# THE FIRST SUSLIN HOMOLOGY GROUP OF A SPLIT SIMPLY CONNECTED SEMISIMPLE ALGEBRAIC GROUP

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Abstract. Let $k$ be a perfect field and $G$ be a $k$-split simply connected semisimple algebraic group. We prove in this article that the first Suslin (singular) homology group $\mathrm{H}_1^{\mathrm{S}}(G)$ of the variety $G$ is isomorphic to $\mathrm{K}_2^M(k)^d$, where $d$ is the number of irreducible components of the root system of $G$.

## 0. Introduction

Algebraic topology is a creation of this first half of the last century. The various (co)homology theories as well as homotopy theory have been (and are still) an indispensable tool in the study of manifolds and topological spaces. Due to this success these techniques have been transfered from the beginnings to other mathematical disciplines like *e.g.* algebraic geometry or algebraic number theory.

However only recently an analogue of singular homology theory, or of homotopy theory has been constructed for algebraic varieties. Suslin has defined a homology theory $\mathrm{H}_*^{\mathrm{S}}(X)$ for algebraic varieties, the so called *Suslin (singular) homology theory*, see the article [19] of Suslin and Voevodsky; and Morel and Voevodsky [17] have defined a homotopy theory for algebraic varieties, the so called $\mathbb{A}^1$-*homotopy theory.*

Till today there are only very few actual computations of Suslin homology. The aim of this work is to compute the first Suslin homology sheaf of a split simply connected semisimple algebraic group $G$. As a corollary we get from this a computation of the first Suslin homology group $\mathrm{H}_1^{\mathrm{S}}(G)$.

To express our result let $\mathrm{Sm}_k$ be the category of smooth $k$-schemes, and $\underline{C}^\bullet(X)$ be the Suslin complex of $X \in \mathrm{Sm}_k$. By definition we have $\mathrm{H}_1^{\mathrm{S}}(X) = \underline{h}_1^{\mathrm{Nis}}(X)(k)$, where $\underline{h}_i^{\mathrm{Nis}}(X)$ denotes the $(-i)$-th (Nisnevich) cohomology sheaf of $\underline{C}^\bullet(X)$. We show in this work:

**Theorem A.** *Let $k$ be a perfect field and $G$ be a $k$-split simply connected semisimple algebraic group. Denote by $d$ the number of irreducible components of the root system of $G$, and $\underline{\mathrm{K}}_2^M$ the unramified Milnor $K$-theory sheaf. Then*

$$\underline{h}_1^{\mathrm{Nis}}(G) \simeq (\underline{\mathrm{K}}_2^M)^d .$$

*In particular we have* $\mathrm{H}_1^{\mathrm{S}}(G) \simeq \mathrm{K}_2^M(k)^d$.

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Our computation is based on the following fact, see Theorem 5.13 for a more general result. Denote $\mathcal{N}\mathrm{Sh}_k^{tr}$ the category of (abelian) Nisnevich sheaves with transfers, and $\mathbb{Z}_{\mathrm{tr}} \in \mathcal{N}\mathrm{Sh}_k^{tr}$ the constant sheaf. If $\underline{h}_0^{\mathrm{Nis}}(X)$ is isomorphic to $\mathbb{Z}_{\mathrm{tr}}$ then there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{N}\mathrm{Sh}_k^{tr}}(\underline{h}_1^{\mathrm{Nis}}(X), \mathcal{F}) \simeq \mathrm{H}^1_{\mathrm{Nis}}(X, \mathcal{F})$$

for all homotopy invariant $\mathcal{F} \in \mathcal{N}\mathrm{Sh}_k^{tr}$. It is a consequence of the Kneser-Tits conjecture (which is true for split simply connected groups, see [12]) that $\underline{h}_0^{\mathrm{Nis}}(G) = \mathbb{Z}_{\mathrm{tr}}$, and therefore to prove Theorem A it is enough to compute $\mathrm{H}^1_{\mathrm{Nis}}(G, \mathcal{F})$ for all homotopy invariant Nisnevich sheaves with transfers $\mathcal{F}$.

To compute the group $\mathrm{H}^1_{\mathrm{Nis}}(X, \mathcal{F})$ we use a theorem of Déglise [5]. There is a functor

$$\mathcal{F} \longmapsto \widehat{\mathcal{F}}_* = \bigoplus_{n\in\mathbb{Z}} \widehat{\mathcal{F}}_n$$

from the category of homotopy invariant Nisnevich sheaves with transfers over $\mathrm{Sm}_k$ to Rost's [18] category of cycle modules over $k$. This functor is in fact an equivalence with a localization of Rost's category of cycle modules over $k$, and we have $\mathrm{H}^i_{\mathrm{Nis}}(X, \mathcal{F}) \simeq \mathrm{H}^i(X, \widehat{\mathcal{F}}_0)$. Here the latter group denotes the $i$-th cohomology group of the cycle complex of $\widehat{\mathcal{F}}_0$. We compute the cohomology of a cycle module as follows.

**Theorem B.** *Let $k$ be an arbitrary field, $G$ a $k$-split simply connected semisimple algebraic group, and $\mathrm{M}_* = \bigoplus_{n\in\mathbb{Z}} \mathrm{M}_n$ a cycle module over $k$. Then we have an isomorphism*

$$\mathrm{H}^1(G, \mathrm{M}_n) \simeq \mathrm{M}_{n-2}(k)^d\,,$$

*where $d$ denotes the number of connected components of the root system of $G$.*

The content of this work is as follows. In Section 3 we recall the definition of Rost's cycle modules. As Brylinski and Deligne [3], who computed $\mathrm{H}^1(G, \mathrm{K}_2^M)$, we use then in Section 4 the Bruhat decomposition to define a filtration on the cycle complex of $\mathrm{M}_n$. This filtration allows to reduce the proof of Theorem B to the computation of certain cohomology groups of complexes of alternating forms on the coroots of $G$. We introduce these complexes abstractly in the first section, and compute the needed cohomology groups in Section 2. In the last section we show that Theorem B implies Theorem A.

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## 1. A complex of alternating forms

**1.1.** *Notations.* Let $\Pi$ be a root system of rank $r$ with base $\Delta = \{\alpha_1, \dots, \alpha_r\}$. We assume that $\Pi$ is reduced in the sense of Bourbaki [2, Chap. VI, §1, No 4], *i.e.* $\mathbb{R}\alpha \cap \Pi = \{\pm\alpha\}$ for all $\alpha \in \Pi$. By $\Pi^+$ and $\Pi^-$ we denote the positive and negative roots, respectively.

Let $\mathbb{Z}^r \simeq X \subset \mathbb{R}^r$ be the weight lattice of $\Pi$, and $Y = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ its $\mathbb{Z}$-dual, *i.e.* $Y$ is the span of the dual roots (coroots) $\alpha^\vee$. If $y^\vee \in Y$ we denote the evaluation of

$y^\vee$ at $x \in X$ by $y^\vee(x)$ (differing from the usual notation $< x, y^\vee >$). In particular, $\alpha_i^\vee \in Y$ is the dual root (coroot) of the simple root $\alpha_i$ (hence we have $\alpha_i^\vee(\alpha_i) = 2$).

We denote $\varpi_1, \ldots, \varpi_r \in X$ the fundamental weights, *i.e.* $\alpha_i^\vee(\varpi_j) = \delta_{ij}$ (Kronecker delta).

Let $\Bbbk$ be a commutative ring (with 1). We set in the following $X_\Bbbk := \Bbbk \otimes_\mathbb{Z} X$ and $Y_\Bbbk := \Bbbk \otimes_\mathbb{Z} Y$. Since $X$ and $Y$ are free $\mathbb{Z}$-modules we have then $Y_\Bbbk = \mathrm{Hom}_\Bbbk(X_\Bbbk, \Bbbk)$ and $X_\Bbbk = \mathrm{Hom}_\Bbbk(Y_\Bbbk, \Bbbk)$. By abuse of notation we denote $x \otimes 1$ and $y^\vee \otimes 1$ by $x$ and $y^\vee$, respectively, for all $x \in X$ and $y^\vee \in Y$.

If $\alpha \in \Pi$ we denote $s_\alpha$ the corresponding reflection, *i.e.* $s_\alpha(x) = x - \alpha^\vee(x) \cdot \alpha$ for all $x \in X$. We denote $s_i = s_{\alpha_i}$ the reflection corresponding to $\alpha_i$ and $W$ the Weyl group of $\Pi$, *i.e.* the group generated by the set of simple reflections $S = \{s_1, \ldots, s_r\}$.

There is a canonical operation of the Weyl group on $Y$ given by

$$(w \cdot \alpha^\vee)(x) \;:=\; \alpha^\vee(w^{-1}x)$$

which identifies $W$ with the Weyl group of the dual root system $\Pi^\vee$.

Denote $\ell(w)$ the length of a word in $W$ (with respect to $S$) and $w_0 \in W$ be the unique longest word. Let $N := \ell(w_0)$ be its length. We set

$$W^{(i)} \;:=\; \{\, w \in W \,|\, \ell(w) = \ell(w_0) - i \,\}$$

for $i = 0, \ldots, \ell(w_0)$. Note that $W^{(1)} = \{\, w_0 s_i \,|\, 1 \le i \le r \,\}$ and $W^{(2)} = \{\, w_0 s_i s_j \,|\, 1 \le i \ne j \le r \,\}$, see [2, Chap. VI, Cor. 3, p. 158].

**1.2.** *Bruhat ordering.* We recall the definition of the Bruhat order on the Weyl group $W$, see Deodhar [10].

Let $T$ be the set of $W$-conjugates of the set of simple reflections $S$, *i.e.* $T$ is the set of reflections $\{\, s_\alpha \,|\, \alpha \in \Pi \,\}$. Then by definition we have $w' < w$ in the Bruhat order if there exists elements

$$w' = w_0, w_1, \ldots, w_{n-1}, w_n = w \;\in W\,,$$

such that

(i) $\ell(w_i) < \ell(w_{i+1})$, and
(ii) there is $t_i \in T$, such that $w_i = w_{i+1} \cdot t_i$

for all $0 \le i \le n-1$.

It is well known that in this case if $w = s_{i_1} \cdot s_{i_2} \cdot \ldots \cdot s_{i_r}$ is a reduced decomposition of $w$ ($s_{i_j} \in S$) then $w'$ is a subword of $w$, and any subword of $w$ in this decomposition is smaller than $w$ in the Bruhat order.

Given $w > w''$ with $\ell(w) = \ell(w'') + 2$ then there are exactly two elements $w_1, w_2 \in W$, such that $w > w_1, w_2 > w''$, see *e.g.* [1, Lem. 2.7.3]. This implies that the only words between $w_0$ and $w_0 s_i s_j$ are $w_0 s_i$ and $w_0 s_j$ for all $1 \le i \ne j \le r$. We remark that $w_0 s_i s_j = w_0 s_j s_i$ if and only if the simple roots $\alpha_i$ and $\alpha_j$ are orthogonal, *i.e.* $\alpha_i^\vee(\alpha_j) = 0$.

**1.3.** *Definition of the complex.* Let $w \in W^{(i)}$ and $w' \in W^{(i+1)}$ with $w' < w$. We define for all $0 \le j \le r$ a homomorphism

$$d^w_{w'} \;:\; \bigwedge^{j} X \longrightarrow \bigwedge^{j-1} X$$

as follows. By definition we have $w = w' \cdot s_\alpha$ for some reflection $s_\alpha$, $\alpha \in \Pi^+$. Denote $\alpha^\vee$ the dual (coroot) of $\alpha \in \Pi^+$. Then we set

$$d^w_{w'}(x_1 \wedge \ldots \wedge x_j) \;:=\; \sum_{i=1}^{j} (-1)^{i-1} \alpha^\vee(x_i) \cdot x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_j \,, \qquad (1)$$

where $\hat{x}$ means that $x$ is omitted. These maps are the components of the differentials of a cohomological complex $K^\bullet(X,m)$:

$$0 \longrightarrow K^{-m}(X,m) \xrightarrow{d^{-m}_{(X,m)}} K^{-(m-1)}(X,m) \xrightarrow{d^{-(m-1)}_{(X,m)}} \cdots$$

$$\cdots \longrightarrow K^{-1}(X,m) \xrightarrow{d^{-1}_{(X,m)}} K^{0}(X,m) \longrightarrow 0$$

with $K^j(X,m)$ in degree $j$, which we define now. Set for $i \geq 0$

$$K^{-m+i}(X,m) \;:=\; \bigoplus_{w \in W^{(i)}} \bigwedge^{m-i} X \,,$$

where $\bigwedge^{m-i}$ denotes the $(m-i)$-th exterior power of the $\mathbb{Z}$-module $X$. The differential $d^{-m+i}_{(X,m)} : K^{-m+i}(X,m) \longrightarrow K^{-m+(i+1)}(X,m)$ is defined as follows. Let $w \in W^{(i)}$ and $w' \in W^{(i+1)}$. Denote $(d^{-m+i}_{(X,m)})^w_{w'}$ the $ww'$-component of it. Then we set

$$(d^{-m+i}_{(X,m)})^w_{w'} \;=\; \begin{cases} 0 & \text{if } w' \not\leq w \\ d^w_{w'} & \text{if } w' \leq w \end{cases}$$

(recall that $w \in W^{(i)} \iff \ell(w) = \ell(w_0) - i$ and so $\ell(w') = \ell(w_0) - i - 1 < \ell(w)$).

**1.4.** We have to show that $K^\bullet(X,m)$ is a complex, *i.e.* $d^{-m+i+1}_{(X,m)} \cdot d^{-m+i}_{(X,m)} = 0$. Let $w \in W^{(i)}$ and $w'' \in W^{(i+2)}$ for some $0 \leq i \leq \ell(w_0) - 2$. Then, see 1.2, there are only two elements $w_1, w_2$ between $w$ and $w''$, and we have $w_1 = w \cdot s_\alpha$ and $w_2 = w \cdot s_\beta$ for some $\alpha, \beta \in \Pi^+$. There exists then $\gamma, \delta \in \Pi^+$, such that $w'' = ws_\alpha s_\gamma = ws_\beta s_\delta$. A straightforward calculation shows that $K^\bullet(X,m)$ is a complex if and only if

$$\alpha^\vee(x) \cdot \gamma^\vee(x') \,-\, \alpha^\vee(x') \cdot \gamma^\vee(x) \;=\; -(\beta^\vee(x) \cdot \delta^\vee(x') \,-\, \beta^\vee(x') \cdot \delta^\vee(x)) \qquad (2)$$

for all $x, x' \in X$. This can be deduced from Pieri's formula.

There exists a $k$-split semisimple algebraic group $G$ with root system $(\Pi, \Delta)$. Denote $\mathbb{T} \subseteq G$ a maximal $k$-split torus and $B \supseteq \mathbb{T}$ the Borel subgroup. Pieri's formula describes the intersection product in the Chow ring $\mathrm{CH}^*(G/B)$.

By Chevalley [4] the graded abelian group

$$\mathrm{CH}^*(G/B) \;=\; \bigoplus_{i=0}^{\ell(w_0)} \mathrm{CH}^i(G/B)$$

is a free abelian group with generators the cycles $[X_v]$, $v \in W$, where $X_v$ is the closure of $B\tilde{v}B/B$ in $G/B$. We denote here and in the following by $\tilde{v}$ a representative of $v \in W \simeq N_G(\mathbb{T})/\mathbb{T}$.

If now $\alpha_j$ is a simple reflection and $w \in W^{(i)} \subset W$ is as above we have the following intersection formula, called Pieri's formula:

$$[X_{w_0 s_j}] \cdot [X_w] \;=\; \sum_{\substack{\rho \,\in\, \Pi^+ \\ \ell(ws_\rho) \,=\, \ell(w) \,-\, 1}} \rho^\vee(\varpi_j)[X_{ws_\rho}]\,,$$

see [9, 4.4, Cor. 2]. Since $\mathrm{CH}^*(G/B)$ is commutative we have

$$[X_{w_0 s_l}] \cdot [X_{w_0 s_j}] \cdot [X_w] \;=\; [X_{w_0 s_j}] \cdot [X_{w_0 s_l}] \cdot [X_w]$$

and therefore

$$\sum_{\substack{\rho,\,\nu \,\in\, \Pi^+ \\ ws_\rho \,\in\, W^{(i-1)} \\ ws_\rho s_\nu \,\in\, W^{(i-2)}}} \nu^\vee(\varpi_l)\rho^\vee(\varpi_j)\,[X_{ws_\rho s_\nu}] \;=\; \sum_{\substack{\rho,\,\nu \,\in\, \Pi^+ \\ ws_\rho \,\in\, W^{(i-1)} \\ ws_\rho s_\nu \,\in\, W^{(i-2)}}} \nu^\vee(\varpi_j)\rho^\vee(\varpi_l)\,[X_{ws_\rho s_\nu}]$$

for all $1 \le j \ne l \le r$. As observed above, see 1.2, there are only two ordered pairs $(s_\rho, s_\nu)$ of reflections, such that $w'' = ws_\rho s_\nu$, and hence by comparing the coefficients of $[X_{w''}]$ in the above equation we get

$$\alpha^\vee(\varpi_l)\gamma^\vee(\varpi_j) \,+\, \beta^\vee(\varpi_l)\delta^\vee(\varpi_j) \;=\; \alpha^\vee(\varpi_j)\gamma^\vee(\varpi_l) \,+\, \beta^\vee(\varpi_j)\delta^\vee(\varpi_l)$$

for all $1 \le j \ne l \le r$. Since the fundamental weights $\varpi_1, \ldots, \varpi_r$ constitute a $\mathbb{Z}$-basis of $X = X(\mathbb{T}) = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_\mathrm{m})$ this proves (2), and so $K^\bullet(X,m)$ is a complex.

**1.5.** *General base ring.* Let $\Bbbk$ be a commutative ring (with 1). We set then $K^\bullet(X,m)_\Bbbk := \Bbbk \otimes_\mathbb{Z} K^\bullet(X,m)$. This is a complex of free $\Bbbk$-modules, and we have

$$K^{-m+i}(X,m)_\Bbbk \;=\; \bigoplus_{w \in W^{(i)}} \Bbbk \otimes_\mathbb{Z} \left( \bigwedge^{m-i} X \right) \;=\; \bigoplus_{w \in W^{(i)}} \bigwedge_{\Bbbk}^{m-i} X_\Bbbk\,.$$

We denote the differential of the complex $K^\bullet(X,m)_\Bbbk$ by $d^\bullet_{(X_\Bbbk,m)}$.

## 2. On the cohomology of $K^\bullet(X,m)_\Bbbk$

**2.1.** *An interpretation of tensor and exterior powers of* $\Bbbk \otimes_\mathbb{Z} X = X_\Bbbk$. We continue with the notations of the last section. In particular $\Bbbk$ is a commutative ring.

In the following we denote by $\bigotimes^i$ (respectively $\bigwedge^i$) the $i$-th tensor (respectively exterior) power over $\Bbbk$. The $i$-th tensor power $\bigotimes^i X_\Bbbk$ is isomorphic to the abelian group of $i$-multilinear maps from $Y_\Bbbk$ to $\Bbbk$:

$$x_1 \otimes \ldots \otimes x_i \;\longrightarrow\; \{\; (y_1^\vee, \ldots, y_i^\vee) \;\mapsto\; \prod_{j=1}^{i} y_j^\vee(x_j) \;\}\,.$$

There is a monomorphism $\bigwedge^i X_\Bbbk \hookrightarrow \bigotimes^i X_\Bbbk$ given by

$$x_1 \wedge \ldots \wedge x_i \;\longmapsto\; \sum_{\sigma \in S_i} \mathrm{sgn}(\sigma) x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \ldots \otimes x_{\sigma(i)}\,,$$

where $S_i$ is the symmetric group on $\{1, \ldots, i\}$ and sgn denotes the signum of the permutation $\sigma$. This map identifies the $i$-th exterior power of $X_\Bbbk$ with the group of alternating $i$-forms on $Y_\Bbbk$.

Note that using these isomorphisms we can consider $\left(\bigotimes^i X_\Bbbk\right) \otimes_\Bbbk \left(\bigwedge^j X_\Bbbk\right)$ as the abelian group of $(i+j)$-multilinear maps on $Y_\Bbbk$ which are alternating in the last $j$ variables. We need this in particular in the case $i = 1$. Then the above monomorphism $\bigwedge^{j+1} X_{\Bbbk} \hookrightarrow \bigotimes^{j+1} X_{\Bbbk}$ factors

$$\bigwedge^{j+1} X_{\Bbbk} \hookrightarrow X_{\Bbbk} \otimes_{\Bbbk} \bigwedge^{j} X_{\Bbbk} \hookrightarrow \bigotimes^{j+1} X_{\Bbbk}\,,$$

where the map on the left hand side is given by

$$x_1 \wedge \ldots \wedge x_j \wedge x_{j+1} \longmapsto \sum_{s=1}^{j+1} (-1)^{s-1} x_s \otimes (x_1 \wedge \ldots \wedge \hat{x_s} \wedge \ldots \wedge x_{j+1})\,. \qquad (3)$$

In other words, this map is the natural injection of the abelian group of $(j+1)$-alternating forms on $Y_{\Bbbk}$ to the abelian group of $(j+1)$-multilinear forms on $Y_{\Bbbk}$ which are alternating in the last $j$ variables.

These identifications will be used without further comment in the following.

**2.2.** There is an isomorphism

$$f_m \,:\, X_{\Bbbk} \otimes_{\Bbbk} \bigwedge^{m-1} X_{\Bbbk} \xrightarrow{\simeq} \bigoplus_{w \in W^{(1)}} \bigwedge^{m-1} X_{\Bbbk} \qquad x \otimes y \longmapsto (\alpha_1^{\vee}(x) y, \ldots, \alpha_r^{\vee}(x) y)\,.$$

We get a commutative diagram

$$\begin{array}{ccccc} \bigwedge^{m} X_{\Bbbk} & \xrightarrow{d^{-m}_{(X_{\Bbbk},m)}} & \bigoplus_{w \in W^{(1)}} \bigwedge^{m-1} X_{\Bbbk} & \xrightarrow{d^{-m+1}_{(X_{\Bbbk},m)}} & \bigoplus_{w \in W^{(2)}} \bigwedge^{m-2} X_{\Bbbk} \\ & \searrow^{\gamma_m} & \simeq \uparrow f_m & \nearrow_{\kappa_{m-1}} & \\ & & X_{\Bbbk} \otimes_{\Bbbk} \bigwedge^{m-1} X_{\Bbbk}\,, & & \end{array} \qquad (4)$$

where $\kappa_{m-1} = d^{-m+1}_{(X_{\Bbbk},m)} \cdot f_m$ and $\gamma_m = f_m^{-1} \cdot d^{-m}_{(X_{\Bbbk},m)}$ is the map

$$x_1 \wedge \ldots \wedge x_m \longmapsto \sum_{j=1}^{m} (-1)^{j-1} x_j \otimes (x_1 \wedge \ldots \wedge \hat{x_j} \wedge \ldots \wedge x_m)\,.$$

Obviously $\gamma_m$ coincides with (3). In particular it is injective. We have proven:

**Lemma.** *We have*

$$\mathrm{H}^{-m}(K^{\bullet}(X,m)_{\Bbbk}) = \begin{cases} 0 & m \neq 0 \\ \Bbbk & m = 0 \end{cases}$$

*for all $m \in \mathbb{Z}$.*

**2.3.** We describe now the homomorphism $\kappa_{m-1} = d^{-m+1}_{(X_{\Bbbk},m)} \cdot f_m$ which we defined by the commutative diagram (4).

As observed above, see 1.2, if $i \neq j$ then $w_0 s_i$ and $w_0 s_j$ are the only elements of $W^{(1)}$ which are bigger than $w_0 s_i s_j \in W^{(2)}$. It follows that $(d^{-m+1}_{(X_{\Bbbk},m)})^{w_0 s_k}_{w_0 s_i s_j} \neq 0$ only if $k \in \{i, j\}$. Let

$$\underline{x} = (x_1^i \wedge \ldots \wedge x_{m-1}^i)_{i=1}^{r} \in \bigoplus_{i=1}^{r} \bigwedge^{m-1} X_{\Bbbk} = \bigoplus_{w \in W^{(1)}} \bigwedge^{m-1} X_{\Bbbk}$$

(recall that $W^{(1)} = \{\, w_0 s_i \,|\, 1 \le i \le r \,\}$). Then the $w_0 s_i s_j$-component of $d^{-m+1}_{(X_\Bbbk, m)}(\underline{x})$ is given by

$$\sum_{k=1}^{m-1} (-1)^{k-1} \alpha_j^\vee(x_k^i) x_1^i \wedge \ldots \wedge \widehat{x_k^i} \wedge \ldots \wedge x_{m-1}^i + \sum_{k=1}^{m-1} (-1)^{k-1} \alpha_i^\vee(s_j x_k^j) x_1^j \wedge \ldots \wedge \widehat{x_k^j} \wedge \ldots \wedge x_{m-1}^j \,.$$

Let now $m \ge 2$ and

$$b \in X_\Bbbk \otimes_\Bbbk \bigwedge^{m-1} X_\Bbbk \xrightarrow{\;\simeq\;} \bigoplus_{w \in W^{(1)}} \bigwedge^{m-1} X_\Bbbk$$

a $m$-multilinear form on $Y_\Bbbk$ which is alternating in the last $(m-1)$ variables, *i.e.* $b(y_1^\vee, \ldots, y_2^\vee, \ldots, y_2^\vee, \ldots) \equiv 0$ for all $y_1^\vee, y_2^\vee \in Y_\Bbbk$ if $m \ge 3$ respectively $b$ is a bilinear form on $Y_\Bbbk$ if $m = 2$. Then the $w_0 s_i s_j$-component of $\kappa_{m-1}(b) = d^{-m+1}_{(X_\Bbbk, m)}(f_m(b))$ is the following $(m-2)$-alternating form on $Y_\Bbbk$:

$$(y_3^\vee, \ldots, y_m^\vee) \longmapsto b(\alpha_i^\vee, \alpha_j^\vee, y_3^\vee, \ldots, y_m^\vee) + b(\alpha_j^\vee, s_j \alpha_i^\vee, y_3^\vee, \ldots, y_m^\vee)\,. \tag{5}$$

Part (ii) of the following theorem is due to Brylinski and Deligne [3, Lem. 4.5].

**2.4. Theorem.** *Let $m \ge 1$ and $K^\bullet(X, m)_\Bbbk$ be as above.*

(i) *If $m \ne 2$ then $\mathrm{H}^{-m+1}(K^\bullet(X, m)_\Bbbk) = 0$.*

(ii) *If $m = 2$ then $\mathrm{H}^{-1}(K^\bullet(X, 2)_\Bbbk) \simeq \Bbbk^d$, where $d$ is the number of irreducible components of the root system $\Pi$. More precisely the $\Bbbk$-module $\mathrm{H}^{-1}(K^\bullet(X, 2)_\Bbbk)$ is isomorphic to the $\Bbbk$-module of $W$-invariant quadratic forms on $Y_\Bbbk = \mathrm{Hom}_\Bbbk(X_\Bbbk, \Bbbk)$.*

*Proof.* We prove (i) first for $m = 1$. We have to show that $X_\Bbbk \xrightarrow{(\alpha_i^\vee)_{i=1}^r} \bigoplus_{i=1}^r \Bbbk$ is surjective. This follows from our assumption, see 1.1, that $X$ is the weight lattice of $\Pi$: There are $\epsilon_j \in X_\Bbbk$, such that $\alpha_i^\vee(\epsilon_j) = \delta_{ij}$ (Kronecker delta).

Let now $m \ge 3$, and $b \in X_\Bbbk \otimes_\Bbbk \bigwedge^{m-1} X_\Bbbk$ in the kernel of $\kappa_{m-1} = d^{-m+1}_{(X_\Bbbk, m)} \cdot f_m$, *i.e.*

$$b(\alpha_i^\vee, \alpha_j^\vee, \ldots) + b(\alpha_j^\vee, s_j \alpha_i^\vee, \ldots) \equiv 0 \tag{6}$$

for all $1 \le i \ne j \le r$, see 2.3. We show that $b$ is in the image of $\partial_m = f_m^{-1} \cdot d^{-m}_{(X_\Bbbk, m)}$. This is by 2.2 equivalent to the assertion that $b$ is an alternating $m$-multilinear form. Note that for $m \ge r + 1 > \operatorname{rank} Y_\Bbbk$ any alternating $m$-multilinear form on $Y_\Bbbk$ is zero and hence this argument implies that $d^{-m+1}_{(X_\Bbbk, m)}$ is injective in this case.

Since $\alpha_1^\vee, \ldots, \alpha_r^\vee \in Y_\Bbbk$ are a basis of the free $\Bbbk$-module $Y_\Bbbk$ it is enough to show that (a) $b(\alpha_i^\vee, \alpha_i^\vee, \ldots) \equiv 0$ for all $1 \le i \le r$, and (b) $b(\alpha_i^\vee, \alpha_j^\vee, \ldots) + b(\alpha_j^\vee, \alpha_i^\vee, \ldots) \equiv 0$ for all $1 \le i \ne j \le r$.

We fix $1 \le j \le r$. By (6) we have

$$0 \equiv b(\alpha_i^\vee, \alpha_j^\vee, \alpha_j^\vee, \ldots) + b(\alpha_j^\vee, s_j \alpha_i^\vee, \alpha_j^\vee, \ldots) \equiv b(\alpha_j^\vee, s_j \alpha_i^\vee, \alpha_j^\vee, \ldots)\,.$$

for all $i \ne j$. Since $s_j \alpha_j^\vee = -\alpha_j^\vee$ and $b$ is alternating in its last $(m-1)$ variables by assumption we have also $0 \equiv b(\alpha_j^\vee, s_j \alpha_j^\vee, \alpha_j^\vee, \ldots)$. It follows $b(\alpha_i^\vee, \alpha_i^\vee, \ldots) \equiv -b(\alpha_i^\vee, \_, \alpha_i^\vee, \ldots) \equiv 0$ because $s_j \alpha_1^\vee, \ldots, s_j \alpha_r^\vee$ is also a $\Bbbk$-basis of $Y_\Bbbk$. We have verified (a).

To check (b) we insert $s_j\alpha_i^\vee = \alpha_i^\vee - \alpha_i^\vee(\alpha_j)\cdot\alpha_j^\vee$ in equation (6) getting

$$\begin{aligned} 0 &\equiv b(\alpha_i^\vee,\alpha_j^\vee,\ldots) \,+\, b(\alpha_j^\vee,\alpha_i^\vee,\ldots) \,-\, \alpha_i^\vee(\alpha_j)b(\alpha_j^\vee,\alpha_j^\vee,\ldots) \\ &\equiv b(\alpha_i^\vee,\alpha_j^\vee,\ldots) \,+\, b(\alpha_j^\vee,\alpha_i^\vee,\ldots) \end{aligned}$$

since the last term in the first line is zero by (a). We have shown (b) and so are done for part (i) of the theorem.

As mentioned above (ii) is proven in [3, Lem. 4.5] for $\Bbbk = \mathbb{Z}$, and the same argument works for any commutative ring $\Bbbk$. But we can also deduce (ii) for any $\Bbbk$ from this case. In fact, by [3, Prop. 4.6] the map

$$\kappa_1 \,:\, X\otimes_{\mathbb{Z}} X \,\longrightarrow\, \bigoplus_{w\in W^{(2)}} \mathbb{Z}$$

is surjective and so splits. Hence $\mathrm{H}^{-1}(K^\bullet(X,2)_\Bbbk) \simeq \mathrm{H}^{-1}(K^\bullet(X,2))\otimes_{\mathbb{Z}}\Bbbk$. □

**2.5. Remarks.**

(i) Using Demazure's [8] description of the ring of invariants of the Weyl group one can give a different proof of Theorem 2.4.

(ii) With this approach it is also possible to compute $\mathrm{H}^i(K^\bullet(X,m)_\Bbbk)$ for all $i,m\in\mathbb{Z}$ if $\Bbbk = \mathbb{Q}$ is the field of rational numbers. This gives a simple explanation for our results $\mathrm{H}^{-1}(K^\bullet(X,2)_\Bbbk)\simeq\Bbbk^d$ and $\mathrm{H}^{-m+1}(K^\bullet(X,m)_\Bbbk) = 0$ for $m\neq 2$: There is no element $\neq 0$ of $X$ invariant under the Weyl group $W$.

## 3. Rost's cycle modules

**3.1.** *Milnor K-theory.* Before we recall the definition of Rost's cycle modules we remind the reader of some properties of Milnor $K$-theory defined in [15].

Let for this $k$ be a field. The Milnor $K$-theory of $k$

$$\mathrm{K}_*^M(k) \,:=\, \bigoplus_{i=1}^{\infty}\mathrm{K}_i^M(k)$$

is the quotient ring of the tensor algebra of $k^\times = k\setminus\{0\}$ over $\mathbb{Z}$ by the ideal generated by all elements $a\otimes(1-a)$, $a\in F^\times\setminus\{1\}$. As usual we denote the image of a tensor $a_1\otimes\ldots\otimes a_i$ in $\mathrm{K}_i^M(k)$ by $\{a_1,\ldots,a_i\}$.

If $E/F$ is a field extension then there is restriction map $\mathrm{K}_*^M(F)\longrightarrow\mathrm{K}_*^M(E)$ and if the extension is finite a corestriction map $\mathrm{K}_*^M(E)\longrightarrow\mathrm{K}_*^M(F)$ satisfying the usual projection formula.

If $v : k^\times\longrightarrow\mathbb{Z}$ is a discrete valuation on the field $k$ with residue field $k(v)$ then there is the so called *second residue homomorphism* $\partial_v : \mathrm{K}_*^M(k)\longrightarrow\mathrm{K}_{*-1}^M(k(v))$ which maps the symbol $\{x,u_2,\ldots,u_i\}$ to $v(x)\cdot\{\bar{u}_2,\ldots,\bar{u}_i\}$ if $v(u_2)=\ldots=v(u_i)=0$, where $\bar{u}$ denotes the image of the $v$-unit $u$ in $k(v)$.

**3.2.** *Rost's cycle modules.* These have been defined by Rost in [18], for which we refer for details and much more information.

A *cycle module* over the field $k$ is a covariant functor

$$\mathfrak{Fields}_k \longrightarrow \mathfrak{GrAb} \qquad E \longmapsto \mathrm{M}_*(E) \,:=\, \bigoplus_{i\in\mathbb{Z}}\mathrm{M}_i(E)$$

$$\iota \,:\, K\longrightarrow L \qquad\longmapsto\qquad \iota_* \,:\, \mathrm{M}_*(K)\longrightarrow\mathrm{M}_*(L)$$

from the category of finitely generated fields over $k$ to the category of graded abelian groups, such that

(a) For any finite extension $\iota : K \hookrightarrow L$ in $\mathfrak{Fields}_k$ there is a corestriction homomorphism
$$\iota^* : \mathrm{M}_*(L) \longrightarrow \mathrm{M}_*(F)\,.$$
(b) The graded abelian group $\mathrm{M}_*(K)$ is a graded $\mathrm{K}^M_*(K)$-module for any $K \in \mathfrak{Fields}_k$.
(c) For any valuation $v$ on $K \in \mathfrak{Fields}_k$ which is trivial on $k \subseteq K$ there is a (second) residue map
$$\partial^{\mathrm{M}}_v \;:\; \mathrm{M}_*(K) \longrightarrow \mathrm{M}_{*-1}(K(v))\,,$$
where $K(v)$ denotes the residue field of $v$.

These data are subject to several axioms, see [18, Sect. 1 and 2]. Most of them corresponds to well known properties of Milnor $K$-theory, which is the prototype of a cycle module.

**3.3.** We will use the following property of the residue map $\partial^{\mathrm{M}}_v$.

Let $\iota : k \hookrightarrow K$ be a field extension with valuation $v$ which is trivial on $k$, and $\mathrm{M}_*$ a cycle module over $k$. Denote $K(v)$ the residue field of $v$. We have then a residue map $\partial^{\mathrm{M}}_v : \mathrm{M}_*(K) \longrightarrow \mathrm{M}_{*-1}(K(v))$ and restriction maps $\iota_* : \mathrm{M}_*(k) \longrightarrow \mathrm{M}_*(K)$ and $\bar{\iota}_* : \mathrm{M}_*(k) \longrightarrow \mathrm{M}_*(K)$, where $\bar{\iota} : k \hookrightarrow K(v)$ denotes the by $\iota$ induced inclusion.

Let $\xi \in \mathrm{M}_j(k)$ and $\{u_1, \ldots, u_i\} \in \mathrm{K}^M_i(K)$ be a symbol. It is a consequence of the axioms [18, R3c and R3f] that
$$\partial^{\mathrm{M}}_v(\{u_1, \ldots, u_i\} \cdot \iota_*(\xi)) \;=\; \partial_v(\{u_1, \ldots, u_i\}) \cdot \bar{\iota}_*(\xi)\,,$$
where $\partial_v : \mathrm{K}^M_*(K) \longrightarrow \mathrm{K}^M_{*-1}(K(v))$ is the second residue homomorphism of Milnor $K$-theory.

**3.4.** *The cycle complex.* Let $X$ be a scheme of finite type over the field $k$ or a localization of such a scheme. We denote $X^{(i)} \subset X$ the set of points of codimension $i$. For any cycle module $\mathrm{M}_*$ over $k$ Rost [18, Sect. 3] defines a complex $\mathrm{C}^\bullet(X, \mathrm{M}_*, n)$ as follows:
$$\mathrm{C}^i(X, \mathrm{M}_*, n) \;:=\; \begin{cases} \bigoplus\limits_{x \in X^{(i)}} \mathrm{M}_{n-i}(k(x)) & 0 \le i \le \dim X \\ \qquad 0 & \text{otherwise,} \end{cases}$$
where $k(x)$ denotes the residue field of $x \in X$. The $xy$-component, $x \in X^{(i)}, y \in X^{(i+1)}$, of the differential $\mathrm{d}^i \;:\; \mathrm{C}^i(X, \mathrm{M}_*, n) \longrightarrow \mathrm{C}^{i+1}(X, \mathrm{M}_*, n)$ is defined to be 0 if $y \notin \overline{\{x\}}$ and if $y \in \overline{\{x\}}$ as the following sum. Let $Z \longrightarrow \overline{\{x\}}$ be the normalization and $y_1, \ldots, y_m \in Z$ the points above $y$. These points define valuations $v_1, \ldots, v_m$ in the function field $k(Z) = k(x)$ of $Z$. Denote the respective residue maps by $\partial^{\mathrm{M}}_i$ and the respective inclusions $k(y) \longrightarrow k(y_i)$ by $\iota_i$. Then $\mathrm{d}^x_y \;:=\; \sum\limits_{i=1}^{m} \iota^*_i \cdot \partial^{\mathrm{M}}_i$.

The $i$-th cohomology group of $\mathrm{M}_n$ over $X$ is then by definition:
$$\mathrm{H}^i(X, \mathrm{M}_n) \;:=\; \mathrm{H}^i(\mathrm{C}^\bullet(X, \mathrm{M}_*, n))\,.$$

**Remarks.**

(i) For $\mathrm{M}_* = \mathrm{K}^M_*$ this complex has been defined by Kato [14].
(ii) There is an isomorphism $\mathrm{CH}^p(X) \simeq \mathrm{H}^n(X, \mathrm{K}^M_n)$ for all $p \ge 0$ and smooth $k$-schemes $X$.

**3.5.** These groups are homotopy invariant. Let $p : E \longrightarrow X$ be an affine fibration. Then the homomorphism $p^* : \mathrm{H}^i(X, \mathrm{M}_n) \longrightarrow \mathrm{H}^i(E, \mathrm{M}_n)$ is an isomorphism for all $i \in \mathbb{Z}$, see [18, Prop. 8.6]. As usual we get from this fact and the long exact cohomology sequence the cohomology of a split torus $\mathbb{T} = \mathbb{G}_{\mathrm{m}}^r(k) = \operatorname{Spec} k[t_1^{\pm}, \ldots, t_r^{\pm}]$:

$$\mathrm{H}^i(\mathbb{T}, \mathrm{M}_n) \simeq \begin{cases} \bigoplus\limits_{j=0}^{n} \left( \bigoplus\limits_{1 \leq s_1 < \ldots < s_j \leq r} \{t_{s_1}, \ldots, t_{s_j}\} \cdot \mathrm{M}_{n-j}(k) \right) & i = 0 \\ 0 & i \neq 0\,. \end{cases} \tag{7}$$

We recall briefly the argument. Let $A = k[t_1^{\pm}, \ldots, t_{r-1}^{\pm}]$ and consider the localization sequence for the open set $X = \operatorname{Spec} A[t_r^{\pm}] \subset \operatorname{Spec} A[t_r]$:

$$0 \longrightarrow \mathrm{H}^0(Y[t_r], \mathrm{M}_n) \longrightarrow \mathrm{H}^0(X, \mathrm{M}_n) \xrightarrow{\delta} \mathrm{H}^0(Y, \mathrm{M}_{n-1}) \longrightarrow$$

$$\mathrm{H}^1(Y[t_r], \mathrm{M}_n) \longrightarrow \mathrm{H}^1(X, \mathrm{M}_n) \longrightarrow \mathrm{H}^1(Y, \mathrm{M}_{n-1}) \longrightarrow \cdots,$$

where $Y = \operatorname{Spec} A$. The homomorphism $\delta$ splits by $x \mapsto \{t_r\} \cdot x$, and by homotopy invariance we have $\mathrm{H}^1(Y[t_r], \mathrm{M}_{n-1}) \simeq \mathrm{H}^1(Y, \mathrm{M}_{n-1})$. This implies the claimed fact by induction.

## 4. The cyclic complex of a cycle module over a semisimple algebraic group

**4.1.** *Notations.* Let $k$ be a field and $G$ a $k$-split simply connected semisimple algebraic group. Let $\mathbb{T}$ be a maximal $k$-split torus of rank $r$ and $B \supseteq \mathbb{T}$ a Borel subgroup. We denote $W = N_G(\mathbb{T})/\mathbb{T}$ the Weyl group, and $\Pi = \Pi(B, \mathbb{T}) \subset \operatorname{Hom}(\mathbb{T}, \mathbb{G}_{\mathrm{m}})$ the associated root system with simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$. Denote $s_1, \ldots, s_r$ the corresponding simple reflections.

Recall the Bruhat decomposition: $G = \bigcup\limits_{w \in W} B\tilde{w}B$, where $\tilde{w} \in N_G(\mathbb{T})$ denotes a representative of $w \in W$. We have $B\tilde{w}B \cap B\tilde{w}'B = \emptyset$ if $w \neq w'$. This disjoint decomposition of $G$ defines a partial order, the so called (strong) Bruhat order, on $W$, see [4]: $w \leq w'$ if and only if $B\tilde{w}B \subseteq \overline{B\tilde{w}'B}$.

We remark that this definition of an partial ordering on $W$ coincides with the algebraic defined ordering in [10], which we have recalled in 1.2. As there we denote $w_0 \in W$ the unique longest word (with respect to the simple reflections $s_1, \ldots, s_r$) and let $N = \ell(w_0)$ be its length. We set also $W^{(i)} := \{\, w \in W \mid \ell(w) = N - i \,\}$.

**4.2.** *A filtration on* $\mathrm{C}^\bullet(G, \mathrm{M}_*, n)$*.* Let $\mathrm{M}_*$ be a cycle module. Following [3] we define a filtration on the cycle complex $\mathrm{C}^\bullet(G, \mathrm{M}_*, n)$. Let $f^pW := \bigcup\limits_{i \geq p} W^{(i)}$ and

$$F^p\, \mathrm{C}^j(G, \mathrm{M}_*, n) := \bigoplus_{w \in f^pW} \Big( \bigoplus_{x \in B\tilde{w}B \cap G^{(j)}} \mathrm{M}_{n-j}(k(x)) \Big)\,.$$

This is a decreasing filtration whose associated graded complex is equal

$$\mathrm{gr}^p\, \mathrm{C}_n^\bullet \;=\; F^p\, \mathrm{C}_n^\bullet / F^{p+1}\, \mathrm{C}_n^\bullet \;\simeq\; \bigoplus_{w \in W^{(p)}} \mathrm{C}^\bullet(B\tilde{w}B, \mathrm{M}_*, n-p)[-p]\,.$$

In particular we have $\operatorname{gr}^p \mathrm{C}_n^i = 0$ for $i < p$ and

$$\mathrm{H}^p(\operatorname{gr}^p \mathrm{C}_n^\bullet) \simeq \bigoplus_{w \in W^{(p)}} \mathrm{H}^0(\mathrm{C}_n^\bullet(B\tilde{w}B, \mathrm{M}_*, n-p)) .$$

**4.3.** Recall now the following facts about the cell $B\tilde{w}B$. Denote $\Pi^+$ and $\Pi^-$ the positive and negative roots, respectively, $U_\alpha$ the radicial subgroup of $\alpha \in \Pi$, and $U_w$ the subgroup generated by all $U_\alpha$ with $\alpha \in \Pi^+ \cap w\Pi^-$ $(w \in W)$.

The product in $G$ induces an isomorphism $U_w \times \tilde{w}\mathbb{T} \times U \xrightarrow{\simeq} B\tilde{w}B$, where $U$ is the unipotent radical of $B$. We get a morphism

$$p_{\tilde{w}} \,:\, B\tilde{w}B \longrightarrow \tilde{w}\mathbb{T} \xrightarrow{\cdot \tilde{w}^{-1}} \mathbb{T} ,$$

where the map on the left hand side is the inverse of the above isomorphism composed with the projection to $\tilde{w}\mathbb{T}$. This is an affine fibration, and therefore by homotopy invariance we have $\mathrm{H}^i(B\tilde{w}B, \mathrm{M}_n) \simeq \mathrm{H}^i(\mathbb{T}, \mathrm{M}_n)$ for all $i \in \mathbb{N}$.

We denote in the following by $x_{\tilde{w}}$ the pull-back of $x \in X$ along $p_{\tilde{w}}$.

**4.4.** It follows by (7) that $\mathrm{H}^{p+q}(\operatorname{gr}^p \mathrm{C}_n^\bullet) = 0$ for $q \neq 0$ and therefore the spectral sequence $E_1^{p,q} = \mathrm{H}^{p+q}(\operatorname{gr}^p \mathrm{C}_n^\bullet)$ degenerates at the $E_1$-term. Hence the embeddings $\operatorname{gr}^p \mathrm{C}_n^p \hookrightarrow \mathrm{C}_n^p$ induce a quasi-isomorphism of the subcomplex

$$\begin{gathered}\bigoplus_{w \in W^{(0)}} \mathrm{H}^0(B\tilde{w}B, \mathrm{M}_n) \xrightarrow{d^0} \bigoplus_{w \in W^{(1)}} \mathrm{H}^0(B\tilde{w}B, \mathrm{M}_{n-1}) \xrightarrow{d^1} \\ \bigoplus_{w \in W^{(2)}} \mathrm{H}^0(B\tilde{w}B, \mathrm{M}_{n-2}) \xrightarrow{d^2} \bigoplus_{w \in W^{(3)}} \mathrm{H}^0(B\tilde{w}B, \mathrm{M}_{n-3}) \longrightarrow \cdots\end{gathered} \tag{8}$$

with $\mathrm{C}^\bullet(G, \mathrm{M}_*, n)$, where $d^p$ is the restriction of the $p$-th differential of the complex $\mathrm{C}^\bullet(G, \mathrm{M}_*, n)$.

**4.5.** We define now a decreasing filtration on the complex (8) as follows. We set $F^p\, \mathrm{H}^0(B\tilde{w}B, \mathrm{M}_m) = 0$ for $p \geq 1$, and for $p \geq 0$ let $F^{-p}\, \mathrm{H}^0(B\tilde{w}B, \mathrm{M}_m)$ be the image of the homomorphism

$$\bigoplus_{i=0}^{p} \big( X^{\otimes i} \otimes_{\mathbb{Z}} \mathrm{M}_{m-i}(k) \big) \longrightarrow \mathrm{H}^0(B\tilde{w}B, \mathrm{M}_m) ,$$

which is given by $(x_1 \otimes \ldots \otimes x_i) \otimes \xi \mapsto \{x_{1\,\tilde{w}}, \ldots, x_{i\,\tilde{w}}\} \cdot \xi$. This defines a decreasing filtration on the complex (8), and by (7) we know that $F^{-r}\, \mathrm{H}^0(B\tilde{w}B, \mathrm{M}_m) = \mathrm{H}^0(B\tilde{w}B, \mathrm{M}_m)$, where $r$ is the rank of $\mathbb{T}$.

The associated spectral sequence $E_1^{p,q}(G, \mathrm{M}_*, n)$ converges to the cohomology of the complex (8) and so also to the cohomology of the cycle complex $\mathrm{C}^\bullet(G, \mathrm{M}_*, n)$.

**4.6.** *The $E_1$-terms of the spectral sequence $E_1^{p,q}(G, \mathrm{M}_*, n)$.* Let $w \in W^{(p)}$ and $w' \in W^{(p+1)}$. If $w' \not< w$ then $B\tilde{w}'B$ is not in the closure of $B\tilde{w}B$ and so the $ww'$-component $(d^p)^w_{w'}$ of

$$d^p \,:\, \bigoplus_{w \in W^{(p)}} \mathrm{H}^0(B\tilde{w}B, \mathrm{M}_{n-p}) \longrightarrow \bigoplus_{w \in W^{(p+1)}} \mathrm{H}^0(B\tilde{w}B, \mathrm{M}_{n-(p+1)})$$

is zero. Otherwise, *i.e.* if $w' < w$, we have by 3.3

$$(d^p)^w_{w'}(\{x_{1\,\tilde{w}}, \ldots, x_{s\,\tilde{w}}\} \cdot \xi) \,=\, \partial^w_{w'}(\{x_{1\,\tilde{w}}, \ldots, x_{s\,\tilde{w}}\}) \cdot \xi \tag{9}$$

for $\xi \in \mathrm{M}_{n-p-s}(k)$, where

$$\partial^w_{w'} \,:\, \mathrm{K}^M_s(k(B\tilde{w}B)) \longrightarrow \mathrm{K}^M_{s-1}(k(B\tilde{w}'B))$$

denotes the second residue map on Milnor $K$-theory induced by the smooth divisor $B\tilde{w}'B \subseteq \overline{B\tilde{w}B}$. Hence we have

$$(d^p)^w_{w'}(F^{-j}\,\mathrm{H}^0(B\tilde{w}B,\mathrm{M}_{n-p})) \,\subseteq\, F^{-j+1}\,\mathrm{H}^0(B\tilde{w}B,\mathrm{M}_{n-p-1})$$

for all $j \in \mathbb{Z}$, and therefore

$$E_1^{p,q}(G,\mathrm{M}_*,n) \,\simeq\, \begin{cases} \left(\bigoplus\limits_{w\in W^{(p+q)}} \left(\bigwedge^{-p} X\right)\right) \otimes_{\mathbb{Z}} \mathrm{M}_{n-q}(k) & p \le 0\,,\ p+q \ge 0 \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

**4.7.** *The differential of the spectral sequence $E_1^{p,q}(G,\mathrm{M}_*,n)$.* To describe the differentials of this spectral sequence it is by (9) above enough to consider the differential $\partial^w_{w'} : \mathrm{K}^M_*(k(B\tilde{w}B)) \longrightarrow \mathrm{K}^M_{*-1}(k(B\tilde{w}'B))$ for $w \in W^{(p)}$ and $w' \in W^{(p+1)}$ with $w' < w$. Then we have, see [10], $w' = w_1 \cdot w_2$ and $w = w_1 \cdot s_{i_0} \cdot w_2$ for some $w_1, w_2 \in W$ with $\ell(w_1)+\ell(w_2) = \ell(w') = \ell(w)-1$ and some $i_0 \in \{1,\ldots,r\}$, *i.e.* $s_{i_0}$ is a simple reflection. Equivalently, we have $w' = w\cdot s_\alpha$, where $s_\alpha = w_2^{-1}\cdot s_{i_0}\cdot w_2$, *i.e.* $\alpha = w_2^{-1}(\alpha_{i_0})$. We denote in this case by $v^w_{w'}$ the valuation of $k(B\tilde{w}B)$ induced by the smooth divisor $B\tilde{w}'B \subset \overline{B\tilde{w}B}$.

It follows from results of Demazure [9], see [3, Lem. 4.2], that the valuation $v^w_{w'}$ of $x_{\tilde{w}}$, $x \in X = \mathrm{Hom}(\mathbb{T},\mathbb{G}_{\mathrm{m}})$, is given as follows: $v^w_{w'}(x_{\tilde{w}}) = \alpha^\vee(x)$. where $\alpha^\vee \in Y = \mathrm{Hom}(\mathbb{G}_{\mathrm{m}},\mathbb{T})$ denotes the dual (coroot) of $\alpha$. If $\alpha^\vee(x) = 0$ then the restriction of $x_{\tilde{w}}$ to $B\tilde{w}'B$ is equal to $c\cdot x_{\tilde{w}'}$ for some $c \in k^\times = k\setminus\{0\}$.

Using these facts we compute the differential $d_1^{p,q}$ of $E_1^{p,q} = E_1^{p,q}(G,\mathrm{M}_*,n)$. For ease of notation (analogous to 4.5) we denote by $F^{-p}\mathrm{K}^M_n(k(B\tilde{w}B))$ the image of the multiplication map $\bigoplus\limits_{i=0}^{p}\left(\left(\bigotimes^i X\right) \otimes_{\mathbb{Z}} \mathrm{K}^M_{n-i}(k)\right) \longrightarrow \mathrm{K}^M_n(k(B\tilde{w}B))$, which sends (as above) $(x_1\otimes\ldots\otimes x_i)\otimes\xi$ to $\{x_{1\,\tilde{w}},\ldots,x_{i\,\tilde{w}}\}\cdot\xi$, and set $F^i\mathrm{K}^M_n(k(B\tilde{w}B)) = 0$ for $i > 0$. With this notation we have the following

**Lemma.** *Let $w > w' \in W$ be as above, i.e. in particular $w' = w\cdot s_\alpha$, $x_1,\ldots,x_p \in X = \mathrm{Hom}(\mathbb{T},\mathbb{G}_{\mathrm{m}})$, and $\xi \in \mathrm{K}^M_{n-p}(k)$. Then we have* $\partial^w_{w'}(\{x_{1\,\tilde{w}}\ \ldots, x_{p\,\tilde{w}}\}\cdot\xi)$

$$\equiv \sum_{l=1}^{p}(-1)^{l-1}\alpha^\vee(x_l)\cdot\left\{x_{1\,\tilde{w}'},\,\ldots\,,\widehat{x_{l\,\tilde{w}'}},\,\ldots\,,x_{p\,\tilde{w}'}\right\}\cdot\xi \bmod F^{-p+2}\mathrm{K}^M_{p-1}(k(B\tilde{w}'B))\,.$$

*Proof.* We prove the lemma by induction on $p \ge 1$. This is obvious for $p = 1$.

Let $p \ge 2$. If $x_{p\,\tilde{w}}$ is a unit at $B\tilde{w'}B$ then we have $\partial^w_{w'}(a\cdot\{x_{p,\tilde{w}}\}) = \partial^w_{w'}(a)\cdot\{x_{p\,\tilde{w'}}\} \bmod F^{-p+2}\mathrm{K}^M_{p-1}(k(B\tilde{w'}B))$ for all $a \in \mathrm{K}^M_{p-1}(k(B\tilde{w}B))$, and so the lemma follows by induction in this case.

We choose now $z \in X$, such that $v^w_{w'}(z) = 1$, *i.e.* $z_{\tilde{w}}$ is a local uniformizer at the divisor $B\tilde{w}'B$. This is possible since we assume $G$ to be simply connected and so $X$ coincides with the weight lattice. Then we have $x_p = z^{i_p}r_p$ and $x_{p-1} = z^{i_{p-1}}r_{p-1}$ for $r_{p-1}, r_p \in X$ with $v^w_{w'}(r_{p-1\,\tilde{w}}) = v^w_{w'}(r_{p\,\tilde{w}}) = 0$ and appropriate $i_{p-1}, i_p \in \mathbb{Z}$.

We have $\{\, x_{1\,\tilde{w}}, \dots, x_{p\,\tilde{w}} \} \;=$

$$\{x_{1\tilde{w}}, \dots, x_{p-1\tilde{w}}, r_{p\tilde{w}}\} + \{x_{1\tilde{w}}, \dots, r_{p-1\tilde{w}}, z_{\tilde{w}}^{i_p}\} + \{x_{1\tilde{w}}, \dots, z_{\tilde{w}}^{i_{p-1}}, z_{\tilde{w}}^{i_p}\}\,.$$

Since $\{z_{\tilde{w}}, z_{\tilde{w}}\} = \{z_{\tilde{w}}, -1\}$ the last summand is mapped by $\partial^w_{w'}$ to zero modulo the subgroup $F^{-p+2}\mathrm{K}^M_{p-1}(k(B\tilde{w}'B))$.

Hence by induction we compute $\partial^w_{w'}(\{x_{1\,\tilde{w}}, \dots, x_{p\,\tilde{w}}\})$

$$\begin{aligned}
&\equiv \sum_{l=1}^{p-1} (-1)^{l-1}\alpha^\vee(x_l)\{x_{1\tilde{w}'}, \dots, \widehat{x_{l\tilde{w}'}}, \dots, x_{p-1\tilde{w}'}, r_{p\tilde{w}'}\} \\
&\qquad - \sum_{l=1}^{p-2} (-1)^{l-1}\alpha^\vee(x_l)\{x_{1\tilde{w}'}, \dots, \widehat{x_{l\tilde{w}'}}, \dots, x_{p-2\tilde{w}'}, z_{\tilde{w}'}^{i_p}, r_{p-1\tilde{w}'}\} \\
&\qquad\qquad - (-1)^{p-2}\alpha^\vee(x_p)\{x_{1\tilde{w}'}, \dots, x_{p-2\tilde{w}'}, r_{p-1\tilde{w}'}\}
\end{aligned}$$

$$\begin{aligned}
&\equiv \sum_{l=1}^{p-1} (-1)^{l-1}\alpha^\vee(x_l)\{x_{1\tilde{w}'}, \dots, \widehat{x_{l\tilde{w}'}}, \dots, x_{p-1\tilde{w}'}, r_{p\tilde{w}'}\} \\
&\qquad + \sum_{l=1}^{p-1} (-1)^{l-1}\alpha^\vee(x_l)\{x_{1\tilde{w}'}, \dots, \widehat{x_{l\tilde{w}'}}, \dots, x_{p-1\tilde{w}'}, z_{\tilde{w}'}^{i_p}\} \\
&\qquad - \sum_{l=1}^{p-1} (-1)^{l-1}\alpha^\vee(x_l)\{x_{1\tilde{w}'}, \dots, \widehat{x_{l\tilde{w}'}}, \dots, x_{p-1\tilde{w}'}, z_{\tilde{w}'}^{i_p}\} \\
&\qquad\quad - \sum_{l=1}^{p-2} (-1)^{l-1}\alpha^\vee(x_l)\{x_{1\tilde{w}'}, \dots, \widehat{x_{l\tilde{w}'}}, \dots, x_{p-2\tilde{w}'}, z_{\tilde{w}'}^{i_p}, r_{p-1\tilde{w}'}\} \\
&\qquad\qquad - (-1)^{p-2}\alpha^\vee(x_p)\{x_{1\tilde{w}'}, \dots, x_{p-2\tilde{w}'}, r_{p-1\tilde{w}'}\}
\end{aligned}$$

$$\begin{aligned}
&\equiv \sum_{l=1}^{p-1} (-1)^{l-1}\alpha^\vee(x_l)\{x_{1\tilde{w}'}, \dots, \widehat{x_{l\tilde{w}'}}, \dots, x_{p-1\tilde{w}'}, x_{p\tilde{w}'}\} \\
&\qquad + \sum_{l=1}^{p-2} (-1)^{l-1}\alpha^\vee(x_l)\{x_{1\tilde{w}'}, \dots, \widehat{x_{l\tilde{w}'}}, \dots, x_{p-2\tilde{w}'}, z_{\tilde{w}'}^{-i_{p-1}}, z_{\tilde{w}'}^{i_p}\} \\
&\qquad\quad - (-1)^{p-2}\alpha^\vee(x_{p-1})\{x_{1\tilde{w}'}, \dots, x_{p-2\tilde{w}'}, z_{\tilde{w}'}^{i_p}\} \\
&\qquad\qquad - (-1)^{p-2}\alpha^\vee(x_p)\{x_{1\tilde{w}'}, \dots, x_{p-2\tilde{w}'}, r_{p-1\tilde{w}'}\}
\end{aligned}$$

modulo $F^{-p+2}\mathrm{K}^M_{p-1}(k(B\tilde{w}'B))$. By the same reasons as above the summand

$$\sum_{l=1}^{p-2} (-1)^{l-1}\alpha^\vee(x_l)\{x_{1\tilde{w}'}, \dots, \widehat{x_{l\tilde{w}'}}, \dots, x_{p-2\tilde{w}'}, z_{\tilde{w}'}^{-i_{p-1}}, z_{\tilde{w}'}^{i_p}\}$$

is zero modulo $F^{-p+2}\mathrm{K}^M_{p-1}(k(B\tilde{w}'B))$ and we are done. □

**4.8.** It follows from this lemma that the following diagram commutes:

$$\begin{array}{ccc}
\left(\bigoplus_{w\in W^{(p+q)}} \bigwedge^{-p} X\right) \otimes_{\mathbb{Z}} \mathrm{M}_{n-q}(k) & \xrightarrow{\;d\;} & \left(\bigoplus_{w\in W^{(p+1+q)}} \bigwedge^{-(p+1)} X\right) \otimes_{\mathbb{Z}} \mathrm{M}_{n-q}(k) \\
\downarrow \simeq & & \downarrow \simeq \\
E_1^{p,q}(G, \mathrm{M}_*, n) & \xrightarrow{\;d_1^{-p,q}\;} & E_1^{p+1,q}(G, \mathrm{M}_*, n)
\end{array}$$

for all $p \leq 0$, where $d = d_{(X,q)}^{-(p+q)} \otimes \mathrm{id}_{\mathrm{M}_{n-q}(k)}$ and the columns are the isomorphisms of (10) in 4.6. Hence the $q$-th line of the spectral sequence $E^1_{p,q}(G, \mathrm{M}_*, n)$ is isomorphic to $K^\bullet(X,q) \otimes_{\mathbb{Z}} \mathrm{M}_{n-q}(k)$, where $K^\bullet(X,q)$ is the complex defined in 1.3.

Therefore by the lemma in 2.2 and Theorem 2.4 we have $E_2^{-p,p+1}(G, \mathrm{M}_*, n) = 0$ except if $p = 1$, and $E_2^{-p,p}(G, \mathrm{M}_*, n) = 0$ except if $p = 0$ ($p \in \mathbb{N}$). Since we have also $E_2^{-p,q}(G, \mathrm{M}_*, n) = 0$ for $p \in \mathbb{N}$ and $q < p$, see (10), it follows

$$E_2^{-p,p+1}(G, \mathrm{M}_*, n) \;=\; E_\infty^{-p,p+1}(G, \mathrm{M}_*, n)$$

and also

$$E_2^{-p,p}(G, \mathrm{M}_*, n) = E_\infty^{-p,p}(G, \mathrm{M}_*, n)$$

for all $p \in \mathbb{N}$. Again by the lemma in 2.2 and Theorem 2.4 we have $E_2^{0,0}(G, \mathrm{M}_*, n) \simeq \mathrm{M}_n(k)$, and $E_2^{-1,2}(G, \mathrm{M}_*, n) \simeq \mathrm{M}_{n-2}(k)^d$, where $d$ is the number of connected components of the Dynkin diagram of $G$.

We have established the following result.

**4.9. Theorem.** *Let $G$ be a simply connected $k$-split semisimple algebraic group and $\mathrm{M}_*$ a cycle module over $k$. Then:*

(i) $\mathrm{H}^0(G, \mathrm{M}_n) \simeq \mathrm{M}_n(k)$.

(ii) $\mathrm{H}^1(G, \mathrm{M}_n) \simeq \mathrm{H}^1(G, \mathrm{K}_2^M) \otimes_{\mathbb{Z}} \mathrm{M}_{n-2}(k) \simeq \mathbb{Z}^d \otimes_{\mathbb{Z}} \mathrm{M}_{n-2}(k)$, *where $d$ is the number of irreducible components of the roots system $\Pi$ of $G$.*

**4.10. Remark.** For $\mathrm{M}_n = \mathrm{K}_2^M$ this has been proven in [3, Prop. 4.6], and with a different method in [11].

## 5. The first Suslin homology group of a simply connected semisimple split algebraic group

**5.1.** The aim of this section is to prove Theorem A of the introduction. Let $k$ be a field.

*We assume in the whole section that our base field $k$ is perfect.*

We denote $\mathrm{Sm}_k$ the category of smooth $k$-schemes, and $\mathrm{Psh}_k$ the category of abelian presheaves on $\mathrm{Sm}_k$.

The sheaves appearing in this section are sheaves with respect to the Nisnevich topology. If not otherwise said by a (pre)sheaf we mean an abelian (pre)sheaf, *i.e.* a (pre)sheaf of abelian groups. We denote the category of (abelian) Nisnevich sheaves by $\mathcal{N}\mathrm{Sh}_k$. Recall that the forgetful functor $\mathcal{N}\mathrm{Sh}_k \longrightarrow \mathrm{Psh}_k$ has a left adjoint $\mathcal{F} \mapsto \mathcal{F}_{\mathrm{Nis}}$.

**5.2.** *Presheaves with transfers.* These (pre)sheaves on $\mathrm{Sm}_k$ have been introduced by Voevodsky, see [21]. We start by recalling the definition of the category $\mathrm{Cor}_k$ of finite correspondences.

Let $X, Y \in \mathrm{Sm}_k$. Then an *elementary correspondence* from $X$ to $Y$ is an irreducible closed subscheme of $X \times_k Y$ which is finite and surjective above an irreducible component of $X$. The group $c(X,Y)$ is the free abelian group generated by all elementary correspondences from $X$ to $Y$. Its elements are called *finite correspondences.*

The *category of finite correspondences* $\mathrm{Cor}_k$ has smooth $k$-schemes as objects and $c(X,Y)$ as group of morphisms $X \longrightarrow Y$. It is an additive category, and there is an obvious functor $\mathrm{Sm}_k \longrightarrow \mathrm{Cor}_k$ which is the identity on objects and sends any morphism to its graph.

A presheaf *with transfers* is a presheaf on $\mathrm{Cor}_k$. We call a presheaf with transfers $\mathcal{F}$ a (Nisnevich) sheaf with transfers if $\mathcal{F}$ is also a Nisnevich sheaf on $\mathrm{Sm}_k$. We denote $\mathrm{Psh}_k^{tr}$ the category of presheaves with transfers and $\mathcal{N}\mathrm{Sh}_k^{tr}$ the category of sheaves with transfers (over $\mathrm{Sm}_k$). Note that if $\mathcal{F}$ is a presheaf with transfers then $\mathcal{F}_{\mathrm{Nis}}$ is a Nisnevich sheaf with transfers, see [21, Chap. 5, Lem. 3.1.6].

The category $\mathcal{N}\mathrm{Sh}_k^{tr}$ possesses a commutative tensor product $\otimes^{\mathcal{N}\mathrm{Sh}_k^{tr}}$ and an inner Hom-functor $\mathcal{H}om_{\mathcal{N}\mathrm{Sh}_k^{tr}}(\,-\,,\,-\,)$, such that

$$\mathcal{H}om_{\mathcal{N}\mathrm{Sh}_k^{tr}}(\mathcal{F},\mathcal{G})(k) \,=\, \mathrm{Hom}_{\mathcal{N}\mathrm{Sh}_k^{tr}}(\mathcal{F},\mathcal{G})$$

for all $\mathcal{F},\mathcal{G} \in \mathcal{N}\mathrm{Sh}_k^{tr}$, see Déglise [5, Prop. 2.2.18].

**Examples.**

(i) The presheaf

$$\mathbb{Z}_{\mathrm{tr}}(X) \,:\, Y \longmapsto c(Y,X)$$

is obviously a presheaf with transfer. It is a sheaf in the étale, and so also in the Nisnevich, topology, see *e.g.* [21, Chap. 5, Lem. 3.1.2]. Note that $\mathbb{Z}_{\mathrm{tr}}(\mathrm{Spec}\,k)$ is the constant sheaf, *i.e.* the Nisnevich sheaf associated to the presheaf $U \mapsto \mathbb{Z}$. We denote it $\mathbb{Z}_{\mathrm{tr}}$.

(ii) Let $\mathrm{M}_*$ be a cycle module over $k$. Then

$$\underline{\mathrm{M}}_n \,:\, Y \longmapsto \mathrm{H}^0(Y,\mathrm{M}_n)$$

is a sheaf with transfers, see [5, Sect. 4.3] or [7]. This sheaf is also homotopy invariant, a notion which we recall now.

**5.3.** *Homotopy invariant (pre)sheaves.* A (pre)sheaf $\mathcal{F}$ is called *homotopy invariant* if for any $X \in \mathrm{Sm}_k$ the projection $X \times_k \mathbb{A}_k^1 \longrightarrow X$ induces an isomorphism

$$\mathcal{F}(X) \xrightarrow{\simeq} \mathcal{F}(X \times_k \mathbb{A}_k^1)\,.$$

We denote the category of homotopy invariant Nisnevich sheaves (respectively presheaves) with transfers on $\mathrm{Sm}_k$ by $\mathcal{N}\mathrm{Sh}_k^{Htr}$ (respectively by $\mathrm{Psh}_k^{Htr}$). The inclusion functor $\mathcal{N}\mathrm{Sh}_k^{tr} \hookrightarrow \mathcal{N}\mathrm{Sh}_k^{Htr}$ has a left adjoint $\mathcal{F} \longmapsto \mathcal{F}^H$, see [5, Prop. 3.1.7].

By a result of Voevodsky [21, Chap. 3, Thm. 5.6] any $\mathcal{F} \in \mathcal{N}\mathrm{Sh}_k^{Htr}$ is strictly homotopy invariant:

$$\mathrm{H}^i_{\mathrm{Nis}}(X,\mathcal{F}) \xrightarrow{\simeq} \mathrm{H}^i_{\mathrm{Nis}}(X \times_k \mathbb{A}_k^1,\mathcal{F})$$

for all $i \geq 0$.

The comparison of the Zariski and Nisnevich cohomology of such sheaves is given by another theorem of Voevodsky [21, Chap. 5, Thm. 3.1.12].

**Theorem.** *Let* $\mathcal{F} \in \mathcal{N}\mathrm{Sh}_k^{Htr}$*. Then we have*

$$\mathrm{H}^i_{\mathrm{Nis}}(X,\mathcal{F}) \,\simeq\, \mathrm{H}^i_{\mathrm{Zar}}(X,\mathcal{F})$$

*for all* $i \geq 0$ *and all* $X \in \mathrm{Sm}_k$.

There is also a tensor product $\otimes^{Htr}$ in $\mathcal{N}\mathrm{Sh}_k^{Htr}$ which is defined by

$$\mathcal{F} \otimes^{Htr} \mathcal{G} \,:=\, (\mathcal{F} \otimes^{tr} \mathcal{G})^H \,,$$

see [5, Prop. 3.1.9].

We will need the following

**5.4. Lemma.** *Let $f\,:\,\mathcal{F} \longrightarrow \mathcal{G}$ be a morphism in $\mathrm{Psh}_k^{Htr}$. Then the induced morphism*

$$f_{\mathrm{Nis}} \,:\, \mathcal{F}_{\mathrm{Nis}} \longrightarrow \mathcal{G}_{\mathrm{Nis}}$$

*is an isomorphism if and only if $f_E : \mathcal{F}(E) \longrightarrow \mathcal{G}(E)$ is an isomorphism for all separable finite type field extensions $E/k$.*

*Proof.* See *e.g.* [5, Lem. 3.3.5]. □

**5.5.** For any $\mathcal{F} \in \mathrm{Psh}_k$ there are presheaves $\mathcal{F}_n$, $n \in \mathbb{Z}$, which are defined as follows.

Let $\mathcal{F} \in \mathrm{Psh}_k$. In analogy to negative $K$-theory the presheaves $\mathcal{F}_{-n}$, $n > 0$, are defined inductively by setting:

$$\mathcal{F}_{-1} \,:\, X \longmapsto \mathrm{Coker}\left(\mathcal{F}(X \times_k \mathbb{A}^1_k) \longrightarrow \mathcal{F}(X \times_k (\mathbb{G}_{\mathrm{m}})_k)\right),$$

and for $n \geq 2$:

$$\mathcal{F}_{-n} \,:=\, (\mathcal{F}_{-(n-1)})_{-1} \,.$$

We have $\mathcal{F}_{-n} \in \mathcal{N}\mathrm{Sh}_k^{Htr}$ if $\mathcal{F} \in \mathcal{N}\mathrm{Sh}_k^{Htr}$, and $\mathcal{F}_{-n} \simeq \mathcal{H}om_{\mathcal{N}\mathrm{Sh}_k^{tr}}(\underline{\mathrm{K}}_n^M, \mathcal{F})$ for all $\mathcal{F} \in \mathcal{N}\mathrm{Sh}_k^{Htr}$, see [5, Sect. 3.4]. It follows

$$\begin{aligned} \mathcal{F}_{-n}(k) &\simeq \mathcal{H}om_{\mathcal{N}\mathrm{Sh}_k^{tr}}(\underline{\mathrm{K}}_n^M, \mathcal{F})(k) \\ &\simeq \mathrm{Hom}_{\mathcal{N}\mathrm{Sh}_k^{tr}}(\underline{\mathrm{K}}_n^M, \mathcal{F}) \,=\, \mathrm{Hom}_{\mathcal{N}\mathrm{Sh}_k^{Htr}}(\underline{\mathrm{K}}_n^M, \mathcal{F}) \end{aligned} \tag{11}$$

for all homotopy invariant sheaves with transfers $\mathcal{F}$.

For $n \geq 0$ we set

$$\mathcal{F}_n \,:=\, \underline{\mathrm{K}}_n^M \otimes^{\mathcal{N}\mathrm{Sh}_k^{Htr}} \mathcal{F}$$

for all $\mathcal{F} \in \mathcal{N}\mathrm{Sh}_k^{Htr}$.

**Remark.** Note that the unramified Milnor sheaf $\underline{\mathrm{K}}_n^M$ does not appear in [5, Sect. 3.4] but a sheaf $\mathcal{S}_t^n \in \mathcal{N}\mathrm{Sh}_k^{Htr}$ which is defined as follows: $\mathcal{S}_t^1 = \mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_{\mathrm{m}} \setminus \{1\})^H$ and $\mathcal{S}_t^n = (\mathcal{S}_t^1)^{\otimes n}$. It follows from a theorem of Suslin and Voevodsky [20, Thm. 3.4] that $\mathcal{S}_t^n \simeq \underline{\mathrm{K}}_n^M$, see [5, Prop. 6.3.20].

**5.6.** *Cycle modules and* $\mathcal{N}\mathrm{Sh}_k^{Htr}$. The main result of Déglise's thesis, see [5, Thm. 6.3.12] and [6], is the following: The functor

$$\widehat{\mathcal{F}}_* \,:\, \mathfrak{Fields}_k \longrightarrow \mathfrak{GrAb} \qquad E \longmapsto \widehat{\mathcal{F}}_*(E) \,=\, \bigoplus_{n\in\mathbb{Z}} \widehat{\mathcal{F}}_n(E)\,,$$

where $\widehat{\mathcal{F}}_n(E) := \mathcal{F}_n(E)$ for all $E \in \mathfrak{Fields}_k$ and $n \in \mathbb{Z}$, is a cycle module in the sense of Rost [18], and the functor $\mathcal{F} \longmapsto \widehat{\mathcal{F}}_*$ is an equivalence between the category $\mathcal{N}\mathrm{Sh}_k^{Htr}$ and a localization of the category of cycle modules over $k$. The inverse of this equivalence is given by $\mathrm{M}_* \longmapsto \underline{\mathrm{M}}_0$.

Moreover, see Déglise [5, Thm. 6.2.3], we have

$$\mathrm{H}^i_{\mathrm{Nis}}(X, \mathcal{F}) \,\simeq\, \mathrm{H}^i(X, \widehat{\mathcal{F}}_0)$$

for all $i \geq 0$ and all $\mathcal{F} \in \mathcal{N}\mathrm{Sh}_k^{Htr}$, where the cohomology on the right hand side is the cohomology of cycle modules as defined in 3.4.

Hence we get from Theorem 4.9:

**5.7. Theorem.** *Let $G$ be a $k$-split simply connected semisimple algebraic group. Then we have*

$$\mathrm{H}^1_{\mathrm{Nis}}(G,\mathcal{F}) \;\simeq\; \mathcal{F}_{-2}(k)^d$$

*for all $\mathcal{F} \in \mathcal{N}\mathrm{Sh}_k^{Htr}$, where $d$ is the number of irreducible components of the root system of $G$.*

**5.8.** *The category* $\mathrm{DM}_-^{eff}(k)$. We denote by $\mathrm{D}_-(\mathcal{N}\mathrm{Sh}_k^{tr})$ the derived category of bounded above complexes of Nisnevich sheaves with transfers. Voevodsky's [21, Chap. 5, Sect. 3.1] category of effective motivic complexes is by definition the full subcategory $\mathrm{DM}_-^{eff}(k) \subseteq \mathrm{D}_-(\mathcal{N}\mathrm{Sh}_k^{tr})$ consisting of complexes whose cohomology sheaves are homotopy invariant.

There is a $t$-structure on the triangulated category $\mathrm{DM}_-^{eff}(k)$ which is the restriction of the natural $t$-structure on $\mathrm{D}_-(\mathcal{N}\mathrm{Sh}_k^{tr})$. The heart of this $t$-structure is equivalent to the abelian category $\mathcal{N}\mathrm{Sh}_k^{Htr}$ via the natural functor $\mathcal{N}\mathrm{Sh}_k^{Htr} \longrightarrow \mathrm{DM}_-^{eff}(k)$, see [21, Chap. 5, Sect. 3.1].

**5.9.** The embedding $\mathrm{DM}_-^{eff}(k) \hookrightarrow \mathrm{D}_-(\mathcal{N}\mathrm{Sh}_k^{tr})$ has a left adjoint. We recall its definition. Let for this

$$\Delta^i \;=\; \Delta^i_k \;:= \mathrm{Spec}\, k[x_0,\ldots,x_i]\Big/(1-\sum_{j=0}^{i} x_j)\,,$$

be the standard $i$-simplex with coface maps

$$\delta^l \,:\, \Delta^{i-1} \;\longrightarrow \Delta^i \quad (a_0,\ldots,a_{i-1}) \;\longmapsto\; (a_0,\ldots,a_{l-1},0,a_l,\ldots,a_{i-1})\,.$$

Given a presheaf $\mathcal{F}$ we have then a complex of presheaves $\underline{C}^\bullet(\mathcal{F})$ which is defined as follows: We set $\underline{C}^i(\mathcal{F}) = 0$ for $i>0$, and define $\underline{C}^{-i}(\mathcal{F})$, $i \geq 0$, as the sheaf

$$U \;\longmapsto\; \mathcal{F}(U \times_k \Delta^i)\,.$$

The differential $\underline{d}^{-i}(\mathcal{F}) \,:\, \underline{C}^{-i}(\mathcal{F}) \longrightarrow \underline{C}^{-i+1}(\mathcal{F})$ is the alternating sum

$$\underline{d}^{-i}(\mathcal{F}) \;:=\; \sum_{l=0}^{i} (-1)^l (\delta^l)^*\,.$$

Following [21, Chap. 5, Sect. 3] we denote the cohomology presheaf of the complex $\underline{C}^\bullet(\mathcal{F})$ in degree $-i$ by $\underline{h}_i(\mathcal{F})$. If $\mathcal{F}$ is a presheaf with transfers then $\underline{h}_i^{\mathrm{Nis}}(\mathcal{F}) := \underline{h}_i(\mathcal{F})_{\mathrm{Nis}}$ is a homotopy invariant Nisnevich sheaf, see [21, Chap. 5, Lem. 3.2.1].

We get a functor

$$\mathcal{N}\mathrm{Sh}_k^{tr} \;\longrightarrow\; \mathrm{DM}_-^{eff}(k)\,, \quad \mathcal{F} \;\longmapsto\; \underline{C}^\bullet(\mathcal{F})\,.$$

By [21, Chap. 5, Prop. 3.2.3] this extends to a functor

$$\mathbf{R}\mathrm{C} \,:\, \mathrm{D}_-(\mathcal{N}\mathrm{Sh}_k^{tr}) \;\longrightarrow\; \mathrm{DM}_-^{eff}(k)$$

which is left adjoint to the embedding $\mathrm{DM}_-^{eff}(k) \hookrightarrow \mathrm{D}_-(\mathcal{N}\mathrm{Sh}_k^{tr})$.

**5.10.** *Suslin complex and Suslin (singular) homology.* Let $X \in \mathrm{Sm}_k$. Applying the functor $\underline{C}^\bullet$ to the sheaf $\mathbb{Z}_{\mathrm{tr}}(X)$ defined in 5.2 we get a complex of Nisnevich sheaves $\underline{C}^\bullet(X) := \underline{C}^\bullet(\mathbb{Z}_{\mathrm{tr}}(X))$:

$$\underline{C}^\bullet(X) : \quad \underline{C}^0(X) \xleftarrow{\underline{d}^{-1}} \underline{C}^{-1}(X) \xleftarrow{\underline{d}^{-2}} \underline{C}^{-2}(X) \xleftarrow{\underline{d}^{-3}} \underline{C}^{-3}(X) \longleftarrow \cdots .$$

This complex is called the *Suslin complex* of $X$, and we denote $\underline{h}_i(X) := \underline{h}_i(\mathbb{Z}_{\mathrm{tr}}(X))$ its cohomology presheaf in degree $-i$, and set as usual $\underline{h}_i^{\mathrm{Nis}}(X) := \underline{h}_i(X)_{\mathrm{Nis}}$.

We are in position to define Suslin (singular) homology.

**Definition.** Let $X \in \mathrm{Sm}_k$ and $i \geq 0$. The abelian group

$$\mathrm{H}_i^{\mathrm{S}}(X) := \underline{h}_i(X)(k)$$

is called the $i$-th *Suslin (singular) homology group* of $X$.

For later use we note the following. Let $X \in \mathrm{Sm}_k$. Then $\mathbf{R}\mathrm{C}(\mathbb{Z}_{\mathrm{tr}}(X)) = \underline{C}^\bullet(X)$, see [21, Chap. 5, Thm. 3.2.6], and therefore we have a natural isomorphism

$$\mathrm{Hom}_{\mathrm{DM}_-^{eff}(k)}(\underline{C}^\bullet(X), \mathcal{F}^\bullet) \simeq \mathrm{Hom}_{\mathrm{D}_-(\mathcal{N}\mathrm{Sh}_k^{tr})}(\mathbb{Z}_{\mathrm{tr}}(X), \mathcal{F}^\bullet) \qquad (12)$$

for all $\mathcal{F}^\bullet \in \mathrm{DM}_-^{eff}(k)$.

**5.11.** *Homological $n$-connected smooth schemes.* We define now homological $n$-connected smooth schemes. Roughly speaking a smooth scheme $X$ is homological $n$-connected if its Suslin complex is trivial in degrees $\geq -n$. It turns out, see the example below, that a split simply connected semisimple algebraic group is 0-connected.

**Definition.** Let $X \in \mathrm{Sm}_k$ and $n \in \mathbb{N}$. We say $X$ is *homological* 0*-connected* if $\underline{h}_0^{\mathrm{Nis}}(X) \simeq \mathbb{Z}_{\mathrm{tr}} = \mathbb{Z}_{\mathrm{tr}}(\mathrm{Spec}\, k)$. More generally we call $X$ *homological $n$-connected* if

(i) $X$ is homological 0-connected, and

(ii) $\underline{h}_i^{\mathrm{Nis}}(X) = 0$ for all $1 \leq i \leq n$.

**5.12. Example.** Let $G$ be a $k$-split simply connected semisimple algebraic group. Then $G$ is homological 0-connected. Denote $e \in G(k)$ the neutral element of $G$. This $k$-rational point induces a morphism $\alpha : \mathbb{Z}_{\mathrm{tr}} = \mathbb{Z}_{\mathrm{tr}}(\mathrm{Spec}\, k) \longrightarrow \underline{C}^\bullet(G)$ which in turn induces a split homomorphism $\alpha_* : \mathbb{Z}_{\mathrm{tr}} \hookrightarrow \underline{h}_0^{\mathrm{Nis}}(G)$. We claim that $\alpha_*$ is surjective and so an isomorphism. This follows from the Kneser–Tits conjecture which is true for such groups, see [12].

Recall the assertion of this conjecture. Denote $G^+(k)$ the subgroup of $G(k)$ generated by all subgroups isomorphic to the additive group $\mathbb{G}_{\mathrm{a}}(k)$. Since (by assumption) the group $G$ is $k$-split simply connected semisimple we have $G^+(k) = G(k)$, see [12, Cor. 5.10].

This implies that given $g \in G(k)$ then there exists a chain of elements $g = g_s, g_{s-1}, \ldots, g_1, g_0 = e$ in $G(k)$, and morphisms $u_i : \mathbb{A}_k^1 \longrightarrow G$, $1 \leq i \leq s$, such that $u_i(0) = g_{i-1}$ and $u_i(1) = g_i$.

Let now $E/k$ be a finite field extension and $g \in G(E)$. Since $E/k$ is finite the $E$-rational point $g$ defines an element $[g] \in \underline{C}^0(G)(k)$. Moreover any element of $\underline{C}^0(G)(k)$ is a sum of such elements for various finite extension fields $E$.

The Kneser–Tits conjecture is true for $G_E := G \times_k E$, too, and therefore there are morphisms $u_i : \Delta_E^1 \simeq \mathbb{A}_E^1 \longrightarrow G_E$, $1 \leq i \leq s$, such that $u_0\delta^0 = e$, $u_i\delta^0 = u_{i-1}\delta^1$, and $u_s\delta^1 = g$. From this it follows $[g] = [e] \bmod \mathrm{Im}\, \underline{d}_k^1$ by the following remark.

Let $E/k$ be as above and $u : \Delta^1_E \longrightarrow G_E$ be a morphism with $E/k$ finite. Set $g_0 = u\delta^0$ and $g_1 = u\delta^1$. Then $[g_0] = [g_1] \bmod \operatorname{Im} \underline{d}^1_k$. In fact, the push-forward along $\Delta^1 \times_k G \times_k E \longrightarrow \Delta^1 \times_k G$ of the graph of $u$ is an elementary cycle $\xi \in \underline{C}^1(G)(k)$ with $\underline{d}^1_k(\xi) = [g_0] - [g_1]$. It follows that $\alpha_{*\,k}$ is surjective and so an isomorphism $\mathbb{Z} \xrightarrow{\simeq} \underline{h}_0^{\mathrm{Nis}}(G)(k) = \mathrm{H}^{\mathrm{S}}_0(G)$.

By the same argument $\alpha_{*\,F} : \mathbb{Z} \simeq \underline{h}_0^{\mathrm{Nis}}(G)(F)$ is a bijection for all $F \in \mathfrak{Fields}_k$ and therefore $\alpha_*$ is isomorphism $\mathbb{Z}_{\mathrm{tr}} \simeq \underline{h}_0^{\mathrm{Nis}}(G)$ by Lemma 5.4.

**5.13. Theorem.** *Let $n \geq 1$ and $X \in \mathrm{Sm}_k$ be homological $(n-1)$-connected. Then there is a natural (in $\mathcal{F}$) isomorphism*

$$\mathrm{Hom}_{\mathcal{N}\mathrm{Sh}_k^{Htr}}(\underline{h}_n^{\mathrm{Nis}}(X), \mathcal{F}) \;\simeq\; \mathrm{H}^n_{\mathrm{Nis}}(X, \mathcal{F})$$

*for all $\mathcal{F} \in \mathcal{N}\mathrm{Sh}_k^{Htr}$.*

*Proof.* By a theorem of Veovodsky [21, Chap. 5, Prop. 3.1.9], there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{D}_-(\mathcal{N}\mathrm{Sh}_k^{tr})}(\mathbb{Z}_{\mathrm{tr}}(X), \mathcal{F}[n]) \simeq \mathrm{H}^n_{\mathrm{Nis}}(X, \mathcal{F})$$

for all $n \in \mathbb{N}$, where we consider $\mathbb{Z}_{\mathrm{tr}}(X)$ as a complex concentrated in degree 0, and $\mathcal{F}[n]$ denotes the $n$-th translate of $\mathcal{F}$, *i.e.* the complex $\mathcal{F}[n]$ is 0 except in degree $n$, where it is equal to $\mathcal{F}$.

Hence by (12) we have a natural isomorphism

$$\mathrm{Hom}_{\mathrm{DM}_-^{eff}(k)}(\underline{C}^\bullet(X), \mathcal{F}[n]) \;\simeq\; \mathrm{H}^n_{\mathrm{Nis}}(X, \mathcal{F})\,.$$

In particular we have $\mathrm{Hom}_{\mathrm{DM}_-^{eff}(k)}(\underline{h}_0^{\mathrm{Nis}}(X), \mathcal{F}[n]) \simeq 0$ by our assumptions that $n \geq 1$ and $\underline{h}_0^{\mathrm{Nis}}(X) \simeq \mathbb{Z}_{\mathrm{tr}}(\operatorname{Spec} k)$. We get from this the theorem by means of elementary homological algebra. □

**5.14. Corollary.** *Let $G$ be a $k$-split simply connected semisimple algebraic group. Let $d$ be the number of connected components of the root system of $G$. Then we have*

$$\underline{h}_1^{\mathrm{Nis}}(G) \;\simeq\; (\underline{\mathrm{K}}_2^M)^d\,,$$

*and so in particular $\mathrm{H}^{\mathrm{S}}_1(G) \simeq \mathrm{K}_2^M(k)^d$.*

*Proof.* By Example 5.12 the $k$-variety $G$ is homological 0-connected, and so we have by the theorem above a natural isomorphism

$$\mathrm{Hom}_{\mathcal{N}\mathrm{Sh}_k^{Htr}}(\underline{h}_1^{\mathrm{Nis}}(G), \mathcal{F}) \;\simeq\; \mathrm{H}^1_{\mathrm{Nis}}(G, \mathcal{F})\,.$$

On the other hand by Theorem 5.7 we know that $\mathrm{H}^1_{\mathrm{Nis}}(G, \mathcal{F}) \simeq \mathcal{F}_{-2}(k)^d$, and by (11) that $\mathrm{Hom}_{\mathcal{N}\mathrm{Sh}_k^{Htr}}((\underline{\mathrm{K}}_2^M)^d, \mathcal{F}) \simeq \mathcal{F}_{-2}(k)^d$. Hence we have by the Yoneda lemma an isomorphism $\underline{h}_1^{\mathrm{Nis}}(G) \simeq (\underline{\mathrm{K}}_2^M)^d$. We are done □

**5.15. Remark.** If we consider a strictly homotopy invariant Nisnevich sheaf $\mathcal{F}$ without transfers then we still have $\mathrm{H}^1_{\mathrm{Zar}}(X, \mathcal{F}) \simeq \mathrm{H}^1_{\mathrm{Nis}}(X, \mathcal{F})$ for all $X \in \mathrm{Sm}_k$ by a theorem of Morel [16], but the group $\mathrm{H}^1_{\mathrm{Zar}}(G, \mathcal{F})$ depends on the type of the split simply connected group $G$. For instance, if $\mathcal{F} = \mathcal{W}$ is the Witt sheaf, *i.e.* the sheaf associated to the presheaf $U \mapsto W(U)$, where $W(U)$ denotes the Witt group of the scheme $U$, and $G$ is a split simply connected simple group then we have, see [13],

$$\mathrm{H}^1_{\mathrm{Zar}}(G,\mathcal{W}) \simeq \begin{cases} W(k) & \text{if } G \text{ is of type } \mathrm{A}_1,\ \mathrm{B}_2, \text{ and } \mathrm{C}_r\ (r \geq 2) \\ 0 & \text{otherwise.} \end{cases}$$

In particular this implies that contrary to the first Suslin (singular) homology group the first $\mathbb{A}^1$-homotopy group of $G$ as defined by Morel and Voevodsky [17] depends on the type of $G$.

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