A dévissage theorem for modular exact categories with weak equivalences

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Abstract

In this note, we will introduce a notion of modularity of exact categories due to Masana Harada [Har05]. The naming is coming from the classical modular lattices theory [Bir48]. We will also state and prove so-called "homotopy Grayson-Staffeldt-Jordan-Hölder theorem" which is implicitly appeared in [Gra87] and [Sta89]. The theorem says contractibility of a simplicial set associated to a certain ordered set. Combining these two idea and utilizing Waldhausen's technique in [Wal85], we will get a dévissage theorem for modular exact categories with weak equivalences which is a generalization of original Quillen's one in [Qui73].

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Introduction

After [Gra87] and [Sta89], it might be well-known to experts that for any exact category satisfying a suitable Jordan-Hölder theorem, the proof of dévissage theorem in Ibid with ap-

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propriate refinements still works fine. See for example [Mit94, p.175 l.5]. The author was learned this fact from Masana Harada and does not know a public reference written the precise statement of it. In this note, we will state and consciously prove a dévissage theorem which is a variant of Harada's version in [Har05], a generalization of Quillen's one in [Qui73] and a different flavour from Yao's one in [Yao95]. The theorem will be used in author's forthcoming paper [Moc08]. Now we will explain the structure of this note. After reviewing the fundamental facts, we will introduce a notion of modularity of exact categories (§2). The concept is due to Masana Harada and the naming is coming from the classical modular lattices theory [Bir48]. We will also explain the dévissage properties (§3). As mentioned above, to do dévissage theorem, we need to state a suitable Jordan-Hölder theorem and we will precisely state so-called "homotopy Grayson-Staffeldt-Jordan-Hölder theorem" (4.3). This theorem has been already implicitly appeared in [Gra87], [Sta89]. The theorem says contractibility of a simplicial set associated to a certain ordered set. Combining the theorem and stereo-type argument about comparing two K-theories initiated by Waldhausen [Wal85, p.346 1.5.9.] and additivity theorem for F-construction (§A), we get the dévissage theorem (4.4). Finally as some authors state a dévissage theorem in a specific context (see for example [BM06]), we also give examples of dévissage type statements (5.2) in a particular occasion which is not obtained from (4.4) directly. In appendix we will give a brief proof of the additivity theorem for F-construction (A.2).

1 Preliminary

In this section, we will briefly review the notations. For example the Grothendieck construction of simplicial sets and F-construction of fully faithful inclusions between exact categories with weak equivalences and so on.

1.1 For any category $\mathcal{C}$ and $\mathcal{D}$, $\mathcal{HOM}(\mathcal{C}, \mathcal{D})$ is the category of functors from $\mathcal{C}$ to $\mathcal{D}$ with natural transformations. In particular $\operatorname{Ar}\mathcal{C} := \mathcal{HOM}([1], \mathcal{C})$ is the category of morphisms of $\mathcal{C}$.

1.2 We review notations and fundamental facts for (bi)simplicial sets.

1.2.1 We recall the Grothendieck construction of simplicial sets. Let X be a simplicial set. The Grothendieck construction of X is the category $\int_{\Delta^{\mathrm{op}}} X$ whose object is a pair $([n], x)$ of non-negative integer n and a n-simplex x of X and whose morphism $\alpha : ([n], x) \to ([m], y)$ is a morphism $\alpha : [m] \to [n]$ in Δ such that $\alpha_*(x) = y$. It is well-known that there is the canonical homotopy equivalence $N \int_{\Delta^{\mathrm{op}}} X \to X$. (c.f. [BK72, p.293 2.6 (iii)], [GJKMSW04, 1.2] [Wal85, p.359]).

1.2.2 We recall that the simplicial version of Quillen's Theorem B [Wal85, p.337 1.4.B]. Let $\Delta[n]$ denote the simplicial set represented by $[n]$. For a n-simplex y of a simplicial set Y, by Yoneda's lemma, it can be considered as a map of simplicial set $y : \Delta[n] \to Y$. For a map of simplicial set $f : X \to Y$ and a n-simplex y of Y, define a simplicial set $f/(n, y)$ as the Cartesian square

$$\begin{array}{ccc} f/(n,y) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \Delta[n] & \xrightarrow[y]{} & Y \end{array} \quad . \tag{1.2.2.1}$$

By Ibid and the argument in p.338 l.7-10, we get the following.

1.2.2.2 Theorem (Simplicial theorem B). *In the notation above, if for every map $a : [0] \to [n]$ and every n-simplex y of Y, the induced map $f/(0, a^*y) \to f/(n, y)$ is a homotopy equivalence then for every (n, y) the Cartesian square 1.2.2.1 is a homotopy Cartesian.*

Recall the following useful realization lemma.

1.2.3 Lemma (Realization Lemma) (c.f. [Seg74, Appendix A], [Wal78, p.164 5.1]). *Let $X_{\bullet\bullet} \to Y_{\bullet\bullet}$ be a map of bisimplicial sets. Suppose that for every n, the map $X_{\bullet n} \to Y_{\bullet n}$ is a homotopy equivalence. Then $X_{\bullet\bullet} \to Y_{\bullet\bullet}$ is a homotopy equivalence.*

1.3 We follow the notations of exact categories in [Kel90]. We will call admissible monomorphisms, admissible epimorphisms and admissible short exact sequences in [Qui73] as *inflations*, *deflations* and *conflations* respectively. For an exact category $\mathcal{E}$, we denote the category of inflations in $\mathcal{E}$ by $\operatorname{Inf}\mathcal{E}$.

1.3.1 For an essentially small exact category $\mathcal{E}$ and an object e in $\mathcal{E}$, $\operatorname{Inf}(\mathcal{E})/e$ is category equivalent to a partially ordered set. Up to isomorphism, this partially ordered set is uniquely determined and we denote it as $\mathcal{P}(e)$.

1.3.2 (c.f. [Wal85, p.336 in the proof of 1.4.3.].) Now we consider a class of weak equivalences $w(\mathcal{E})$ in an exact category $\mathcal{E}$. Fix a non-negative integer m. Consider the full subcategory $\mathcal{E}(m, w)$ of $\mathcal{HOM}([m], \mathcal{E})$ of those functors which take values in $w(\mathcal{E})$. We can make $\mathcal{E}(m, w)$ into an exact category by defining the conflations to be term-wised conflations in $\mathcal{E}$.

1.3.3 (F-construction). (c.f. [Wal85, p.344]) Let $\iota : (\mathcal{B}, w(\mathcal{B})) \hookrightarrow (\mathcal{A}, w(\mathcal{A}))$ be a fully faithful inclusion morphism between exact categories with weak equivalences. $F_\bullet(\mathcal{A}, \mathcal{B})$ be a full sub simplicial exact category with weak equivalences of $\mathcal{HOM}([-], \mathcal{A})$ defined as follows. For each n, $F_n(\mathcal{A}, \mathcal{B})$ is the category whose objects are the sequence of inflations in $\mathcal{A}$,

$$a_0 \rightarrowtail a_1 \rightarrowtail \cdots \rightarrowtail a_n$$

such that for every pair $0 \leqq i \leqq j \leqq n$, the object a_j/a_i is isomorphic to an object in $\mathcal{B}$. We now explain the structure of exact category with weak equivalences on $F_n(\mathcal{A}, \mathcal{B})$. We will write the pull-back of $S_n\iota : S_n \mathcal{B} \to S_n \mathcal{A}$ along $\partial_0 : S_{n+1} \mathcal{A} \to S_n \mathcal{A}$ as $S_n(\iota)$. Then it has the natural structure of exact category with weak equivalences induced from $S_{n+1} \mathcal{A}$. A functor $[n] \ni k \mapsto (0 \leqq k+1) \in \operatorname{Ar}[n+1]$ induces a category equivalence $S_n(\iota) \xrightarrow{\sim} F_n(\mathcal{A}, \mathcal{B})$. Therefore $F_n(\mathcal{A}, \mathcal{B})$ inherits a structure of exact category with weak equivalences from $S_n(\iota)$. In particular $F_n \mathcal{A} := F_n(\mathcal{A}, \mathcal{A})$ is equivalent to $S_{n+1} \mathcal{A}$ as an exact category with weak equivalences.

2 Modularity of exact categories

In this section, we intend to give a sufficient condition of $\mathcal{E}$ that makes $\mathcal{P}(e)$ (1.3.1) a modular lattice for each e. So first we will review the definition of modular lattices.

2.1 Definition A *lattice* is a partially ordered set such that for each elements a and b, its sup and inf written by $a \cup b$ and $a \cap b$ respectively exist. A lattice is called *modular* if for any elements $x \leqq z$ and y, the following identity called *modular law* holds.

$$x \cup (y \cap z) = (x \cup y) \cap z. \tag{2.1.1}$$

2.2 Definition An exact category $\mathcal{E}$ is called *modular* if it satisfies the following two conditions.

2.2.1 For any inflations $x \rightarrowtail z$, $y \rightarrowtail z$ in $\mathcal{E}$, there exists a fiber product $x \times_z y$ and if the canonical morphisms $x \times_z y \to x$, $x \times_z y \to y$ and $x \coprod_{x \times_z y} y \to z$ are inflations.

2.2.2 In the following commutative diagram of inflations,

$$\begin{array}{ccc} x & \rightarrowtail & y \\ \downarrow & & \downarrow \\ z & \rightarrowtail & w \end{array}$$

the canonical morphism $x \to z \times_w y$ is an inflation.

2.2.3 Remarks If $\mathcal{E}$ is an essentially small modular exact category, then for each object e in $\mathcal{E}$, $\mathcal{P}(e)$ is a lattice with $x \cup y = x \times_z y$ and $x \cap y = x \coprod_{x \times_z y} y$ for each inflations $x \rightarrowtail z$ and $y \rightarrowtail z$. Then by modularity and [Kel90, p.406 step 1], we have the following biCartesian square:

$$\begin{array}{ccc} x \cap y & \rightarrowtail & x \\ \downarrow & & \downarrow \\ y & \rightarrowtail & x \cup y \end{array} \quad . \tag{2.2.3.1}$$

Therefore we get the following so-called "the second isomorphism theorem identity"

$$x/(x \cap y) \xrightarrow{\sim} (x \cup y)/y. \tag{2.2.3.2}$$

2.3 Lemma *Let $\mathcal{E}$ be a modular exact category. Then for any inflations $x \rightarrowtail z \rightarrowtail a \leftarrowtail w \leftarrowtail y$, the canonical induced morphisms $x \times_a y \to z \times_a w$ and $x \oplus y/x \times_a y \to z \oplus w/z \times_a w$ are inflations. Moreover, putting $x \cap y := x \times_a y$ and $x \cup y := x \oplus y/x \cap y$ and so on, we have a conflation*

$$z \cap w/x \cap y \rightarrowtail z/x \oplus w/y \twoheadrightarrow z \cup w/x \cup y. \tag{2.3.1}$$

2.3.2 *Proof.* First applying the diagram below to 2.2.2

$$\begin{array}{ccc} x \times_a y & \rightarrowtail & z \\ \downarrow & & \downarrow \\ y & \rightarrowtail & a \end{array} \quad ,$$

we learn that the induced morphism $x \times_a y \to z \times_a w$ is an inflation. Since by 2.2.1, the canonical morphism $z \coprod_{z \times_a w} w \to v$ is an inflation, a fortiori a monomorphism in $\mathcal{E}$, we learn that there is a canonical isomorphism $x \times_{z \coprod_{z \times_a w} w} y \xrightarrow{\sim} x \times_a y$. Applying 2.2.1 to the following Cartesian square

$$\begin{array}{ccc} x \times_a y & \rightarrowtail & z \\ \downarrow & & \downarrow \\ y & \rightarrowtail & z \coprod_{z \times_a w} w \end{array},$$

we learn that the canonical morphism $x \coprod_{x \times_a y} y \to z \coprod_{z \times_a w} w$ is an inflation.
Next consider the following commutative diagram

$$\begin{array}{ccccc} x \cap y & \rightarrowtail & x \oplus y & \twoheadrightarrow & x \cup y \\ \downarrow & & \downarrow & & \downarrow \\ z \cap w & \rightarrowtail & z \oplus w & \twoheadrightarrow & z \cup w \\ \downarrow & & \downarrow & & \downarrow \\ z \cap w / x \cap y & \longrightarrow & z/x \oplus w/y & \longrightarrow & z \cup w / x \cup y \end{array}$$

where the first and the second lines are conflations by 2.2.3.1, the first and the third vertical lines are also by the first paragraph. Easy inspection shows that the second vertical line is also. Therefore by 3×3 lemma in exact categories (More precisely, first using [Kel90, p.410 A.3], we may assume $\mathcal{E}$ is small. Next embedding it in the abelian category of left exact functors on $\mathcal{E}$, we may consider 3×3 lemma in it.), we get the conflation (2.3.1). **Q.E.D.**

The naming of modularity is coming from the following result.

2.3.3 Corollary *For any essentially small modular exact category $\mathcal{E}$ and any object e in $\mathcal{E}$, $\mathcal{P}(e)$ is a modular lattice.*

2.3.3.1 *Proof.* In 2.3.1, putting $w = y$, we learn that for each $x \rightarrowtail z$ and y, the conditions $x \cap y = z \cap y$ and $x \cup y = z \cup y$ imply $x = z$. Now the result is followed from the Dedekind criterion below. **Q.E.D.**

2.3.3.1.1 Lemma (Dedekind criterion). *A lattice L is modular if and only if the following condition is verified.*
For each elements $a \leqq b$, c in L, conditions $a \cup c = b \cup c$ and $a \cap c = b \cap c$ imply $a = b$.

2.3.3.1.1.1 *Proof of **Lemma** 2.3.3.1.1* First we assume that L is not modular. Then there are element $x < z$ and y such that $x \cup (y \cap z) < (x \cup y) \cap z$. Put $a = x \cup (y \cap z)$ and $b = (x \cup y) \cap z$. Then we have $a < b$ and $a \cap y = b \cap y$ and $a \cup y = b \cup y$.
Next we assume that there are elements $a < b$ and c such that $a \cap c = b \cap c$ and $a \cup c = b \cup c$. Then $a \cup (b \cap c) = a \neq b = (a \cup c) \cap b$. **Q.E.D.**

2.4 Definition An exact category with weak equivalences $(\mathcal{E}, w)$ is called *finely modular* if $\mathcal{E}$ is modular and if $(\mathcal{E}, w)$ satisfies the following *strong cogluing axiom*.

2.4.1 If the diagram below

$$\begin{array}{ccccc} x & \rightarrowtail & y & \leftarrowtail & z \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ x' & \rightarrowtail & y' & \leftarrowtail & z' \end{array}$$

where vertical arrows are weak equivalences, the induced canonical morphism

$$a \times_b c : x \times_y z \to x' \times_{y'} z'$$

is a weak equivalence.

For example, a modular exact category is finely modular by taking the class of isomorphisms as the class of weak equivalences.

2.5 Lemma *Let $(\mathcal{E}, w)$ be a finely modular exact category with weak equivalences. Then for each non-negative integer m, $\mathcal{E}(m, w)$* (1.3.2) *is modular.*

2.5.1 *Proof.* Recall that an inflation in $\mathcal{E}(m, w)$ is a term-wised inflations in $\mathcal{E}$ (1.3.2). Only non-trivial part is existence of a fiber product $x \times_z y$ for any inflations $x \rightarrowtail z$ and $y \rightarrowtail z$. Since $\mathcal{E}$ is modular, it exists term-wisely and therefore exists in $\mathcal{HOM}([m], \mathcal{E})$. By the strong cogluing axiom, it is actually in $\mathcal{E}(m, w)$. **Q.E.D.**

3 Dévissage properties

We will consider the (strong) dévissage properties of exact categories (with weak equivalences). First let $\mathcal{A}$ be an exact category and $\mathcal{B}$ its full sub exact category.

3.1 Definition The inclusion functor $\iota : \mathcal{B} \hookrightarrow \mathcal{A}$ is satisfying the *dévissage properties* if it satisfies the following two properties.

3.1.1 Dév 1 $\mathcal{B}$ is closed under the admissible sub and quotient objects. This means that for a conflation in $\mathcal{A}$

$$x \rightarrowtail y \twoheadrightarrow z,$$

if y is isomorphic to an object in $\mathcal{B}$, then x and z are.

3.1.2 Dév 2 For any object a in $\mathcal{A}$, there is a finite filtration of inflations

$$* = a_0 \rightarrowtail a_1 \rightarrowtail a_2 \rightarrowtail \cdots \rightarrowtail a_n = a$$

such that a_i/a_{i-1} is isomorphic to an object in $\mathcal{B}$ for each i.

3.2 Definition Now we assume that $\mathcal{A}$ and $\mathcal{B}$ possess classes of weak equivalences w and v respectively and ι is a morphism of exact categories with weak equivalences. ι is satisfies the *strong dévissage condition* if for each non-negative integer m, the canonical inclusion induced from ι, $\mathcal{A}(m, w) \hookrightarrow \mathcal{B}(m, v)$ (For the definition of $\mathcal{A}(m, w)$ and $\mathcal{B}(m, v)$, see 1.3.2.) satisfies the dévissage properties.

3.3 Lemma *Let $\mathcal{A} \hookrightarrow \mathcal{B}$ be a fully faithful exact inclusion. If $\iota : \mathcal{A} \hookrightarrow \mathcal{B}$ is satisfying the dévissage properties and $\mathcal{B}$ is modular, then for each non-negative integer n, $S_n\iota : S_n \mathcal{A} \hookrightarrow S_n \mathcal{B}$ also satisfies the dévissage properties.*

3.3.1 *Proof.* We may check that $F_n\iota : F_n \mathcal{A} \hookrightarrow F_n \mathcal{B}$ satisfies the dévissage properties for each n. ($S_0 \mathcal{A} \hookrightarrow S_0 \mathcal{B}$ trivially satisfies the dévissage properties.) Let b be an element in $F_n \mathcal{B}$. Since an admissible sub or quotients of b is a term-wised those one in $\mathcal{B}$, the condition 3.1.1 is trivial. Fix a filtration $b(n) = b(n)_p \leftarrowtail b(n)_{p-1} \leftarrowtail \cdots \leftarrowtail b(n)_0 = *$ such that $b(n)_i/b(n)_{i-1}$ is isomorphic to an object in $\mathcal{A}$. From now on we may consider in $\mathcal{P}(b(n))$ (For the definition of $\mathcal{P}(b(n))$, see 1.3.1). We put $b(k)_i := b(k) \cap b(n)_i$. Then by 2.2.1, for each k and i the induced morphism $b(k)_i \to b(k)_{i+1}$ is an inflation and therefore b_i is in $F_n \mathcal{B}$. Next we will prove that $b_i \to b_{i+1}$ is an inflation in $F_n \mathcal{B}$ for each i. For each i, k, notice that the following diagram is a Cartesian square

$$\begin{array}{ccc} b(k)_i & \rightarrowtail & b(k+1)_i \\ \downarrow & & \downarrow \\ b(k)_{i+1} & \rightarrowtail & b(k+1)_{i+1} \end{array}$$

and therefore by 2.2.1 again, we learn that the canonical morphism $b(k)_{i+1} \coprod_{b(k)_i} b(k+1)_i \to b(k+1)_{i+1}$ is an inflation. Finally we will prove that for each i, b_{i+1}/b_i is isomorphic to an object in $F_n \mathcal{A}$. Fix integer i and k and notice that $b(k)_i = b(k)_{i+1} \cap b(n)_i$ and therefore there is an inflation $b(k)_{i+1}/b(k)_i = b(k)_{i+1}/b(k)_{i+1} \cap b(n)_i \xrightarrow{\sim} b(k)_{i+1} \cup b(n)_i/b(n)_i \rightarrowtail b(n)_{i+1}/b(n)_i$ by 2.2.3.2. By 3.1.1, we learn that $b(k)_{i+1}/b(k)_i$ is isomorphic to an object in $\mathcal{A}$. **Q.E.D.**

3.3.1.1 Corollary *Let $(\mathcal{A}, v) \hookrightarrow (\mathcal{B}, w)$ be a fully faithful inclusion morphism of exact categories with weak equivalence. If $\iota : \mathcal{A} \hookrightarrow \mathcal{B}$ is satisfying the strong dévissage condition (3.2) and $(\mathcal{B}, w)$ is finely modular (2.4), then for each non-negative integer n, $S_n\iota : S_n \mathcal{A} \hookrightarrow S_n \mathcal{B}$ also satisfies the strong dévissage condition.*

3.3.1.1.1 *Proof.* For each non-negative integer m, $\mathcal{B}(m, w)$ is modular by 2.5. Applying 3.3 to the inclusion $\mathcal{A}(m, v) \hookrightarrow \mathcal{B}(m, w)$, we learn that for each non-negative integer n, $S_n\iota : S_n(\mathcal{A}(m, v)) \hookrightarrow S_n(\mathcal{B}(m, w))$ is satisfying the dévissage properties. Now we have an identity $S_n(\mathcal{A}(m, v)) = (S_n \mathcal{A})(m, vS_n \mathcal{A})$ and similar one also holds to $\mathcal{B}$. Therefore we get the result. **Q.E.D.**

4 Main theorems

We will state a Jordan-Hölder type theorem. To make the theorem belongs to purely order sets theory, we will abstract the argument in [Gra87] and [Sta89].

4.1 Notations We will first prepare the notations. Let S be an ordered set with $1 = \sup S$ and $0 = \inf S$ and an operation $+ : S \times S \to S$ satisfies the following conditions.

4.1.1 For each $x \leqq y$ and $z \leqq w$ in S, we have $x + z \leqq y + w$.

4.1.2 For each x in S, we have $x + 1 = 1$.

4.1.3 For each x in S, we have $x + 0 = x$.

We also fix a subset $B \subset \mathrm{Ar}\, S$. We will denote an element $(x, y) \in B$ by y/x. Assume that B satisfies the following two conditions.

4.1.4 For each $x \in S$, $x/x \in B$.

4.1.5 For each y/x, $w/z \in B$, $(y + w)/(x + z) \in B$.

By 4.1.4 and 4.1.5, B also satisfies the following.

4.1.6 For each $y/x \in B$ and $z \in S$, $(y + z)/(x + z) \in B$.

We denote the nerve of S by NS and let NS^B is a sub simplicial set of NS whose k-simplexes $x_0 \leqq x_1 \leqq \cdots \leqq x_k$ satisfying the condition that x_j/x_i is in B for each $0 \leqq i \leqq j \leqq k$. For well-definedness of NS^B, we are using 4.1.4.

4.2 Examples Let $\mathcal{A}$ be a modular exact category and $\mathcal{B} \hookrightarrow \mathcal{A}$ be a fully faithful exact inclusion satisfying 3.1.1. Fix an object a in $\mathcal{A}$. Now $\mathcal{P}(a)$ is a lattice with $\sup \mathcal{P}(a) = a$ and $\inf \mathcal{P}(b) = *$. We define $+ : \mathcal{P}(a) \times \mathcal{P}(a) \to \mathcal{P}(a)$ as $\cup$. Then the operation $+$ satisfies the conditions 4.1.1, 4.1.2 and 4.1.3. Moreover if we put

$$\underline{\mathcal{B}} := \{x/y \in \mathrm{Ar}\, \mathcal{P}(a); x/y \text{ is isomorphic to an object in } \mathcal{B}.\},$$

then $\underline{\mathcal{B}}$ satisfies the conditions 4.1.4, 4.1.5.

4.2.1 *Proof.* The condition 4.1.1, 4.1.2 and 4.1.3 is easy consequences of the lattice operation $\cup$. Since $\mathcal{B}$ possesses the zero object, 4.1.4 trivially holds for $\underline{\mathcal{B}}$. Finally we prove the condition 4.1.5 for $\underline{\mathcal{B}}$. Let fix inflations $x \rightarrowtail z \rightarrowtail b \leftarrowtail w \leftarrowtail y$ in $\mathcal{A}$ such that z/x, w/y are in $\underline{\mathcal{B}}$. By 3.1.1 and the deflation $(z \cup x)/(w \cup y) \twoheadleftarrow z/x \oplus w/y$ coming from 2.3.1, we learn that $(z+x)/(w+y)$ is isomorphic to an object in $\mathcal{B}$. **Q.E.D.**

4.3 Theorem (Homotopy Grayson-Staffeldt-Jordan-Hölder theorem). *In the notation 4.1, assume that* 1 *has a finite filtration* $1 = b_p \geqq b_{p-1} \geqq b_{p-2} \geqq \cdots \geqq b_0 = 0$ *such that* b_i/b_{i-1} *is in* B *for each* i*. Then* NS^B *is contractible.*

4.3.1 *Proof of 4.3* Define a simplicial mapping $F_i : NS^B \to NS^B$ for each i $(0 \leqq i \leqq p)$ which sends a k-simplex $x_0 \leqq \cdots \leqq x_k$ to $x_0 + b_i \leqq \cdots \leqq x_k + b_i$ which is actually in NS^B by the condition 4.1.6. Therefore F_i is well-defined. F_0 is the identity map by 4.1.3 and F_p is constant by 4.1.2. A simplicial homotopy between F_i and F_{i+1} can be constructed as follows. For a morphism $\alpha : [k] \to [1]$ like $\alpha^{-1}(0) = \{0, \cdots, t\}$, and a k simplex $x_0 \leqq \cdots \leqq x_k$, its value is

$$x_0 + b_{i-1} \leqq \cdots \leqq x_t + b_{i-1} \leqq x_{t+1} + b_i \leqq \cdots \leqq x_k + b_i$$

which is actually in NS^B by 4.1.5 and 4.1.6. Therefore it is well-defined. **Q.E.D.**

4.4 Theorem (Dévissage theorem) *Let* $(\mathcal{A}, w)$ *be an essentially small finely modular exact category with weak equivalences (2.4) and* $\iota : \mathcal{B} \hookrightarrow \mathcal{A}$ *a full sub exact category with weak equivalences. Assume that* ι *is satisfying the strong dévissage condition (3.2). Then the canonical map* $wS_\bullet\iota : wS_\bullet\,\mathcal{B} \to wS_\bullet\,\mathcal{A}$ *is a homotopy equivalence.*

4.4.1 *Proof.* We may assume $\mathcal{A}$ is small. By [Wal85, p.344, p.345 1.5.7], we have a sequence of homotopy type of a fibration

$$wS_\bullet\,\mathcal{B} \overset{wS_\bullet\iota}{\to} wS_\bullet\,\mathcal{A} \to wS_\bullet F_\bullet(\mathcal{A}, \mathcal{B}).$$

Fix non-negative integers n and m. We have the following identities.

$$N_m wS_n F_\bullet(\mathcal{A}, \mathcal{B}) \overset{\sim}{\to} f_\bullet(S_n\,\mathcal{A}(m, wS_n\,\mathcal{A}), S_n\,\mathcal{B}(m, wS_n\,\mathcal{B})) \overset{\sim}{\to} f_\bullet(S_n(\mathcal{A}(m, w)), S_n(\mathcal{B}(m, w)))$$

where $f_\bullet$ denote the simplicial set of objects of $F_\bullet$ and for the definition $\mathcal{A}(m,w)$ and so on see 1.3.2. By the realization lemma 1.2.3, and replacing $\mathcal{A}(m,w)$ and $\mathcal{B}(m,w)$ with $\mathcal{A}$ and $\mathcal{B}$ respectively, we may just check the following claim.

4.4.1.1 Claim For a small modular exact category $\mathcal{A}$ and $\iota : \mathcal{B} \hookrightarrow \mathcal{A}$ a full sub exact category. Assume that ι is satisfying the dévissage properties. Then for each non-negative integer n, $f_\bullet(S_n \mathcal{A}, S_n \mathcal{B})$ is contractible.

If $n = 0$, this claim is trivial. By A.3.4.1, we may assume $n = 1$. We now consider $\int_{\Delta^{\mathrm{op}}} f_\bullet(\mathcal{A},\mathcal{B})$ the Grothendieck construction of $f_\bullet(\mathcal{A},\mathcal{B})$ (1.2.1) and there is a functor

$$L : \int_{\Delta^{\mathrm{op}}} f_\bullet(\mathcal{A},\mathcal{B}) \to \mathrm{Inf}(\mathcal{A})$$

defined as follows. For each object $([n], a)$ in $\int_{\Delta^{\mathrm{op}}} f_\bullet(\mathcal{A},\mathcal{B})$, we put $L([n],a) := a_n$ and for each morphism $\alpha : ([n],a) \to ([m],a')$, we put $L(\alpha) := (a_n = a'_{\alpha(n)} \rightarrowtail a_m)$. We intend to prove that L is a homotopy equivalence by using Quillen theorem A. Since $\mathrm{Inf}(\mathcal{A})$ has an initial object, it is contractible and therefore $\int_{\Delta^{\mathrm{op}}} f_\bullet(\mathcal{A},\mathcal{B})$ is also. Let a be an object in $\mathcal{A}$ and consider the left fiber category L/a. Inspection shows that L/a is equivalent to the category $\int_{\Delta^{\mathrm{op}}} N\,\mathcal{P}(a)^{\underline{\mathcal{B}}}$ where $N\,\mathcal{P}(a)^{\underline{\mathcal{B}}}$ is just a simplicial set in 4.2 and contractible by 4.3. Therefore we get the result. **Q.E.D.**

4.4.2 Corollary *Let $\mathcal{A}$ be an essentially small modular exact category (2.2) and $\iota : \mathcal{B} \hookrightarrow \mathcal{A}$ a full sub exact category. Assume that ι is satisfying the dévissage properties (3.1). Then the canonical map $iS_\bullet\iota : iS_\bullet \mathcal{B} \to iS_\bullet \mathcal{A}$ is a homotopy equivalence.*

4.4.2.1 *Proof.* Now if $\mathcal{A}$ and $\mathcal{B}$ are considered as exact categories with weak equivalences by taking the class of isomorphisms, then ι satisfies the strong dévissage condition. For we have a category equivalence $\mathcal{A}(m,i) \ni x \mapsto x(0) \in \mathcal{A}$ and similarly $\mathcal{B}$'s one for each m. Moreover $(\mathcal{A}, i)$ is finely modular. So it is just the corollary of 4.4. **Q.E.D.**

5 Final remarks

In this section, we give examples of dévissage type theorems which can not be obtained from 4.4 directly. From now on the author assumes that readers are familiar with the preprint [Moc07].

5.1 Notations Let R be a commutative discrete valuation rings and π its uniformizer element. Put $k = R/\pi$. Now consider the following two exact categories (with weak equivalences). Let $\mathbf{Kos}^1(R)$ (resp. $\mathbf{Kos}^1(R)_{\mathrm{red}}$) be the the full subcategory of complexes of finitely generated free R-modules such that $X_i = 0$ for $i \neq 0, 1$ and its boundary morphism $d^X : X_1 \to X_0$ is injective and whose homology groups are torsion R-modules (resp. annihilated by π.) (In a preprint version of [Moc07], the author stupidly missed the word "free"in the definition of $\mathbf{Kos}^1(R)$. So the main theorem in [Moc07] seemed to be trivial.) One can prove that $\mathbf{Kos}^1(R)$ and $\mathbf{Kos}^1(R)_{\mathrm{red}}$ is an exact category in the natural way. (In particular, we can show that $\mathbf{Kos}^1(R)_{\mathrm{red}}$ is semi-simple. But we will not use this fact below.) We will denote the class of quasi-isomorphisms in $\mathbf{Kos}^1(R)$ and $\mathbf{Kos}^1(R)_{\mathrm{red}}$ by the same letter q.

5.2 Claim *In the notations above, the inclusion functor $\iota : \boldsymbol{Kos}^1(R)_{\mathrm{red}} \hookrightarrow \boldsymbol{Kos}^1(R)$ does not satisfies the dévissage conditions. But we have the homotopy equivalences*

$$iS_\bullet\iota : iS_\bullet \mathbf{Kos}^1(R)_{\mathrm{red}} \to iS_\bullet \mathbf{Kos}^1(R), \tag{5.2.1}$$

$$qS_\bullet\iota : qS_\bullet \mathbf{Kos}^1(R)_{\mathrm{red}} \to qS_\bullet \mathbf{Kos}^1(R). \tag{5.2.2}$$

5.2.3 *Proof.* For example, a complex $[R \overset{\pi^2}{\to} R]$ has no filtration such like a 3.1.2. Let $\mathcal{M}^1(R)$ be the category of finitely generated torsion R-modules and $\mathcal{M}(R)$ (resp. $\mathcal{M}(k)$) the category of finitely generated R (resp. k)-modules. Then we have the following commutative diagrams

$$\begin{array}{ccc} iS_\bullet \mathbf{Kos}^1(R)_{\mathrm{red}} & \longrightarrow & iS_\bullet \mathbf{Kos}^1(R) \\ \downarrow & & \downarrow \\ iS_\bullet \mathcal{M}(R) \times iS_\bullet \mathcal{M}(k) & \longrightarrow & iS_\bullet \mathcal{M}(R) \times iS_\bullet \mathcal{M}^1(R) \end{array} \qquad \begin{array}{ccc} qS_\bullet \mathbf{Kos}^1(R)_{\mathrm{red}} & \longrightarrow & qS_\bullet \mathbf{Kos}^1(R) \\ \downarrow & & \downarrow \\ iS_\bullet \mathcal{M}(k) & \longrightarrow & iS_\bullet \mathcal{M}^1(R) \end{array}$$

where the vertical maps are homotopy equivalents by the (variant of) 0.1. and 3.2., 3.3. in [Moc07]. Now applying the dévissage theorem to bottom lines, we get the results. **Q.E.D.**

A Additivity theorem for F-construction

In this appendix, we will give a brief proof of the additivity theorem for F-construction. The result might be well-known to experts. But the author does not know the reference. The author will just reproduces the proofs of S-construction case in [Wal85]. See also [McC93], [GJKMSW04].

A.1 Notations We will call a category with cofibrations and weak equivalences as a Waldhausen category. For a Waldhausen category $(\mathcal{C}, w)$ and its sub Waldhausen categories $\mathcal{A}$ and $\mathcal{B}$, let $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ be the Waldhausen category of cofibration sequences in $\mathcal{C}$ whose sub and quotients are in $\mathcal{A}$ and $\mathcal{B}$, respectively. (See [Wal85, p.325].) Let $\mathcal{C}$ be a full sub Waldhausen category of $\mathcal{C}$ and $\mathcal{A}'$, $\mathcal{B}'$ full sub Waldhausen categories of $\mathcal{A} \cap \mathcal{C}'$, $\mathcal{B} \cap \mathcal{C}'$, respectively. We will also adapt F-construction (1.3.3) to Waldhausen categories. Recall that $f_\bullet$ is a simplicial set of objects of $F_\bullet$ (See 4.4.1.) To simplify the notations, from now on, we will assume that every Waldhausen category is small and a specific zero object in a Waldhausen category is denoted by the common letter $*$.

A.2 Theorem (Additivity theorem). *The following conditions are equivalent and verified.*

A.2.1 *The following projection is a homotopy equivalence,*

$$wF_\bullet(E(\mathcal{A}, \mathcal{C}, \mathcal{B}), E(\mathcal{A}', \mathcal{C}', \mathcal{B}')) \ni A \rightarrowtail B \twoheadrightarrow C \mapsto (A, B) \in wF_\bullet(\mathcal{A}, \mathcal{A}') \times wF_\bullet(\mathcal{B}, \mathcal{B}').$$

A.2.2 *The following projection is a homotopy equivalence,*

$$wF_\bullet(S_2\,\mathcal{C}, S_2\,\mathcal{C}') \ni A \rightarrowtail B \twoheadrightarrow C \mapsto (A, B) \in wF_\bullet(\mathcal{C}, \mathcal{C}') \times wF_\bullet(\mathcal{C}, \mathcal{C}').$$

A.2.3 *The following projection is a homotopy equivalence,*

$$f_\bullet(E(\mathcal{A},\mathcal{C},\mathcal{B}),E(\mathcal{A}',\mathcal{C}',\mathcal{B}')) \ni A \rightarrowtail B \twoheadrightarrow C \mapsto (A,B) \in f_\bullet(\mathcal{A},\mathcal{A}') \times f_\bullet(\mathcal{B},\mathcal{B}').$$

A.2.4 *The following projection is a homotopy equivalence,*

$$f_\bullet(S_2\,\mathcal{C},S_2\,\mathcal{C}') \ni A \rightarrowtail B \twoheadrightarrow C \mapsto (A,B) \in f_\bullet(\mathcal{C},\mathcal{C}') \times f_\bullet(\mathcal{C},\mathcal{C}').$$

Proving the equivalence of four assertions above is almost similar to S-construction case [Wal85, p.331 1.3.2]. (To prove that the assertion A.2.2 implies that of A.2.1, we need to more equivalent assertions.) To prove that the assertion A.2.2 implies A.2.4, we will use the following useful lemma.

A.3 Lemma (c.f. [Wal85, p.335 1.4.1 and its corollary], [GSVW92, p.261 2.12], [Sch87, p.272 1.2], [HS85, p.416 3.6])

A.3.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with cofibrations and $\mathcal{A}'$, $\mathcal{B}'$ its full sub categories with cofibrations, respectively and let $\phi_i : \mathcal{A} \to \mathcal{B}$ $(i = 0,\ 1.)$ be exact functors which send $\mathcal{A}'$ to $\mathcal{B}'$. Assume there is an natural equivalence between ϕ_0 and ϕ_1, then it induces a simplicial homotopy between $f_\bullet\phi_0$ and $f_\bullet\phi_1$.*

A.3.2 *For an exact fully faithful inclusion of small categories with cofibrations $\mathcal{C} \hookrightarrow \mathcal{D}$, the canonical map $f_\bullet(\mathcal{D},\mathcal{C}) \to iF_\bullet(\mathcal{D},\mathcal{C})$ is a homotopy equivalence.*

A.3.3 *Proof.* Let $\Phi : \mathcal{A} \times [1] \to \mathcal{B}$ be a natural transformation between ϕ_0 and ϕ_1. Then for each $a : [n] \to [1]$ and a n-simplex A of $f_\bullet(\mathcal{A},\mathcal{A}')$, we associate a n-simplex in $f_\bullet(\mathcal{B},\mathcal{B}')$ defined by the following formula

$$[n] \overset{(A,a)}{\to} \mathcal{A} \times [1] \overset{\Phi}{\to} \mathcal{B}\,.$$

This correspondence gives a simplicial homotopy between $f_\bullet\phi_0$ and $f_\bullet\phi_1$. Next let $i_\bullet\,\mathcal{C}$ be the simplicial category $[m] \mapsto \mathcal{C}(m,i)$ (1.3.2). Then for each m, the canonical diagonal functor $\mathcal{C} \to i_m\,\mathcal{C}$ is a category equivalence and therefore by the firsr paragraph, we get the homotpoy equivalence $f_\bullet(\mathcal{C},\mathcal{D}) \to i_m F_\bullet(\mathcal{C},\mathcal{D})$. By the realization lemma 1.2.3, we get the result. **Q.E.D.**

A.3.4 Corollary (c.f. [Wal78, p.184]) *For any Waldhausen category $(\mathcal{C},w)$ and its full sub Waldhausen category $\mathcal{D}$ and any positive integer n, we have an canonical homotopy equivalence*

$$f_\bullet(S_n\,\mathcal{C},S_n\,\mathcal{D}) \to f_\bullet(\mathcal{C},\mathcal{D})^n. \tag{A.3.4.1}$$

A.3.4.2 *Proof.* Noticing that the identity $S_{n+1}\,\mathcal{C} \overset{\sim}{\to} E(S_1\,\mathcal{C},S_{n+1}\,\mathcal{C},S_n\,\mathcal{C})$, (See Ibid l.6 for exact categories case.) we get the result by induction on n and A.2.3. **Q.E.D.**

The rest of this note is devoted to giving a brief proof of the assertion A.2.4.

A.4 Sublemma (c.f. [Wal85, p.337 sublemma].) *In the notation A.2.4, the map*

$$g : f_\bullet(S_2\,\mathcal{C},S_2\,\mathcal{C}') \ni (A \rightarrowtail C \twoheadrightarrow B) \mapsto A \in f_\bullet(\mathcal{C},\mathcal{C}'),$$

satisfies the hypothesis of 1.2.2.2. Moreover the map

$$f_\bullet(\mathcal{C}, \mathcal{C}') \ni B \mapsto (* \rightarrowtail B \twoheadrightarrow B) \in g/(0, *)$$

is a homotopy equivalence. We conclude that we have the following fibration up to homotopy

$$f_\bullet(\mathcal{C}, \mathcal{C}') \to f_\bullet(S_2\,\mathcal{C}, S_2\,\mathcal{C}') \to f_\bullet(\mathcal{C}, \mathcal{C}'). \tag{A.4.1}$$

Since g has a section $A \mapsto (A \rightarrowtail A \twoheadrightarrow *)$, the fibration sequence (A.4.1) is split and thus we get the result. Now we are concentrating to prove A.4. Let $S'_2\,\mathcal{C}$ (resp. $S'_2\,\mathcal{C}'$) be the subcategory of cofibrations of $S_2\,\mathcal{C}$ (resp. $S_2\,\mathcal{C}'$) whose objects are cofibration sequences $* \rightarrowtail C \twoheadrightarrow B$. Then by A.3.1, the category equivalence $j : \mathcal{C} \ni \mathrm{B} \mapsto (* \rightarrowtail B \twoheadrightarrow B) \in S'_2\,\mathcal{C}$ induces a homotopy equivalences

$$j_* : f_\bullet(\mathcal{C}, \mathcal{C}') \to f_\bullet(S'_2\,\mathcal{C}, S'_2\,\mathcal{C}') = g/(0, *).$$

Hence we get the second assertion in A.4. Now fix a positive integer n and a n-simplex A' of $f_\bullet(\mathcal{C}, \mathcal{C}')$. For each i, let $v_i : [0] \to [n]$ be a map such that $v_i(0) = i$. Our purpose is proving that for each i,

$$v_{i*} : g/(0, *) \to g/(n, A')$$

is a homotopy equivalence. Now a m-simplex of $g/(n, A')$ may be considered as a pair

$$(A \rightarrowtail B \twoheadrightarrow C, u : [m] \to [n])$$

consisting of a cofibration sequence in $F_m(\mathcal{C}, \mathcal{C}')$ and an ordered map u subject to the condition that $A' \circ u = A$. Now consider a map $p : g/(n, A') \ni (A \rightarrowtail C \twoheadrightarrow B) \mapsto B \in f_\bullet(\mathcal{C}, \mathcal{C}')$ which is left inverse to each of the composede map $v_{i*} \circ j_*$. Since we have already known that the map j_* is a homotopy equivalence, easy observation shows that it suffices to check that the

$$v_{n*} j_* p : g/(n, A') \to g/(n, A')$$

is homotopic to the identity map on $g/(n, A')$. Now imitating the argument that a lifting of contraction of $\Delta[n]$ to its last vertex in Ibid p.339, we define a simplicial homotopy between them as follows. For each $v : [m] \to [1]$, we define an association

$$(A \rightarrowtail B \twoheadrightarrow C, u : [m] \to [n]) \mapsto (\bar{A} \rightarrowtail \bar{B} \twoheadrightarrow \bar{C}, \bar{u} : [m] \to [n]).$$

Now explain the notations above. Here $\bar{u} := w \circ (u, v) : [m] \to [n] \times [1] \to [n]$ and where $w(j, 0) = j$ and $w(j, 1) = n$. We put $\bar{A} := A' \circ \bar{u} : [m] \to [n] \to \mathcal{C}$. Since for each j in $[m]$, we have an inequality $u(j) \leqq \bar{u}(j)$, we have a natural transformation

$$(u : [m] \to [n]) \to (\bar{u} : [m] \to [n])$$

and it induces a unique map $A \to \bar{A}$ in $F_m(\mathcal{C}, \mathcal{C}')$. Now $\bar{B}$ and $\bar{C}$ are defined by the following pushout diagram as carefully choosing coproduct objects as in Ibid p.340:

$$\begin{array}{ccccc} A & \rightarrowtail & C & \twoheadrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \wr \\ \bar{A} & \rightarrowtail & \bar{C} & \twoheadrightarrow & \bar{B} \end{array} .$$

This defines a desired simplicial homotopy. This complete the proof of the A.4. **Q.E.D.**

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