Chow motives without projectivity

by

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June 20, 2008

Abstract

In [Bo2], Bondarko recently defined the notion of weight structure, and proved that the category $DM_{gm}(k)$ of geometrical motives, as defined and studied by Voevodsky, Suslin and Friedlander [VSF], is canonically equipped with such a structure. Building on this result, and under a condition on the weights avoided by the boundary motive [W1], we describe a method to construct intrinsically in $DM_{gm}(k)$ a motivic version of interior cohomology of smooth, but possibly non-projective schemes. In a sequel to this work [W2], this method will be applied to Shimura varieties.

Keywords: weight structures, weight filtrations, Chow motives, geometrical motives, motives for modular forms, boundary motive, interior motive.

Math. Subj. Class. (2000) numbers: 14F42 (14F20, 14F25, 14F30, 14G35, 18E30, 19E15).

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0 Introduction

The full title of this work would be "Approximation of motives of varieties which are smooth, but not necessarily projective, by Chow motives, using homological rather than purely geometrical methods". This might give a better idea of our program — but as a title, it appeared too long. So there.

One way to place the problem historically is to start with the question asked by Serre [Se, p. 341], whether the "virtual motive" $\chi_c(X)$ of an arbitrary variety X over a fixed base field k can be (well) defined in the Grothendieck group of the category $CHM^{eff}(k)$ of effective *Chow motives.* When k admits resolution of singularities, Gillet and Soulé [GiSo] (see also [GuNA] when $char(k) = 0$) provided an affirmative answer. In fact, their solution yields much more information: they define the *weight complex* $W(X)$ ($h_c(X)$ in [GuNA]) in the category of complexes over $CHM^{eff}(k)$, well defined up to canonical homotopy equivalence. By definition, $\chi_c(X)$ equals the class of $W(X)$ in $K_0\big(CHM^{eff}(k)\big)$. Thus, given any representative

$$M^\bullet : \qquad \ldots \longrightarrow M^n \longrightarrow M^{n+1} \longrightarrow \ldots$$

of $W(X)$, we have the fomula $\chi_c(X) = \sum_n (-1)^n [M^n]$.

Consider the fully faithful embedding ι of $CHM^{eff}(k)$ into $DM^{eff}_{gm}(k)$, the category of effective *geometrical motives*, as defined and studied by Voevodsky, Suslin and Friedlander [VSF]. As observed in [GiSo], localization for both $\chi_c(X)$ and the *motive with compact support* $M^c_{gm}(X)$ of X shows that the element $\chi_c(X)$ is mapped to the class of $M^c_{gm}(X)$ under $K_0(\iota)$. Gillet and Soulé went on and asked [GiSo, p. 153] whether

$$K_0(\iota) : K_0\big(CHM^{eff}(k)\big) \longrightarrow K_0\big(DM^{eff}_{gm}(k)\big)$$

is an isomorphism.

Just as Serre, Gillet and Soulé provoked much more than just an affirmative answer to their question. Bondarko defined in [Bo2] the notion of *weight structure* on a triangulated category $\mathcal{C}$. He proved that the inclusion of the *heart* of the weight structure into $\mathcal{C}$ induces an isomorphism on the level of Grothendieck groups, whenever the weight structure is bounded, and the heart is pseudo-Abelian. The definitions will be recalled in our Section 1; let us just note that shift by $[k]$, for $k \in \mathbb{Z}$, adds k to the weight of an object of $\mathcal{C}$. According to one of the main results of [loc. cit.] (recalled in Theorem 1.13), $DM_{gm}^{eff}(k)$ carries a canonical weight structure, which is indeed bounded, and admits $CHM^{eff}(k)$ as its heart. By definition, this means that among all geometrical motives, Chow motives distinguish themselves as being the motives which are pure of weight zero.

In particular, this gives an intrinsic characterization of objects of the category $DM_{gm}^{eff}(k)$ belonging to $CHM^{eff}(k)$. We think of this insight as nothing less but revolutionary.

To come back to the beginning, the component M^n of the weight complex $W(X)$ can be considered as a "$\mathrm{Gr}_n W(X)$" with respect to the weight structure. In the context of Bondarko's theory, the formula for the "virtual motive" of X thus reads

$$\chi_c(X) = \sum_n (-1)^n \, \mathrm{Gr}_n W(X) \; .$$

Basically, our approach to construct a Chow motive out of X, when X is smooth, is very simple: according to Bondarko (see Corollary 1.14), the *motive* $M(X)$ of X is of weights ≤ 0. We would like to consider $\mathrm{Gr}_0 M(X)$, the "quotient" of $M(X)$ of maximal weight zero.

However, one of the main subtleties of the notion of weight structure is that "the" *weight filtration* of an object is almost never unique. (In the example of the weight complex, this corresponds to the fact that $W(X)$ is only well-defined up to homotopy.) Hence "$\mathrm{Gr}_n M_{gm}(X)$" is not well-defined, and the above approach cannot work as stated. In fact, for any smooth compactification $\widetilde{X}$ of X, the Chow motive $M_{gm}(\widetilde{X})$ occurs as "$\mathrm{Gr}_0 M_{gm}(X)$" for a suitable weight filtration!

Our main contribution is to identify a criterion assuring existence and unicity of a "best choice" of $\mathrm{Gr}_0 M_{gm}(X)$. It is best understood in the abstract setting created by Bondarko, that is, in the context of a weight structure on a triangulated category $\mathcal{C}$. Let M be an object of $\mathcal{C}$ of non-positive weights. Then Bondarko's axioms imply that if M admits a weight filtration in which the adjacent weight -1 does not occur, then the associated $\mathrm{Gr}_0 M$ is unique up to unique isomorphism. In the motivic context, this observation leads to our criterion on weights avoided by the *boundary motive* $\partial M_{gm}(X)$

of X, introduced and studied in [W1]. The behavior of the realizations of the Chow motive $Gr_0 \, M_{gm}(X)$ motivates its name: we chose to call it the *interior motive* of X.

Let us now give a more detailed description of the content of this article. Section 1 claims no originality whatsoever. We give Bondarko's definition of weight structures (Definition 1.1) and review the results from [Bo2] needed in the sequel. We treat particularly carefully the phenomenon which will turn out to be the main theme of this article, namely *the absence of certain weights*. Thus, we introduce the central notion of *weight filtration avoiding weights* $m, m+1, \ldots, n-1, n$, for fixed integers $m \leq n$. Bondarko's axioms imply that whenever such a filtration exists, it behaves functorially (Proposition 1.7). In particular, for any fixed object, it is unique up to unique isomorphism (Corollary 1.9). We conclude Section 1 with Bondarko's application of his theory to geometrical motives, which we already mentioned before (Theorem 1.13, Corollary 1.14).

Section 2 is the technical center of this article. It is devoted to a further study of functoriality of weight filtrations avoiding certain weights. We work in the context of an abstract weight structure $w = (\mathcal{C}_{w \leq 0}, \mathcal{C}_{w \geq 0})$ on a triangulated category $\mathcal{C}$. We first show (Proposition 2.2) that the inclusion ι_- of the heart $\mathcal{C}_{w=0}$ into the full sub-category $\mathcal{C}_{w \leq 0, \neq -1}$ of objects of non-positive weights $\neq -1$ admits a left adjoint

$$\mathrm{Gr}_0 : \mathcal{C}_{w \leq 0, \neq -1} \longrightarrow \mathcal{C}_{w=0} \; .$$

There is a dual version of this statement for non-negative weights $\neq 1$. We then consider the situation which will be of interest in our application to motives. We thus fix a morphism $u : M_- \to M_+$ between objects $M_- \in \mathcal{C}_{w \leq 0}$ and $M_+ \in \mathcal{C}_{w \geq 0}$, and a cone $C[1]$ of u. While the axioms characterizing weight structures easily show that u can be factored through *some* object of $\mathcal{C}_{w=0}$, our aim is to do so *in a canonical way*. We therefore formulate Assumption 2.3: the object C is without weights -1 and 0. Theorem 2.4 states that Assumption 2.3 not only allows to factor u as desired; in addition this factorization is through an object which is simultaneously identified with $\mathrm{Gr}_0 \, M_-$ and with $\mathrm{Gr}_0 \, M_+$. As a formal consequence of this, and of the functoriality properties of Gr_0 from Proposition 2.2, we get a statement on abstract factorization of u (Corollary 2.5), whose rigidity may appear surprising at first sight, given that we work in a triangulated category: whenever $u : M_- \to N \to M_+$ factors through an object N of $\mathcal{C}_{w=0}$, then $\mathrm{Gr}_0 \, M_- = \mathrm{Gr}_0 \, M_+$ is canonically identified with a direct factor of N, admitting in addition a canonical direct complement.

The reader willing to turn directly to the application of these results to geometrical motives may choose to skip Section 3, in which Scholl's construction of motives for modular forms [Scho1] is discussed at length. As in [loc. cit.], we consider a self-product X_n^k of the universal elliptic curve X_n over a modular curve. We show (Theorem 3.3, Corollary 3.4, Corollary 3.6) that certain direct factors $M_{gm}(X_n^k)^e$ of the motive $M_{gm}(X_n^k)$ and $M_{gm}^c(X_n^k)^e$ of the motive with compact support $M_{gm}^c(X_n^k)$, together with the canonical morphism $u : M_{gm}(X_n^k)^e \to M_{gm}^c(X_n^k)^e$, satisfy the conclusions of Theorem 2.4. In particular, the Chow motives $\mathrm{Gr}_0 M_{gm}(X_n^k)^e$ and $\mathrm{Gr}_0 M_{gm}^c(X_n^k)^e$ are defined, and canonically isomorphic. In fact, they are both canonically isomorphic to the motive denoted ${}_n^k\mathcal{W}$ in [Scho1]. We insist on giving a proof of the above statements which is independent of the theory developed in Section 2, that is, we prove the conclusions of Theorem 2.4 without first checking Assumption 2.3. Instead, we use the detailed analysis from [Scho1] of the geometry of the boundary of a "good choice" of smooth compactification of X_n^k. If one forgets about the language of weight structures, whose use could not be completely avoided, Section 3 is thus technically independent of the material preceding it. The reader may find an interest in the re-interpretation of Scholl's construction in the context of Voevodsky's geometrical motives, which were not yet defined at the time when [Scho1] was written. This concerns in particular the exact triangle

$$M_{gm}\big(S_n^\infty\big)(k+1)[k+1] \longrightarrow M_{gm}(X_n^k)^e \longrightarrow {}_n^k\mathcal{W} \longrightarrow M_{gm}\big(S_n^\infty\big)(k+1)[k+2]$$

from Corollary 3.4 ($S_n^\infty :=$ the cuspidal locus of the modular curve). As a by-product of Scholl's analysis, we prove that the triangle is defined using $\mathbb{Z}[1/(2n \cdot k!)]$-coefficients. An interpretation of Beilinson's *Eisenstein symbol* [B] in this context (namely, as a splitting of this triangle) is clearly desirable.

Section 4 is devoted to the application of the results from Section 2 to geometrical motives. As $u : M_- \to M_+$, we take the canonical morphism $M_{gm}(X)^e \to M_{gm}^c(X)^e$, for a fixed smooth variety X over k, and a fixed idempotent e. Then the role of the object C from Section 2 is canonically played by $\partial M_{gm}(X)^e$, the e-part of the boundary motive. In this context, Assumption 2.3 reads as follows: the object $\partial M_{gm}(X)^e$ is without weights -1 and 0 (Assumption 4.2). Our main result Theorem 4.3 is then simply the translation of the sum of the results from Section 2 into this particular motivic context. Thus, the Chow motive $\mathrm{Gr}_0 M_{gm}(X)^e = \mathrm{Gr}_0 M_{gm}^c(X)^e$ is defined. Furthermore (Corollary 4.6), we get the motivic version of abstract factorization: whenever $\widetilde{X}$ is a smooth compactification of X, then $\mathrm{Gr}_0 M_{gm}(X)^e$ is canonically a direct factor of the Chow motive $M_{gm}(\widetilde{X})$, with a canonical direct complement. We then study the implications of these results for the Hodge theoretic and ℓ-adic realizations. Theorems 4.7 and 4.8 state that they are equal to the respective e-parts of interior cohomology of X. Abstract factorization allows to say more about the quality of the Galois representation on the ℓ-adic realization of $\mathrm{Gr}_0 M_{gm}(X)^e$ (Theorem 4.14). For example, simple semi-stable reduction of *some* smooth compactification of X implies that the representation is semi-stable.

To conclude, we get back to Scholl's construction. We show (Remark 4.17) that essentially all results of Section 3 can be deduced (just) from Assumption 4.2, and the theory developed in Section 4. We consider that in spite of the technical independence of Section 3, there is a good reason to include that material in this article: for higher dimensional Shimura varieties, "good choices" of smooth compactifications as the one used in [Scho1] may simply not be available. Therefore, the purely geometrical strategy of proof of the results of Section 3 can safely be expected not to be generalizable. We think that a promising way to generalize is via a verification of Assumption 4.2 by other than purely geometrical means. We refer to [W2] for the development of such an alternative in a context including that of (powers of universal elliptic curves over) modular curves.

I wish to thank M.V. Bondarko, F. Déglise, A. Deitmar, O. Gabber, B. Kahn, A. Mokrane, C. Soulé and J. Tilouine for useful discussions and comments.

Notation and conventions: k denotes a fixed perfect base field, Sch/k the category of separated schemes of finite type over k, and $Sm/k \subset Sch/k$ the full sub-category of objects which are smooth over k. When we assume k to admit resolution of singularities, then it will be in the sense of [FV, Def. 3.4]: (i) for any $X \in Sch/k$, there exists an abstract blow-up $Y \to X$ [FV, Def. 3.1] whose source Y is in Sm/k, (ii) for any $X, Y \in Sm/k$, and any abstract blow-up $q : Y \to X$, there exists a sequence of blow-ups $p : X_n \to \ldots \to X_1 = X$ with smooth centers, such that p factors through q. Let us note that the main reason for us to suppose k to admit resolution of singularities is to have the motive with compact support satisfy localization [V1, Prop. 4.1.5].

As far as motives are concerned, the notation of this paper follows that of [V1]. We refer to [W1, Sect. 1] for a review of this notation, and in particular, of the definition of the categories $DM_{gm}^{eff}(k)$ and $DM_{gm}(k)$ of (effective) geometrical motives over k, and of the motive $M_{gm}(X)$ and the motive with compact support $M_{gm}^c(X)$ of $X \in Sch/k$. Let F be a commutative flat $\mathbb{Z}$-algebra, i.e., a commutative unitary ring whose additive group is without torsion. The notation $DM_{gm}^{eff}(k)_F$ and $DM_{gm}(k)_F$ stands for the pseudo-Abelian completion of $DM_{gm}^{eff}(k) \otimes_{\mathbb{Z}} F$ and $DM_{gm}(k) \otimes_{\mathbb{Z}} F$, respectively. Similarly, let us denote by $CHM^{eff}(k)$ and $CHM(k)$ the categories opposite to the categories of (effective) Chow motives, and by $CHM^{eff}(k)_F$ and $CHM(k)_F$ the pseudo-Abelian completion of the category $CHM^{eff}(k) \otimes_{\mathbb{Z}} F$ and $CHM(k) \otimes_{\mathbb{Z}} F$, respectively. Using [V2, Cor. 2] ([V1, Cor. 4.2.6] if k admits resolution of singularities), we canonically identify $CHM^{eff}(k)_F$ and $CHM(k)_F$ with a full additive sub-category of $DM_{gm}^{eff}(k)_F$ and $DM_{gm}(k)_F$, respectively.

1 Weight structures

In this section, we review definitions and results of Bondarko's recent paper [Bo2].

Definition 1.1. Let $\mathcal{C}$ be a triangulated category. A *weight structure on* $\mathcal{C}$ is a pair $w = (\mathcal{C}_{w \le 0}, \mathcal{C}_{w \ge 0})$ of full sub-categories of $\mathcal{C}$, such that, putting

$$\mathcal{C}_{w \le n} := \mathcal{C}_{w \le 0}[n] \quad , \quad \mathcal{C}_{w \ge n} := \mathcal{C}_{w \ge 0}[n] \quad \forall\, n \in \mathbb{Z} \, ,$$

the following conditions are satisfied.

(1) The categories $\mathcal{C}_{w \le 0}$ and $\mathcal{C}_{w \ge 0}$ are Karoubi-closed (i.e., closed under retracts formed in $\mathcal{C}$).

(2) (Semi-invariance with respect to shifts.) We have the inclusions

$$\mathcal{C}_{w \le 0} \subset \mathcal{C}_{w \le 1} \quad , \quad \mathcal{C}_{w \ge 0} \supset \mathcal{C}_{w \ge 1}$$

of full sub-categories of $\mathcal{C}$.

(3) (Orthogonality.) For any pair of objects $M \in \mathcal{C}_{w \le 0}$ and $N \in \mathcal{C}_{w \ge 1}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(M, N) = 0 \, .$$

(4) (Weight filtration.) For any object $M \in \mathcal{C}$, there exists an exact triangle

$$A \longrightarrow M \longrightarrow B \longrightarrow A[1]$$

in $\mathcal{C}$, such that $A \in \mathcal{C}_{w \le 0}$ and $B \in \mathcal{C}_{w \ge 1}$.

By condition 1.1 (2),

$$\mathcal{C}_{w \le n} \subset \mathcal{C}_{w \le 0}$$

for negative n, and

$$\mathcal{C}_{w \ge n} \subset \mathcal{C}_{w \ge 0}$$

for positive n. There are obvious analogues of the other conditions for all the categories $\mathcal{C}_{w \le n}$ and $\mathcal{C}_{w \ge n}$. In particular, they are all Karoubi-closed, and any object $M \in \mathcal{C}$ is part of an exact triangle

$$A \longrightarrow M \longrightarrow B \longrightarrow A[1]$$

in $\mathcal{C}$, such that $A \in \mathcal{C}_{w \le n}$ and $B \in \mathcal{C}_{w \ge n+1}$. By a slight generalization of the terminology introduced in condition 1.1 (4), we shall refer to any such exact triangle as a weight filtration of M.

Remark 1.2. (a) Our convention concerning the sign of the weight is actually opposite to the one from [Bo2, Def. 1.1.1], i.e., we exchanged the roles of $\mathcal{C}_{w \le 0}$ and $\mathcal{C}_{w \ge 0}$.
(b) Note that in condition 1.1 (4), "the" weight filtration is not assumed to

be unique.
(c) Condition 1.1 (1) implies that any object of $\mathcal{C}$ isomorphic to an object in $\mathcal{C}_{w\le 0}$ (resp. in $\mathcal{C}_{w\ge 0}$) lies in $\mathcal{C}_{w\le 0}$ (resp. in $\mathcal{C}_{w\ge 0}$).
(d) Recall the notion of *t-structure* on a triangulated category $\mathcal{C}$ [BBD, Déf. 1.3.1]. It consists of a pair $t = (\mathcal{C}^{t\le 0}, \mathcal{C}^{t\ge 0})$ of full sub-categories satisfying formal analogues of conditions 1.1 (2)–(4), but putting

$$\mathcal{C}^{t\le n} := \mathcal{C}^{t\le 0}[-n] \quad , \quad \mathcal{C}^{t\ge n} := \mathcal{C}^{t\ge 0}[-n] \quad \forall\, n \in \mathbb{Z}\,.$$

Note that in the context of t-structures, the analogues of the exact triangles in 1.1 (4) are then unique up to unique isomorphism, and that the analogue of condition 1.1 (1) is formally implied by the others.

The following is contained in [Bo2, Def. 1.2.1].

Definition 1.3. Let $w = (\mathcal{C}_{w\le 0}, \mathcal{C}_{w\ge 0})$ be a weight structure on $\mathcal{C}$. The *heart* of w is the full additive sub-category $\mathcal{C}_{w=0}$ of $\mathcal{C}$ whose objects lie both in $\mathcal{C}_{w\le 0}$ and in $\mathcal{C}_{w\ge 0}$.

Among the basic properties developed in [Bo2], let us note the following.

Proposition 1.4. *Let $w = (\mathcal{C}_{w\le 0}, \mathcal{C}_{w\ge 0})$ be a weight structure on $\mathcal{C}$,*

$$L \longrightarrow M \longrightarrow N \longrightarrow L[1]$$

an exact triangle in $\mathcal{C}$.
(a) If both L and N belong to $\mathcal{C}_{w\le 0}$, then so does M.
(b) If both L and N belong to $\mathcal{C}_{w\ge 0}$, then so does M.

Proof. This is the content of [Bo2, Prop. 1.3.1 3]. **q.e.d.**

The reader may wonder whether there is an easy criterion on a given sub-category of a triangulated category to be the heart of a suitable weight structure. Bondarko has results [Bo2, Thm. 4.3.2] answering this question. For our purposes, the result with the most restrictive finiteness condition will be sufficient.

Proposition 1.5. *Let $\mathcal{H}$ be a full additive sub-category of a triangulated category $\mathcal{C}$. Suppose that $\mathcal{H}$ generates $\mathcal{C}$, i.e., $\mathcal{C}$ is the smallest full triangulated sub-category containing $\mathcal{H}$.*
(a) If there is a weight structure on $\mathcal{C}$ whose heart contains $\mathcal{H}$, then it is unique. In this case, the heart is equal to the Karoubi envelope *of $\mathcal{H}$, i.e., the category of retracts of $\mathcal{H}$ in $\mathcal{C}$.*
(b) The following conditions are equivalent.

(i) There is a weight structure on $\mathcal{C}$ whose heart contains $\mathcal{H}$.

(ii) $\mathcal{H}$ is negative*, i.e.,*

$$\mathrm{Hom}_{\mathcal{C}}\big(A, B[i]\big) = 0$$

for any two objects A, B of $\mathcal{H}$, and any integer $i > 0$.

Proof. Condition (ii) on $\mathcal{H}$ is clearly necessary for $\mathcal{H}$ to belong to the heart, given orthogonality 1.1 (3). As for (a), note that by Proposition 1.4, there is only one possible definition of the category $\mathcal{C}_{w\le 0}$ (resp. $\mathcal{C}_{w\ge 0}$): it is necessarily the full sub-category of successive extensions of objects of the form $A[n]$, for $A \in \mathcal{H}$ and $n \le 0$ (resp. $n \ge 0$).

The main point is to show that under condition (ii), the above construction indeed yields a weight structure on $\mathcal{C}$. We refer to [Bo2, Thm. 4.3.2 II] for details. **q.e.d.**

For the rest of this section, we consider a fixed weight structure w on a triangulated category $\mathcal{C}$.

Definition 1.6. Let $M \in \mathcal{C}$, and $m \le n$ two integers (which may be identical). A *weight filtration of M avoiding weights $m, m+1, \ldots, n-1, n$* is an exact triangle

$$M_{\le m-1} \longrightarrow M \longrightarrow M_{\ge n+1} \longrightarrow M_{\le m-1}[1]$$

in $\mathcal{C}$, with $M_{\le m-1} \in \mathcal{C}_{w\le m-1}$ and $M_{\ge n+1} \in \mathcal{C}_{w\ge n+1}$.

The following is central for everything to follow.

Proposition 1.7. *Assume that $m \le n$, and that $M, N \in \mathcal{C}$ admit weight filtrations*

$$M_{\le m-1} \stackrel{x_-}{\longrightarrow} M \stackrel{x_+}{\longrightarrow} M_{\ge n+1} \longrightarrow M_{\le m-1}[1]$$

and

$$N_{\le m-1} \stackrel{y_-}{\longrightarrow} N \stackrel{y_+}{\longrightarrow} N_{\ge n+1} \longrightarrow N_{\le m-1}[1]$$

avoiding weights $m, \ldots, n$. Then any morphism $M \to N$ in $\mathcal{C}$ extends uniquely to a morphism of exact triangles

$$\begin{array}{ccccccc} M_{\le m-1} & \longrightarrow & M & \longrightarrow & M_{\ge n+1} & \longrightarrow & M_{\le m-1}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ N_{\le m-1} & \longrightarrow & N & \longrightarrow & N_{\ge n+1} & \longrightarrow & N_{\le m-1}[1] \end{array}$$

Proof. This follows from [Bo2, Lemma 1.5.1 2]. For the convenience of the reader, let us recall the proof. Let $\alpha \in \mathrm{Hom}_{\mathcal{C}}(M, N)$. The composition $y_+ \circ \alpha \circ x_- : M_{\le m-1} \to N_{\ge n+1}$ is zero by orthogonality 1.1 (3): $m-1$ is strictly smaller than $n+1$. Hence $\alpha \circ x_-$ factors through $N_{\le m-1}$. We claim that this factorization is unique. Indeed, the error term comes from $\mathrm{Hom}_{\mathcal{C}}(M_{\le m-1}, N_{\ge n+1}[-1])$. But this group is trivial, thanks to orthogonality, and our assumption on the weights: the object $N_{\ge n+1}[-1]$ lies in

$$\mathcal{C}_{w\ge n+1}[-1] = \mathcal{C}_{w\ge n} ,$$

and $m-1$ is still stricly smaller than n. Similarly, the composition $y_+ \circ \alpha$ factors uniquely through $M_{\ge n+1}$. **q.e.d.**

Remark 1.8. Note that the hypothesis of Proposition 1.7 does not imply unicity of weight filtrations in the (more general) sense of 1.1 (4). For example, assume that $m = n = -1$, and let

$$(*) \qquad M_{\leq -2} \longrightarrow M \longrightarrow M_{\geq 0} \longrightarrow M_{\leq -2}[1]$$

be a weight filtration avoiding weight -1. Choose any object M_0 in $\mathcal{C}_{w=0}$ and replace $M_{\geq 0}$ by $M_0 \oplus M_{\geq 0}$, and $M_{\leq -2}$ by $M_0[-1] \oplus M_{\leq -2}$. Arguing as in the proof of Proposition 1.7, one shows that any weight filtration of M is isomorphic to one obtained in this way. Thus, the exact triangle $(*)$ satisfies a minimality property among all weight filtrations of M.

Corollary 1.9. *Assume that $m \leq n$. Then if $M \in \mathcal{C}$ admits a weight filtration avoiding weights $m, \ldots, n$, it is unique up to unique isomorphism.*

Definition 1.10. Assume that $m \leq n$. We say that $M \in \mathcal{C}$ *does not have weights $m, \ldots, n$*, or that M *is without weights $m, \ldots, n$*, if it admits a weight filtration avoiding weights $m, \ldots, n$.

Let us now state what we consider as one of the main results of [Bo2].

Theorem 1.11. *Assume that the triangulated category $\mathcal{C}$ is generated by its heart $\mathcal{C}_{w=0}$.*
(a) The pseudo-Abelian completion $\mathcal{C}'_{w=0}$ of $\mathcal{C}_{w=0}$ generates the pseudo-Abelian completion $\mathcal{C}'$ of $\mathcal{C}$.
(b) There is a weight structure w' on $\mathcal{C}'$, uniquely characterized by any of the following conditions.

(i) The weight structure w' extends w.

(ii) The heart of w' equals $\mathcal{C}'_{w=0}$.

(iii) The heart of w' contains $\mathcal{C}'_{w=0}$.

Proof. This is [Bo2, Prop. 5.2.2]. Let us describe the main steps of the proof. Recall that by [BaSchl, Thm. 1.5], the category $\mathcal{C}'$ is indeed triangulated. The criterion from Proposition 1.5 implies the existence of a weight structure w' on the full triangulated sub-category $\mathcal{D}$ of $\mathcal{C}'$ generated by $\mathcal{C}'_{w=0}$ (hence containing $\mathcal{C}$), and uniquely characterized by condition (iii), hence also by (i) or (ii). The claim then follows from [Bo2, Lemma 5.2.1], which states that $\mathcal{D}$ is pseudo-Abelian, and hence equal to $\mathcal{C}'$. **q.e.d.**

Remark 1.12. Note that given Proposition 1.5, part (b) of Theorem 1.11 follows formally from its part (a). One may see Theorem 1.11 (a) as a generalization of [BaSchl, Cor. 2.12], which states that the pseudo-Abelian completion of the bounded derived category $D^b(\mathcal{A})$ of an exact category $\mathcal{A}$ equals the bounded derived category $D^b(\mathcal{A}')$ of the pseudo-Abelian completion $\mathcal{A}'$ of $\mathcal{A}$.

For our purposes, the main application of the preceding is the following (cmp. [Bo2, Sect. 6]).

Theorem 1.13. *Let F be a commutative flat $\mathbb{Z}$-algebra, and assume k to admit resolution of singularities.*
(a) There is a canonical weight structure on the category $DM_{gm}^{eff}(k)_F$. It is uniquely characterized by the requirement that its heart equal $CHM^{eff}(k)_F$.
(b) There is a canonical weight structure on the category $DM_{gm}(k)_F$, extending the weight structure from (a). It is uniquely characterized by the requirement that its heart equal $CHM(k)_F$.
(c) Statements (a) and (b) hold without assuming resolution of singularities provided F is a $\mathbb{Q}$-algebra.

Proof. For $F = \mathbb{Z}$ and k of characteristic zero, this is the content of [Bo2, Sect. 6.5 and 6.6]:

(1) As in [Bo2], denote by DM^s the full triangulated sub-category of $DM_{gm}^{eff}(k)$ generated by the motives $M_{gm}(X)$ [V1, Def. 2.1.1] of objects X of Sm/k, by J_0 the full additive sub-category of DM^s generated by $M_{gm}(X)$ for X smooth and projective, and by J_0' the Karoubi envelope of J_0. Thus, $DM_{gm}^{eff}(k)$ is the pseudo-Abelian completion of DM^s, and $CHM^{eff}(k)$ is the pseudo-Abelian completion of both J_0 and J_0'.

(2) We need two of the main results from [V1]. First, by [loc. cit.], Cor. 3.5.5, the additive category J_0 generates the triangulated category DM^s. Next, by [loc. cit.], Cor. 4.2.6, the category J_0 is negative:

$$\mathrm{Hom}_{DM^s}\big(A, B[i]\big) = 0$$

for any two objects A, B of J_0, and any integer $i > 0$.

(3) By Proposition 1.5, there is a weight structure on DM^s, uniquely characterized by the fact that J_0 is contained in the heart. Furthermore, the heart equals the Karoubi envelope J_0'.

(4) By Theorem 1.11, the pseudo-Abelian completion $CHM^{eff}(k)$ of J_0 generates the pseudo-Abelian completion $DM_{gm}^{eff}(k)$ of DM^s (let us remark that this is stated, but not proved in [V1, Cor. 3.5.5]). Thus, part (a) of our claim holds for $F = \mathbb{Z}$.

(5) Recall that $CHM(k)$ and $DM_{gm}(k)$ are obtained from $CHM^{eff}(k)$ and $DM_{gm}^{eff}(k)$ by inverting an object, namely the Tate object, with respect to the tensor structures. Hence $CHM(k)$ generates the triangulated category $DM_{gm}(k)$. Its negativity follows immediately from that of $CHM^{eff}(k)$. Thus, we may again apply Proposition 1.5. The resulting weight structure extends the one on $DM_{gm}^{eff}(k)$: in fact, its restriction to $DM_{gm}^{eff}(k)$ is a weight structure, whose heart equals $CHM^{eff}(k)$. This proves part (b) of our claim for $F = \mathbb{Z}$.

If F is flat over $\mathbb{Z}$, then the same proof works. (1') Replace DM^s by $DM^s \otimes_{\mathbb{Z}} F$ and J_0 by $J_0 \otimes_{\mathbb{Z}} F$. (2') The two results cited in (2) formally imply that $J_0 \otimes_{\mathbb{Z}} F$ generates $DM^s \otimes_{\mathbb{Z}} F$, and that $J_0 \otimes_{\mathbb{Z}} F$ is negative.

Steps (3') and (4') are formally identical to (3) and (4), proving part (a) of the claim. Step (5') shows part (b), once we observe that $CHM(k)_F$ and $DM_{gm}(k)_F$ are obtained from $CHM^{eff}(k)_F$ and $DM_{gm}^{eff}(k)_F$ by inverting the Tate object.

As for part (c) of our claim, everything reduces to showing analogues of the two statements made in step (2). By [A, Cor. 18.1.1.2], the additive category $J_0 \otimes_{\mathbb{Z}} F$ generates the triangulated category $DM^s \otimes_{\mathbb{Z}} F$. The argument uses alterations à la de Jong; since this involves finite extensions of fields, whose degrees need to be inverted, one requires F to be a $\mathbb{Q}$-algebra. The generalization of [V1, Cor. 4.2.6] to arbitrary fields [V2, Cor. 2] shows that the category J_0 is negative. Hence so is $J_0 \otimes_{\mathbb{Z}} F$. **q.e.d.**

The following is the content of [Bo1, Thm. 6.2.1 1 and 2].

Corollary 1.14. *Assume k to admit resolution of singularities. Let X in Sch/k be of (Krull) dimension d.*
(a) The motive with compact support $M_{gm}^c(X)$ lies in

$$DM_{gm}^{eff}(k)_{w\geq 0} \cap DM_{gm}^{eff}(k)_{w\leq d} \ .$$

(b) If $X \in Sm/k$, then the motive $M_{gm}(X)$ lies in

$$DM_{gm}^{eff}(k)_{w\geq -d} \cap DM_{gm}^{eff}(k)_{w\leq 0} \ .$$

Proof. (a) We proceed by induction on d. If $d = 0$, then $M_{gm}^c(X)$ is an effective Chow motive, hence of weight 0 by Theorem 1.13 (a).

For $d \geq 1$, Nagata's theorem on the existence of a compactification of X, and resolution of singularities imply that there is an open dense subscheme U of X admitting a smooth compactification $\widetilde{X}$. Denote by Z the complement of U in X, and by Y the complement of U in $\widetilde{X}$ (both with the reduced scheme structure). By localization for the motive with compact support [V1, Prop. 4.1.5], there are exact triangles

$$M_{gm}^c(Z) \longrightarrow M_{gm}^c(X) \longrightarrow M_{gm}^c(U) \longrightarrow M_{gm}^c(Z)[1] \ .$$

and

$$M_{gm}^c(Y) \longrightarrow M_{gm}^c(\widetilde{X}) \longrightarrow M_{gm}^c(U) \longrightarrow M_{gm}^c(Y)[1] \ .$$

By induction,

$$M_{gm}^c(Y), M_{gm}^c(Z) \in DM_{gm}^{eff}(k)_{w\geq 0} \cap DM_{gm}^{eff}(k)_{w\leq d-1} \ .$$

Therefore,

$$M_{gm}^c(Y)[1], M_{gm}^c(Z) \in DM_{gm}^{eff}(k)_{w\geq 0} \cap DM_{gm}^{eff}(k)_{w\leq d} \ .$$

Given that $M_{gm}^c(\widetilde{X})$ is of weight 0, Proposition 1.4 shows first that

$$M_{gm}^c(U) \in DM_{gm}^{eff}(k)_{w\geq 0} \cap DM_{gm}^{eff}(k)_{w\leq d} \ ,$$

and then that

$$M^c_{gm}(X) \in DM^{eff}_{gm}(k)_{w\geq 0} \cap DM^{eff}_{gm}(k)_{w\leq d} \ .$$

(b) By [V1, Theorem 4.3.7 1 and 2], the category $DM_{gm}(k)$ is rigid tensor triangulated. The claim thus follows formally from (a), and from the following observations: (i) assuming (as we may) X to be of pure dimension d, the motive $M_{gm}(X)$ is dual to $M^c_{gm}(X)(-d)[-2d]$ [V1, Theorem 4.3.7 3], (ii) the object $\mathbb{Z}(-d)[-2d]$ is a Chow motive, (iii) the heart of the weight structure on $DM_{gm}(k)$ is stable under duality, hence for any natural number n, induction on n shows that the dual of an object of the intersection $DM_{gm}(k)_{w\geq 0} \cap DM_{gm}(k)_{w\leq n}$ belongs to $DM_{gm}(k)_{w\geq -n} \cap DM_{gm}(k)_{w\leq 0}$, (iv) the weight structure on $DM^{eff}_{gm}(k)$ is induced from the weight structure on $DM_{gm}(k)$ (Theorem 1.13 (b)). **q.e.d.**

Remark 1.15. (a) Corollary 1.14 (a) and its proof should be compared to the construction of the weight complex $W(X)$ from [GiSo, Sect. 2.1]. Let $\widetilde{j} : X_\bullet \to \widetilde{X}_\bullet$ be a smooth hyper-envelope (in the sense of [loc. cit.]) of a compactification of X. Then it should be possible to employ [V1, Thm. 4.1.2] in order to show that the motive with compact support $M^c_{gm}(X)$ is isomorphic to the (opposite of the) complex $S(\widetilde{j})$ from [GiSo, Sect. 2.1] representing $W(X)$. The statement

$$M^c_{gm}(X) \in DM^{eff}_{gm}(k)_{w\geq 0} \cap DM^{eff}_{gm}(k)_{w\leq d}$$

from Corollary 1.14 (a) should be compared to [GiSo, Thm. 2 (i)].
(b) Similarly, the construction from [GuNA, Thm. (5.10) (3)] of the object $h(X)$ should be compared to Corollary 1.14 (b).

Corollary 1.16. *Assume k to admit resolution of singularities. Suppose given a direct factor M of $M_{gm}(X)$, for $X \in Sm/k$, which is abstractly isomorphic to a direct factor of $M^c_{gm}(Y)$, for some $Y \in Sch/k$. Then M is an effective Chow motive.*

2 Weight zero

Throughout this section, we fix a weight structure $w = (\mathcal{C}_{w\leq 0}, \mathcal{C}_{w\geq 0})$ on a triangulated category $\mathcal{C}$. Anticipating the situation which will be of interest in our applications, we formulate Assumption 2.3 on the cone of a morphism u in $\mathcal{C}$. As we shall see (Theorem 2.4), this hypothesis ensures in particular unique factorization of u through an object of the heart $\mathcal{C}_{w=0}$.

Definition 2.1. Denote by $\mathcal{C}_{w\leq 0, \neq -1}$ the full sub-category of $\mathcal{C}_{w\leq 0}$ of objects without weight -1, and by $\mathcal{C}_{w\geq 0, \neq 1}$ the full sub-category of $\mathcal{C}_{w\geq 0}$ of objects without weight 1.

Proposition 2.2. *(a) The inclusion of the heart* $\iota_- : \mathcal{C}_{w=0} \hookrightarrow \mathcal{C}_{w\leq 0,\neq -1}$ *admits a left adjoint*

$$\mathrm{Gr}_0 : \mathcal{C}_{w\leq 0,\neq -1} \longrightarrow \mathcal{C}_{w=0} \ .$$

On objects, it is given by sending M *to the term* $M_{\geq 0}$ *of a weight filtration*

$$M_{\leq -2} \longrightarrow M \longrightarrow M_{\geq 0} \longrightarrow M_{\leq -2}[1]$$

avoiding weight -1. *The composition* $\mathrm{Gr}_0 \circ \iota_-$ *equals the identity on* $\mathcal{C}_{w=0}$.
(b) The inclusion of the heart $\iota_+ : \mathcal{C}_{w=0} \hookrightarrow \mathcal{C}_{w\geq 0,\neq 1}$ *admits a right adjoint*

$$\mathrm{Gr}_0 : \mathcal{C}_{w\geq 0,\neq 1} \longrightarrow \mathcal{C}_{w=0} \ .$$

On objects, it is given by sending M *to the term* $M_{\leq 0}$ *of a weight filtration*

$$M_{\leq 0} \longrightarrow M \longrightarrow M_{\geq 2} \longrightarrow M_{\leq 0}[1]$$

avoiding weight 1. *The composition* $\mathrm{Gr}_0 \circ \iota_+$ *equals the identity on* $\mathcal{C}_{w=0}$.

Proof. Given Proposition 1.7 and Corollary 1.9, all that remains to be proved is that the objects $M_{\geq 0}$ (in (a)) resp. $M_{\leq 0}$ (in (b)) actually do lie in $\mathcal{C}_{w=0}$. But this follows from Proposition 1.4. **q.e.d.**

Let us now fix the following data.

(1) A morphism $u : M_- \to M_+$ in $\mathcal{C}$ between $M_- \in \mathcal{C}_{w\leq 0}$ and $M_+ \in \mathcal{C}_{w\geq 0}$.

(2) An exact triangle

$$C \stackrel{v_-}{\longrightarrow} M_- \stackrel{u}{\longrightarrow} M_+ \stackrel{v_+}{\longrightarrow} C[1]$$

in $\mathcal{C}$. Thus, the object $C[1]$ is a fixed choice of cone of u.

We make the following rather restrictive hypothesis.

Assumption 2.3. The object C is without weights -1 and 0, i.e., it admits a weight filtration

$$C_{\leq -2} \stackrel{c_-}{\longrightarrow} C \stackrel{c_+}{\longrightarrow} C_{\geq 1} \stackrel{\delta_C}{\longrightarrow} C_{\leq -2}[1]$$

avoiding weights -1 and 0.

The validity of this assumption is independent of the choice of C. Here is our main technical tool.

Theorem 2.4. *Fix the data (1), (2), and suppose Assumption 2.3.*
(a) The object M_- *is without weight* -1, *and* M_+ *is without weight* 1.
(b) The morphisms $v_- \circ c_- : C_{\leq -2} \to M_-$ *and* $\pi_0 : M_- \to \mathrm{Gr}_0 M_-$ *resp.* $i_0 : \mathrm{Gr}_0 M_+ \to M_+$ *and* $(c_+[1]) \circ v_+ : M_+ \to C_{\geq 1}[1]$ *can be canonically extended to exact triangles*

$$(3) \qquad C_{\leq -2} \stackrel{v_- c_-}{\longrightarrow} M_- \stackrel{\pi_0}{\longrightarrow} \mathrm{Gr}_0 M_- \stackrel{\delta_-}{\longrightarrow} C_{\leq -2}[1]$$

and

$$(4) \qquad C_{\geq 1} \xrightarrow{\delta_+} \mathrm{Gr}_0\, M_+ \xrightarrow{i_0} M_+ \xrightarrow{(c_+[1])v_+} C_{\geq 1}[1] \ .$$

Thus, (3) *is a weight filtration of* M_- *avoiding weight* -1*, and* (4) *is a weight filtration of* M_+ *avoiding weight* 1*.*

(c) There is a canonical isomorphism $\mathrm{Gr}_0\, M_- \xrightarrow{\sim} \mathrm{Gr}_0\, M_+$*. As a morphism, it is uniquely determined by the property of making the diagram*

$$\begin{array}{ccc} M_- & \xrightarrow{u} & M_+ \\ {\scriptstyle \pi_0}\downarrow & & \uparrow{\scriptstyle i_0} \\ \mathrm{Gr}_0\, M_- & \longrightarrow & \mathrm{Gr}_0\, M_+ \end{array}$$

commute. Its inverse makes the diagram

$$\begin{array}{ccc} C_{\geq 1} & \xrightarrow{\delta_C} & C_{\leq -2}[1] \\ {\scriptstyle \delta_+}\downarrow & & \uparrow{\scriptstyle \delta_-} \\ \mathrm{Gr}_0\, M_+ & \longrightarrow & \mathrm{Gr}_0\, M_- \end{array}$$

commute.

Proof. We start by choosing and fixing exact triangles

$$(3') \qquad C_{\leq -2} \xrightarrow{v_- c_-} M_- \xrightarrow{\pi'_0} G_- \xrightarrow{\delta_-} C_{\leq -2}[1]$$

and

$$(4') \qquad C_{\geq 1} \xrightarrow{\delta_-} G_+ \xrightarrow{i'_0} M_+ \xrightarrow{(c_+[1])v_+} C_{\geq 1}[1] \ .$$

Thus, G_- is a cone of $v_- \circ c_-$, and G_+ a cone of $c_+ \circ v_+[-1]$. Observe first that by Proposition 1.4,

$$G_- \in \mathcal{C}_{w \leq 0} \quad \text{and} \quad G_+ \in \mathcal{C}_{w \geq 0} \ .$$

Given this, the existence of *some* isomorphism $G_- \cong G_+$ clearly implies parts (a) and (b) of our statement.

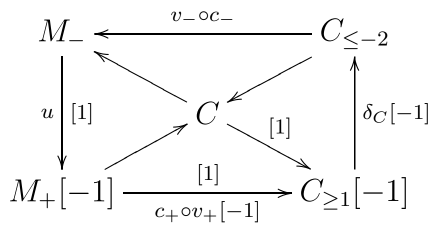
Let us now show that there is an isomorphism $\alpha : G_- \xrightarrow{\sim} G_+$ making the diagrams

$$\begin{array}{ccc} M_- & \xrightarrow{u} & M_+ \\ {\scriptstyle \pi'_0}\downarrow & & \uparrow{\scriptstyle i'_0} \\ G_- & \xrightarrow{\alpha} & G_+ \end{array}$$

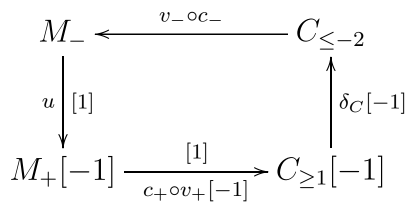
and

$$\begin{array}{ccc} C_{\geq 1} & \xrightarrow{\delta_C} & C_{\leq -2}[1] \\ {\scriptstyle \delta_+}\downarrow & & \uparrow{\scriptstyle \delta_-} \\ G_+ & \xrightarrow{\alpha^{-1}} & G_- \end{array}$$

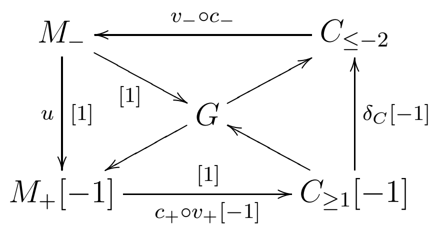
commute. To see this, consider the following.



The three arrows marked [1] link the source to the shift by [1] of the target, the upper and lower triangles are commutative, and the left and right triangles are exact. In the terminology of [BBD, Sect. 1], this is a *calotte inférieure*, which thanks to the axiom TR4' of triangulated categories in the formulation of [BBD, 1.1.6] can be completed to an octahedron. In particular, its contour



is part of a *calotte supérieure*:



Here, the upper and lower triangles are exact, and the left and right triangles are commutative. Hence the same object $G[1]$ can be chosen as cone of $v_- \circ c_-$ and of $c_+ \circ v_+[-1]$, and in addition, such that the morphisms u and δ_C factor through $G[1]$. For our fixed choices of cones, this means precisely that there is an isomorphism $G_- \cong G_+$ factorizing both u and δ_C.

As observed before, this implies (a) and (b). It remains to show the unicity statement from (c). For this, use the exact triangle (3) and apply Proposition 2.2 (b) to see that

$$\mathrm{Hom}_{\mathcal{C}}\big(M_-, M_+\big) = \mathrm{Hom}_{\mathcal{C}}\big(\mathrm{Gr}_0\, M_-, M_+\big) = \mathrm{Hom}_{\mathcal{C}_{w=0}}\big(\mathrm{Gr}_0\, M_-, \mathrm{Gr}_0\, M_+\big) \ .$$

Under this identification, a morphism $M_- \to M_+$ is sent to its unique factorization $\mathrm{Gr}_0\, M_- \to \mathrm{Gr}_0\, M_+$. **q.e.d.**

Given Theorem 2.4 (c), we may and do identify $\mathrm{Gr}_0\, M_-$ and $\mathrm{Gr}_0\, M_+$.

Corollary 2.5. *Fix the data (1), (2), and suppose Assumption 2.3. Let* $M_- \to N \to M_+$ *be a factorization of* u *through an object* N *of* $\mathcal{C}_{w=0}$. *Then*

$\mathrm{Gr}_0 M_- = \mathrm{Gr}_0 M_+$ *is canonically identified with a direct factor of* N*, admitting a canonical direct complement.*

Proof. By Proposition 2.2, the morphism $M_- \to N$ factors uniquely through $\mathrm{Gr}_0 M_-$, and $N \to M_+$ factors uniquely through $\mathrm{Gr}_0 M_+$. The composition $\mathrm{Gr}_0 M_- \to N \to \mathrm{Gr}_0 M_+$ is therefore a factorization of u. By Theorem 2.4 (c), it thus equals the canonical identification $\mathrm{Gr}_0 M_- = \mathrm{Gr}_0 M_+$. Hence $\mathrm{Gr}_0 M_- = \mathrm{Gr}_0 M_+$ is a retract of N. Consider a cone of $\mathrm{Gr}_0 M_- \to N$ in $\mathcal{C}$:

$$\mathrm{Gr}_0 M_- \longrightarrow N \longrightarrow P \longrightarrow \mathrm{Gr}_0 M_-[1]$$

The exact triangle is split in the sense that $P \to \mathrm{Gr}_0 M_-[1]$ is zero. Hence the morphism $N \to P$ admits a right inverse $P \to N$, unique up to morphisms $P \to \mathrm{Gr}_0 M_-$. There is therefore a unique right inverse $i : P \to N$ such that its composition with the projection $p : N \to \mathrm{Gr}_0 M_+ = \mathrm{Gr}_0 M_-$ is zero. The image of i is then a kernel of p, whose existence is thus established. This is the canonical complement of $\mathrm{Gr}_0 M_- = \mathrm{Gr}_0 M_+$ in N.

As a retract of N, the object P belongs to $\mathcal{C}_{w=0}$ (condition 1.1 (1)).

q.e.d.

For future use, we check the compatibility of Assumption 2.3 with tensor products. Assume therefore that our category $\mathcal{C}$ is tensor triangulated. Thus, a bilinear bifunctor

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

is given, and it is assumed to be triangulated in both arguments. Assume also that the weight structure w is compatible with $\otimes$, i.e., that

$$\mathcal{C}_{w \le 0} \otimes \mathcal{C}_{w \le 0} \subset \mathcal{C}_{w \le 0} \quad \text{and} \quad \mathcal{C}_{w \ge 0} \otimes \mathcal{C}_{w \ge 0} \subset \mathcal{C}_{w \ge 0} \; .$$

It follows that the heart $\mathcal{C}_{w=0}$ is a tensor category. Now fix a second set of data as above.

(1') A morphism $u' : M'_- \to M'_+$ in $\mathcal{C}$ between $M'_- \in \mathcal{C}_{w \le 0}$ and $M'_+ \in \mathcal{C}_{w \ge 0}$.

(2') An exact triangle

$$C' \stackrel{v'_-}{\longrightarrow} M'_- \stackrel{u'}{\longrightarrow} M'_+ \stackrel{v'_+}{\longrightarrow} C'[1] \; .$$

Fix an exact triangle

$$D \longrightarrow M_- \otimes M'_- \stackrel{u \otimes u'}{\longrightarrow} M_+ \otimes M'_+ \longrightarrow D[1] \; .$$

Proposition 2.6. *If* C *and* C' *are without weights* -1 *and* 0*, then so is* D*. In other words, the validity of Assumption 2.3 for* u *and* u' *implies the validity of Assumption 2.3 for* $u \otimes u'$*.*

Proof. We leave it to the reader to first construct an exact triangle

$$D \longrightarrow M_+ \otimes C' \stackrel{v_+\otimes v'_-}{\longrightarrow} C \otimes M'_-[1] \longrightarrow D[1] ,$$

i.e., to show that $D[1]$ is isomorphic to the cone of the morphism $v_+ \otimes v'_-$. Then, consider the morphisms

$$\delta_- \otimes v'_- c'_- : \mathrm{Gr}_0 M_- \otimes C'_{\leq -2} \longrightarrow C_{\leq -2} \otimes M'_-[1]$$

and

$$(c_+[1])v_+ \otimes \delta'_+ : M_+ \otimes C'_{\geq 1} \longrightarrow C_{\geq 1} \otimes \mathrm{Gr}_0 M'_+[1]$$

(notation as in Theorem 2.4). They are completed to give exact triangles

$$D_{\leq -2} \longrightarrow \mathrm{Gr}_0 M_- \otimes C'_{\leq -2} \longrightarrow C_{\leq -2} \otimes M'_-[1] \longrightarrow D_{\leq -2}[1]$$

and

$$D_{\geq 1} \longrightarrow M_+ \otimes C'_{\geq 1} \longrightarrow C_{\geq 1} \otimes \mathrm{Gr}_0 M'_+[1] \longrightarrow D_{\leq 1}[1] .$$

By compatibility of w and $\otimes$, and by Proposition 1.4, the object $D_{\leq -2}$ is of weights ≤ -2, and $D_{\geq 1}$ is of weights ≥ 1. Finally, it remains to construct an exact triangle

$$D_{\leq -2} \longrightarrow D \longrightarrow D_{\geq 1} \longrightarrow D_{\leq -2}[1] .$$

We leave this to the reader (hint: use Theorem 2.4 (c)). **q.e.d.**

Corollary 2.7. *Under the hypotheses of Proposition 2.6, the canonical morphisms*

$$\mathrm{Gr}_0\big(M_- \otimes M'_-\big) \longrightarrow \mathrm{Gr}_0 M_- \otimes \mathrm{Gr}_0 M'_-$$

and

$$\mathrm{Gr}_0 M_+ \otimes \mathrm{Gr}_0 M'_+ \longrightarrow \mathrm{Gr}_0\big(M_+ \otimes M'_+\big)$$

are isomorphisms.

3 Example: motives for modular forms

In his article [Scho1], Scholl constructs the Grothendieck motive $M(f)$ for elliptic normalized newforms f of fixed level n and weight $w = k + 2$, for positive integers $n \geq 3$ and $k \geq 1$. It is a direct factor of a Grothendieck motive, which underlies a Chow motive denoted ${}^k_n\mathcal{W}$ in [loc. cit.] (this Chow motive depends only on n and k).

In order to establish the relation of ${}^k_n\mathcal{W}$ to the theory of weights, let us begin by setting up the notation. It is identical to the one introduced in [Scho1], up to one exception: the letter M used in [loc. cit.] to denote certain sub-schemes of the modular curve will change to S in order to avoid confusion with the motivic notation used earlier in the present paper. Thus, for our fixed $n \geq 3$ and $k \geq 1$, let $S_n \in Sm/\mathbb{Q}$ denote the modular curve parametrizing elliptic curves with level n structure, $j : S_n \hookrightarrow \overline{S}_n$ its smooth compactification, and S_n^∞ the complement of S_n in $\overline{S}_n$. Thus, S_n^∞ is of dimension zero. Write $X_n \to S_n$ for the universal elliptic curve, and $\overline{X}_n \to \overline{S}_n$ for the universal generalized elliptic curve. Thus, $\overline{X}_n$ is smooth and proper over $\mathbb{Q}$. The k-fold fibre product $\overline{X}_n^k := \overline{X}_n \times_{\overline{S}_n} \times \ldots \times_{\overline{S}_n} \overline{X}_n$ of $\overline{X}_n$ over $\overline{S}_n$ is singular for $k \geq 2$, and can be desingularized canonically [D1, Lemmas 5.4, 5.5] (see also [Scho1, Sect. 3]). Denote by $\overline{\overline{X}}_n^k$ this desingularization ($\overline{\overline{X}}_n^k = \overline{X}_n^k$ for $k = 1$). Write X_n^k for the k-fold fibre product X_n over S_n. The symmetric group $\mathfrak{S}_k$ acts on $\overline{X}_n^k$ by permutations, the k-th power of the group $\mathbb{Z}/n\mathbb{Z}$ by translations, and the k-th power of the group μ_2 by inversion in the fibres. Altogether [Scho1, Sect. 1.1.1], this gives a canonical action of the semi-direct product

$$\Gamma_k := \left((\mathbb{Z}/n\mathbb{Z})^2 \rtimes \mu_2\right)^k \rtimes \mathfrak{S}_k$$

by automorphisms on $\overline{X}_n^k$. By the canonical nature of the desingularisation, this extends to an action of Γ_k by automorphisms on $\overline{\overline{X}}_n^k$. Of course, this action respects the open sub-scheme X_n^k of $\overline{\overline{X}}_n^k$.

As in [Scho1, Sect. 1.1.2], let $\varepsilon : \Gamma_k \to \{\pm 1\}$ be the morphism which is trivial on $(\mathbb{Z}/n\mathbb{Z})^{2k}$, is the product map on μ_2^k, and is the sign character on $\mathfrak{S}_k$.

Definition 3.1. (a) Let F denote the $\mathbb{Z}$-algebra $\mathbb{Z}[1/(2n \cdot k!)]$.
(b) Let e denote the idempotent in the group ring $F[\Gamma_k]$ associated to ε :

$$e := \frac{1}{(2n^2)^k \cdot k!} \sum_{\gamma \in \Gamma_k} \varepsilon(\gamma)^{-1} \cdot \gamma = \frac{1}{(2n^2)^k \cdot k!} \sum_{\gamma \in \Gamma_k} \varepsilon(\gamma) \cdot \gamma$$

(observe that $\varepsilon^{-1} = \varepsilon$).

Let M be an object of an F-linear pseudo-Abelian category. If M comes equipped with an action of the group Γ_k, let us agree to denote by M^e the direct factor of M on which Γ_k acts via ε (in other words, the image of e). Let us define the object ${}_n^k\mathcal{W}$ as in [Scho1, 1.2.2].

Definition 3.2. Denote by ${}_n^k\mathcal{W} := M_{gm}\big(\overline{\overline{X}}_n^k\big)^e \in DM_{gm}^{eff}(\mathbb{Q})_F$ the image of the idempotent e on $M_{gm}\big(\overline{\overline{X}}_n^k\big)$.

Given that $\overline{\overline{X}}_n^k$ is smooth and proper, we see that ${}_n^k\mathcal{W}$ is an effective Chow motive over $\mathbb{Q}$. As above, denote by $M_{gm}\big(X_n^k\big)^e$ and $M_{gm}^c\big(X_n^k\big)^e$ the images of e on $M_{gm}\big(X_n^k\big)$ and $M_{gm}^c\big(X_n^k\big)$, respectively. The following can be seen as a translation into the language of geometrical motives of the detailed analysis from [Scho1, Sect. 2, 3] of the geometry of the boundary of $\overline{\overline{X}}{}_n^k$.

Theorem 3.3. *(a) The motive $M_{gm}^c\big(X_n^k\big)^e$ is without weight 1. In particular, the object $\mathrm{Gr}_0\, M_{gm}^c\big(X_n^k\big)^e$ is defined.*
(b) The restriction

$$j_n^{k,*} : {}_n^k\mathcal{W} = M_{gm}\big(\overline{\overline{X}}{}_n^k\big)^e \longrightarrow M_{gm}^c\big(X_n^k\big)^e$$

induced by the open immersion j_n^k of X_n^k into $\overline{\overline{X}}{}_n^k$ factors canonically through an isomorphism

$$\mathrm{Gr}_0\, j_n^{k,*} : {}_n^k\mathcal{W} \xrightarrow{\ \sim\ } \mathrm{Gr}_0\, M_{gm}^c\big(X_n^k\big)^e \, .$$

(c) There is an exact triangle in $DM_{gm}^{eff}(\mathbb{Q})_F$

$$C_k \xrightarrow{\delta_+} \mathrm{Gr}_0\, M_{gm}^c(X_n^k)^e \xrightarrow{i_0} M_{gm}^c(X_n^k)^e \xrightarrow{p_+} C_k[1] \, ,$$

where

$$C_k = M_{gm}\big(S_n^\infty\big)[k]$$

is pure of weight k. The exact triangle is canonical up to a replacement of the triple of morphisms (δ_+, i_0, p_+) by $((-1)^k\delta_+, i_0, (-1)^k p_+)$.

Duality for smooth schemes [V1, Thm. 4.3.7 3] immediately implies the following.

Corollary 3.4. *(a) The motive $M_{gm}\big(X_n^k\big)^e$ is without weight -1. In particular, the object $\mathrm{Gr}_0\, M_{gm}\big(X_n^k\big)^e$ is defined.*
(b) The morphism

$$j_n^k : M_{gm}\big(X_n^k\big)^e \longrightarrow M_{gm}\big(\overline{\overline{X}}{}_n^k\big)^e = {}_n^k\mathcal{W}$$

factors canonically through an isomorphism

$$\mathrm{Gr}_0\, j_n^k : \mathrm{Gr}_0\, M_{gm}\big(X_n^k\big)^e \xrightarrow{\ \sim\ } {}_n^k\mathcal{W} \, .$$

(c) There is an exact triangle in $DM_{gm}^{eff}(\mathbb{Q})_F$

$$C_{-(k+1)} \xrightarrow{\iota_-} M_{gm}(X_n^k)^e \xrightarrow{\pi_0} \mathrm{Gr}_0\, M_{gm}(X_n^k)^e \xrightarrow{\delta_-} C_{-(k+1)}[1] \, ,$$

where

$$C_{-(k+1)} = M_{gm}\big(S_n^\infty\big)(k+1)[k+1]$$

is pure of weight $-(k+1)$. The exact triangle is canonical up to a replacement of the triple of morphisms $(\iota_-, \pi_0, \delta_-)$ by $((-1)^k\iota_-, \pi_0, (-1)^k\delta_-)$.

Remark 3.5. It is well known that the motive $M_{gm}\big(S_n^\infty\big)$ is isomorphic to a finite sum of copies of $M_{gm}(\mathbf{Spec}\,\mathbb{Q}(\mu_n))$.

Note that Theorem 3.3 (b) and Corollary 3.4 (b) together imply the following.

Corollary 3.6. *The canonical morphism* $M_{gm}\big(X_n^k\big)^e \to M_{gm}^c\big(X_n^k\big)^e$ *factors canonically through an isomorphism*

$$\mathrm{Gr}_0\, M_{gm}\big(X_n^k\big)^e \xrightarrow{\ \sim\ } \mathrm{Gr}_0\, M_{gm}^c\big(X_n^k\big)^e \; .$$

For any object M of $DM_{gm}^{eff}(\mathbb{Q})_F$, define motivic cohomology

$$H^r_{\mathcal{M}}\big(M, F(s)\big) := \mathrm{Hom}_{DM_{gm}^{eff}(\mathbb{Q})_F}\big(M, \mathbb{Z}(s)[r]\big) \; .$$

When $M = M_{gm}(Y)$ for a scheme $Y \in Sm/\mathbb{Q}$, this gives motivic cohomology $H^r_{\mathcal{M}}\big(Y, \mathbb{Z}(s)\big)$ of Y, tensored with $F = \mathbb{Z}[1/(2n \cdot k!)]$. Thus for example,

$$H^{k+1}_{\mathcal{M}}\big(C_{-(k+1)}, F(k+1)\big) = H^0_{\mathcal{M}}\big(S_n^\infty, \mathbb{Z}(0)\big) \otimes_{\mathbb{Z}} F \; .$$

Similarly,

$$H^{k+2}_{\mathcal{M}}\big(C_{-(k+1)}, F(k+\ell+2)\big) = H^1_{\mathcal{M}}\big(S_n^\infty, \mathbb{Z}(\ell+1)\big) \otimes_{\mathbb{Z}} F$$

for any integer ℓ. We get the following refinement of [Scho1, Cor. 1.4.1].

Corollary 3.7. *Let* $\ell \geq 0$ *be a second integer. Then the kernel of the morphism*

$$H_{\mathcal{M}}(\iota_-) : \big(H^{k+2}_{\mathcal{M}}\big(X_n^k, \mathbb{Z}(k+\ell+2)\big) \otimes_{\mathbb{Z}} F\big)^e \longrightarrow H^1_{\mathcal{M}}\big(S_n^\infty, \mathbb{Z}(\ell+1)\big) \otimes_{\mathbb{Z}} F$$

equals $H^{k+2}_{\mathcal{M}}\big(\mathrm{Gr}_0\, M_{gm}(X_n^k)^e, F(k+\ell+2)\big)$.

Proof. This follows from the exact triangle of Corollary 3.4 (c), and from the vanishing of $H^0_{\mathcal{M}}\big(S_n^\infty, \mathbb{Z}(\ell+1)\big)$ (since $\ell+1 \geq 1$). **q.e.d.**

Recall that following ideas of Beilinson [B], this result can be employed as follows: using the *Eisenstein symbol* defined in [loc. cit.] one constructs elements in $\big(H^{k+2}_{\mathcal{M}}\big(X_n^k, \mathbb{Z}(k+\ell+2)\big) \otimes_{\mathbb{Z}} \mathbb{Q}\big)^e$. By Corollary 3.7, linear combinations of such elements vanishing under $H_{\mathcal{M}}(\iota_-)$ lie in the sub-$\mathbb{Q}$-vector space $H^{k+2}_{\mathcal{M}}\big(\mathrm{Gr}_0\, M_{gm}(X_n^k)^e, \mathbb{Q}(k+\ell+2)\big)$. It can then be shown that there are sufficiently many such linear combinations, and that they have the relation to the leading coefficient at $s = -\ell$ of the L-function of $\mathrm{Gr}_0\, M_{gm}(X_n^k)^e$, predicted by Beilinson's conjecture. For details, see forthcoming work of Scholl [Scho2].

Proof of Theorem 3.3. Denote by $\overline{\overline{X}}_n^{k,\infty}$ the complement of the smooth scheme X_n^k in the smooth and proper scheme $\overline{\overline{X}}_n^k$. Localization for the motive with compact support [V1, Prop. 4.1.5] shows that there is a canonical exact triangle

$$M_{gm}\big(\overline{\overline{X}}_n^k\big)^e \xrightarrow{j_n^{k,*}} M_{gm}^c(X_n^k)^e \longrightarrow M_{gm}\big(\overline{\overline{X}}_n^{k,\infty}\big)^e[1] \longrightarrow M_{gm}\big(\overline{\overline{X}}_n^k\big)^e[1] \; .$$

Following the strategy from [Scho1], we shall show the following claim.

$$(C) \qquad M_{gm}\big(\overline{\overline{X}}_n^{k,\infty}\big)^e = M_{gm}^c\big(\overline{\overline{X}}_n^{k,\infty}\big)^e \xrightarrow{\ \sim\ } M_{gm}^c\big(S_n^\infty\big)[k] = M_{gm}\big(S_n^\infty\big)[k]$$

canonically up to a sign $(-1)^k$. In particular, the motive $M_{gm}(\overline{\overline{X}}_n^{k,\infty})^e$ is pure of weight $k \geq 1$. Claim (C) implies that the above exact triangle is a weight filtration of $M_{gm}^c(X_n^k)^e$ avoiding weight 1. Furthermore, the restriction $j_n^{k,*}$ identifies $M_{gm}(\overline{\overline{X}}_n^k)^e$ with $\mathrm{Gr}_0 M_{gm}^c(X_n^k)^e$.

To show claim (C), observe first that the motivic version of [Scho1, Statement 1.3.0] remains valid: for any $S \in Sm/\mathbb{Q}$, there is a decomposition in $DM_{gm}^{eff}(\mathbb{Q})_F$ (in fact, already in $DM_{gm}^{eff}(\mathbb{Q})_{\mathbb{Z}[1/2]}$)

$$M_{gm}^c(\mathbb{G}_m \times_{\mathbb{Q}} S) \cong M_{gm}^c(S)(1)[2] \oplus M_{gm}^c(S)[1] ,$$

such that inversion $x \mapsto x^{-1}$ on $\mathbb{G}_m$ acts on the first factor by $+1$, and on the second by -1. The projection onto the first factor is canonical, and the projection onto the second factor is canonical up to a sign. Proof: localization [V1, Prop. 4.1.5] for the inclusion of $\mathbb{G}_m$ into the projective line; the choice of the second projection is equivalent to the choice of one of the residue morphisms to 0 or ∞. Fix one of the two choices of projection

$$\pi_- : M_{gm}^c(\mathbb{G}_m) \longrightarrow\!\!\!\!\!\rightarrow M_{gm}^c(\mathbf{Spec}\,\mathbb{Q})[1]$$

(and use the same notation for $M_{gm}^c(\mathbb{G}_m \times_{\mathbb{Q}} S) \longrightarrow\!\!\!\!\!\rightarrow M_{gm}^c(S)[1]$ obtained by base change via S). Then to define the morphism

$$j^{0,*} : M_{gm}^c(\overline{\overline{X}}_n^{k,\infty})^e \longrightarrow M_{gm}^c(S_n^\infty)[k] ,$$

consider the following.

(i) The intersection $\overline{\overline{X}}_n^{k,\infty,\mathrm{reg}}$ of $\overline{\overline{X}}_n^{k,\infty}$ with the non-singular part $\overline{X}_n^{k,\mathrm{reg}}$ of $\overline{X}_n^k$. Explanation: by [Scho1, Thm. 3.1.0 ii)], the desingularization $\overline{\overline{X}}_n^k \longrightarrow\!\!\!\!\!\rightarrow \overline{X}_n^k$ is an isomorphism over $\overline{X}_n^{k,\mathrm{reg}}$. Thus, $\overline{\overline{X}}_n^{k,\infty,\mathrm{reg}}$ is an open sub-scheme of $\overline{\overline{X}}_n^{k,\infty}$,

(ii) the neutral component $\overline{\overline{X}}_n^{k,\infty,0}$ of $\overline{\overline{X}}_n^{k,\infty,\mathrm{reg}}$, i.e., its intersection with the Néron model of X_n^k. Thus, $\overline{\overline{X}}_n^{k,\infty,0}$ is an open sub-scheme of $\overline{\overline{X}}_n^{k,\infty,\mathrm{reg}}$,

(iii) the identification of $\overline{\overline{X}}_n^{k,\infty,0}$ with the k-th power (over the base S_n^∞) of the neutral component $\overline{\overline{X}}_n^{1,\infty,0}$ for $k = 1$. The latter can be identified with $\mathbb{G}_m \times_{\mathbb{Q}} S_n^\infty$, canonically up to an automorphism $x \mapsto x^{-1}$. Hence $\overline{\overline{X}}_n^{k,\infty,0} \cong \mathbb{G}_m^k \times_{\mathbb{Q}} S_n^\infty$, canonically up to an automorphism

$$(x_1, \ldots, x_k) \longmapsto (x_1, \ldots, x_k)^{-1} .$$

The three steps (i)–(iii) give an open immersion

$$j^0 : \mathbb{G}_m^k \times_{\mathbb{Q}} S_n^\infty \hookrightarrow \overline{\overline{X}}_n^{k,\infty} ,$$

which by contravariance of M_{gm}^c induces a morphism

$$j^{0,*} : M_{gm}^c\bigl(\overline{\overline{X}}_n^{k,\infty}\bigr)^e \longrightarrow M_{gm}^c\bigl(\mathbb{G}_m^k \times_{\mathbb{Q}} S_n^\infty\bigr) \; .$$

Its composition with the k-th power of π_- gives the desired morphism

$$M_{gm}^c\bigl(\overline{\overline{X}}_n^{k,\infty}\bigr)^e \longrightarrow M_{gm}^c\bigl(S_n^\infty\bigr)[k] \; ,$$

equally denoted $j^{0,*}$, and canonical up to a sign $(-1)^k$. In the context of twisted Poincaré duality theories (H^*, H_*), Scholl's main technical result [Scho1, Thm. 1.3.3] is equivalent to stating that the morphism induced by $j^{0,*}$ on the level of the theory H_* is an isomorphism.

Our observation is simply that the same proof as the one given in [loc. cit.] runs through, with (H^*, H_*) replaced by (M_{gm}, M_{gm}^c).

More precisely, [Scho1, Lemma 1.3.1] holds for M_{gm}^c, and hence the proof of [Scho1, Prop. 2.4.1] runs through for M_{gm}^c. The latter result implies that the schemes occurring in a suitable stratification of the complement of $\overline{\overline{X}}_n^{k,\infty,\mathrm{reg}}$ in $\overline{\overline{X}}_n^{k,\infty}$ all have trivial $M_{gm}^{c,e}$ (cmp. [Scho1, proof of Thm. 3.1.0 ii)]). This shows that step (i) induces an isomorphism

$$M_{gm}^c\bigl(\overline{\overline{X}}_n^{k,\infty}\bigr)^e \stackrel{\sim}{\longrightarrow} M_{gm}^c\bigl(\overline{\overline{X}}_n^{k,\infty,\mathrm{reg}}\bigr)^e \; .$$

To deal with step (ii), one shows that the group Γ_k acts transitively on the set of components of the singular part of $\overline{\overline{X}}_n^{k,\infty,\mathrm{reg}}$, and that the stabilizer of each component admits a subgroup of order two acting trivially on the component, but having trivial intersection with the kernel of ε (cmp. [Scho1, proof of Thm. 3.1.0 iii)]). This shows first that the singular part of $\overline{\overline{X}}_n^{k,\infty,\mathrm{reg}}$ does not contribute to $M_{gm}^{c,e}$, and then that

$$M_{gm}^c\bigl(\overline{\overline{X}}_n^{k,\infty,\mathrm{reg}}\bigr)^e \longrightarrow M_{gm}^c\bigl(\overline{\overline{X}}_n^{k,\infty,0}\bigr)^{e'}$$

is an isomorphism, where e' denotes the projection onto the eigenspace for the restriction of ε to the subgroup $\mu_2^k \rtimes \mathfrak{S}_k$ of Γ_k. To conclude, we apply the motivic version of [Scho1, Lemma 1.3.1] to see that $\pi_-^{\otimes k}$ induces an isomorphism $M_{gm}^c\bigl(\mathbb{G}_m^k \times_{\mathbb{Q}} S_n^\infty\bigr)^{e'} \stackrel{\sim}{\longrightarrow} M_{gm}^c\bigl(S_n^\infty\bigr)[k]$. **q.e.d.**

Remark 3.8. (a) The proof of Theorem 3.3 shows that the open immersion of the non-singular part $\overline{X}_n^{k,\mathrm{reg}}$ of $\overline{X}_n^k$ into $\overline{\overline{X}}_n^k$ induces an isomorphism

$$M_{gm}\bigl(\overline{\overline{X}}_n^k\bigr)^e \stackrel{\sim}{\longrightarrow} M_{gm}^c\bigl(\overline{X}_n^{k,\mathrm{reg}}\bigr)^e \; .$$

In particular, $M_{gm}^c\bigl(\overline{X}_n^{k,\mathrm{reg}}\bigr)^e$ is a Chow motive. By duality for smooth schemes [V1, Theorem 4.3.7 3],

$$M_{gm}\bigl(\overline{X}_n^{k,\mathrm{reg}}\bigr)^e \longrightarrow M_{gm}\bigl(\overline{\overline{X}}_n^k\bigr)^e$$

is an isomorphism, too, and hence so is the canonical morphism

$$M_{gm}\big(\overline{X}_n^{k,\mathrm{reg}}\big)^e \longrightarrow M_{gm}^c\big(\overline{X}_n^{k,\mathrm{reg}}\big)^e \; .$$

The construction of the motives $M(f)$ can therefore also be done using the smooth non-proper scheme $\overline{X}_n^{k,\mathrm{reg}}$ instead of $\overline{\overline{X}}_n^k$.

(b) A slightly closer look at the proof of [Scho1, Thm. 3.1.0 ii)] reveals that the open immersion of $\overline{X}_n^{k,\mathrm{reg}}$ into $\overline{X}_n^k$ also induces an isomorphism

$$M_{gm}\big(\overline{X}_n^k\big)^e \xrightarrow{\ \sim\ } M_{gm}^c\big(\overline{X}_n^{k,\mathrm{reg}}\big)^e \; .$$

By (a), $M_{gm}\big(\overline{X}_n^k\big)^e = M_{gm}^c\big(\overline{X}_n^k\big)^e$ is a Chow motive. The construction of the motives $M(f)$ can therefore also be done using the non-smooth (for $k \geq 2$) proper scheme $\overline{X}_n^k$ instead of $\overline{\overline{X}}_n^k$.

4 Weights, boundary motive and interior motive

This section contains our main result (Theorem 4.3). We list its main consequences, and define in particular the motivic analogue of (certain direct factors of) interior cohomology (Definition 4.9). Throughout, we assume k to admit resolution of singularities.

Let us fix $X \in Sm/k$. The *boundary motive* $\partial M_{gm}(X)$ of X [W1, Def. 2.1] fits into a canonical exact triangle

$$(*) \qquad \partial M_{gm}(X) \longrightarrow M_{gm}(X) \longrightarrow M_{gm}^c(X) \longrightarrow \partial M_{gm}(X)[1]$$

in $DM_{gm}^{eff}(k)$. The algebra of *finite correspondences* $c(X,X)$ acts on $M_{gm}(X)$ [V1, p. 190]. Denote by ${}^t c(X,X)$ the transposed algebra: a cycle $\mathfrak{Z}$ on $X \times_k X$ lies in ${}^t c(X,X)$ if and only if ${}^t\mathfrak{Z} \in c(X,X)$. The definition of composition of correspondences [loc. cit.] shows that the intersection $c(X,X) \cap {}^t c(X,X)$ acts on $M_{gm}^c(X)$.

Definition 4.1. (a) Define the algebra $c_{1,2}(X,X)$ as the intersection of the algebras $c(X,X)$ and ${}^t c(X,X)$. As an Abelian group, $c_{1,2}(X,X)$ is thus free on the symbols (Z), where Z runs through the integral closed subschemes of $X \times_k X$, such that both projections to the components X are finite on Z, and map Z surjectively to a connected component of X. Multiplication in $c_{1,2}(X,X)$ is defined by composition of correspondences as in [V1, p. 190].

(b) Denote by ^{t} the canonical anti-involution on $c_{1,2}(X,X)$ mapping a cycle $\mathfrak{Z}$ to ${}^t\mathfrak{Z}$.

It results directly from the definitions that the algebra $c_{1,2}(X,X)$ acts on the triangle $(*)$ in the sense that it acts on the three objects, and the morphisms are $c_{1,2}(X,X)$-equivariant. Denote by $\bar{c}_{1,2}(X,X)$ the quotient of $c_{1,2}(X,X)$ by the kernel of this action. Fix a commutative flat $\mathbb{Z}$-algebra F, and an idempotent e in $\bar{c}_{1,2}(X,X) \otimes_{\mathbb{Z}} F$. Denote by $M_{gm}(X)^e$, $M^c_{gm}(X)^e$ and $\partial M_{gm}(X)^e$ the images of e on $M_{gm}(X)$, $M^c_{gm}(X)$ and $\partial M_{gm}(X)$, respectively, considered as objects of the category $DM^{eff}_{gm}(k)_F$. We are ready to set up the data (1), (2) considered in Section 2, for $\mathcal{C} := DM^{eff}_{gm}(k)_F$.

(1) The morphism u is the morphism $M_{gm}(X)^e \to M^c_{gm}(X)^e$. By Corollary 1.14 and condition 1.1 (1), the object $M_{gm}(X)^e$ belongs indeed to $DM^{eff}_{gm}(k)_{F,w\le 0}$, and $M^c_{gm}(X)^e$ to $DM^{eff}_{gm}(k)_{F,w\ge 0}$.

(2) Our choice of cone of u is $\partial M_{gm}(X)^e[1]$, together with the exact triangle

$$\partial M_{gm}(X)^e \xrightarrow{v_-} M_{gm}(X)^e \xrightarrow{u} M^c_{gm}(X)^e \xrightarrow{v_+} \partial M_{gm}(X)^e[1]$$

in $DM^{eff}_{gm}(k)_F$ induced by $(*)$.

Observe that our data (1), (2) are stable under the natural action of

$$GCen_{\bar{c}_{1,2}(X,X)}(e) := \left\{ z \in \bar{c}_{1,2}(X,X) \otimes_{\mathbb{Z}} F \ , \ ze = eze \right\} .$$

In particular, they are stable under the action of the centralizer $Cen_{\bar{c}_{1,2}(X,X)}(e)$ of e in $\bar{c}_{1,2}(X,X) \otimes_{\mathbb{Z}} F$. In this context, Assumption 2.3 reads as follows.

Assumption 4.2. The direct factor $\partial M_{gm}(X)^e$ of the boundary motive of X is without weights -1 and 0.

Thus, we may and do fix a weight filtration

$$C_{\le -2} \xrightarrow{c_-} \partial M_{gm}(X)^e \xrightarrow{c_+} C_{\ge 1} \xrightarrow{\delta_C} C_{\le -2}[1]$$

avoiding weights -1 and 0. Theorem 2.4, Corollary 2.5 and the adjunction property from Proposition 2.2 then give the following.

Theorem 4.3. *Fix the data (1), (2), and suppose Assumption 4.2.*
(a) The motive $M_{gm}(X)^e$ is without weight -1, and the motive $M^c_{gm}(X)^e$ is without weight 1. In particular, the effective Chow motives $\mathrm{Gr}_0 M_{gm}(X)^e$ and $\mathrm{Gr}_0 M^c_{gm}(X)^e$ are defined, and they carry a natural action of $GCen_{\bar{c}_{1,2}(X,X)}(e)$.
(b) There are canonical exact triangles

$$(3) \qquad C_{\le -2} \xrightarrow{v_- c_-} M_{gm}(X)^e \xrightarrow{\pi_0} \mathrm{Gr}_0 M_{gm}(X)^e \xrightarrow{\delta_-} C_{\le -2}[1]$$

and

$$(4) \qquad C_{\ge 1} \xrightarrow{\delta_+} \mathrm{Gr}_0 M^c_{gm}(X)^e \xrightarrow{i_0} M^c_{gm}(X)^e \xrightarrow{(c_+[1])v_+} C_{\ge 1}[1] ,$$

which are stable under the natural action of $GCen_{\bar{c}_{1,2}(X,X)}(e)$.
(c) There is a canonical isomorphism $\mathrm{Gr}_0 M_{gm}(X)^e \xrightarrow{\sim} \mathrm{Gr}_0 M^c_{gm}(X)^e$ in

$CHM^{eff}(k)_F$. As a morphism, it is uniquely determined by the property of making the diagram

$$\begin{array}{ccc} M_{gm}(X)^e & \xrightarrow{u} & M^c_{gm}(X)^e \\ {\scriptstyle \pi_0}\downarrow & & \uparrow{\scriptstyle i_0} \\ \mathrm{Gr}_0\, M_{gm}(X)^e & \longrightarrow & \mathrm{Gr}_0\, M^c_{gm}(X)^e \end{array}$$

commute; in particular, it is $GCen_{\bar{c}_{1,2}(X,X)}(e)$-equivariant. Its inverse makes the diagram

$$\begin{array}{ccc} C_{\ge 1} & \xrightarrow{\delta_C} & C_{\le -2}[1] \\ {\scriptstyle \delta_+}\downarrow & & \uparrow{\scriptstyle \delta_-} \\ \mathrm{Gr}_0\, M^c_{gm}(X)^e & \longrightarrow & \mathrm{Gr}_0\, M_{gm}(X)^e \end{array}$$

commute.

(d) Let $N \in CHM(k)_F$ be a Chow motive. Then π_0 and i_0 induce isomorphisms

$$\mathrm{Hom}_{CHM(k)_F}\big(\mathrm{Gr}_0\, M_{gm}(X)^e, N\big) \xrightarrow{\ \sim\ } \mathrm{Hom}_{DM_{gm}(k)_F}\big(M_{gm}(X)^e, N\big)$$

and

$$\mathrm{Hom}_{CHM(k)_F}\big(N, \mathrm{Gr}_0\, M^c_{gm}(X)^e\big) \xrightarrow{\ \sim\ } \mathrm{Hom}_{DM_{gm}(k)_F}\big(N, M^c_{gm}(X)^e\big)\ .$$

(e) Let $M_{gm}(X)^e \to N \to M^c_{gm}(X)^e$ be a factorization of u through a Chow motive $N \in CHM(k)_F$. Then $\mathrm{Gr}_0\, M_{gm}(X)^e = \mathrm{Gr}_0\, M^c_{gm}(X)^e$ is canonically a direct factor of N, with a canonical direct complement.

We explicitly mention the following immediate consequence of Theorem 4.3.

Corollary 4.4. *Fix X and e, and suppose that $\partial M_{gm}(X)^e = 0$, i.e., that*

$$u : M_{gm}(X)^e \xrightarrow{\ \sim\ } M^c_{gm}(X)^e\ .$$

Then $M_{gm}(X)^e \cong M^c_{gm}(X)^e$ are effective Chow motives.

Of course, this also follows from Corollary 1.16. The author knows of no proof of Corollary 4.4 "avoiding weights" when $e \ne 1$. (For $e = 1$, we leave it to the reader to show (using for example [V1, Cor. 4.2.5]) that the assumption $\partial M_{gm}(X) = 0$ is equivalent to X being proper.)

Remark 4.5. It is not difficult to see that Assumption 4.2 is actually implied by parts (a) and (c) of Theorem 4.3.

Henceforth, we identify $\mathrm{Gr}_0\, M_{gm}(X)^e$ and $\mathrm{Gr}_0\, M^c_{gm}(X)^e$ via the canonical isomorphism of Theorem 4.3 (c).

Corollary 4.6. *In the situation considered in Theorem 4.3, let $\widetilde{X}$ be any smooth compactification of X. Then $\mathrm{Gr}_0\, M_{gm}(X)^e$ is canonically a direct factor of the Chow motive $M_{gm}(\widetilde{X})$, with a canonical direct complement.*

Proof. Indeed, the morphism u factors canonically through $M_{gm}(\widetilde{X})$:

$$M_{gm}(X)^e \hookrightarrow M_{gm}(X) \longrightarrow M_{gm}(\widetilde{X}) \longrightarrow M_{gm}^c(X) \twoheadrightarrow M_{gm}^c(X)^e \ .$$

Hence we may apply Theorem 4.3 (e). **q.e.d.**

Recall that the category $CHM(k)_F$ is pseudo-Abelian. Thus, the construction of a sub-motive of $M_{gm}(\widetilde{X})$ does not *a priori* necessitate the *identification*, but only the *existence* of a complement. In our situation, Corollary 4.6 states that the complement of $\operatorname{Gr}_0 M_{gm}(X)^e$ *is* canonical. This shows that Assumption 4.2 is indeed rather restrictive, an observation confirmed by part (c) of the following results on the Hodge theoretic and ℓ-adic realizations ([H, Cor. 2.3.5, Cor. 2.3.4 and Corrigendum]; see [DGo, Sect. 1.5] for a simplification of this approach). They can be seen as applications of the *cohomological weight spectral sequence* [Bo2, Thm. 2.4.1, Rem. 2.4.2] in a very special case.

Theorem 4.7. *Keep the situation considered in Theorem 4.3. Assume that k can be embedded into the field $\mathbb{C}$ of complex numbers. Fix one such embedding. Let H^* be the cohomological Hodge theoretic realization, i.e., the functor on $DM_{gm}^{eff}(k)_F$ given by Betti cohomology of the topological space of $\mathbb{C}$-valued points, tensored with $\mathbb{Q} \otimes_{\mathbb{Z}} F$, and with its natural mixed Hodge structure. Let $n \in \mathbb{N}$.*

(a) The morphisms π_0 and i_0 induce isomorphisms

$$H^n\big(\operatorname{Gr}_0 M_{gm}(X)^e\big) \stackrel{\sim}{\longrightarrow} W_n H^n\big(M_{gm}(X)^e\big) = \big(W_n H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Z}} F\big)^e$$

and

$$\frac{\big(H_c^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Z}} F\big)^e}{\big(W_{n-1} H_c^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Z}} F\big)^e} = \frac{H^n(M_{gm}^c(X)^e)}{W_{n-1} H^n(M_{gm}^c(X)^e)} \stackrel{\sim}{\longrightarrow} H^n\big(\operatorname{Gr}_0 M_{gm}(X)^e\big).$$

Here, W_r denotes the r-th filtration step of the weight filtration of a mixed Hodge structure (thus, the weights of $H^n(X(\mathbb{C}), \mathbb{Q})$ are $\geq n$, and those of $H_c^n(X(\mathbb{C}), \mathbb{Q})$ are $\leq n$).

(b) The isomorphisms of (a) identify $H^n(\operatorname{Gr}_0 M_{gm}(X)^e)$ with the image of the natural morphism

$$\big(H_c^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Z}} F\big)^e \longrightarrow \big(H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Z}} F\big)^e \ .$$

(c) The image of $(H_c^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Z}} F)^e$ in $(H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Z}} F)^e$ equals the lowest weight filtration step W_n of $(H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Z}} F)^e$.

The reader should be aware that the algebra $\bar{c}_{1,2}(X, X)$ acts contravariantly on Betti cohomology $H^n(X(\mathbb{C}), \mathbb{Q})$. The same remark applies of course to the ℓ-adic realization, which we consider now.

Theorem 4.8. *Keep the situation considered in Theorem 4.3, and fix a prime $\ell > 0$. Assume that k is finitely generated over its prime field, and of characteristic zero. Let H^* be the cohomological ℓ-adic realization, i.e., the* *functor on $DM^{eff}_{gm}(k)_F$ given by ℓ-adic cohomology of the base change to a fixed algebraic closure $\bar{k}$ of k, tensored with $\mathbb{Q}_\ell \otimes_{\mathbb{Z}} F$, and with its natural action of the absolute Galois group G_k of k. Let $n \in \mathbb{N}$.*

(a) The morphisms π_0 and i_0 induce isomorphisms

$$H^n\big(\mathrm{Gr}_0 M_{gm}(X)^e\big) \xrightarrow{\sim} W_n H^n\big(M_{gm}(X)^e\big) = \big(W_n H^n(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}} F\big)^e$$

and

$$\frac{\big(H^n_c(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}} F\big)^e}{\big(W_{n-1} H^n_c(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}} F\big)^e} = \frac{H^n(M^c_{gm}(X)^e)}{W_{n-1} H^n(M^c_{gm}(X)^e)} \xrightarrow{\sim} H^n\big(\mathrm{Gr}_0 M_{gm}(X)^e\big) \; .$$

Here, W_r denotes the r-th filtration step of the weight filtration of a G_k-module, and $X_{\bar{k}}$ denotes the base change $X \otimes_k \bar{k}$ of X to $\bar{k}$ (thus, the weights of $H^n(X_{\bar{k}}, \mathbb{Q}_\ell)$ are $\geq n$, and those of $H^n_c(X_{\bar{k}}, \mathbb{Q}_\ell)$ are $\leq n$).

(b) The isomorphisms of (a) identify $H^n(\mathrm{Gr}_0 M_{gm}(X)^e)$ with the image of the natural morphism

$$\big(H^n_c(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}} F\big)^e \longrightarrow \big(H^n(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}} F\big)^e \; .$$

(c) The image of $(H^n_c(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}} F)^e$ in $(H^n(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}} F)^e$ equals the lowest weight filtration step W_n of $(H^n(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}} F)^e$.

Definition 4.9. Fix the data (1), (2), and suppose Assumption 4.2. We call $\mathrm{Gr}_0 M_{gm}(X)^e$ the *e-part of the interior motive of X*.

This terminology is motivated by parts (b) of Theorems 4.7 and 4.8, which show that the realizations of $\mathrm{Gr}_0 M_{gm}(X)^e$ are classes of complexes, whose cohomology equals the part of the interior cohomology of X fixed by e.

Remark 4.10. The author ignores whether for general $Y \in Sm/k$ it is possible (or even reasonable to expect) to find a complex computing interior (Betti or ℓ-adic) cohomology of Y, and through which the natural morphism $R\Gamma_c(Y) \to R\Gamma(Y)$ factors.

Remark 4.11. When k is a number field, Theorems 4.7 (b) and 4.8 (b) tell us in particular that the L-function of the Chow motive $\mathrm{Gr}_0 M_{gm}(X)^e$ is computed via (the e-part of) interior cohomology of X.

Example 4.12. Let C be a smooth projective curve, and $P \in C$ a k-rational point. Put $X := C - P$. Localization [V1, Prop. 4.1.5] shows that there is an exact triangle

$$M_{gm}(P) \longrightarrow M_{gm}(C) \longrightarrow M^c_{gm}(X) \longrightarrow M_{gm}(P)[1] \; .$$

The morphism $M_{gm}(P) \to M_{gm}(C)$ is split; hence $M^c_{gm}(X)$ is a direct factor of $M_{gm}(C)$. It is therefore a Chow motive. By duality [V1, Theorem 4.3.7 3], the same is then true for $M_{gm}(X)$. In particular, both $M_{gm}(X)$ and $M^c_{gm}(X)$ are pure of weight zero. But the morphism $u : M_{gm}(X) \to M^c_{gm}(X)$ is not an isomorphism (look at degree 0 or -2, or check that the conclusions of

Theorems 4.7 (c) and 4.8 (c) do not hold). Therefore, Assumption 4.2 is not fulfilled for $e = 1$. Of course, this can be seen directly: the boundary motive $\partial M_{gm}(X)$ has a weight filtration

$$\mathbb{Z}(1)[1] \longrightarrow \partial M_{gm}(X) \longrightarrow \mathbb{Z}(0) \longrightarrow \mathbb{Z}(1)[2]$$

(which is necessarily split since there are no non-trivial morphisms from $\mathbb{Z}(0)$ to $\mathbb{Z}(1)[2]$). Orthogonality 1.1 (3) then shows that there are no non-trivial morphisms from an object of weights ≤ -2 to $\partial M_{gm}(X)$, and no non-trivial morphisms from $\partial M_{gm}(X)$ to an object of weights ≥ 1. The object $\partial M_{gm}(X)$ being non-trivial, we conclude that it does not admit a weight filtration avoiding weights -1 and 0.

More generally, we have the following, which again illustrates just how restrictive Assumption 4.2 is.

Proposition 4.13. *Fix data (1) and (2) as before. Assume that X admits a smooth compactification $\widetilde{X}$ such that the complement $Y = \widetilde{X} - X$ is smooth. Then the following statements are equivalent.*

(i) Assumption 4.2 is valid, i.e., the object $\partial M_{gm}(X)^e$ is without weights -1 and 0.

(ii) The object $\partial M_{gm}(X)^e$ is trivial (hence the conclusion of Corollary 4.4 holds).

Proof. Statement (ii) clearly implies (i). In order to show that it is implied by (i), let us show that the hypothesis on $\widetilde{X}$ and Y forces the boundary motive $\partial M_{gm}(X)$ to lie in the intersection

$$DM^{eff}_{gm}(k)_{w \ge -1} \cap DM^{eff}_{gm}(k)_{w \le 0} \ .$$

By orthogonality 1.1 (3), the same is then true for its direct factor $\partial M_{gm}(X)^e$. Thus, the only way for $\partial M_{gm}(X)^e$ to avoid weights -1 and 0 is to be trivial (again by orthogonality).

In order to show our claim, apply [W1, Prop. 2.4] to see that $\partial M_{gm}(X)$ is isomorphic to the shift by $[-1]$ of a choice of cone of the canonical morphism

$$M_{gm}(Y) \oplus M_{gm}(X) \longrightarrow M_{gm}(\widetilde{X}) \ .$$

In particular, there is a morphism $c_+ : \partial M_{gm}(X) \to M_{gm}(Y)$, and an exact triangle

$$(W) \qquad C_- \longrightarrow \partial M_{gm}(X) \xrightarrow{c_+} M_{gm}(Y) \longrightarrow C_-[1] \ ,$$

where C_- equals the shift by $[-1]$ of a cone of

$$M_{gm}(X) \longrightarrow M_{gm}(\widetilde{X}) \ .$$

By assumption, Y is smooth and proper, hence $M_{gm}(Y)$, as a Chow motive, is pure of weight 0. Duality for smooth schemes [V1, Theorem 4.3.7 3] shows that C_- is pure of weight -1. Hence (W) is a weight filtration of $\partial M_{gm}(X)$ by objects of weights -1 and 0. Proposition 1.4 then shows that $\partial M_{gm}(X)$ belongs indeed to

$$DM_{gm}^{eff}(k)_{w\geq -1} \cap DM_{gm}^{eff}(k)_{w\leq 0} .$$

q.e.d.

Proof of Theorems 4.7 and 4.8. Consider the exact triangle

$$(3) \qquad C_{\leq -2} \longrightarrow M_{gm}(X)^e \xrightarrow{\pi_0} \operatorname{Gr}_0 M_{gm}(X)^e \longrightarrow C_{\leq -2}[1]$$

from Theorem 4.3 (b). Recall that as suggested by the notation, the motive $C_{\leq -2}$ is of weights ≤ -2. The cohomological functor H^* transforms it into a long exact sequence

$$H^{n-1}\big(C_{\leq -2}\big) \longrightarrow H^n\big(\operatorname{Gr}_0 M_{gm}(X)^e\big) \xrightarrow{H^n(\pi_0)} H^n\big(M_{gm}(X)^e\big) \longrightarrow H^n\big(C_{\leq -2}\big) .$$

The essential information we need to use is that H^n transforms Chow motives into objects which are pure of weight n, for any $n \in \mathbb{N}$. Since $C_{\leq -2}$ admits a filtration whose cones are Chow motives sitting in degrees ≥ 2, the object $H^n(C_{\leq -2})$ admits a filtration whose graded pieces are of weights $\geq n+2$. Since our coefficients are $\mathbb{Q}$-vector spaces, there are no non-trivial morphisms between objects of disjoint weights. The above long exact sequence then shows that

$$H^n(\pi_0) : H^n\big(\operatorname{Gr}_0 M_{gm}(X)^e\big) \longrightarrow H^n\big(M_{gm}(X)^e\big)$$

is injective, and its image is identical to the part of weight n of $H^n(M_{gm}(X)^e)$. This shows the part of claim (a) concerning π_0.

Now recall that the realization functor is compatible with the tensor structures [H, Cor. 2.3.5, Cor. 2.3.4], and sends the Tate motive $\mathbb{Z}(1)$ to the dual of the Tate object [H, Thm. 2.3.3]. It follows that it is compatible with duality. Since on the one hand, $M_{gm}(X)$ and $M_{gm}^c(X)$ are in duality [V1, Theorem 4.3.7 3], and on the other hand, the same is true for Betti, resp. ℓ-adic cohomology and Betti, resp. ℓ-adic cohomology with compact support, we see that H^n sends the motive with compact support $M_{gm}^c(X)$ to cohomology with compact support H_c^n of X.

Now repeat the above argument for the exact triangle

$$(4) \qquad C_{\geq 1} \longrightarrow \operatorname{Gr}_0 M_{gm}^c(X)^e \xrightarrow{i_0} M_{gm}^c(X)^e \longrightarrow C_{\geq 1}[1]$$

from Theorem 4.3 (b). This shows the remaining part of claim (a).

Claims (b) and (c) follow, once we observe that the composition of $H^n(i_0)$ and $H^n(\pi_0)$ equals the canonical morphism from cohomology with compact support to cohomology without support. **q.e.d.**

Corollary 4.6 allows to say more about the étale realizations.

Theorem 4.14. *Keep the situation considered in Theorem 4.3, and fix a prime $\ell > 0$. Assume that k is the quotient field of a Dedekind domain A of* *characteristic zero, and fix a non-zero prime ideal $\mathfrak{p}$ of A. Let p denote its residue characteristic.*

(a) Assume that $p \neq \ell$. Then a sufficient condition for the ℓ-adic realization $H^*(\mathrm{Gr}_0 M_{gm}(X)^e)$ to be unramified at $\mathfrak{p}$ is the existence of some smooth compactification of X having good reduction at $\mathfrak{p}$. A sufficient condition for $H^*(\mathrm{Gr}_0 M_{gm}(X)^e)$ to be semi-stable at $\mathfrak{p}$ is the existence of some smooth compactification of X having simple semi-stable reduction at $\mathfrak{p}$.

(b) Assume that $p = \ell$, and that the residue field $A/\mathfrak{p}$ is perfect. Then a sufficient condition for the p-adic realization $H^*(\mathrm{Gr}_0 M_{gm}(X)^e)$ to be crystalline at $\mathfrak{p}$ is the existence of some smooth compactification of X having good reduction at $\mathfrak{p}$. A sufficient condition for $H^*(\mathrm{Gr}_0 M_{gm}(X)^e)$ to be semi-stable at $\mathfrak{p}$ is the existence of some smooth compactification of X having simple semi-stable reduction at $\mathfrak{p}$.

Note that one may be able to choose different compactifications for different primes $\mathfrak{p}$.

Proof of Theorem 4.14. By Corollary 4.6, $\mathrm{Gr}_0 M_{gm}(X)^e$ is a direct factor of the motive $M_{gm}(\widetilde{X})$ of any smooth compactification $\widetilde{X}$ of X. Hence $H^*(\mathrm{Gr}_0 M_{gm}(X)^e)$ is a direct factor of the cohomology of any such $\widetilde{X}$.

Part (a) uses the spectral sequence [D2, Scholie 2.5] relating cohomology with coefficients in the vanishing cycle sheaves ψ^q to cohomology of the generic fibre $\widetilde{X}$. By [RZ, Kor. 2.25] ([D2, Thm. 3.3] when $p = 0$), our assumption on the reduction of $\widetilde{X}$ implies that the inertia group acts trivially on the ψ^q. It therefore acts unipotently on the cohomology of $\widetilde{X}$. Part (b) follows from the C_{st}-conjecture, proved by Tsuji [T, Thm. 0.2] (see also [N] for a proof via K-theory). **q.e.d.**

By [V1, Thm. 4.3.7], the category $DM_{gm}(k)_F$ is a rigid tensor triangulated category. Furthermore, if X is of pure dimension n, then the objects $M_{gm}(X)$ and $M^c_{gm}(X)(-n)[-2n]$ are canonically dual to each other. By [W1, Thm. 6.1], the boundary motive $\partial M_{gm}(X)$ is canonically dual to $\partial M_{gm}(-n)[-(2n-1)]$. Furthermore, these dualities fit together to give an identification of the dual of the exact triangle

$$(*) \qquad \partial M_{gm}(X) \longrightarrow M_{gm}(X) \longrightarrow M^c_{gm}(X) \longrightarrow \partial M_{gm}(X)[1]$$

and the exact triangle $(*)(-n)[-2n]$. The construction of the duality isomorphism [loc. cit.] shows that the dual of the action of the algebra $\bar{c}_{1,2}(X,X)$ on $(*)$ equals the natural (anti-)action given by the composition of the canonical action on $(*)(-n)[-2n]$, preceded by the anti-involution ^{t}. Consider the idempotent ${}^te$ (i.e., the transposition of e).

Proposition 4.15. *(a) Assumption 4.2 is equivalent to any of the following statements.*

(i) Both $\partial M_{gm}(X)^e$ and $\partial M_{gm}(X)^{{}^te}$ are without weight -1.

(ii) Both $\partial M_{gm}(X)^e$ *and* $\partial M_{gm}(X)^{{}^te}$ *are without weight* 0.

In particular, Assumption 4.2 is satisfied for e if and only if it is satisfied for the transposition ${}^te$.
(b) Assume that e is symmetric, i.e., that $e = {}^te$. *Then Assumption 4.2 is equivalent to any of the following statements.*

(i) $\partial M_{gm}(X)^e$ *is without weight* -1.

(ii) $\partial M_{gm}(X)^e$ *is without weight* 0.

Proof. Indeed, $\partial M_{gm}(X)^e$ is dual to $\partial M_{gm}(X)^{{}^te}(-n)[-(2n-1)]$. Now observe that $\mathbb{Z}(-n)[-(2n-1)]$ is pure of weight 1. Then use unicity of weight filtrations avoiding weight -1 resp. weight 0 (Corollary 1.9). **q.e.d.**

Example 4.16. Assume that an abstract group G acts by automorphisms on X.
(a) The action of G translates into a morphism of algebras $\mathbb{Z}[G] \to c_{1,2}(X,X)$. This morphism transforms the natural anti-involution * of $\mathbb{Z}[G]$ induced by $g \mapsto g^{-1}$ into the anti-involution ^{t} of $c_{1,2}(X,X)$.
(b) Let F be a flat $\mathbb{Z}$-algebra, and e an idempotent in $F[G]$. The morphism of (a) then allows to consider the image of e (equally denoted by e) in $c_{1,2}(X,X)$, then in $\bar{c}_{1,2}(X,X)$, and to ask whether Assumption 4.2 is valid for e.
(c) As a special case of (b), consider the case when G is finite, its order r is invertible in F, and e is the idempotent in $F[G]$ associated to a character ε on G with values in the multiplicative group F^*:

$$e = \frac{1}{r}\sum_{g\in G} \varepsilon(g)^{-1}\cdot g \; .$$

Observe that the idempotent $e \in \bar{c}_{1,2}(X,X)$ is symmetric if $\varepsilon^{-1} = \varepsilon$.
(d) Let us consider the situation from Section 3, and show that Assumption 4.2 is satisfied for $X = X_n^k \in Sm/\mathbb{Q}$ and

$$e = \frac{1}{(2n^2)^k \cdot k!}\sum_{\gamma\in\Gamma_k} \varepsilon(\gamma)^{-1}\cdot \gamma \; .$$

As in Section 3, denote by $\overline{\overline{X}}_n^{k,\infty}$ the complement of X_n^k in $\overline{\overline{X}}_n^k$. By [W1, Prop. 2.4], the object $\partial M_{gm}(X_n^k)^e$ is canonically isomorphic to the shift by $[-1]$ of a canonical choice of cone of the canonical morphism

$$M_{gm}\big(\overline{\overline{X}}_n^{k,\infty}\big)^e \oplus M_{gm}(X_n^k)^e \longrightarrow M_{gm}\big(\overline{\overline{X}}_n^k\big)^e \; .$$

In particular, there is a canonical morphism $c_+ : \partial M_{gm}(X_n^k)^e \to M_{gm}\big(\overline{\overline{X}}_n^{k,\infty}\big)^e$, and an exact triangle

$$C_- \longrightarrow \partial M_{gm}(X_n^k)^e \stackrel{c_+}{\longrightarrow} M_{gm}\big(\overline{\overline{X}}_n^{k,\infty}\big)^e \longrightarrow C_-[1] \; ,$$

where C_- equals the shift by $[-1]$ of a cone of

$$j_n^k : M_{gm}(X_n^k)^e \longrightarrow M_{gm}\big(\overline{\overline{X}}_n^k\big)^e \ .$$

By Corollary 3.4 (b) and (c),

$$C_- \cong M_{gm}\big(S_n^\infty\big)(k+1)[k+1]$$

is pure of weight $-(k+1)$. It follows from this and from Corollary 1.14 (a) that the exact triangle

$$C_- \longrightarrow \partial M_{gm}(X_n^k)^e \xrightarrow{c_+} M_{gm}\big(\overline{\overline{X}}_n^{k,\infty}\big)^e = M_{gm}^c\big(\overline{\overline{X}}_n^{k,\infty}\big)^e \longrightarrow C_-[1]$$

is a weight filtration of $\partial M_{gm}(X_n^k)^e$ avoiding weights $-k, \ldots, -1$, and hence in particular, avoiding weight -1 (since $k \geq 1$). Our claim then follows from Proposition 4.15 (b) (observe that e is symmetric).

(Alternatively, use Claim (C) of the proof of Theorem 3.3, to see directly that the last term of the weight filtration of $\partial M_{gm}(X_n^k)^e$,

$$M_{gm}\big(\overline{\overline{X}}_n^{k,\infty}\big)^e \cong M_{gm}\big(S_n^\infty\big)[k]$$

is pure of weight $k \geq 1$.)

Remark 4.17. Let us agree to forget the results from Section 3, and see what the theory developed in the present section implies in the situation studied in Example 4.16 (d). We only use the validity of Assumption 4.2.

(a) It follows formally from Theorem 4.3 (a)–(c) that $\mathrm{Gr}_0 M_{gm}(X_n^k)^e$ and $\mathrm{Gr}_0 M_{gm}^c(X_n^k)^e$ are defined, and canonically isomorphic, and that there are exact triangles

$$C_- \longrightarrow M_{gm}(X_n^k)^e \xrightarrow{\pi_0} \mathrm{Gr}_0 M_{gm}(X_n^k)^e \xrightarrow{\delta_-} C_-[1]$$

and

$$M_{gm}\big(\overline{\overline{X}}_n^{k,\infty}\big)^e \xrightarrow{\delta_+} \mathrm{Gr}_0 M_{gm}^c(X_n^k)^e \xrightarrow{i_0} M_{gm}^c(X_n^k)^e \longrightarrow M_{gm}\big(\overline{\overline{X}}_n^{k,\infty}\big)^e[1] \ .$$

(b) It follows formally from Corollary 4.6 that $\mathrm{Gr}_0 M_{gm}(X_n^k)^e$ is a direct factor of ${}^k_n\mathcal{W} = M_{gm}(\overline{\overline{X}}_n^k)^e$, with a canonical complement. Call this complement N. It follows formally from Theorems 4.7 (b) and 4.8 (b) that the realizations of the motive $\mathrm{Gr}_0 M_{gm}\big(X_n^k\big)^e$ equal interior cohomology, i.e., the image of the morphism

$$H_c^n\big(X_n^k\big)^e \longrightarrow H^n\big(X_n^k\big)^e \ .$$

By [Scho1, Sect. 1.2.0, Thm. 1.2.1, Sect. 1.3.4], the same is true for ${}^k_n\mathcal{W}$. Therefore, the complement N has trivial realizations. Thus, its underlying Grothendieck motive is trivial. This means that the construction of the Grothendieck motives for modular forms $M(f)$ can be done replacing the Chow motive ${}^k_n\mathcal{W}$ by $\mathrm{Gr}_0 M_{gm}\big(X_n^k\big)^e$.

Remark 4.18. (a) Of course, none of the implications listed in the preceding remark are new: they are all consequences of Theorem 3.3, Corollary 3.4 and Corollary 3.6, whose proof involves the geometry of the boundary of the smooth compactification $\overline{\overline{X}}_n^k$ of X_n^k. Observe that some of these results were even used in our proof 4.16 (d) of the validity of Assumption 4.2. In other words, we applied the strategy from Remark 4.5, and proved Assumption 4.2 via parts (a) and (c) of Theorem 4.3.

(b) For Shimura varieties of higher dimension, Hecke-equivariant smooth compactifications (like $\overline{\overline{X}}_n^k$ in the case of powers of the universal elliptic curve over a modular curve) are not known (and maybe not reasonable to expect) to exist. In this more general setting, a different strategy of proof of Assumption 4.2 "avoiding geometry as far as possible" would therefore be of interest. Such a strategy, totally disjoint from the one from Remark 4.5, will be developed in [W2].

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