

Higher K -theory of Koszul cubes

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Abstract

In this paper, for noetherian commutative ring with a unit A and for a certain A -regular sequence f_1, \dots, f_p , we define $\mathbf{Kos}(A; f_1, \dots, f_p)$ the category of Koszul cubes associated to f_1, \dots, f_p which has a natural Waldhausen categorical structure. We also give a new algorithm of resolution process of modules by Koszul cubes. By the virtue of the algorithm, we get the comparison theorem between the K -theory of $\mathbf{Kos}(A; f_1, \dots, f_p)$ with that of $\mathbf{Wt}(\text{Spec } A \text{ on } V(f_1, \dots, f_p))$ defined in [HM08]. The theorem is a generalization of a result in [PID]. We also define the Waldhausen category of reduced Koszul cubes associated to f_1, \dots, f_p and prove that a dévissage theorem for Koszul cubes on regular rings which is a generalization of a result in [Dev]. As their corollary, we get a weakly affirmative answer of the question in [DVR].

Introduction

In the paper [DVR, 1.6], the author asked the following question.

Question. *For any regular local ring R and non negative integer n , is $K_n(\mathcal{M}^p(R))$ generated by pure weight p objects? Here pure weight p objects mean Koszul complexes $\text{Kos}(f_1, \dots, f_p)$ where f_1, \dots, f_p forms a R -regular sequence and $\mathcal{M}^p(R)$ is the category of finitely generated R -modules M with $\text{Codim}_{\text{Spec } R} M \geq p$.*

The aim of this note, we give a weakly affirmative answer of this question. Being inspired by the works of Gillet and Soulé [GS87], of Diekert [Die86], and of Grayson [Gra92], to study purity of modules (or action of Adams operators to modules), we prefer to replace the category of certain modules with the category of cubes in the category of appropriate modules. More precisely, for a commutative noetherian ring A and for an A -regular sequence f_1, \dots, f_p satisfying a suitable condition (see 3.1), we define $\mathbf{Kos}(A; f_1^p)$ the category of Koszul cubes associated to f_1, \dots, f_p which is the full subcategory of the category of contravariant functors from $[1]^p$ to the category of finitely generated (projective) A -modules, where $[1]$ is the ordered set $\{0, 1\}$. For example, the following p -th cube is a typical example of an object in $\mathbf{Kos}(A; f_1^p)$.

Example (Example 2.3). The (*typical*) Koszul cube associated with f_1, \dots, f_p is the p -th cube in $\mathcal{M}(A)(= \mathcal{M}^0(A))$ denoted by $\text{Kos}_A(f_1, \dots, f_p)$ or shortly $\text{Kos}_A(f_1^p)$ and defined by as follows:

$$((i_1, i_2, \dots, \overset{j}{1}, \dots, i_p) \rightarrow (i_1, i_2, \dots, \overset{j}{0}, \dots, i_p)) \mapsto (A \xrightarrow{f_j} A).$$

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If we consider a p -th cube as a multi-complex, we can take its total complex. For example, the total complex of the typical Koszul cube $\mathbf{Kos}_A(\mathfrak{f}_1^p)$ is just the usual Koszul complex associated to f_1, \dots, f_p . The category $\mathbf{Kos}(A; \mathfrak{f}_1^p)$ also has natural Waldhausen categorical structure and the first main theorem is the following:

Theorem (Corollary 3.4.4.2). *There is a canonical homotopy equivalence*

$$wS_{\bullet} \mathbf{Kos}(A; \mathfrak{f}_1^p) \rightarrow iS_{\bullet} \mathbf{Wt}(\text{Spec } A \text{ on } V(f_1, \dots, f_p))$$

where $\mathbf{Wt}(\text{Spec } A \text{ on } V(f_1, \dots, f_p))$ is the exact category of Thomason-Trobaugh pure weight perfect A -Modules supported on $V(f_1, \dots, f_p)$ defined in [HM08]. (See also 3.2.)

It is a highly generalization of Theorem 0.1 in [PID]. Actually the proof of theorem above in this paper is done by extending arguments in [PID]. The other ingredient of the proof is giving a quite elementary, but new algorithm of resolution process of modules by finite direct sum of typical Koszul cubes. (See Theorem 3.4).

We also define $\mathbf{Kos}_{\text{red}}(A; \mathfrak{f}_1^p)$ the category of reduced Koszul cubes associated to f_1, \dots, f_p which is the full subcategory of $\mathbf{Kos}(A; \mathfrak{f}_1^p)$. $\mathbf{Kos}_{\text{red}}(A; \mathfrak{f}_1^p)$ also has natural Waldhausen categorical structure and the typical Koszul cube $\mathbf{Kos}_A(\mathfrak{f}_1^p)$ is contained in $\mathbf{Kos}_{\text{red}}(A; \mathfrak{f}_1^p)$. The second main theorem is the following:

Theorem (Corollary 4.2.3.3.2). *In the notation above, moreover if we assume that A is regular, then the canonical inclusion functor $\iota : \mathbf{Kos}_{\text{red}}(A; \mathfrak{f}_1^p) \hookrightarrow \mathbf{Kos}(A; \mathfrak{f}_1^p)$ induces the following homotopy equivalences*

$$\begin{aligned} iS_{\bullet} \iota : iS_{\bullet} \mathbf{Kos}_{\text{red}}(A; \mathfrak{f}_1^p) &\rightarrow iS_{\bullet} \mathbf{Kos}(A; \mathfrak{f}_1^p) \\ wS_{\bullet} \iota : wS_{\bullet} \mathbf{Kos}_{\text{red}}(A; \mathfrak{f}_1^p) &\rightarrow wS_{\bullet} \mathbf{Kos}(A; \mathfrak{f}_1^p). \end{aligned}$$

It is a highly generalization of Claim 5.2 in [Dev]. Actually the proof of theorem above in this paper is done by imitating arguments in the proof of Claim 5.2 in [Dev]. As pointed out in Ibid, since in general ι is not satisfying the dévissage conditions in [Qui73] or [Dev], we can not obtain the theorem above as a direct corollary of the dévissage theorem in [Dev]. In this paper, to prove the theorem above, we will utilize the split fibration theorem 2.7.1 which is the generalization of Lemma 3.3 in [PID].

If R is regular ring, by [DVR, Claim in the proof of Proposition 1.2] and [HM08, 6.3], we have a canonical isomorphisms

$$\begin{aligned} K(\mathbf{Wt}^p(\text{Spec } R)) &\xrightarrow{\sim} K(\mathcal{M}^p(R)), \\ K(\mathbf{Wt}^p(\text{Spec } R)) &\xrightarrow{\sim} \lim_{Y \subset \text{Spec } R} K(\mathbf{Wt}(\text{Spec } R \text{ on } Y)), \end{aligned}$$

where the taking over the regular closed immersion of codimension $\geq p$. Therefore combining the main theorems and the isomorphisms above, we could get a “weakly” affirmative answer of the Question above. That is,

$K_n(\mathcal{M}^p(R))$ is actually generated by reduced Koszul cubes associated to regular sequences.

Now we explain the meaning of the proviso “weakly”. First notice that as in the paper [DHM85], if R is not regular, in general, the Question above is false even though we replace $\mathcal{M}^p(R)$ with $\mathbf{Wt}^p(R)$ and R is a Cohen-Macaulay local ring. So in this sense, $\mathbf{Kos}(A; \mathfrak{f}_1^p)$ is

actually bigger than our tantalizing category. But we can also say that our approach also gives a reduction of the Question above to the classification of (reduced) Koszul cubes over regular local rings.

Now we give a guide for the structure of this paper. In §1, we give a resolution theorem for Waldhausen categories which is a generalization of original Quillen's one in [Qui73]. In the next section §2, we develop a theory of semi-direct products of exact categories which is initiated in [PID]. In the third section §3, by utilizing results in previous sections, we define the categories $\mathbf{Kos}(A; f_1^p)$ and $\mathbf{Kos}_{\text{red}}(A; f_1^p)$, give the algorithm of resolution process as mentioned above and as its corollary we get the first main result. In the final section §4, assuming the regularity of A , we will prove a dévissage theorem for Koszul cubes.

Conventions. Throughout this paper, we use the letter A to denote a commutative noetherian ring with a unit. For exact categories, we follow the notations in [Kel90] and for algebraic K -theory of categories with cofibrations and weak equivalences, we follow the notations in [Wal85]. We will call a category with cofibrations and weak equivalences a Waldhausen category.

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1 A resolution theorem for Waldhausen categories

(1.0) In this section, we will prove the resolution theorem for Waldhausen categories by improving the proof for exact categories in [Sta89]. This theorem is a generalization of Quillen's original one in [Qui73]. Let (\mathcal{C}, w) be a category with cofibrations and weak equivalences structure and \mathcal{D} a full subcategory of \mathcal{C} closed under extensions in \mathcal{C} . We will define the (strong) resolution conditions of inclusion functor $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ 1.3, 1.4. In this situation, \mathcal{D} inherits the natural subcategory with cofibrations and weak equivalences and if \mathcal{C} is essentially small, then the canonical map induced by ι , $wS_{\bullet} \mathcal{D} \rightarrow wS_{\bullet} \mathcal{C}$ is a homotopy equivalence 1.5. In this section, from now on, let (\mathcal{C}, w) be a category with cofibrations and weak equivalences.

(1.1) Definition (c.f. [Wal85, p.336 in the proof of 1.4.3].) Fix a non-negative m . Let us consider the full subcategory $\mathcal{C}(m, w)$ of $\mathcal{HOM}([m], \mathcal{C})$ of those functors which take values in $w(\mathcal{C})$. We can make $\mathcal{C}(m, w)$ into a category with cofibrations by defining the cofibrations to be term-wised cofibrations in \mathcal{C} .

(1.2) Notation. Let \mathcal{D} be a full subcategory of \mathcal{C} . \mathcal{D} is said to be *closed under the extensions* in \mathcal{C} if for a cofibration sequence $x \rightarrow y \rightarrow z$ in \mathcal{C} , x and z are isomorphic to objects in \mathcal{D} , then y is also. In this case, \mathcal{D} is a category with cofibrations with weak equivalences by declaring that a morphism $x \rightarrow y$ in \mathcal{D} is a cofibration in \mathcal{D} if it is a cofibration in \mathcal{C} and if y/x is isomorphic to an object in \mathcal{D} and that a morphism $x \rightarrow y$ in \mathcal{D} is a weak equivalence in \mathcal{D} if it is a weak equivalence in \mathcal{C} . In this section, from now on, let \mathcal{D} be a full subcategory of \mathcal{C} closed under the extensions.

(1.2.1) In this case, for any non-negative integer m , inclusion $\mathcal{D}(m, w) \hookrightarrow \mathcal{C}(m, w)$ is closed under extensions in $\mathcal{C}(m, w)$.

(1.2.2) In this situation, we can define a category $\mathcal{C}_{\mathcal{D}}$ as follows. The class of objects of $\mathcal{C}_{\mathcal{D}}$ is same as that of \mathcal{C} . A morphism $x \rightarrow y$ in $\mathcal{C}_{\mathcal{D}}$ is a cofibration in \mathcal{C} such that y/x is isomorphic to an object in \mathcal{D} . One can easily prove that morphisms in $\mathcal{C}_{\mathcal{D}}$ is closed under the compositions, namely it is actually a category by the virtue of the assumption 1.2.

Notice that there is a natural inclusion functor $j : \text{Cof } \mathcal{D} \rightarrow \mathcal{C}_{\mathcal{D}}$. Here $\text{Cof } \mathcal{D}$ is the category of cofibrations in \mathcal{D} .

(1.3) **Definition** An inclusion functor $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ is said to be *satisfying the resolution condition* if it satisfies the following condition.

(1.3.1) For any x in \mathcal{C} , there are objects y, z in \mathcal{D} and a cofibration sequence $x \twoheadrightarrow y \twoheadrightarrow z$.

(1.3.2) **Lemma** ([Gra87, Proof of 4.1.], [Sta89, p.524]) *If $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ is satisfying the resolution condition, then $\mathcal{C}_{\mathcal{D}}$ is contractible.*

(1.3.2.1) *Proof.* Since $\text{Cof } \mathcal{D}$ has the initial object, it is contractible. We intend to apply Quillen's Theorem A to $j : \text{Cof } \mathcal{D} \rightarrow \mathcal{C}_{\mathcal{D}}$ and then we will get the result. Fix an object a in $\mathcal{C}_{\mathcal{D}}$ and objects x and y in \mathcal{D} such that there is a cofibration sequence $a \twoheadrightarrow x \twoheadrightarrow y$. Now we will prove that a/j is contractible. To do so, consider an object $a \twoheadrightarrow b$ in a/j . Since \mathcal{D} is closed under the extensions in \mathcal{C} , by the following push out diagram

$$\begin{array}{ccccc} a & \twoheadrightarrow & x & \twoheadrightarrow & y \\ \downarrow & & \downarrow & & \parallel \\ b & \twoheadrightarrow & b \coprod_a x & \twoheadrightarrow & y \end{array}$$

where the square \star is a coCartesian, we can take $b \coprod_a x$ in \mathcal{D} . Now there are natural transformations

$$((a \twoheadrightarrow b) \mapsto (a \twoheadrightarrow b)) \twoheadrightarrow ((a \twoheadrightarrow b) \mapsto (a \twoheadrightarrow b \coprod_a x)) \leftarrow ((a \twoheadrightarrow b) \mapsto (a \twoheadrightarrow x))$$

between the identity functor and a constant functor on a/j . Therefore a/j is contractible. **Q.E.D.**

(1.4) **Definition.** An inclusion functor $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ is said to be *satisfying the strong resolution condition* if for any non-negative integer m , $\mathcal{D}(m, w) \hookrightarrow \mathcal{C}(m, w)$ satisfies the resolution condition.

(1.5) **Theorem (Resolution theorem).** *In the notation above, if \mathcal{C} is essentially small, then the canonical map induced by ι , $wS_{\bullet} \mathcal{D} \rightarrow wS_{\bullet} \mathcal{C}$ is a homotopy equivalence.*

(1.5.1) *Proof.* We may assume \mathcal{C} is small. By [Wal85, p.344, p.345 1.5.7], we have a sequence of homotopy type of a fibration

$$wS_{\bullet} \mathcal{D} \xrightarrow{wS_{\bullet} \iota} wS_{\bullet} \mathcal{C} \rightarrow wS_{\bullet} F_{\bullet}(\mathcal{C}, \mathcal{D}).$$

Fix non-negative integers n and m . We have the following identities.

$$N_m wS_n F_{\bullet}(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} f_{\bullet}(S_n \mathcal{C}(m, wS_n \mathcal{C}), S_n \mathcal{B}(m, wS_n \mathcal{D})) \xrightarrow{\sim} f_{\bullet}(S_n(\mathcal{C}(m, w)), S_n(\mathcal{D}(m, w)))$$

where f_{\bullet} denote the simplicial set of objects of F_{\bullet} and for the definition $\mathcal{C}(m, w)$ and so on see 1.1. By the realization lemma [Seg74, Appendix A] and [Wal78, 5.1], and by replacing $\mathcal{C}(m, w)$ and $\mathcal{D}(m, w)$ with \mathcal{C} and \mathcal{D} respectively, we may just check the following claim.

(1.5.1.1) Claim. For a small category with cofibrations \mathcal{C} and $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ a full sub category closed under extensions. Assume that ι is satisfying the resolution conditions. Then for each non-negative integer n , $f_\bullet(S_n \mathcal{C}, S_n \mathcal{D})$ is contractible.

If $n = 0$, this claim is trivial. By the additivity theorem for f -construction in [Dev, Appendix], we may assume $n = 1$. Now $f_\bullet(\mathcal{C}, \mathcal{D})$ is just a nerve of $\mathcal{C}_{\mathcal{D}}$ in 1.2.2 and therefore we get the result by 1.3.2. **Q.E.D.**

2 Semi-direct products of exact categories

(2.0) In this section, we will establish the theory of semi-direct products of exact categories (with weak equivalences) which is a generalization in [PID, §3].

(2.1) Notation. For a category \mathcal{C} and non-negative integer n , we put

$$\mathbf{Cub}^n(\mathcal{C}) := \mathcal{HOM}([1]^n)^{\text{op}}, \mathcal{C}$$

and call it *the category of n -cubes in \mathcal{C}* .

(2.1.1) For any abelian (resp. exact) category \mathcal{C} and non-negative integer n , $\mathbf{Cub}^n(\mathcal{C})$ is an abelian category (resp. exact category by defining the conflations to be term-wised conflations in \mathcal{C}).

(2.1.2) By the exponential law, we have an identity

$$\mathbf{Cub}^{n+m}(\mathcal{C}) \xrightarrow{\sim} \mathbf{Cub}^n(\mathbf{Cub}^m(\mathcal{C})).$$

Moreover if \mathcal{C} is an abelian (resp. exact) category, then the isomorphism above is an exact functor.

(2.2) Let n be a positive integer. For an n -th cube y in an abelian category \mathcal{A} and an integer $0 \leq j \leq n$, we define the $(n-1)$ -th cube $H_j^- y$ by the formula

$$(H_j^- y)_{(i_0, \dots, i_{n-1})} := \text{Coker}(y_{(i_0, \dots, \overset{j}{0}, \dots, i_{n-1})} \rightarrow y_{(i_0, \dots, \overset{j}{1}, \dots, i_{n-1})}).$$

We call $H_j^- y$ *j -direction homology of y* .

(2.3) Example (Koszul cubes). Let f_1, \dots, f_r be elements in A . The (*typical*) Koszul cube associated with f_1, \dots, f_r is the r -th cube in $\mathcal{M}(A)$ denoted by $\text{Kos}_A(f_1, \dots, f_r)$ or shortly $\text{Kos}_A(f_1^r)$ and defined by as follows:

$$((i_1, i_2, \dots, \overset{j}{1}, \dots, i_r) \rightarrow (i_1, i_2, \dots, \overset{j}{0}, \dots, i_r)) \mapsto (A \xrightarrow{f_j} A).$$

(2.4) The following lemma is often used when we are treating with cubes. Its proof is very easy.

(2.4.1) **Lemma (Cube lemma).** *For the diagram in the category \mathcal{C} :*

$$\begin{array}{ccccc}
 a & \xrightarrow{\quad} & & \xrightarrow{\quad} & b \\
 & \swarrow & & \searrow & \\
 & & x & \xrightarrow{\quad} & y \\
 & & \downarrow & & \downarrow \\
 & & z & \xrightarrow{\quad} & w \\
 & \swarrow & & \searrow & \\
 c & \xrightarrow{\quad} & & \xrightarrow{\quad} & d
 \end{array} ,$$

assume that the morphism \vec{wd} is a monomorphism (resp. \vec{xa} is an epimorphism) and every squares except $xywz$ (resp. $abdc$) are commutative. Then $xywz$ (resp. $abdc$) is also commutative.

(2.5) Let \mathcal{A} be an abelian category and \mathcal{E}, \mathcal{F} full sub exact categories of \mathcal{A} such that inclusion functors $\mathcal{E} \hookrightarrow \mathcal{A}$ and $\mathcal{F} \hookrightarrow \mathcal{A}$ are exact and reflect exactness. We want to define the category $\mathcal{F} \times \mathcal{E}$ as follows. $\mathcal{F} \times \mathcal{E}$ is the full subcategory those of complexes $x = [x_1 \xrightarrow{d^x} x_0]$ in $\mathbf{Cub}^1(\mathcal{E})$ such that its boundary morphism d^x is a monomorphism in \mathcal{A} and $\text{Coker } d^x$ is in \mathcal{F} . If \mathcal{F} satisfies a suitable condition, then we can make $\mathcal{F} \times \mathcal{E}$ into an exact category in the natural way. More precisely we have the following proposition.

(2.5.1) **Proposition.** *In the notations above, if \mathcal{F} satisfies the following condition 2.5.1.1 or 2.5.1.2, then $\mathcal{F} \times \mathcal{E}$ is an exact category such that inclusion functor $\iota : \mathcal{F} \times \mathcal{E} \hookrightarrow \mathbf{Cub}^1(\mathcal{A})$ is exact and reflects exactness. Moreover $H_0 : \mathcal{F} \times \mathcal{E} \rightarrow \mathcal{F}$ is exact. Let*

$$a \twoheadrightarrow b \twoheadrightarrow c$$

be an exact sequence in \mathcal{A} .

(2.5.1.1) \mathcal{F} is closed under extensions, that is, if a and c are isomorphic to objects in \mathcal{F} respectively, then b is also.

(2.5.1.2) \mathcal{F} is closed under admissible sub and quotient objects, that is, if b is isomorphic to an object in \mathcal{F} , then a and c are also.

(2.5.1.3) *Proof.* We may assume that \mathcal{E} and \mathcal{F} are closed under the isomorphisms in \mathcal{A} . That is, if a is an object in \mathcal{A} which is isomorphic to an object in \mathcal{E} (resp. \mathcal{F}), then a is also in \mathcal{E} (resp. \mathcal{F}). We declare that a sequence $x \rightarrow y \rightarrow z$ in $\mathcal{F} \times \mathcal{E}$ is a conflation if it is exact in $\mathbf{Cub}^1(\mathcal{A})$. Obviously a split short exact sequence in $\mathcal{F} \times \mathcal{E}$ is a conflation. We intend to prove that the class of inflations (resp. deflations) is closed under the composition and co-base change (resp. base change) along arbitrary morphisms. We just check for the inflations case. To prove for the deflations case is similar. For a pair of composable inflations $x \twoheadrightarrow y \twoheadrightarrow z$, by snake lemma in $\mathbf{Cub}^1(\mathcal{A})$, we have a short exact sequence

$$y/x \twoheadrightarrow z/x \twoheadrightarrow z/y$$

in $\mathbf{Cub}^1(\mathcal{A})$. Applying the snake lemma to the following diagram

$$\begin{array}{ccccc} (y/x)_1 & \xrightarrow{\quad} & (z/x)_1 & \twoheadrightarrow & (z/y)_1 \\ d^{y/x} \downarrow & & \downarrow d^{z/x} & & \downarrow d^{z/y} \\ (y/x)_0 & \xrightarrow{\quad} & (z/x)_0 & \twoheadrightarrow & (z/y)_0 \end{array} ,$$

we learn that $d^{z/x}$ is a monomorphism in \mathcal{A} . Considering the following two short exact sequences in \mathcal{A}

$$\begin{aligned} \mathbf{H}_0(y/x) &\twoheadrightarrow \mathbf{H}_0(z/x) \twoheadrightarrow \mathbf{H}_0(z/y), \\ \mathbf{H}_0(x) &\twoheadrightarrow \mathbf{H}_0(z) \twoheadrightarrow \mathbf{H}_0(z/x), \end{aligned}$$

we notice that $\mathbf{H}_0(z/x)$ is actually in \mathcal{F} . Therefore the short exact sequence

$$x \twoheadrightarrow z \twoheadrightarrow z/x$$

is a conflation in $\mathcal{F} \times \mathcal{E}$. Hence the class of inflations are closed under the composition. Next let us consider morphisms $y \leftarrow x \twoheadrightarrow z$ in $\mathcal{F} \times \mathcal{E}$. Consider the coproduct $y \coprod_x z$ in $\mathbf{Cub}^1(\mathcal{A})$. Then since inflations in \mathcal{E} is closed under the co-base change by arbitrary morphisms, $(y \coprod_x z)_i$ is in \mathcal{E} for $i = 0, 1$. Applying the snake lemma to the following diagram

$$\begin{array}{ccccc} y_1 & \xrightarrow{\quad} & (y \coprod_x z)_1 & \twoheadrightarrow & (z/x)_1 \\ d^y \downarrow & & \downarrow d^{y \coprod_x z} & & \downarrow d^{z/x} \\ y_0 & \xrightarrow{\quad} & (y \coprod_x z)_0 & \twoheadrightarrow & (z/x)_0 \end{array} ,$$

we learn that $d^{y \coprod_x z}$ is a monomorphism in \mathcal{A} and get the following diagram

$$\begin{array}{ccccc} \mathbf{H}_0(x) & \xrightarrow{\quad} & \mathbf{H}_0(z) & \twoheadrightarrow & \mathbf{H}_0(z/x) \\ \downarrow & \star & \downarrow & & \parallel \\ \mathbf{H}(y) & \xrightarrow{\quad} & \mathbf{H}_0(y \coprod_x z) & \twoheadrightarrow & \mathbf{H}_0(z/x) \end{array}$$

where the horizontal lines are short exact sequences in \mathcal{A} . Thus \star is a coCartesian square and since the class of inflations in \mathcal{F} is closed under co-base change by arbitrary morphisms, $\mathbf{H}_0(y \coprod_x z)$ is in \mathcal{F} . We conclude that $y \coprod_x z$ is in $\mathcal{F} \times \mathcal{E}$ and

$$y \twoheadrightarrow y \coprod_x z \twoheadrightarrow z/x$$

is a conflation in $\mathcal{F} \times \mathcal{E}$. **Q.E.D.**

(2.5.2) In the situation 2.5.1, moreover assume that \mathcal{F} has the subcategory $w\mathcal{F}$ containing the class of isomorphisms in \mathcal{F} . We define the subcategory $w(\mathcal{F} \times \mathcal{E})$ of $\mathcal{F} \times \mathcal{E}$ as follows. A morphism $f : x \rightarrow y$ in $\mathcal{F} \times \mathcal{E}$ is in $w(\mathcal{F} \times \mathcal{E})$ if and only if $\mathbf{H}_0(f) : \mathbf{H}_0(x) \rightarrow \mathbf{H}_0(y)$ is in $w\mathcal{F}$. Then the isomorphisms in $\mathcal{F} \times \mathcal{E}$ is in $w(\mathcal{F} \times \mathcal{E})$. Moreover we can easily get the following proposition.

(2.5.2.1) Proposition. *In the notation above, if $w(\mathcal{F})$ satisfies the gluing (resp. cogluing, saturational, extensional) axiom, then $w(\mathcal{F} \times \mathcal{E})$ is also. In particular if (\mathcal{F}, w) is an exact category with weak equivalences, then $(\mathcal{F} \times \mathcal{E}, w(\mathcal{F} \times \mathcal{E}))$ is also and the functor*

$$\mathbf{H}_0 : (\mathcal{F} \times \mathcal{E}, w(\mathcal{F} \times \mathcal{E})) \rightarrow (\mathcal{F}, w)$$

is a morphism of Waldhausen categories.

(2.5.2.2) Proposition. *In the notation 2.5.2.1, moreover assume that (\mathcal{F}, w) is a Waldhausen category and \mathcal{F} is contained in \mathcal{E} , then the functor H_0 induces a homotopy equivalence*

$$wS_{\bullet}H_0 : wS_{\bullet}\mathcal{F} \times \mathcal{E} \rightarrow wS_{\bullet}\mathcal{F}.$$

(2.5.2.2.1) Proof. Define a morphism of Waldhausen categories $s : (\mathcal{F}, w) \rightarrow (\mathcal{F} \times \mathcal{E}, w)$ by $x \mapsto [0 \rightarrow x]$. Then obviously we have the equality $H_0 \circ s = \text{id}$. Moreover there is a natural weak equivalence $\text{id} \rightarrow s \circ H_0$ defined by the canonical morphisms

$$\begin{array}{ccc} \left[\begin{array}{c} x_1 \\ d^x \downarrow \\ x_0 \end{array} \right] & \rightarrow & \left[\begin{array}{c} 0 \\ \downarrow \\ H_0(x) \end{array} \right] \end{array}$$

for any object x in $\mathcal{F} \times \mathcal{E}$. Therefore by [Wal85, p.330 1.3.1], we learn that $wS_{\bullet}H_0$ is a homotopy equivalence. **Q.E.D.**

(2.5.3) Next let \mathcal{E}_i and \mathcal{F}_i ($i = 1, 2$) be full sub exact categories of \mathcal{A} such that inclusion functors $\mathcal{E}_i \hookrightarrow \mathcal{A}$ and $\mathcal{F}_i \hookrightarrow \mathcal{A}$ are exact and reflect exactness. Moreover we assume that inclusion functors $\mathcal{E}_1 \hookrightarrow \mathcal{E}_2$ and $\mathcal{F}_1 \hookrightarrow \mathcal{F}_2$ are closed under the extensions. Moreover we suppose that \mathcal{F}_i satisfies a condition 2.5.1.1 or 2.5.1.2. Then by 2.5.1, $\mathcal{F}_i \times \mathcal{E}_i$ is an exact category. Some closed conditions are preserved by semi-direct products construction. More precisely we have the following.

(2.5.3.1) *Assume that inclusion functors $\mathcal{E}_1 \hookrightarrow \mathcal{E}_2$ and $\mathcal{F}_1 \hookrightarrow \mathcal{F}_2$ are closed under the extensions. That is, for a conflation*

$$x \twoheadrightarrow y \twoheadrightarrow z$$

in \mathcal{E}_2 (resp. \mathcal{F}_2) if x and z are isomorphic to objects in \mathcal{E}_1 (resp. \mathcal{F}_1), then y is also. Then $\mathcal{F}_1 \times \mathcal{E}_1 \hookrightarrow \mathcal{F}_2 \times \mathcal{E}_2$ is also closed under the extensions.

(2.5.3.2) *Assume that inclusion functors $\mathcal{E}_1 \hookrightarrow \mathcal{E}_2$ and $\mathcal{F}_1 \hookrightarrow \mathcal{F}_2$ are closed under taking the kernel. That is, for a short exact sequence*

$$x \twoheadrightarrow y \twoheadrightarrow z$$

in \mathcal{A} if z is isomorphic to an object in \mathcal{E}_2 (resp. \mathcal{F}_2) and y is isomorphic to an object in \mathcal{E}_1 (resp. \mathcal{F}_1), then x is also isomorphic to an object in \mathcal{E}_1 (resp. \mathcal{F}_1). Then $\mathcal{F}_1 \times \mathcal{E}_1 \hookrightarrow \mathcal{F}_2 \times \mathcal{E}_2$ is also closed under taking the kernel.

(2.5.3.3) Proof of 2.5.3.1 and 2.5.3.2. Let us consider a short exact sequence in $\mathbf{Cub}^1(\mathcal{A})$

$$\begin{array}{ccccc} x_1 & \twoheadrightarrow & y_1 & \twoheadrightarrow & z_1 \\ d^x \downarrow & & \downarrow d^y & & \downarrow d^z \\ x_0 & \twoheadrightarrow & y_1 & \twoheadrightarrow & z_1 \end{array} .$$

If d^z and d^x (resp. d^y) are monomorphisms, then d^y (resp. d^x) is also. In this case, observing the 3×3 commutative diagram below,

$$\begin{array}{ccccc}
x_1 & \xrightarrow{\quad} & y_1 & \longrightarrow & z_1 \\
d^x \downarrow & & \downarrow d^y & & \downarrow d^z \\
x_0 & \xrightarrow{\quad} & y_1 & \longrightarrow & z_1 \\
\downarrow & & \downarrow & & \downarrow \\
\Downarrow & & \Downarrow & & \Downarrow \\
H_0(x) & \xrightarrow{\quad} & H_0(y) & \longrightarrow & H_0(z)
\end{array} ,$$

we learn that if the condition 2.5.3.1 (resp. 2.5.3.2) is verified and if z is isomorphic to an object in $\mathcal{F}_1 \times \mathcal{E}_1$ (resp. $\mathcal{F}_2 \times \mathcal{E}_2$) and if x (resp. y) is isomorphic to an object in $\mathcal{F}_1 \times \mathcal{E}_1$, then y (resp. x) is also isomorphic to an object in $\mathcal{F}_1 \times \mathcal{E}_1$. **Q.E.D.**

(2.5.3.3.1) Remarks. 2.5.3.1, 2.5.3.2 and dual of 2.5.3.2 imply the following statements.:

(2.5.3.3.1.1) Assume that inclusion functors $\mathcal{E}_1 \hookrightarrow \mathcal{E}_2$ and $\mathcal{F}_1 \hookrightarrow \mathcal{F}_2$ are closed under admissible sub and quotients, then $\mathcal{E}_1 \times \mathcal{F}_1$ is also closed in $\mathcal{E}_2 \times \mathcal{F}_2$.

(2.5.3.3.1.2) Let \mathcal{A} be an abelian category and \mathcal{E}, \mathcal{F} full subcategory of \mathcal{A} closed under the extensions in \mathcal{A} . Then $\mathcal{F} \times \mathcal{E}$ is closed under the extensions in $\mathbf{Cub}^1(\mathcal{A})$.

(2.6) Theorem. Let \mathcal{A} be an abelian category, $\mathcal{E}_1 \hookrightarrow \mathcal{E}_2$ be a fully faithful exact functor between full sub exact categories of \mathcal{A} and \mathcal{F} a full sub exact category of \mathcal{A} . Moreover let w be a class of weak equivalences of \mathcal{F} satisfying the cogluing axiom. Assume that the following conditions:

(2.6.1) \mathcal{E}_1 is closed under the extensions in \mathcal{E}_2 .

(2.6.2) Let

$$x \twoheadrightarrow y \twoheadrightarrow z \tag{2.6.2.1}$$

be a conflation in \mathcal{A} . Assume that y is isomorphic to an object in \mathcal{E}_1 and z is isomorphic to an object in \mathcal{E}_2 or \mathcal{F} . Then x is isomorphic to an object in \mathcal{E}_1 .

(2.6.3) For any non-negative integer m and an object z in $\mathcal{HOM}([m], \mathcal{E}_2)$, there is an object y in $\mathcal{HOM}([m], \mathcal{E}_1)$ and deflation $y \twoheadrightarrow z$.

Then the canonical exact functor $\mathcal{F} \times \mathcal{E}_1 \hookrightarrow \mathcal{F} \times \mathcal{E}_2$ satisfies dual of the strong resolution condition 1.4. In particular we have the homotopy equivalence

$$wS_{\bullet} \mathcal{F} \times \mathcal{E}_1 \xrightarrow{\sim} wS_{\bullet} \mathcal{F} \times \mathcal{E}_2.$$

(2.6.4) Proof. Let us fix a non-negative integer m . We intend to prove that

$$\mathcal{F} \times \mathcal{E}_1(m, w) \hookrightarrow \mathcal{F} \times \mathcal{E}_2(m, w)$$

satisfies dual of the resolution condition 1.3. For each x in $\mathcal{F} \times \mathcal{E}_2(m, w)$, by 2.6.3, we have an object y_0 in $\mathcal{HOM}([m], \mathcal{E}_1)$ and a deflation $y_0 \twoheadrightarrow x_0$. Then for each $0 \leq i \leq m$, we put

$$\begin{aligned}
y_1(i) &:= \text{Ker}(y_1(i) \twoheadrightarrow x_0(i) \twoheadrightarrow H_0 x(i)), \text{ and} \\
z_1(i) &:= z_0(i) := \text{Ker}(y_0(i) \twoheadrightarrow x_0(i)).
\end{aligned}$$

We have the following commutative diagram:

$$\begin{array}{ccccc}
z_1(i) & \xrightarrow{\gamma} & y_1(i) & \longrightarrow & x_1(i) \\
\parallel & & \downarrow & & \downarrow \\
z_0(i) & \xrightarrow{\gamma} & y_0(i) & \longrightarrow & x_0(i) \\
& & \downarrow & & \downarrow \\
& & \Downarrow & & \Downarrow \\
& & H_0 x(i) & \equiv & H_0 x(i) .
\end{array}$$

By 2.6.2, we have y and z are in $\mathcal{HOM}([m], \mathcal{F} \times \mathcal{E}_1)$. Since $y \rightarrow x$ is a quasi-isomorphism and all $z(i)$ is an acyclic complex for each $0 \leq i \leq m$, we learn that y and z are in $\mathcal{F} \times \mathcal{E}_1(m, w)$. Therefore we get the result. **Q.E.D.**

(2.6.5) Corollary (Abstract weight declension theorem). *In the notation 2.6, moreover we assume that \mathcal{F} is a full sub exact category of \mathcal{E}_2 . Then the exact functor*

$$H_0 : (\mathcal{F} \times \mathcal{E}_1, w) \rightarrow (\mathcal{F}, w)$$

induces a homotopy equivalence

$$wS_{\bullet} H_0 : wS_{\bullet} \mathcal{F} \times \mathcal{E}_1 \rightarrow wS_{\bullet} \mathcal{F} .$$

(2.6.5.1) Proof. We have the factorization

$$wS_{\bullet} H_0 : wS_{\bullet} \mathcal{F} \times \mathcal{E}_1 \xrightarrow{I} wS_{\bullet} \mathcal{F} \times \mathcal{E}_2 \xrightarrow{II} wS_{\bullet} \mathcal{F} .$$

where the maps I and II are homotopy equivalences by 2.6 and 2.5.2.2 respectively. **Q.E.D.**

(2.7) Let \mathcal{A} be an abelian category and $\mathcal{E} \hookrightarrow \mathcal{F} \hookrightarrow \mathcal{A}$ full sub exact categories such that inclusion functors $\mathcal{E} \hookrightarrow \mathcal{F}$ and $\mathcal{F} \hookrightarrow \mathcal{A}$ are exact and reflect exactness. Assume that \mathcal{F} is satisfying a condition 2.5.1.1 or 2.5.1.2. Moreover assume that \mathcal{F} has a class of weak equivalences $w\mathcal{F}$ which is satisfying the axiom of weak equivalences in [Wal85]. We put $v := w\mathcal{F} \cap \mathcal{E}$ and it is a class of weak equivalences in \mathcal{E} . We define a new class of weak equivalences $lv(\mathcal{F} \times \mathcal{E})$ in $\mathcal{F} \times \mathcal{E}$ as follows. A morphism $f : x \rightarrow y$ in $\mathcal{F} \times \mathcal{E}$ is in $lv(\mathcal{F} \times \mathcal{E})$ if and only if $f_i : x_i \rightarrow y_i$ for $i = 0, 1$ is in $v\mathcal{E}$. We will call a morphism in $lv(\mathcal{F} \times \mathcal{E})$ a *level weak equivalence*. We can easily check that $(\mathcal{F} \times \mathcal{E}, lv)$ is a Waldhausen category.

(2.7.1) Proposition (Split fibration theorem). *In the notation above, we have a sequence of homotopy type of of a split fibration*

$$vS_{\bullet} \mathcal{E} \rightarrow lvS_{\bullet} \mathcal{F} \times \mathcal{E} \rightarrow wS_{\bullet} \mathcal{F} .$$

(2.7.1.1) Proof. Let us denote the category of acyclic complexes in $\mathcal{F} \times \mathcal{E}$ by $(\mathcal{F} \times \mathcal{E})^q$. Since $(\mathcal{F} \times \mathcal{E})^q$ is closed under extensions in $\mathcal{F} \times \mathcal{E}$, it has natural Waldhausen categorical structure and obviously the association $x \mapsto [x \xrightarrow{\text{id}_x} x]$ gives a equivalence as Waldhausen categories between \mathcal{E} and $(\mathcal{F} \times \mathcal{E})^q$. On the other hand, there is a category equivalence as Waldhausen categories

$$\mathcal{F} \times \mathcal{E} \xrightarrow{\sim} E((\mathcal{F} \times \mathcal{E})^q, \mathcal{F} \times \mathcal{E}, \mathcal{F}) \quad ,$$

$$x \mapsto \left(\begin{array}{ccc} \left[\begin{array}{c} x_1 \\ \text{id}_{x_1} \downarrow \\ x_1 \end{array} \right] & \xrightarrow{\text{id}_{x_1}} & \left[\begin{array}{c} x_1 \\ d^x \downarrow \\ x_0 \end{array} \right] & \rightarrow & \left[\begin{array}{c} 0 \\ \downarrow \\ \text{H}_0(x) \end{array} \right] \\ & \rightarrow & & \rightarrow & \end{array} \right).$$

where $\text{E}((\mathcal{F} \times \mathcal{E})^q, \mathcal{F} \times \mathcal{E}, \mathcal{F})$ is the exact category of conflations $x \rightarrow y \rightarrow z$ in $\mathcal{F} \times \mathcal{E}$ such that x is in $(\mathcal{F} \times \mathcal{E})^q$ and z is in \mathcal{F} . Moreover $\text{E}((\mathcal{F} \times \mathcal{E})^q, \mathcal{F} \times \mathcal{E}, \mathcal{F})$ has a natural class of weak equivalences lv induced from $\mathcal{F} \times \mathcal{E}$. Hence by the additivity theorem, we get the result. **Q.E.D.**

3 Koszul cubes

(3.1) Notations. From now on, as A -sequence f_1, \dots, f_q , we mean an A -regular sequence satisfies the following condition:

For each $0 \leq r \leq q$, $A/(f_1, f_2, \dots, f_r)A$ is separated for (f_1, \dots, f_q) -adic topology.

The condition above is verified for the following each cases:

(3.1.1) f_1, \dots, f_q are contained in the Jacobson radical of A .

(3.1.2) A is a graded ring and all f_i are homogeneous of positive degree.

In this case, for any $\sigma \in \mathfrak{S}_q$, $f_{\sigma(1)}, f_{\sigma(2)}, \dots, f_{\sigma(q)}$ is an A -regular sequence, where \mathfrak{S}_q is the q -th symmetric group.

(3.2) Definition. Let us fix an A -sequence $f_1, \dots, f_q, \dots, f_{q+k}$ and an integer $0 \leq p \leq \infty$.

(3.2.1) Let us denote the category of finitely generated A -modules M such that $\text{Projdim}_A M \leq p$ and $\text{Supp } M \subset V(f_1, \dots, f_q)$ by $\mathcal{M}(A \text{ on } (f_1, \dots, f_q))(p)$ or shortly $\mathcal{M}(A \text{ on } \mathfrak{f}_1^q)(p)$. Since this category is closed under the extensions in $\mathcal{M}(A)$, it can be considered as an exact category in the natural way.

(3.2.2) An A -module M in $\mathcal{M}(A \text{ on } \mathfrak{f}_1^q)(p)$ is said to be *reduced* if $f_i M = 0$ for each $1 \leq i \leq q$. We will write the full subcategory of reduced modules in $\mathcal{M}(A \text{ on } \mathfrak{f}_1^q)(p)$ by $\mathcal{M}_{\text{red}}(A \text{ on } \mathfrak{f}_1^q)(p)$. Since obviously $\mathcal{M}_{\text{red}}(A \text{ on } \mathfrak{f}_1^q)(p)$ is closed under taking sub and quotients, applying Lemma 3.2.7 below, we learn that $\mathcal{M}_{\text{red}}(A \text{ on } \mathfrak{f}_1^q)(p)$ can be considered as an exact category.

(3.2.3) We put $\mathcal{M}_?(A \text{ on } \mathfrak{f}_1^q) := \mathcal{M}_?(A \text{ on } \mathfrak{f}_1^q)(\infty)$ and $\mathbf{Wt}_?(A \text{ on } \mathfrak{f}_1^q) := \mathcal{M}_?(A \text{ on } \mathfrak{f}_1^q)(q)$ where $? = \emptyset$ or red . A module in $\mathbf{Wt}(A \text{ on } \mathfrak{f}_1^q)$ (resp. $\mathbf{Wt}_{\text{red}}(A \text{ on } \mathfrak{f}_1^q)$) is said to be a *pure weight module* (resp. *reduced pure weight module*) supported on $V(\mathfrak{f}_1^q)$ (compare [HM08]).

(3.2.4) For $? = \emptyset$ or red , inductively on k , we define the category $\mathcal{M}_?(A \text{ on } \mathfrak{f}_1^q; \mathfrak{f}_{q+1}^{q+k})(p)$ which is the full subcategory of $\mathbf{Cub}^k(\mathcal{M}(A))$ as follows. First for $k = 1$, we put

$$\mathcal{M}_?(A \text{ on } \mathfrak{f}_1^q; \mathfrak{f}_{q+1}^{q+1})(p) := \mathcal{M}_?(A \text{ on } \mathfrak{f}_1^{q+1})(p+1) \times \mathcal{M}_?(A \text{ on } \mathfrak{f}_1^q)(p).$$

For $k \geq 2$, we put

$$\mathcal{M}_?(A \text{ on } \mathfrak{f}_1^q; \mathfrak{f}_{q+1}^{q+k})(p) := \mathcal{M}_?(A \text{ on } \mathfrak{f}_1^{q+1}; \mathfrak{f}_{q+2}^{q+k})(p+1) \times \mathcal{M}_?(A \text{ on } \mathfrak{f}_1^q; \mathfrak{f}_{q+2}^{q+k})(p).$$

By the virtue of 2.5.3.3.1.2, we inductively learn that $\mathcal{M}(A \text{ on } \mathfrak{f}_1^q; \mathfrak{f}_{q+1}^{q+k})(p)$ is closed under the extensions in $\mathbf{Cub}^k(\mathcal{M}(A))$ and therefore it has a natural exact categorical structure.

Next by the virtue of 2.5.3.3.1.1, we inductively learn that $\mathcal{M}_{\text{red}}(A \text{ on } \mathfrak{f}_1^q; \mathfrak{f}_{q+1}^{q+k})(p)$ is closed under taking admissible sub, quotient and finite direct sums in $\mathcal{M}(A \text{ on } \mathfrak{f}_1^q; \mathfrak{f}_{q+1}^{q+k})(p)$ and therefore by 3.2.7, it has a natural exact categorical structure.

For $? = \emptyset$ or red, we put $\mathbf{Wt}_?(A \text{ on } \mathfrak{f}_1^q; \mathfrak{f}_{q+1}^{q+k}) := \mathcal{M}_?(A \text{ on } \mathfrak{f}_1^q; \mathfrak{f}_{q+1}^{q+k})(q)$.

(3.2.5) Moreover we put $\mathbf{Kos}(A; \mathfrak{f}_1^q) := \mathbf{Wt}(A; \mathfrak{f}_1^q)$ (resp. $\mathbf{Kos}_{\text{red}}(A; \mathfrak{f}_1^q) := \mathbf{Wt}_{\text{red}}(A; \mathfrak{f}_1^q)$) and call it *the exact category of* (resp. *reduced*) *Koszul q -th cubes associated with an A -sequence \mathfrak{f}_1^q* . For $? = \emptyset$ or red, $\mathbf{Kos}_?(A; \mathfrak{f}_1^q)$ also has the class of weak equivalences induced from the class of isomorphisms in $\mathbf{Wt}_?(A \text{ on } \mathfrak{f}_1^q)$ by 2.5.2 inductively.

(3.2.6) Remark. The definition above is compatible with that in [HM08]. To prove this we shall notice that torsion and projective dimension are coincided for a finitely generated modules over a noetherian ring (see [Wei94, Proposition 4.1.5]) and that in the notation above, $\text{Spec } A/\mathfrak{f}_1^q \hookrightarrow \text{Spec } A$ is a regular closed immersion.

(3.2.7) Lemma. *Let \mathcal{E} be an exact category and \mathcal{F} its full subcategory which satisfies the following two conditions.*

(3.2.7.1) \mathcal{F} is closed under taking finite direct sum. In particular \mathcal{F} has a zero object 0.

(3.2.7.2) \mathcal{F} is closed under taking admissible sub and quotient objects in \mathcal{E} . That is, for a conflation in \mathcal{E}

$$x \twoheadrightarrow y \twoheadrightarrow z,$$

if y is isomorphic to an object in \mathcal{F} , then x and z are also.

Then we can consider \mathcal{F} as an exact category by declaring that a sequence $x \twoheadrightarrow y \twoheadrightarrow z$ is a conflation in \mathcal{F} if and only if it is in \mathcal{E} .

(3.2.7.3) Proof. For simplicity, we may suppose that \mathcal{F} is closed under isomorphisms in \mathcal{E} . Namely for an object x in \mathcal{F} and an object y in \mathcal{E} , if $x \xrightarrow{\sim} y$, then y is also in \mathcal{F} . We may just check that the class of inflations (resp. deflations) in \mathcal{F} is closed under composition and co-base (resp. base) change by arbitrary morphisms. We only check for the inflation case. For the deflation case, almost same arguments are working.

Let $x \twoheadrightarrow y$ and $y \twoheadrightarrow z$ be inflations in \mathcal{F} . By 3.2.7.2, z/x can be taken in \mathcal{F} . Therefore the sequence

$$x \twoheadrightarrow z \twoheadrightarrow z/x$$

is a conflation in \mathcal{E} and the composition $x \twoheadrightarrow z$ is an inflation in \mathcal{F} .

Next consider the following commutative diagram in \mathcal{E} :

$$\begin{array}{ccccc} x & \twoheadrightarrow & y & \twoheadrightarrow & z \\ \downarrow & & \star & & \downarrow \\ x' & \twoheadrightarrow & y' & \twoheadrightarrow & z \end{array} \quad \begin{array}{c} \parallel \\ \parallel \\ \parallel \end{array}$$

where the square \star is coCartesian and we assume that x, y, z and x' are in \mathcal{F} . Then by [Kel90, p.406 step 1], we have a conflation

$$x \twoheadrightarrow x' \oplus y \twoheadrightarrow y'.$$

By 3.2.7.1, $x' \oplus y$ is in \mathcal{F} and by 3.2.7.2, y' is also in \mathcal{F} . **Q.E.D.**

(3.3) Notice that for a short exact sequence of A -modules

$$0 \rightarrow N \rightarrow N' \rightarrow N'' \rightarrow 0, \quad (3.3.1)$$

we can easily prove the following assertions by utilizing Ext-criterion of projective dimensions. (see [Wei94, pd Lemma 4.1.6])

(3.3.2) If $\text{Projdim}_A N'' \leq n+1$ and $\text{Projdim}_A N' \leq n$, then $\text{Projdim}_A N \leq n$.

(3.3.3) If $\text{Projdim}_A N \leq n$ and $\text{Projdim}_A N'' \leq n$, then $\text{Projdim}_A N' \leq n$.

(3.3.3.1) Proposition. *For an A -sequence f_1, \dots, f_r and any integer $r \leq n < \infty$ and $? = \emptyset$ or red, $\mathcal{M}_?(A \text{ on } \mathfrak{f}_1^r)(n)$ is closed under extensions and taking kernels in $\mathcal{M}_?(A \text{ on } \mathfrak{f}_1^r)(n+1)$. Moreover the inclusion functor $\mathcal{M}_?(A \text{ on } \mathfrak{f}_1^r)(n) \hookrightarrow \mathcal{M}_?(A \text{ on } \mathfrak{f}_1^r)(n+1)$ is satisfying dual of the resolution condition 1.3.1. In particular, we have a homotopy equivalences*

$$iS_\bullet \mathbf{Wt}_?(A \text{ on } \mathfrak{f}_1^r) \rightarrow iS_\bullet \mathcal{M}_?(A \text{ on } \mathfrak{f}_1^r)(n).$$

(3.3.3.1.1) Proof. To check closed conditions, we may only notice that projective dimension conditions and they are easily checked by 3.3.2 and 3.3.3. For any $M \in \mathcal{M}(A \text{ on } \mathfrak{f}_1^r)(n)$, there are positive integers t_s such that $f_s^{t_s} M = 0$ for any $1 \leq s \leq r$. We put $g_s := f_s^{t_s}$ and $B := A/(g_1, \dots, g_r)$. Therefore we can consider M as a finitely generated B -module and there is a surjection from finitely generated free B -module N to M . Since $\text{Projdim}_A N = r$, we get the result by 3.3.2 for $? = \emptyset$. If M is reduced, then we can take $t_s = 1$. Now recalling the remark in 3.2.2, we also get the result for $? = \text{red}$. **Q.E.D.**

(3.3.3.1.2) Corollary. *For an A -sequence f_1, \dots, f_r and any integer $1 \leq s \leq r$ and $? = \emptyset$ or red, $\mathbf{Wt}_?(A \text{ on } \mathfrak{f}_1^s; \mathfrak{f}_{s+1}^r)$ is closed under the extensions and the kernels in $\mathcal{M}_?(A \text{ on } \mathfrak{f}_1^s; \mathfrak{f}_{s+1}^r)(s+1)$.*

(3.3.3.1.2.1) Proof. By the virtue of 2.5.3 and induction, we may only check that $\mathbf{Wt}_?(A \text{ on } \mathfrak{f}_1^s)$ is closed under the extensions and the kernels in $\mathcal{M}_?(A \text{ on } \mathfrak{f}_1^s)(s+1)$ and it were done in 3.3.3.1. **Q.E.D.**

(3.4) Theorem (Koszul resolution theorem). *Let n be non-negative integers and p, k positive integers. Let us fix an A -sequence $f_1, \dots, f_p, \dots, f_{p+k}$ and an object*

$$z \in \mathcal{HOM}([n], \mathcal{M}(A \text{ on } \mathfrak{f}_1^p; \mathfrak{f}_{p+1}^{p+k})(p+1)).$$

(3.4.1) *Then for each $1 \leq s \leq p+k$, there is a non-negative integer m_s such that $f_s^{m_s} z_i(j) = 0$ for any $i \in [1]^k$ and any $0 \leq j \leq n$. We put $g_s = f_s^{m_s}$ for each $1 \leq s \leq p+k$.*

(3.4.2) *There is an object*

$$y \in \mathcal{HOM}([n], \mathbf{Wt}(A \text{ on } \mathfrak{f}_1^p; \mathfrak{f}_{p+1}^{p+k}))$$

such that for each $0 \leq i \leq n$, $y(i)$ is the following form:

$$y(i) = \bigoplus_{t \in [1]^k} \text{Kos}_{A/\mathfrak{g}_1^t}((\mathfrak{g}_{p+1}^t)^{p+k})^{\oplus l_t(i)}$$

and there is a deflation $y \twoheadrightarrow z$. Here the notation $(\mathfrak{g}^t)_{u}^{u+v}$ means $(g_u^{t_u}, g_{u+1}^{t_{u+1}}, \dots, g_{u+v}^{t_{u+v}})$.

(3.4.3) Proof. Since $z_i(j)$ is in $\mathcal{M}(A \text{ on } \mathfrak{f}_1^p)$, the assertion 3.4.1 is trivial. We are concentrating on proving the assertion 3.4.2. From now on, we put $B = A/(g_1, \dots, g_p)$.

We first prove that for the case $n = 0$ by induction on k . In this case we may regard z as an object in $\mathcal{M}(A \text{ on } \mathfrak{f}_1^p; \mathfrak{f}_{p+1}^{p+k})(r)$. Let us assume that $k = 1$. Then z can be considered as a complex $[z_1 \xrightarrow{d^z} z_0]$ in $\mathcal{M}(A \text{ on } \mathfrak{f}_1^p)(r)$. Since z_0 is a finitely generated B -module, there is an integer l_1 and a surjection $B^{\oplus l_1} \twoheadrightarrow z_0$. Then there is a surjection $(B/(g_{p+1}))^{\oplus l_1} \twoheadrightarrow H_0 z$ which makes diagram below commutative:

$$\begin{array}{ccc} B^{\oplus l_1} & \longrightarrow & z_0 \\ \downarrow & & \downarrow \\ (B/(g_{p+1}))^{\oplus l_1} & \longrightarrow & H_0 z \end{array}$$

where vertical maps are canonical surjections. Therefore we have a map $B^{\oplus l_1} \rightarrow z_1$ which makes the diagram below commutative:

$$\begin{array}{ccc} B^{\oplus l_1} & \longrightarrow & z_1 \\ g_{p+1} \downarrow & & \downarrow d^z \\ B^{\oplus l_1} & \longrightarrow & z_0. \end{array}$$

Then by adding $\text{Kos}_B(1)^{\oplus l_0}$ for a suitable l_0 to $\text{Kos}_B(g_{p+1})$, we get a term-wised surjection morphism

$$\text{Kos}_B(1)^{\oplus l_0} \oplus \text{Kos}_B(g_{p+1})^{\oplus l_1} \twoheadrightarrow z. \quad (3.4.3.1)$$

Then by 3.3.3.1.2, we learn that the map (3.4.3.1) is a deflation. Next we assume that the theorem is true for the case of k . Let us fix an object

$$z \in \mathcal{M}(A \text{ on } \mathfrak{f}_1^p; \mathfrak{f}_{p+1}^{p+k+1})(r).$$

We can consider z as a complex $[z_1 \xrightarrow{d^z} z_0]$ in $\mathcal{M}(A \text{ on } \mathfrak{f}_1^p; \mathfrak{f}_{p+2}^{p+k+1})(r)$. By inductive hypothesis, there is a deflation

$$\bigoplus_{t \in [1]^k} \text{Kos}_B((\mathfrak{g}^t)_{p+2}^{p+k+1})^{\oplus l_{1,t}} \twoheadrightarrow z_0.$$

Therefore by cube lemma 2.4.1, we get a term-wised surjection morphism.

$$\bigoplus_{t \in [1]^k} \text{Kos}_{B/(g_{p+1})}((\mathfrak{g}^t)_{p+2}^{p+k+1})^{\oplus l_{1,t}} \twoheadrightarrow H_0 z$$

which makes the diagram below commutative:

$$\begin{array}{ccc} \bigoplus_{t \in [1]^k} \text{Kos}_B((\mathfrak{g}^t)_{p+2}^{p+k+1})^{\oplus l_{1,t}} & \longrightarrow & z_1 \\ \downarrow & & \downarrow \\ \bigoplus_{t \in [1]^k} \text{Kos}_{B/(g_{p+1})}((\mathfrak{g}^t)_{p+2}^{p+k+1})^{\oplus l_{1,t}} & \longrightarrow & H_0 z \end{array}$$

where the vertical maps are canonical projections. Therefore we get a map

$$\bigoplus_{\mathfrak{t} \in [1]^k} \text{Kos}_B((\mathfrak{g}^{\mathfrak{t}})_{p+2}^{p+k+1})^{\oplus l_{1,\mathfrak{t}}} \rightarrow z_1$$

which makes the diagram below commutative:

$$\begin{array}{ccc} \bigoplus_{\mathfrak{t} \in [1]^k} \text{Kos}_B((\mathfrak{g}^{\mathfrak{t}})_{p+2}^{p+k+1})^{\oplus l_{1,\mathfrak{t}}} & \longrightarrow & z_1 \\ g_{p+1} \downarrow & & \downarrow d^z \\ \bigoplus_{\mathfrak{t} \in [1]^k} \text{Kos}_B((\mathfrak{g}^{\mathfrak{t}})_{p+2}^{p+k+1})^{\oplus l_{1,\mathfrak{t}}} & \twoheadrightarrow & z_0. \end{array}$$

By induction hypothesis, there is a term-wised surjective morphism

$$\bigoplus_{\mathfrak{t}' \in [1]^k} \text{Kos}_B((\mathfrak{g}^{\mathfrak{t}'}_{p+2})^{p+k+2})^{\oplus l_{0,\mathfrak{t}'}} \oplus \bigoplus_{\mathfrak{t} \in [1]^k} \text{Kos}_B((\mathfrak{g}^{\mathfrak{t}}_{p+2})^{p+k+2})^{\oplus l_{1,\mathfrak{t}}} \twoheadrightarrow z_1$$

which makes the diagram below commutative:

$$\begin{array}{ccc} \bigoplus_{\mathfrak{t}' \in [1]^k} \text{Kos}_B((\mathfrak{g}^{\mathfrak{t}'}_{p+2})^{p+k+2})^{\oplus l_{0,\mathfrak{t}'}} \oplus \bigoplus_{\mathfrak{t} \in [1]^k} \text{Kos}_B((\mathfrak{g}^{\mathfrak{t}}_{p+2})^{p+k+1})^{\oplus l_{1,\mathfrak{t}}} & \twoheadrightarrow & z_1 \\ \begin{pmatrix} 1 & 0 \\ 0 & g_{p+1} \end{pmatrix} \downarrow & & \downarrow d^z \\ \bigoplus_{\mathfrak{t}' \in [1]^k} \text{Kos}_B((\mathfrak{g}^{\mathfrak{t}'}_{p+2})^{p+k+2})^{\oplus l_{0,\mathfrak{t}'}} \oplus \bigoplus_{\mathfrak{t} \in [1]^k} \text{Kos}_B((\mathfrak{g}^{\mathfrak{t}}_{p+2})^{p+k+1})^{\oplus l_{1,\mathfrak{t}}} & \twoheadrightarrow & z_0. \end{array}$$

Thus by 3.3.3.1.2, we learn that the theorem is true for the case of $k+1$ and therefore we get the conclusion for the case of $n=0$. Next we consider the case of general n . For each z and each $0 \leq i \leq n$, by the previous argument, we have $y(i) = \bigoplus_{\mathfrak{t} \in [1]^k} \text{Kos}_B((\mathfrak{g}^{\mathfrak{t}})_{p+1}^{p+k})^{\oplus l_{\mathfrak{t}}(i)}$ for suitable $l_{\mathfrak{t}}(i)$ and a deflation $y(i) \twoheadrightarrow z(i)$. So we may just prove that for each $0 \leq i \leq n-1$, there is a map $y(i) \rightarrow y(i+1)$ which makes diagram below commutative:

$$\begin{array}{ccc} y(i) & \longrightarrow & y(i+1) \\ \downarrow & & \downarrow \\ z(i) & \longrightarrow & z(i+1). \end{array}$$

For each $\mathfrak{t} \in [1]^k$, we will define a map $y(i)_{\mathfrak{t}} \rightarrow y(i+1)_{\mathfrak{t}}$ by induction on the number of 0 in the co-ordinate \mathfrak{t} . For $\mathfrak{t} = (1, 1, \dots, 1)$, since in the diagram of B -modules below, $y(i)_{1,1,\dots,1}$ is projective B -module,

$$\begin{array}{ccc} y(i)_{1,1,\dots,1} & \twoheadrightarrow & z(i)_{1,1,\dots,1} \\ \downarrow & & \downarrow \\ y(i+1)_{1,1,\dots,1} & \twoheadrightarrow & z(i+1)_{1,1,\dots,1} \end{array}$$

there is a dotted map which makes diagram above commutative. Next assume that for $\mathfrak{s} \in [1]^k$, there is a map $y(i)_{\mathfrak{s}} \rightarrow y(i+1)_{\mathfrak{s}}$ which is compatible with other already defined maps $y(i)_{\mathfrak{t}} \rightarrow y(i+1)_{\mathfrak{t}}$, and assume that j -th co-ordinate of \mathfrak{s} is 1. Then we can define a map $H_{\bar{j}}(y(i))_{\mathfrak{s}} \rightarrow H_{\bar{j}}(y(i+1))_{\mathfrak{s}}$ which makes diagram below commutative

$$\begin{array}{ccc} y(i)_{\mathfrak{s}} & \longrightarrow & y(i+1)_{\mathfrak{s}} \\ \downarrow & & \downarrow \\ H_{\bar{j}}(y(i))_{\mathfrak{s}} & \longrightarrow & H_{\bar{j}}(y(i+1))_{\mathfrak{s}} \end{array}$$

where for the notation $H_{\bar{j}}$, see 2.2 and an index \mathfrak{s}^j is same as \mathfrak{s} except j -th co-ordinate. Therefore for $\mathfrak{s}' := \mathfrak{s} - (0, 0, \dots, \overset{j}{1}, \dots, 0)$, we can find a map $y(i)_{\mathfrak{s}'} \rightarrow y(i+1)_{\mathfrak{s}'}$ which makes diagram below commutative

$$\begin{array}{ccc} y(i)_{\mathfrak{s}'} & \longrightarrow & y(i+1)_{\mathfrak{s}'} \\ \downarrow & & \downarrow \\ y(i)_{\mathfrak{s}} & \longrightarrow & y(i+1)_{\mathfrak{s}} \end{array}$$

Then by induction and the cube lemma 2.4.1, we get a desired map $y(i) \rightarrow y(i+1)$. **Q.E.D.**

(3.4.4) Corollary (Local weight theorem). *For any A -sequence f_1, \dots, f_r and any integer $1 \leq s \leq r$ and $? = \emptyset$ or red, we have a homotopy equivalence*

$$wS \bullet \mathbf{Wt}_?(A \text{ on } f_1^s; f_{s+1}^r) \rightarrow iS \bullet \mathbf{Wt}_?(A \text{ on } f_1^r).$$

(3.4.4.1) Proof. By induction, we may only prove that

$$wS \bullet \mathbf{Wt}_?(A \text{ on } f_1^s; f_{s+1}^r) \rightarrow wS \bullet \mathbf{Wt}_?(A \text{ on } f_1^{s+1}; f_{s+2}^r)$$

is a homotopy equivalence. We intend to use 2.6.5 for

$$\begin{aligned} \mathcal{E}_1 &:= \mathbf{Wt}_?(A \text{ on } f_1^s; f_{s+2}^r) \\ \mathcal{E}_2 &:= \mathcal{M}_?(A \text{ on } f_1^s; f_{s+2}^r)(s+1) \\ \mathcal{F} &:= \mathbf{Wt}_?(A \text{ on } f_1^{s+1}; f_{s+2}^r). \end{aligned}$$

Since $\mathcal{E}_2 = \mathcal{M}_?(A \text{ on } (f_1^s, f_{s+2}^r); f_{s+3}^r) \times \mathbf{Wt}_?(A \text{ on } f_1^{s+1}; f_{s+3}^r)$ and $\mathcal{F} = \mathbf{Wt}_?(A \text{ on } f_1^{s+2}; f_{s+3}^r) \times \mathbf{Wt}_?(A \text{ on } f_1^{s+1}; f_{s+3}^r)$, \mathcal{F} is contained in \mathcal{E}_2 . We need to check the three conditions in 2.6. The conditions 2.6.1 and 2.6.2 are followed from 3.3.3.1.2. Finally the condition 2.6.3 for our \mathcal{E}_1 and \mathcal{E}_2 is followed from 3.4. Therefore we get the result. **Q.E.D.**

(3.4.4.2) Corollary *In the notation above, we have a canonical homotopy equivalence*

$$wS \bullet \mathbf{Kos}_?(A; f_1^r) \rightarrow iS \bullet \mathbf{Wt}_?(A \text{ on } f_1^r).$$

(3.4.4.2.1) Proof. It is just a special case of 3.4.4. **Q.E.D.**

4 A dévissage theorem for K -theory of Koszul cubes on regular rings

(4.0) In this section, we will assume that A is regular and the global homological dimension of A is n . Let us fix an A sequence f_1, \dots, f_p . The aim of this section is to prove a dévissage theorem for Koszul cubes on A 4.2.3.3.2.

(4.1) **Lemma.** For $n \leq q \leq \infty$, $\mathcal{M}(A \text{ on } \mathfrak{f}_1^p)(q)$ is an abelian category.

(4.1.1) *Proof.* We may ignore the projective dimension condition. Then $\mathcal{M}(A \text{ on } \mathfrak{f}_1^p)(q)$ is obviously closed under taking sub and quotient objects in the abelian category $\mathcal{M}(A)$. Hence we get the result. **Q.E.D.**

(4.2) Now we first intend to apply a dévissage theorem for pure weight module on A . Let us start from recalling the dévissage conditions from [Qui73] or [Dev]. Let \mathcal{A} be an exact category and $\iota : \mathcal{B} \hookrightarrow \mathcal{A}$ a full sub exact category. Assume that ι is exact and reflects exactness. ι is said to be *satisfying the dévissage conditions* if it is satisfying the following two conditions.

(4.2.1) \mathcal{B} is closed under the admissible sub and quotient objects. This means that for a conflation in \mathcal{A}

$$x \twoheadrightarrow y \twoheadrightarrow z,$$

if y is isomorphic to an object in \mathcal{B} , then x and z are also in.

(4.2.2) For any object a in \mathcal{A} , there is a finite filtration of inflations

$$* = a_0 \twoheadrightarrow a_1 \twoheadrightarrow a_2 \twoheadrightarrow \dots \twoheadrightarrow a_n = a$$

such that a_i/a_{i-1} is isomorphic to an object in \mathcal{B} for each i where $*$ is the specific zero object in \mathcal{A} .

(4.2.3) **Proposition** For $q \geq n$, the inclusion functor

$$\iota : \mathcal{M}_{\text{red}}(A \text{ on } \mathfrak{f}_1^p)(q) \hookrightarrow \mathcal{M}(A \text{ on } \mathfrak{f}_1^p)(q)$$

is satisfying the dévissage conditions. In particular, ι induces a homotopy equivalence

$$iS_{\bullet} \iota : iS_{\bullet} \mathcal{M}_{\text{red}}(A \text{ on } \mathfrak{f}_1^p)(q) \xrightarrow{\sim} iS_{\bullet} \mathcal{M}(A \text{ on } \mathfrak{f}_1^p)(q).$$

(4.2.3.1) *Proof.* For the condition 4.2.1, we have already noticed it in 3.2.2. Now let us consider the condition 4.2.2. We may ignore the projective dimension condition. Let us fix an A -module M in $\mathcal{M}(A \text{ on } \mathfrak{f}_1^p)(q)$. For each $\mathbf{t} = (t_1, \dots, t_p) \in \mathbb{N}^p$, we put $M_{\mathbf{t}} := f_1^{t_1} f_2^{t_2} \dots f_p^{t_p} M$. Then considering the dictionary order relation on \mathbb{N}^p , we get the filtration $\{M_{\mathbf{t}}\}$. Since M is finitely generated and supported in $V(\mathfrak{f}_1^p)$ the filtration is finite and one can easily check that it is desired one. **Q.E.D.**

(4.2.3.2) **Corollary.** In the notation 4.2.3, the inclusion functor

$$\iota : \mathbf{Wt}_{\text{red}}(A \text{ on } \mathfrak{f}_1^p) \hookrightarrow \mathbf{Wt}(A \text{ on } \mathfrak{f}_1^p)$$

induces homotopy equivalences.

$$iS_{\bullet} \iota \mathbf{Wt}_{\text{red}}(A \text{ on } \mathfrak{f}_1^p) \xrightarrow{\sim} \mathbf{Wt}(A \text{ on } \mathfrak{f}_1^p).$$

(4.2.3.2.1) *Proof.* It is just a consequence of 3.3.3.1 and 4.2.3. **Q.E.D.**

(4.2.3.3) **Corollary.** For an A -sequence $f_1, \dots, f_q, \dots, f_{q+k}$ and integer $q \leq p$, the inclusion functor

$$\mathcal{M}_{\text{red}}(A \text{ on } \mathfrak{f}_1^q; \mathfrak{f}_{q+1}^{q+k})(p) \hookrightarrow \mathcal{M}(A \text{ on } \mathfrak{f}_1^q; \mathfrak{f}_{q+1}^{q+k})(p)$$

induces a homotopy equivalence

$$iS_{\bullet} \mathcal{M}_{\text{red}}(A \text{ on } \mathfrak{f}_1^q; \mathfrak{f}_{q+1}^{q+k})(p) \rightarrow iS_{\bullet} \mathcal{M}(A \text{ on } \mathfrak{f}_1^q; \mathfrak{f}_{q+1}^{q+k})(p).$$

(4.2.3.3.1) *Proof.* We intend to prove by induction on k . First notice that by 3.3.3.1 and 4.2.3.2, we have a homotopy equivalence

$$iS_{\bullet} \mathcal{M}_{\text{red}}(A \text{ on } \mathfrak{f}_1^q)(p) \rightarrow \mathcal{M}(A \text{ on } \mathfrak{f}_1^q)(p).$$

Next since for $? = \emptyset$ and red,

$$\mathcal{M}_?(A \text{ on } \mathfrak{f}_1^q; \mathfrak{f}_{q+1}^{q+k})(p) = \mathcal{M}_?(A \text{ on } \mathfrak{f}_1^{q+1}; \mathfrak{f}_{q+2}^{q+k})(p) \times \mathcal{M}_?(A \text{ on } \mathfrak{f}_1^q; \mathfrak{f}_{q+2}^{q+k})(p)$$

by 2.7.1, inductive argument is working. **Q.E.D.**

(4.2.3.3.2) **Corollary.** In the notation 4.2.3, the canonical inclusion functor

$$\iota : \mathbf{Kos}_{\text{red}}(A; \mathfrak{f}_1^p) \hookrightarrow \mathbf{Kos}(A; \mathfrak{f}_1^p)$$

induces the following homotopy equivalences

$$\begin{aligned} iS_{\bullet} \iota &: iS_{\bullet} \mathbf{Kos}_{\text{red}}(A; \mathfrak{f}_1^p) \rightarrow iS_{\bullet} \mathbf{Kos}(A; \mathfrak{f}_1^p) \\ wS_{\bullet} \iota &: wS_{\bullet} \mathbf{Kos}_{\text{red}}(A; \mathfrak{f}_1^p) \rightarrow wS_{\bullet} \mathbf{Kos}(A; \mathfrak{f}_1^p). \end{aligned}$$

(4.2.3.3.2.1) *Proof.* The first one is just a special case of 4.2.3.3. The second one is just a consequence of 3.4.4.2 and 4.2.3.2. **Q.E.D.**

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