist elements  $\lambda_{i_1, \dots, i_s} \in K_*^M(k_0)/p$  such that for all field extension  $k/k_0$  and all  $r$ -tuple  $(a_1, \dots, a_r) \in (K_1^M(k)/p)^r$ ,

$$\varphi(a_1, \dots, a_r) = \sum_{1 \leq i_1 < \dots < i_s \leq r} \lambda_{i_1, \dots, i_s} \cdot \{a_{i_1}, \dots, a_{i_s}\}.$$

Assume  $\varphi$  is 0. Fix a subset  $1 \leq i_1 < \dots < i_s \leq r$  and consider the field  $k = k_0(t_{i_1}, \dots, t_{i_s})$  where  $t_{i_1}, \dots, t_{i_s}$  are indeterminates. Let  $a$  be the element in  $(K_1^M(k)/p)^r$  with entry  $\{t_{i_q}\}$  in the  $i_q^{\text{th}}$  coordinate for all  $q$  and zero elsewhere. Consider also the residue maps corresponding to the local parameters  $t_{i_q}$ ,  $\partial_q : K_*^M(k(t_{i_1}, \dots, t_{i_q}))/p \rightarrow K_{*-1}^M(k(t_{i_1}, \dots, t_{i_{q-1}}))/p$ . Then,  $\lambda_{i_1, \dots, i_s} = \partial_s \circ \dots \circ \partial_1(\varphi(x)) = 0$ .  $\square$

As a consequence of the very nice form of the operations  $K_1^M/p \rightarrow K_*^M/p$  over  $k_0$ , we get

**Corollary 3.5.** *Let  $k_0$  be any field. Then, specialization maps commute with operations  $\varphi : K_1^M/p \rightarrow K_*^M/p$  over  $k_0$ .*

**Proof.** For  $K$  and  $v$  as in Definition 3.3, specialization maps  $s_\pi$  are  $K_*^M(k_0)/p$ -linear for any choice of uniformizer  $\pi$ , as easily seen from their definition.  $\square$

**Remark 3.6.** In [9], Theorem 3.4 is proven in Galois cohomology for base fields  $k_0$  of characteristic different from  $p$ . The proof relies on the fact that it is possible to show first that operations  $H^1(-, \mathbf{Z}/p) \rightarrow H^*(-, \mathbf{Z}/p)$  over  $k_0$  commute with specialization maps in the above sense. Roughly, this is done by proving that the specialization maps admit right inverses that are induced by some inclusion of fields. Here, we first determine all the operations and obtain *a posteriori* that the operations commute with specialization maps. Also, using the Faddeev exact sequence for Galois cohomology with finite coefficients ([10, Cor. 6.9.3]) and thanks to Kummer theory, the proof of Theorem 3.4 translates *mutatis mutandis* to the case of operations  $H^1(-, \mathbf{Z}/p) \rightarrow H^*(-, \mathbf{Z}/p)$  over  $k_0$ . Thus, this gives a new proof of [9, Theorem 16.4].

## 3.2 Operations in Milnor $K$ -theory mod $p$

In this section, we determine the group of operations  $K_i^M/p \rightarrow K_*^M/p$  over any field  $k_0$ , except in the case when  $p = 2$  and  $-1$  is not a square in  $k_0$ . The proof of this last case is postponed to the next section. Thus we prove here Theorem 1, and Theorem 2 in the particular case when  $-1$  is a square in  $k_0$ . Proposition 3.7 covers the cases when divided powers are well-defined on  $K_i^M(k_0)/p$  (as in Proposition 2.2) and Proposition 3.11 deals with the case where  $p$  and  $i$  are both odd.

**Proposition 3.7.** *Let  $k_0$  be a field and  $p$  a prime number. The algebra of operations  $K_i^M(k)/p \rightarrow K_*^M(k)/p$  commuting with field extensions over  $k_0$  is*

- If  $i = 0$ , the free  $K_*^M(k_0)/p$ -module of rank  $p$  of functions  $\mathbf{F}_p \rightarrow K_*^M(k_0)/p$ .

- If  $i = 1$ , the free  $K_*^M(k_0)/p$ -module of rank 2, generated by  $\gamma_0$  and  $\gamma_1$ .
- If  $i$  is even  $\geq 2$  and  $p$  is an odd prime, the free  $K_*^M(k_0)/p$ -module generated by the divided powers  $\gamma_n$  for  $n \geq 0$  and the action of one of its element  $(y_0, y_1, \dots)$  is given by
$$x \mapsto y_0 + y_1 \cdot x + y_2 \cdot \gamma_2(x) + \dots + y_l \cdot \gamma_l(x) + \dots$$
- If  $i \geq 2$ ,  $p = 2$  and  $-1$  is a square in  $k_0^\times$ , likewise, the free  $K_*^M(k_0)/2$ -module generated by the divided powers  $\gamma_n$  for  $n \geq 0$ .

**Remark 3.8.** Assume  $k_0$  is a field of characteristic  $\neq p$  and  $k$  is any extension of  $k_0$ . Via the Galois symbol  $K_i^M(k)/p \rightarrow H^i(k, \mathbf{Z}/p(i))$  and under the assumption of the Bloch-Kato conjecture, this gives all operations  $H^i(-, \mathbf{Z}/p(i)) \rightarrow H^*(-, \mathbf{Z}/p(*))$  over  $k_0$  (by  $H^i(k, \mathbf{Z}/p(i))$ ), we mean the étale cohomology of the field  $k$  with values in  $\mu_p^{\otimes i}$ . Since  $k_0$  has characteristic not  $p$ , we get by Galois descent that  $H^i(k, \mathbf{Z}/p(j)) = H^i(k, \mathbf{Z}/p(i)) \otimes \mu_p^{\otimes(j-i)}$ . Thus, if  $p$  is an odd prime,  $i \geq 2$  is even and if  $j$  is any integer, there is a well-defined divided power operation  $\gamma_n : H^i(-, \mathbf{Z}/p(j)) \rightarrow H^{ni}(-, \mathbf{Z}/p(ni + j - i))$  over  $k_0$ .

**Proof.** In the first case ( $i = 0$ ), the functor  $K_0^M/p$  is just the constant functor with value  $\mathbf{F}_p$ , hence the result. The second case is Theorem 3.4. We now restrict our attention to the two last cases, that is either  $i$  even and  $p$  odd, or  $p = 2$  and  $k_0$  has a square-root of  $-1$ . By Proposition 2.2 the given maps define operations. Therefore, the inclusion " $\supseteq$ " holds in each case if we can prove that the algebra of such operations is a free module over  $K_*^M(k_0)/p$ . Let  $\psi = y_l \cdot \gamma_l + y_{i_1} \cdot \gamma_{i_1} + \dots + y_{i_r} \cdot \gamma_{i_r}$  with  $l < i_1 < \dots < i_r$  be an operation on  $K_i^M(k)/p$  which is zero, with minimal  $l$  such that  $y_l \neq 0$ . We are going to see that this is contradictory. If  $l = 0$ ,  $\psi(0) = y_0 = 0$ . If  $l \geq 1$ , let's consider the field extension  $F = k_0(x_{j,k})_{1 \leq j \leq i, 1 \leq k \leq l}$  in  $il$  indeterminates over  $k_0$  and the element  $x = \sum_{k=1}^l \{x_{1,k}, \dots, x_{i,k}\}$  of length at most  $l$  in  $K_i^M(F)/p$ , then  $\psi(x) = y_l \cdot \gamma_l(x) = y_l \cdot \{x_{1,1}, \dots, x_{i,l}\}$ , which is zero only if  $y_l = 0$  by Proposition 3.1.

It is thus enough to prove that given an operation  $\varphi : K_i^M(k)/p \rightarrow K_j^M(k)/p$ ,  $\varphi$  must be of the form  $\varphi = y_0 + y_1 \cdot \text{id} + y_2 \cdot \gamma_2 + \dots + y_l \cdot \gamma_l + \dots$ .

Consider as above the element  $x = \sum_{k=1}^l \{x_{1,k}, \dots, x_{i,k}\} \in K_i^M(F)/p$  of length  $\leq l$ . Define the set  $E_{i,l} = \{(m, k), 1 \leq m \leq i, 1 \leq k \leq l\}$  and equip it with the lexicographic order :  $(m, k) \leq (m', k')$  if either  $k < k'$  or  $k = k'$  and  $m \leq m'$ . Consider an element  $a$  in  $\mathcal{P}(E_{i,l})$  the set of subsets of  $E_{i,l}$ . We will write  $a_x = \prod_{(m,k) \in a} \{x_{m,k}\}$  for the ordered product of the  $\{x_{m,k}\}$ 's for  $(m, k) \in a$ , and if  $a$  is the empty set,  $a_x = 1$ . With our notations, Theorem 3.4 tells us that there are unique elements  $c_a \in K_{j-\#(a)}^M(k_0)/p$  for  $a \in E_{i,l}$  that make the equality

$$\varphi(x) = \sum_{a \in E_{i,l}} c_a \cdot a_x$$

true for all elements  $x$  as above. We want to prove that the only non-zero terms in this sum are the ones which correspond to "concatenation" of the symbols  $s_k =$

$\{x_{1,k}, \dots, x_{i,k}\}$ . Precisely, let  $A_{i,l}$  be the subset of  $\mathcal{P}(E_{i,l})$  consisting of elements  $a$  such that if  $(m, k) \in a$  for some  $m$  and  $k$ , then  $(m', k) \in a$  for all integer  $m' \in [1, i]$ . Also, let  $B_{i,l}$  be the complement of  $A_{i,l}$  in  $\mathcal{P}(E_{i,l})$ . Notice that elements in  $A_{i,l}$  have cardinal a multiple of  $i$ .

**Lemma 3.9.**  $c_a \neq 0$  implies  $a \in A_{i,l}$ .

**Proof of lemma 3.9.** We will proceed by induction on  $\#(a)$ . Let  $P_n$  be the proposition "For all  $a \in B_{i,l}$  such that  $\#(a) \leq n$ ,  $c_a = 0$ ".

$P_0$  is true since if  $\#(a) = 0$ ,  $a$  is not in  $B_{i,l}$ . Now, assume  $P_n$  is true for some  $n$ , and let's prove that  $P_{n+1}$  holds. Let  $a \in B_{i,l}$  be of cardinal  $n + 1$ . Consider the element  $x = \sum_{k=1}^l \{x_{1,k}, \dots, x_{i,k}\}$  where  $x_{m,k} = 1$  if  $(m, k) \notin a$ . To be precise, this means that we are considering the image of  $\sum_{k=1}^l \{x_{1,k}, \dots, x_{i,k}\}$  under the successive application of the specialization maps  $s_{x_{m,k}}$  for  $(m, k) \notin a$ . Then, by induction hypothesis,

$$\varphi(x) = \sum_{a' \subseteq a} c_{a'} \cdot a'_x = c_a \cdot a_x + \sum_{\substack{a' \subsetneq a \\ a' \in A_{i,l}}} c_{a'} \cdot a'_x.$$

Also,

$$\bigcup_{\substack{a' \subsetneq a \\ a' \in A_{i,l}}} a'$$

because  $A_{i,l}$  is stable under union. So now, consider an element  $(m, k) \in a - \left( \bigcup_{a' \subsetneq a, a' \in A_{i,l}} a' \right)$  for some  $m$  and  $k$ , then there exists an  $m'$  such that  $(m', k) \notin a$ . As  $x_{m',k} = 1$ , we see that  $\varphi(x)$  doesn't depend on  $x_{m,k}$ . The only term in  $\varphi(x)$  where  $x_{m,k}$  appears is  $c_a \cdot a_x$ . Therefore, for any field extension  $k/k_0$ , and for any values taken in  $K_1^M(k)/p$  assigned to the elements  $x_{m,k}$  for  $(m, k)$  belonging to  $a$ , we must have  $c_a \cdot a_x = 0$ . Proposition 3.1 implies  $c_a = 0$ .  $\square$

Hence we obtain

$$\varphi(x) = \sum_{a \in A_{i,l}} c_a \cdot a_x.$$

**Lemma 3.10.** If  $a, a' \in A_{i,l}$  are such that  $\#(a) = \#(a') = ri \in \mathbf{Z}i$ , then  $c_a = c_{a'} \in K_{j-ri}^M(k_0)/p$ .

**Proof of lemma 3.10.** It suffices to impose  $l - r$  of the symbols appearing in the decomposition of  $x$  to be 0 and to use the commutativity of addition.  $\square$

All in all, noticing that  $\sum_{a \in A_{i,l}, \#(a)=ri} a_x = \gamma_r(x)$ , we obtain the existence of elements  $y_r \in K_{j-ri}^M(k_0)/p$  such that for all  $x \in K_i^M(k)/p$ ,

$$\varphi(x) = \sum_{r \geq 0} y_r \cdot \gamma_r(x).$$

$\square$

**Proposition 3.11.** *In the case where  $i$  and  $p$  are odd, the algebra of operations is the free  $K_*^M(k_0)/p$ -module of rank 2 generated by  $\gamma_0$  and  $\gamma_1$ .*

**Proof.** The module is clearly free and its elements do define operations. So now, let  $\varphi$  be an operation  $K_i^M(k)/p \rightarrow K_j^M(k)/p$  over  $k_0$  and let  $x = s_1 + \dots + s_l \in K_i^M(k)/p$  be an element of length at most  $l$ . With the same notations as in Theorem 1, we have  $\varphi(x) = \sum_{a \in E_{i,l}} c_a \cdot a_x$  and Lemma 3.9 applies. So, actually,  $\varphi(x) = \sum_{a \in A_{i,l}} c_a \cdot a_x$ . We want to show that  $c_a = 0$  as soon as  $\#(a) > i$ . It is possible to write

$$\varphi(s_1 + \dots + s_l) = s_1 \cdot s_2 \cdot \varphi_0(s_3 + \dots + s_l) + s_1 \cdot \varphi_1(s_3 + \dots + s_l) + s_2 \cdot \varphi_2(s_3 + \dots + s_l).$$

If  $s_1$  and  $s_2$  are permuted, we should obtain the same result. Subtracting both identities and considering that  $s_1 \cdot s_2 = -s_2 \cdot s_1$ , we get the equality

$$2s_1 \cdot s_2 \cdot \varphi_0(s_3 + \dots + s_l) + (s_1 - s_2) \cdot (\varphi_1(s_3 + \dots + s_l) - \varphi_2(s_3 + \dots + s_l)) = 0.$$

Setting  $s_2 = 0$  gives  $s_1 \cdot (\varphi_1(s_3 + \dots + s_l) - \varphi_2(s_3 + \dots + s_l)) = 0$  for all  $s_1, s_3, \dots, s_l$  and Proposition 3.1 implies  $\varphi_1(s_3 + \dots + s_l) = \varphi_2(s_3 + \dots + s_l)$ . Hence, we are led to the equality

$$2s_1 \cdot s_2 \cdot \varphi_0(s_3 + \dots + s_l) = 0$$

for all symbols  $s_1, \dots, s_l$ . Since 2 is invertible in  $K_0^M(k_0)/p = \mathbf{F}_p$ , Proposition 3.1 implies  $\varphi_0(s_3 + \dots + s_l) = 0$  for all  $s_3, \dots, s_l$  that is  $\varphi_0 = 0$ , since  $l$  was arbitrary. The result follows by taking all the different pairs of symbols in place of  $s_1$  and  $s_2$ .  $\square$

### 3.3 Operations in Milnor $K$ -theory mod 2

We finish proving Theorem 2 by considering the remaining case, that is  $p = 2$  and  $-1$  is not necessarily a square in the base field. Let  $k_0$  be any field and consider again, as in section 2.4, the map  $\tau_i : K_*^M(k_0)/2 \rightarrow K_*^M(k_0)/2$ ,  $x \mapsto \{-1\}^{i-1} \cdot x$ .

**Proposition 3.12.** *The algebra of operations  $K_i^M(k)/2 \rightarrow K_*^M(k)/2$  over  $k_0$  commuting with field extensions is*

- If  $i = 0$ , the free  $K_*^M(k_0)/2$ -module of rank 2 of functions  $\mathbf{F}_2 \rightarrow K_*^M(k_0)/2$ .
- If  $i = 1$ , the free  $K_*^M(k_0)/2$ -module of rank 2, generated by  $\gamma_0$  and  $\gamma_1$ .
- If  $i \geq 2$ , the  $K_*^M(k_0)/2$ -module

$$K_*^M(k_0)/2 \cdot \gamma_0 \oplus K_*^M(k_0)/2 \cdot \gamma_1 \oplus \bigoplus_{n \geq 2} \text{Ker}(\tau_i) \cdot \gamma_n.$$

**Proof.** The cases  $i = 0$  and  $i = 1$  have already been treated. Now, suppose  $i \geq 2$ . Given  $l \geq 2$  and  $y_l \in \text{Ker}(\tau_i)$ , the map  $y_l \cdot \gamma_l$  is a well-defined operation by Proposition 2.9. The inclusion " $\supseteq$ " holds for the very same reason as in the proof of Proposition 3.7.

The proof of Proposition 3.7 shows that if  $\varphi$  is an operation  $K_i^{\mathbb{M}}(k)/2 \rightarrow K_j^{\mathbb{M}}(k)/2$ , then necessarily there exist elements  $y_r \in K_{j-r_i}^{\mathbb{M}}(k_0)/2$  such that  $\varphi = y_0 + y_1 \cdot \gamma_1 + \dots + y_l \cdot \gamma_l + \dots$ . All we have to prove is that necessarily, for a given integer  $l \geq 2$ ,  $y_l$  must satisfy  $y_l \cdot \{-1\}^{i-1} = 0$ . Let  $x = s_1 + \dots + s_l$  be a sum of  $l \geq 2$  symbols. Suppose  $s_1 = \{x_1, \dots, x_{i-1}, x_i\}$  and  $s_2 = \{x_1, \dots, x_{i-1}, y_i\}$ . Then

$$\varphi(x) = y_0 + y_1 \cdot x + \dots + y_l \cdot \gamma_l(s_1 + s_2 + \dots + s_l)$$

but also

$$\varphi(x) = y_0 + y_1 \cdot x + \dots + y_{l-1} \cdot \gamma_{l-1}(\{x_1, \dots, x_{i-1}, x_i y_i\} + s_3 + \dots + s_l)$$

It is easy to check that the difference of these two equalities lead to the equality

$$y_l \cdot \{x_1, \dots, x_{i-1}, x_i, x_1, \dots, x_{i-1}, y_i\} \cdot s_3 \cdot \dots \cdot s_l = 0.$$

Recall that  $\{x, x\} = \{x, -1\}$  for all  $x \in k^\times$  and that  $K_*^{\mathbb{M}}(k)/2$  is a commutative algebra. Hence  $y_l$  must satisfy  $y_l \cdot \{-1\}^{i-1} \cdot \{x_1, \dots, x_{i-1}, x_i, y_i\} \cdot s_3 \cdot \dots \cdot s_l = 0$ . Proposition 3.1 then implies that necessarily  $y_l \cdot \{-1\}^{i-1} = 0$ .  $\square$

As a consequence of Theorem 1 and Theorem 2, we get

**Corollary 3.13.** *Let  $k_0$  be any field and let  $i$  be an integer. Then, the operations  $\varphi : K_i^{\mathbb{M}}/p \rightarrow K_*^{\mathbb{M}}/p$  over  $k_0$  commute with specialization maps.*

$\square$

### 3.4 Operations in integral Milnor $K$ -theory

Operations  $K_i^{\mathbb{M}} \rightarrow K_*^{\mathbb{M}}$  over  $k_0$  are not as nice as in the mod  $p$  case since such operations are not determined by the image of a transcendental element. For example, let  $a$  and  $b$  be distinct elements in  $K_1^{\mathbb{M}}(k_0)$  (for  $k_0$  not the field with only 2 elements). Consider the operation  $\varphi : K_1^{\mathbb{M}} \rightarrow K_2^{\mathbb{M}}$  that assigns to each element  $t$  transcendental over  $k_0$  the value  $\{t\} \cdot a$  and to each element  $e$  algebraic over  $k_0$  the value  $\{e\} \cdot b$ . This is a well-defined operation since for any extension  $k$  of  $k_0$ , and any transcendental element  $t \in k$  and any algebraic element  $e \in k$ , we have  $\{t\} \neq \{e\}$  in  $K_1^{\mathbb{M}}(k)$ . Also, such an operation is not of the form described in Theorem 3.4 because of Proposition 3.1.

Nonetheless, the image  $\varphi(\{t\})$  of any transcendental element  $t$  over  $k_0$  determines the image  $\varphi(\{u\})$  of any other transcendental element  $u$  over  $k_0$ , via the obvious isomorphism  $k_0(t) \simeq k_0(u)$ . Also, it seems natural to impose the operations  $\varphi : K_1^{\mathbb{M}} \rightarrow K_*^{\mathbb{M}}$  to commute with specialization maps, in which case the image of any algebraic element over  $k_0$  is determined by  $\varphi(\{t\})$ .

**Definition 3.14.** Let  $k_0$  be any field. We say that an operation  $\varphi : K_i^{\mathbb{M}} \rightarrow K_*^{\mathbb{M}}$  over  $k_0$  commutes with specialization maps if  $\varphi$  satisfies the conclusion of Definition 3.3. In particular,  $\varphi$  commutes with specialization maps only if for any extension  $k/k_0$ , any  $t$  transcendental over  $k$  and for any closed point  $P$  in  $\mathbf{P}_k^1$ , we have a commutative diagram

$$\begin{array}{ccc}
K_i^M(k(t)) & \xrightarrow{\varphi} & K_*^M(k(t)) \\
s_\pi \downarrow & & s_\pi \downarrow \\
K_i^M(\kappa_P) & \xrightarrow{\varphi} & K_*^M(\kappa_P)
\end{array}$$

where  $\kappa_P$  denotes the residue field of  $k(t)$  with respect to the valuation  $v_P$  corresponding to the polynomial  $P$ , and  $\pi$  is any uniformizer for the valuation  $v_P$ .

Before we describe operations commuting with specialization maps, we need to see that residue maps and specialization maps are well-behaved with respect to transcendental field extensions. We also need to relate the specialization maps  $s_\pi$  and  $s_{\pi'}$  for two different choices of uniformizers  $\pi$  and  $\pi'$ .

**Lemma 3.15.** *For any field  $F$  and any  $u$  transcendental over  $F$ , let  $\iota_u : K_*^M(F) \rightarrow K_*^M(F(u))$  be the injective map induced by the inclusion of field  $F \subset F(u)$ . Let  $k$  be a field and  $P$  a closed point in the projective line  $\mathbf{P}_k^1$  with residue field  $\kappa_P$ . Let  $v_P$  be the valuation on  $k(t)$  corresponding to  $P$  and let  $\pi$  be a local parameter for  $v_P$  in  $k(t)$ .*

*Then, the valuation  $v_P$  extends naturally to a valuation, that we still write  $v_P$ , on  $k(u)(t)$ , and  $\pi$  seen as an element in  $k(u)(t)$  defines a uniformizer for  $v_P$  in  $k(u)(t)$ . Moreover, the residue map  $\partial_{v_P}$  and the specialization map  $s_\pi$  commute with  $\iota_u$ . Precisely, the following diagrams commute :*

$$\begin{array}{ccc}
K_*^M(k(t)) & \xrightarrow{\iota_u} & K_*^M(k(u, t)) \\
\partial_{v_P} \downarrow & & \partial_{v_P} \downarrow \\
K_{*-1}^M(\kappa_P) & \xrightarrow{\iota_u} & K_{*-1}^M(\kappa_P(u))
\end{array}
\qquad
\begin{array}{ccc}
K_*^M(k(t)) & \xrightarrow{\iota_u} & K_*^M(k(u, t)) \\
s_\pi \downarrow & & s_\pi \downarrow \\
K_*^M(\kappa_P) & \xrightarrow{\iota_u} & K_*^M(\kappa_P(u))
\end{array}$$

**Proof.** The fact that  $v_P$  and  $\pi$  extend respectively to a valuation and to a uniformizer on  $k(u)(t)$  is straightforward. The commutativity of the diagrams is an immediate consequence of the definition of the residue map and of the specialization map.  $\square$

**Lemma 3.16.** *Let  $k$  be a field and  $x$  be an element in  $K_*^M(k(t))$ . If  $P$  is a closed point in  $\mathbf{P}_k^1 - \{\infty\}$  and  $Q \in k(t)$  is such that  $v_P(Q) = 0$ , then we have the formula*

$$s_{PQ}(x) = s_P(x) - s_P(\{-Q(t)\}) \cdot \partial_{v_P}(x).$$

**Proof.** Under the assumption made on  $Q$ , the element  $PQ$  is a uniformizer for the valuation  $v_P$ . Hence, we have  $s_{PQ}(x) = \partial_{v_P}(\{-PQ\} \cdot x) = s_P(x) + \partial_{v_P}(\{-Q\} \cdot x)$ . It is thus enough to show that  $\partial_{v_P}(\{-Q\} \cdot x) = -s_P(\{-Q\}) \cdot \partial_{v_P}(x)$ . The element  $Q$  being a unit in the ring  $\{a \in k(t), v_P(a) \geq 0\}$ , this follows from the very definition of the residue and specialization maps.  $\square$

**Proposition 3.17.** *Let  $k_0$  be a field.*

- *The algebra of operations  $K_0^M(k) \rightarrow K_*^M(k)$  over  $k_0$  commuting with specialization maps is the free  $K_*^M(k_0)$ -module of countable rank of functions  $\mathbf{Z} \rightarrow K_*^M(k_0)$ .*

- The algebra of operations  $K_1^M(k) \rightarrow K_*^M(k)$  over  $k_0$  commuting with specialization maps is the free  $K_*^M(k_0)$ -module generated by  $\gamma_0$  and  $\gamma_1$ .

**Proof.** For the first statement,  $K_0^M$  is the constant functor with value  $\mathbf{Z}$ . Also, inclusion of fields and specialization maps induce the identity on the  $K_0^M$ -groups of fields. Hence the result.

For the second statement, such an operation  $\varphi : K_1^M(k) \rightarrow K_*^M(k)$  is determined by the image of  $\{t\} \in K_1^M(k_0(t))$  as discussed above. Let  $\varphi : K_1^M \rightarrow K_*^M$  be an operation over  $k_0$  commuting with specialization maps. We are going to show that, for  $t$  a transcendental element over  $k_0$ ,  $\varphi(\{t\})$  is unramified outside  $\{0, \infty\}$ . By Milnor's exact sequence, this will prove the Proposition.

Let's consider the function field in one indeterminate  $k = k_0(u)$  and a monic irreducible polynomial  $P \in k_0[t]$  that we can also see as a monic irreducible polynomial in  $k[t]$  with coefficients in  $k_0$ , via the obvious inclusion of  $k_0$  into  $k$ . Let  $v_P$  denote the valuation on  $k(t)$  corresponding to  $P$ . The choice of a uniformizer  $\pi \in k(t)$  is equivalent to the choice of an element  $Q \in k(t)$  such that  $v_P(Q) = 0$ , by setting  $\pi_Q = PQ$ . Let  $\alpha$  be the image of  $t$  in the residue field  $\kappa_P = k[t]/P$ . By definition of  $\varphi$ , we have a commutative diagram for any  $Q \in k_0(u, t)$  such that  $v_P(Q) = 0$

$$\begin{array}{ccc} K_1^M(k_0(u, t)) & \xrightarrow{\varphi} & K_*^M(k_0(u, t)) \\ s_{\pi_Q} \downarrow & & s_{\pi_Q} \downarrow \\ K_1^M(\kappa_P(u)) & \xrightarrow{\varphi} & K_*^M(\kappa_P(u)) \end{array}$$

For any field  $F$ , let  $\iota_u : K_*^M(F) \rightarrow K_*^M(F(u))$  be the injective map induced by the inclusion of field  $F \subset F(u)$ . For  $\{t\} \in K_1^M(k_0(t))$ , we then have  $s_{\pi_Q} \circ \varphi \circ \iota_u(\{t\}) = \varphi \circ s_{\pi_Q} \circ \iota_u(\{t\})$ . If  $P \neq t$ , Lemma 3.16 says that, on the one hand,  $\varphi \circ s_{\pi_Q} \circ \iota_u(\{t\}) = \iota_u(\varphi(\{\alpha\}))$  and on the other hand  $s_{\pi_Q} \circ \varphi \circ \iota_u(\{t\}) = s_{\pi_Q} \circ \iota_u \circ \varphi(\{t\}) = \{-Q^{-1}(\alpha)\} \cdot \partial_{v_P}(\iota_u \circ \varphi(\{t\})) + s_P(\iota_u(\varphi(\{t\})))$ . Also, since  $P$  has its coefficients in  $k_0$ , Lemma 3.15 implies that  $\partial_{v_P}(\iota_u \circ \varphi(\{t\})) = \iota_u(\partial_{v_P} \circ \varphi(\{t\}))$  and also that  $s_P(\iota_u \circ \varphi(\{t\})) = \iota_u(s_P \circ \varphi(\{t\}))$ . All in all, we have

$$\iota_u(\varphi(\{\alpha\})) = \{-Q^{-1}(\alpha)\} \cdot \iota_u(\partial_{v_P} \circ \varphi(\{t\})) + \iota_u(s_P(\varphi(\{t\}))) \in K_*^M(\kappa_P(u)).$$

Basically, we have just been expressing the fact that apart from  $Q$ , everything exists before adjoining that indeterminate  $u$ . So now, let  $Q$  be the constant polynomial in  $k_0(u)[t]$  equal to  $-u^{-1}$  and consider the residue map  $\partial_u : K_*^M(\kappa_P(u)) \rightarrow K_{*-1}^M(\kappa_P)$ . Applying  $\partial_u$  to the above equality, and using the injectivity of  $\iota_u$ , we get

$$\partial_{v_P} \circ \varphi(\{t\}) = 0.$$

□

**Proposition 3.18.** *Let  $k_0$  be any field and  $i$  an integer  $\geq 2$ .*

- If  $i$  is even, the algebra of operations  $K_i^{\text{M}}(k) \rightarrow K_*^{\text{M}}(k)$  over  $k_0$  commuting with specialization maps is the  $K_*^{\text{M}}(k_0)$ -module

$$K_*^{\text{M}}(k_0) \oplus K_*^{\text{M}}(k_0) \cdot \text{id} \oplus \bigoplus_{n \geq 2} \text{Ker}(\tau_i) \cdot \gamma_n.$$

- If  $i$  is odd, the algebra of operations  $K_i^{\text{M}}(k) \rightarrow K_*^{\text{M}}(k)$  over  $k_0$  commuting with specialization maps is the  $K_*^{\text{M}}(k_0)$ -module

$$K_*^{\text{M}}(k_0) \oplus K_*^{\text{M}}(k_0) \cdot \text{id} \oplus \bigoplus_{n \geq 2} {}_2\text{Ker}(\tau_i) \cdot \gamma_n.$$

**Proof.** The proof is the same as for Theorems 1 and 2. An operation  $\varphi : K_i^{\text{M}} \rightarrow K_*^{\text{M}}$  over  $k_0$  is necessarily a sum of divided power operations and it is well-defined if and only if these are weak divided power operations as in section 2.5.  $\square$

When  $\text{char } k_0 = 2$ , the maps  $\tau_i$  are zero for  $n \geq 2$ . Also, in [11], Izhboldin proves that if  $k_0$  has characteristic  $p$  then the Milnor  $K$ -groups  $K_n^{\text{M}}(k_0)$  have no  $p$ -torsion (result conjectured by Tate). Hence, when  $\text{char } k_0 = 2$ ,  ${}_2K_*^{\text{M}}(k_0) = 0$ , the above Proposition becomes

**Proposition 3.19.** *Let  $k_0$  be a field of characteristic 2 and  $i$  an integer  $\geq 2$ .*

- If  $i$  is even, the algebra of operations  $K_i^{\text{M}}(k) \rightarrow K_*^{\text{M}}(k)$  over  $k_0$  commuting with specialization maps is the free  $K_*^{\text{M}}(k_0)$ -module

$$K_*^{\text{M}}(k_0) \oplus K_*^{\text{M}}(k_0) \cdot \text{id} \oplus \bigoplus_{n \geq 2} K_*^{\text{M}}(k_0) \cdot \gamma_n.$$

- If  $i$  is odd, the algebra of operations  $K_i^{\text{M}}(k) \rightarrow K_*^{\text{M}}(k)$  over  $k_0$  commuting with specialization maps is the free  $K_*^{\text{M}}(k_0)$ -module

$$K_*^{\text{M}}(k_0) \oplus K_*^{\text{M}}(k_0) \cdot \text{id}.$$

$\square$

## 4 Operations in Milnor $K$ -theory of a smooth scheme

In this last section, we generalize the results about operations in Milnor  $K$ -theory of fields to the case of regular schemes over a field  $k$ . We are first interested in the Milnor  $K$ -theory of a regular  $k$ -scheme defined as the kernel of the first map in the Gersten complex. Such a definition coincides with Rost's Chow groups with coefficients as constructed in [22]. Indeed, for  $X$  a regular  $k$ -scheme of dimension  $d$ , with Rost's notations, we have  $K_n^{\text{M}}(X) = A_d(X, n-d)$  where  $A_p(X, q)$  is the  $p^{\text{th}}$  homology group of the Gersten complex  $C_*(X, q)$  defined by  $C_p(X, q) = \bigoplus_{x \in X_{(p)}} K_{p+q}^{\text{M}}(k(x))$  and  $X_{(p)}$  denotes the  $p$ -dimensional

points in  $X$ . It is then a fact that  $K_n^M$  defines a contravariant functor from the category of smooth  $k$ -schemes to the category of groups. As in the case of fields, we are able to determine all operations  $K_n^M/p \rightarrow K_*^M/p$  over a smooth  $k$ -scheme  $X$ . In view of the Gersten complex, we can write  $K_n^M(X) = H^0(X, \mathcal{K}_n^M)$ . It is then possible, under the assumption of the Bloch-Kato conjecture, to relate for  $p \neq \text{char } k$  the Milnor  $K$ -group  $K_n^M(X)/p$  and the unramified cohomology group  $H^0(X, \mathcal{H}_{\text{ét}}^n(\mathbf{Z}/p))$ , and thus to describe all the operations on the unramified cohomology of smooth schemes over  $k$ .

We are then interested in the Milnor  $K$ -theory  $\bar{K}_n^M(A)$  of a ring  $A$  defined as the tensor algebra of the units in  $A$  subject to the Steinberg relations. This defines a covariant functor from the category of rings to the category of sets. If  $k$  is an infinite field and if  $A$  is a regular semi-local  $k$ -algebra, we are also able to determine all operations  $\bar{K}_n^M \rightarrow \bar{K}_*^M$  over  $A$ .

#### 4.1 The unramified case

Let  $X$  be a regular (in codimension-1) scheme, and denote by  $X^{(r)}$  the set of codimension- $r$  points in  $X$ . If  $x$  is a codimension-0 point of  $X$ , e.g. the generic point of  $X$  if  $X$  is irreducible, then the codimension-1 points in the closure of  $x$  define discrete valuations on the function field  $k(x)$  of  $x$ , and thus residue maps on the Milnor  $K$ -theory of  $k(x)$ . We define the Milnor  $K$ -theory of the scheme  $X$  to be

$$K_n^M(X) = \text{Ker} \left( \bigoplus_{x \in X^{(0)}} K_n^M(k(x)) \xrightarrow{\partial} \bigoplus_{y \in X^{(1)}} K_{n-1}^M(k(y)) \right).$$

In particular, this definition makes sense for regular rings. For a regular scheme  $X$  assumed to be irreducible and for any  $i \geq 1$ , an element  $x \in K_i^M(X)$  is an element of  $K_i^M(k(X))$  which is unramified along all codimension-1 points of  $X$ , i.e. which has zero residue for all residue maps corresponding to codimension-1 points in  $X$ . We say that an element  $x \in K_i^M(k(X))$  is unramified if it belongs to  $K_i^M(X)$ . It is therefore possible to write an element  $x$  of  $K_i^M(X)$  as a sum of symbols  $s_k = \{x_{1,k}, \dots, x_{i,k}\}$ ,  $1 \leq k \leq l$ , with all the  $x_{j,k}$ 's in  $k(X)^\times$ . Given an integer  $n$  and an element  $y_n \in K_*^M(X)$ , an  $n^{\text{th}}$  divided power of  $x$  written as a sum  $\sum_{k=1}^l s_k$  is

$$y_n \cdot \gamma_n(x) = y_n \cdot \sum_{1 \leq l_1 < \dots < l_n \leq l} s_{l_1} \cdot \dots \cdot s_{l_n}.$$

**Lemma 4.1.** [10, Prop 7.1.7] *Let  $K$  be a field equipped with a discrete valuation  $v$ , and let  $\mathcal{O}_K$  be its ring of integers and  $\kappa$  be its residue field. Then,  $\text{Ker} (K_n^M(K) \xrightarrow{\partial_v} K_{n-1}^M(\kappa))$  is generated as a group by symbols of the form  $\{x_1, \dots, x_n\}$  where the  $x_i$ 's are units in  $\mathcal{O}_K$  for all  $i$ .*

Let's mention that this lemma implies the following.

**Corollary 4.2.** *Let  $X$  be a regular scheme. The cup-product on  $K_*^M(k(X))$  endows the group  $K_*^M(X) = \bigoplus_{n \geq 0} K_n^M(X)$  with a ring structure.*

**Proposition 4.3.** *If  $X$  is a regular scheme, divided powers are well-defined on  $K_i^M(X)/p$  in the following cases :*

- *if  $i = 0$  or  $i = 1$ .*
- *if  $p = 2$ ,  $i \geq 2$  and  $y_n \in \text{Ker}(\tau_i : K_*^M(X)/2 \rightarrow K_*^M(X)/2, x \mapsto \{-1\}^{i-1} \cdot x)$ .*
- *if  $p$  is odd and  $i \geq 2$  is even, and  $y_n$  is any element in  $K_*^M(X)/p$ .*

**Proof.** For simplicity, assume  $X$  is irreducible with field of rational functions  $k(X)$ . By the results of section 2, it suffices to check that if  $x = \sum_{k=1}^l s_k \in K_i^M(k(X))$  is unramified then  $\gamma_n(x) := \sum_{1 \leq l_1 < \dots < l_n \leq l} s_{l_1} \cdot \dots \cdot s_{l_n} \in K_i^M(k(X))$  is unramified. So, let  $y$  be a codimension-1 point in  $X$  with local ring  $\mathcal{O}_{X,y}$ , and let  $a$  and  $b$  be symbols in  $K_i^M(k(X))$  unramified along  $y$ . Then, thanks to Lemma 4.1, we can write  $a$  as a sum of symbols  $\{a_1, \dots, a_n\}$  and  $b$  as a sum of symbols  $\{b_1, \dots, b_n\}$  with  $a_1, \dots, a_n, b_1, \dots, b_n$  being units in  $\mathcal{O}_{X,y}$ . It is then clear that  $a \cdot b$  is also unramified along  $y$  which finishes the proof.  $\square$

The definition of Milnor  $K$ -theory we gave is functorial with respect to open immersions of regular schemes. Indeed, if  $U \hookrightarrow X$  is an open immersion of regular scheme, the group homomorphism  $K_n^M(X) \rightarrow K_n^M(U)$  is just defined by restriction. Indeed, divisors on  $U$  map injectively into the set of divisors on  $X$ , and thus an element  $x$  unramified along divisors in  $X$  will surely be unramified along divisors in  $U$ . We define the Zariski sheaf  $\mathcal{K}_*^M$  on  $X$  to be

$$U \mapsto K_*^M(U)$$

for any Zariski open subset  $U$  of  $X$ . Clearly, we have  $K_i^M(X) = H^0(X, \mathcal{K}_i^M)$ . By a map of sheaves  $\varphi : \mathcal{K}_i^M/p \rightarrow \mathcal{K}_*^M/p$ , we mean a map that commutes with open immersions, i.e. if  $U \hookrightarrow X$  is an open immersion of regular schemes, the following diagram commutes

$$\begin{array}{ccc} K_i^M(X) & \xrightarrow{\varphi} & K_*^M(X) \\ \downarrow & & \downarrow \\ K_i^M(U) & \xrightarrow{\varphi} & K_*^M(U) \end{array}$$

As a straightforward consequence of the above, we have

**Proposition 4.4.** *Let  $p$  be a prime number,  $S$  be a ring and  $X$  a regular  $S$ -scheme. Then, there exist divided powers of sheaves of sets on  $X$*

$$\gamma_n : \mathcal{K}_i^M/p \rightarrow \mathcal{K}_*^M/p$$

*in the following cases :*

- *if  $i = 0$  or  $i = 1$ .*
- *if  $p = 2$ ,  $i \geq 2$  and  $-1$  is a square in  $S$ .*
- *if  $p$  is odd and  $i \geq 2$  is even.*

**Proof.** This is clear from the definitions and Proposition 4.3.  $\square$

**Remark 4.5.** In the case when  $p = 2, i \geq 2$  and  $-1$  is not a square in  $S$ , it is still possible to define some operations  $\mathcal{K}_i^M/2 \rightarrow \mathcal{K}_*^M/2$ . Indeed, if  $\tau_i : K_*^M(S)/2 \rightarrow K_*^M(S)/2, x \mapsto \{-1\}^{i-1} \cdot x$  and if  $y_n \in \text{Ker } \tau_i$ , then we have an operation of sheaves on  $X, y_n \cdot \gamma_n : \mathcal{K}_i^M/2 \rightarrow \mathcal{K}_*^M/2$ .

We have seen that  $K_n^M$  is functorial with respect to open immersions of regular schemes. If  $k$  is a field, it is actually functorial with respect to any map between smooth  $k$ -schemes. Given a map  $f : Y \rightarrow X$  between smooth  $k$ -schemes, Rost constructs in [22, section 12] a pull-back group homomorphism  $f^* : K_n^M(X) \rightarrow K_n^M(Y)$ , and shows that it is functorial. In particular, if  $f : Y \rightarrow X$  is a dominant map of smooth  $k$ -schemes, it induces an embedding of the field of functions  $k(X)$  of  $X$  into the function field  $k(Y)$  of  $Y$ , and the map  $f^* : K_n^M(X) \rightarrow K_n^M(Y)$  is induced by the map  $i : K_n^M(k(X)) \rightarrow K_n^M(k(Y))$  coming from the inclusion of fields  $k(X) \hookrightarrow k(Y)$  ([22, Lemma 12.8.]), so that an unramified element of  $K_n^M(k(X))$  will map to an unramified element of  $K_n^M(k(Y))$  under  $i$ .

Let  $k$  be a field and  $X$  be a smooth  $k$ -scheme. Let's denote by  $\text{Sm}_X$  the category of smooth  $k$ -schemes with a morphism to  $X$  and with morphisms being morphisms of  $k$ -schemes respecting the  $X$ -structure, i.e. commutative diagrams

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & Z \\ & \searrow & \swarrow \\ & X & \end{array}$$

In particular, if  $X$  is irreducible, the spectrum of its field of rational functions belongs to  $\text{Sm}_X$ . The map  $K_n^M/p : \text{Sm}_X \rightarrow \text{Sets}$  is a contravariant functor, and we define an operation over a smooth scheme  $X$  to be a natural transformation from the functor  $K_n^M/p : \text{Sm}_X \rightarrow \text{Sets}$  to the functor  $K_n^M/p : \text{Sm}_X \rightarrow \mathbf{F}_p - \text{Algebras}$ . Under these assumptions, all the results concerning fields translate to the case of smooth  $k$ -schemes and it is possible to describe all such operations. First, we show that divided powers commute with Rost's pullback map  $f^* : K_n^M(X) \rightarrow K_n^M(Y)$ , for a morphism  $f : Y \rightarrow X$ .

**Lemma 4.6.** *Let  $X$  and  $Y$  be smooth schemes over a field  $k$ , and let  $f : Y \rightarrow X$  be a morphism. The pullback maps  $f^* : K_n^M(X)/p \rightarrow K_n^M(Y)/p$  commute with the divided powers of Proposition 4.3.*

**Proof.** The morphism  $f : Y \rightarrow X$  factors through  $Y \xrightarrow{i} Y \times X \xrightarrow{\pi} X$ , where  $i$  is the closed immersion  $i(y) = (y, f(y))$  and  $\pi$  is the projection  $\pi(y, x) = x$ . By functoriality, we have  $f^* = i^* \circ \pi^*$ . Divided powers commute with  $\pi^*$  because  $\pi$  is a dominant map and as such induces a map on Milnor  $K$ -theory coming from an inclusion of fields, see above. It remains to show that divided powers commute with the pullback maps induced by closed immersions. Let  $i : Z \hookrightarrow X$  be a closed immersion of codimension  $c$ . By [22, Corollary 12.4.],  $i^* : K_n^M(Z) \rightarrow K_n^M(X)$  is the restriction of a composition of specialization maps  $s_1 \circ \dots \circ s_c : K_n^M(k(X)) \rightarrow K_n^M(k(Z))$ . But we already know that divided powers commute with specialization maps.  $\square$

**Theorem 4.7.** *Let  $k$  be any field,  $p$  be a prime number and  $X$  be a smooth scheme over  $k$ . Operations  $K_i^M/p \rightarrow K_*^M/p$  over the smooth  $k$ -scheme  $X$  are spanned as a  $K_*^M(X)/p$ -module by the divided power operations of Proposition 4.3.*

**Proof.** Lemma 4.6 shows that divided powers are indeed operations over  $X$  and that so are their  $K_*^M(X)/p$ -span.

For simplicity, assume  $X$  is irreducible. An operation  $\varphi : K_i^M/p \rightarrow K_*^M/p$  over  $X$  induces naturally an operation  $\bar{\varphi}$  over the field  $k(X)$  of rational functions on  $X$ . By Theorems 1 and 2, the operation  $\bar{\varphi}$  is a sum of divided power operations with coefficients in  $K_*^M(k(X))/p$ . For any irreducible smooth scheme  $Y$  with field of rational functions  $k(Y)$ , let's write  $\iota$  for the inclusion of  $K_*^M(Y)/p$  inside  $K_*^M(k(Y))/p$ . There exist elements  $y_0, \dots, y_n$  in  $K_*^M(k(X))/p$  such that for all smooth scheme  $Y$  over  $X$  and for all  $x \in K_i^M(Y)/p$ , we have  $\varphi \circ \iota(x) = \sum_{k=0}^n y_k \cdot \gamma_k(x)$ . We also have, by definition of an operation,  $\varphi \circ \iota(x) = \iota \circ \varphi(x)$  for all  $x$ . If we can prove that the  $y_k$ 's are actually in  $K_*^M(X)/p$ , then we will be done. First,  $y_0$  is indeed in  $K_*^M(X)/p$ . This is because  $\varphi(0) = y_0$  must be in  $K_*^M(X)/p$ . Suppose we have shown that  $y_0, \dots, y_{l-1}$  are in  $K_*^M(X)/p$  and let's show that  $y_l$  is in  $K_*^M(X)/p$ . Let  $Y = \text{Spec } \mathcal{O}_X[t_{j,k}]_{1 \leq j \leq i, 1 \leq k \leq l}$  be the smooth scheme  $X \times \mathbf{A}^{il}$ . Then, the field of rational functions of  $Y$  is  $k(X)(t_{j,k})_{1 \leq j \leq i, 1 \leq k \leq l}$  and if  $x = \sum_{k=1}^l \{t_{1,k}, \dots, t_{i,k}\}$ ,  $\varphi(x) = y_0 + y_1 \cdot x + \dots + y_l \cdot \gamma_l(x)$ . Therefore,  $y_l \cdot \gamma_l(x) = y_l \cdot \{t_{1,1}, \dots, t_{i,1}, \dots, t_{1,l}, \dots, t_{i,l}\}$  must be in  $K_*^M(Y)/p$ .

Also, the closed subschemes  $Z$  in  $X$  correspond bijectively to the closed subschemes of the form  $Z \times_X Y$  in  $Y$ . Let  $u$  be a codimension-1 point in  $X$  with residue field  $k(u)$ , then  $u$  corresponds to the codimension-1 point  $v = u \times_X Y$  in  $Y$ . The residue at  $v$  of  $y_l \cdot \gamma_l(x)$  considered as an element of  $K_*^M(k(Y))/p$  is 0 since  $y_l \cdot \gamma_l(x) \in K_*^M(Y)/p$ . We also have  $\partial_v(y_l \cdot \gamma_l(x)) = \partial_v(y_l) \cdot \gamma_l(x)$  in  $K_*^M(k(u)(t_{j,k})_{1 \leq j \leq i, 1 \leq k \leq l})/p$ . This in turn implies, by Proposition 3.1, that  $\partial_u(y_l) = 0 \in K_*^M(k(u))/p$ . Thus, by definition of the Milnor  $K$ -theory of a scheme, we get  $y_l \in K_*^M(X)/p$ .  $\square$

**Remark 4.8.** Actually, if  $X$  is a regular scheme over  $k$  and if  $\pi : X \times \mathbf{A}^r \rightarrow X$  is the first projection or more generally if  $\pi$  is an affine bundle over  $X$ , then the induced homomorphism  $\pi^* : K_n^M(X) \rightarrow K_n^M(X \times \mathbf{A}^r)$  is an isomorphism. See [22, Prop. 8.6.].

Let  $k$  be a field and  $p$  a prime number different from the characteristic of  $k$ . We define  $\mathcal{H}^i(\mathbf{Z}/p)$  to be the Zariski sheaf on the category of smooth schemes over  $k$  corresponding to the Zariski presheaf  $U \mapsto H_{\text{et}}^i(U, \mathbf{Z}/p(i))$ . If  $X$  is a smooth scheme over  $k$ , the unramified cohomology of  $X$  is defined to be  $H^0(X, \mathcal{H}^i(\mathbf{Z}/p))$ . This group is birationally invariant when  $X$  is proper over  $k$ , see Theorem 4.1.1 and Remark 4.1.3 of [4]. It is worth saying that our unramified cohomology group is not the same as the one considered by Colliot-Thélène in [4]. The unramified cohomology is then clearly functorial with respect to morphisms of smooth  $k$ -schemes. Under the assumption of the Bloch-Kato conjecture, the sheaf  $\mathcal{K}_i^M/p$  maps isomorphically to the sheaf  $\mathcal{H}^i(\mathbf{Z}/p)$ . Indeed, there is a morphism of exact Gersten complexes

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{K}_i^{\mathbf{M}}/p & \longrightarrow & \bigoplus_{x \in X^{(0)}} K_i^{\mathbf{M}}(k(x))/p & \xrightarrow{\partial} & \bigoplus_{x \in X^{(1)}} K_{i-1}^{\mathbf{M}}(k(x))/p \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{H}^i(\mathbf{Z}/p) & \longrightarrow & \bigoplus_{x \in X^{(0)}} H_{et}^i(k(x), \mathbf{Z}/p(i)) & \xrightarrow{\partial} & \bigoplus_{x \in X^{(1)}} H_{et}^{i-1}(k(x), \mathbf{Z}/p(i-1))
\end{array}$$

where the bottom residue map has a description in terms of Galois cohomology as the edge homomorphisms of a Hochschild-Serre spectral sequence. It is a fact (see e.g. [10, section 6.8]) that both residue maps are compatible with the Galois symbol. Hence the claimed isomorphism of sheaves. In particular, both sheaves have same global sections, i.e. there is an isomorphism

$$K_i^{\mathbf{M}}(X)/p \xrightarrow{\simeq} H^0(X, \mathcal{H}^i(\mathbf{Z}/p)).$$

Moreover, it can be shown that this isomorphism is compatible with the pull-back map  $f^*$  induced by any morphism  $f : X \rightarrow Y$  between smooth  $k$ -schemes. This proves

**Theorem 4.9.** *Let  $k$  be a field and  $p$  be a prime number different from the characteristic of  $k$ . The algebra of operations in unramified cohomology  $H^0(-, \mathcal{H}^i(\mathbf{Z}/p)) \rightarrow H^0(-, \mathcal{H}^*(\mathbf{Z}/p))$  is spanned as a  $H_{et}^*(k, \mathbf{Z}/p(*))$ -module by the divided powers of Proposition 4.3.*

## 4.2 Operations in Milnor $K$ -theory of rings

A natural question is to ask if  $K_n^{\mathbf{M}}(A)$ , for a ring  $A$ , can be presented with generators and relations. This is optimistic for a general  $A$ . However, notice that for a domain  $A$ , the natural map  $(A^\times)^{\otimes n} \rightarrow K_n^{\mathbf{M}}(F)$  factors through

$$\bar{K}_n^{\mathbf{M}}(A) =_{\text{def}} \left( A^\times \otimes \dots \otimes A^\times \right) / St_A,$$

where  $St_A$  is that ideal in  $A^\times \otimes \dots \otimes A^\times$  generated by elements of the form  $a \otimes (1 - a)$  with  $a, 1 - a \in A^\times$ . A ring homomorphism  $A \rightarrow B$  induces a ring homomorphism  $\bar{K}_*^{\mathbf{M}}(A) \rightarrow \bar{K}_*^{\mathbf{M}}(B)$  and this is functorial.

If  $A$  is an excellent ring, there is a Gersten complex

$$0 \rightarrow \bar{K}_n^{\mathbf{M}}(A) \rightarrow \bigoplus_{x \in A^{(0)}} K_i^{\mathbf{M}}(k(x)) \xrightarrow{\partial} \bigoplus_{x \in A^{(1)}} K_{i-1}^{\mathbf{M}}(k(x)) \xrightarrow{\partial} \dots,$$

where  $A^{(r)}$  is the set of codimension- $r$  points in  $\text{Spec } A$  and  $\kappa(x)$  is the residue field at  $x$ . In order to determine all operations  $\bar{K}_i^{\mathbf{M}}/p \rightarrow \bar{K}_*^{\mathbf{M}}/p$  over a "nice" ring  $A$ , we will be concerned with the exactness of that complex. Let  $A$  be an essentially smooth semi-local  $k$ -algebra, where by essentially smooth we mean that  $A$  is a localization of a smooth affine  $k$ -algebra. Elbaz-Vincent and Mueller-Stach have established the exactness of the complex

$$\bar{K}_n^{\mathbf{M}}(A) \rightarrow \bigoplus_{x \in A^{(0)}} K_i^{\mathbf{M}}(k(x)) \xrightarrow{\partial} \bigoplus_{x \in A^{(1)}} K_{i-1}^{\mathbf{M}}(k(x)) \xrightarrow{\partial} \dots$$

in the case when  $A$  has infinite residue field (see [5] and [16]). This last condition has been removed by Kerz in [15]. Moreover, in [14], Kerz shows that the Gersten complex is also exact at the first place when  $A$  has infinite residue field. All in all,

**Theorem 4.10** (Gabber, Elbaz-Vincent, Mueller-Stach, Kerz). *Let  $A$  be an essentially smooth semi-local algebra over a field  $k$  with quotient field  $F$ . If  $A$  has infinite residue field, the Gersten complex is exact, and in particular*

$$\bar{K}_n^M(A) = \text{Im} \left( A^\times \otimes \dots \otimes A^\times \longrightarrow K_n^M(F) \right) = \text{Ker} \left( K_n^M(F) \xrightarrow{\partial} \bigoplus_{x \in A^{(1)}} K_{n-1}^M(k(x)) \right).$$

Without assuming  $A$  has infinite residue field,

$$\text{Im} \left( A^\times \otimes \dots \otimes A^\times \longrightarrow K_n^M(F) \right) = \text{Ker} \left( K_n^M(F) \xrightarrow{\partial} \bigoplus_{x \in A^{(1)}} K_{n-1}^M(k(x)) \right).$$

From now on,  $k$  is an infinite field and  $A$  is a fixed essentially smooth semi-local  $k$ -algebra. Let's denote by  $\mathcal{C}_A$  the category of rings over  $A$  (i.e. the rings  $R$  with a morphism  $R \rightarrow A$ ) with morphisms compatible with the structure maps to  $A$ . Note that fields containing  $A$  are objects in the category  $\mathcal{C}_A$ . The map  $\bar{K}_n^M/p : \mathcal{C}_A \rightarrow \mathbf{Sets}$  is a functor, and an operation over the regular semi-local domain  $A$  is a natural transformation from the functor  $\bar{K}_n^M/p : \mathcal{C}_A \rightarrow \mathbf{Sets}$  to the functor  $\bar{K}_*^M/p : \mathcal{C}_A \rightarrow \mathbf{F}_p\text{-Algebras}$ . Under these assumptions, all the results concerning fields translate to the case of essentially smooth semi-local  $k$ -algebras and it is possible to describe all such operations.

**Theorem 4.11.** *Let  $k$  be an infinite field and  $A$  be an essentially smooth semi-local  $k$ -algebra. Operations  $\bar{K}_i^M/p \rightarrow \bar{K}_*^M/p$  over  $A$  are in the  $K_*^M(A)/p$ -span of divided powers.*

**Proof.** Indeed, by Theorem 4.10, the functors  $\bar{K}_*^M/p$  and  $K_*^M/p$  agree on essentially smooth semi-local  $k$ -algebras. The proof of Theorem 4.7 applied to  $\text{Spec } A$  shows that an operation over  $A$  must be in the span of divided power operations as in Proposition 4.3. It remains to say that such operations do exist. For this purpose, it is enough to check that  $\{b, b\} = \{-1, b\} \in \bar{K}_*^M(B)$  for all  $k$ -algebra  $B$  and all  $b \in B^\times$ . This is the case because  $k$  is infinite. Indeed, the relation  $\{b, -b\} = 0 \in \bar{K}_*^M(B)$  holds for all  $b \in B^\times$  whenever the field  $k$  is infinite, see e.g. [21].  $\square$

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