

**NORM VARIETIES AND THE CHAIN LEMMA
(AFTER MARKUS ROST)**

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The goal of this paper is to present proofs of two results of Markus Rost, the Chain Lemma 0.1 and the Norm Principle 0.3. These are the steps needed to complete the published verification of the *Bloch-Kato conjecture*, that the norm residue maps are isomorphisms $K_n^M(k)/p \xrightarrow{\cong} H_{\text{et}}^n(k, \mathbb{Z}/p)$ for every prime p , every n and every field k containing $1/p$.

Throughout this paper, p is a fixed odd prime, and k is a field of characteristic 0, containing the p -th roots of unity. We fix an integer $n \geq 2$ and an n -tuple (a_1, \dots, a_n) of units in k , such that the symbol $\{a\}$ is nontrivial in the Milnor K -group $K_n^M(k)/p$.

Associated to this data are several notions. A field F over k is a *splitting field* for $\{a\}$ if $\{a\}_F = 0$ in $K_n^M(F)/p$. A variety X over k is called a *splitting variety* if its function field is a splitting field; X is *p -generic* if any splitting field F has a finite extension E/F of degree prime to p with $X(E) \neq \emptyset$. A *Norm variety* for $\{a\}$ is a smooth projective p -generic splitting variety for $\{a\}$ of dimension $p^{n-1} - 1$.

The following sequence of theorems reduces the Bloch-Kato conjecture to the Chain Lemma 0.1 and the Norm Principle 0.3; the notion of a *Rost variety* is defined in 0.5 below; the definition of a *Rost motive* is given in [14] and [15], and will not be needed in this paper.

- (0) The Chain Lemma 0.1 and the Norm Principle 0.3 hold; this is proven here.
- (1) Given (0), Rost varieties exist; this is Theorem 0.7 below, and is proven in [10, p. 253].
- (2) If Rost varieties exist then Rost motives exist; this is proven in [15].
- (3) If Rost motives exist then Bloch-Kato is true; this is proven in [13] and [14].

Here is the statement of the Chain Lemma, which we quote from [10, 5.1] and prove in §5. A field is *p -special* if p divides the order of every finite field extension.

Theorem 0.1 (Rost's Chain Lemma). *Let $\{a\} \in K_n^M(k)/p$ be a nontrivial symbol, where k is a p -special field. Then there exists a smooth projective cellular variety S/k and a collection of invertible sheaves $J = J_1, J'_1, \dots, J_{n-1}, J'_{n-1}$ equipped with nonzero p -forms $\gamma = \gamma_1, \gamma'_1, \dots, \gamma_{n-1}, \gamma'_{n-1}$ satisfying the following conditions.*

- (1) $\dim S = p(p^{n-1} - 1) = p^n - p$;
- (2) $\{a_1, \dots, a_n\} = \{a_1, \dots, a_{n-2}, \gamma_{n-1}, \gamma'_{n-1}\} \in K_n^M(k(S))/p$,
 $\{a_1, \dots, a_{i-1}, \gamma_i\} = \{a_1, \dots, a_{i-2}, \gamma_{i-1}, \gamma'_{i-1}\} \in K_i^M(k(S))/p$ for $2 \leq i < n$.
In particular, $\{a_1, \dots, a_n\} = \{\gamma, \gamma'_1, \dots, \gamma'_{n-1}\} \in K_n^M(k(S))/p$;
- (3) $\gamma \notin \Gamma(S, J)^{\otimes(-p)}$, as is evident from (2);
- (4) for any $s \in V(\gamma_i) \cup V(\gamma'_i)$, the field $k(s)$ splits $\{a_1, \dots, a_n\}$;
- (5) $I(V(\gamma_i)) + I(V(\gamma'_i)) \subseteq p\mathbb{Z}$ for all i , as follows from (4);
- (6) $\deg(c_1(J)^{\dim S})$ is relatively prime to p .

Rost's Norm Principle concerns the group $\overline{A}_0(X, \mathcal{K}_1)$, which we now define.

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Definition 0.2. (Rost, [5]) For any regular scheme X , the group $A_0(X, \mathcal{K}_1)$ is defined to be the group generated by symbols $[x, \alpha]$, where x is a closed point of X and $\alpha \in k(x)^\times$, modulo the relations (i) $[x, \alpha][x, \alpha'] = [x, \alpha\alpha']$ and (ii) for every point y of dimension 1 the image of the tame symbol $K_2(k(y)) \rightarrow \oplus k(x)^\times$ is zero.

The functor $A_0(X, \mathcal{K}_1)$ is covariant in X for proper maps, because it is isomorphic to the motivic homology group $H_{-1, -1}(X) = \text{Hom}_{DM}(\mathbb{Z}, M(X)(1)[1])$ (see [10, 1.1]). It is also the K -cohomology group $H^d(X, \mathcal{K}_{d+1})$, where $d = \dim(X)$.

The reduced group $\overline{A}_0(X, \mathcal{K}_1)$ is defined to be the quotient of $A_0(X, \mathcal{K}_1)$ by the difference of the two projections from $A_0(X \times X, \mathcal{K}_1)$. As observed in [10, 1.2], there is a well defined map $N : \overline{A}_0(X, \mathcal{K}_1) \rightarrow k^\times$ sending $[x, \alpha]$ to the norm of α .

Theorem 0.3 (Norm Principle). *Suppose that k is a p -special field and that X is a Norm variety for some nontrivial symbol $\{\underline{a}\}$. Let $[z, \beta] \in \overline{A}_0(X, \mathcal{K}_1)$ be such that $[k(z) : k] = p^\nu$ for $\nu > 1$. Then there exists a point $x \in X$ with $[k(x) : k] = p$ and $\alpha \in k(x)^\times$ such that $[z, \beta] = [x, \alpha]$ in $\overline{A}_0(X, \mathcal{K}_1)$.*

We will prove the Norm Principle 0.3 in section 9 below.

Our proofs of these two results are based on 1998 Rost's preprint [7], his web site [6] and Rost's lectures [Rost] in 1999-2000 and 2005. The idea for writing up these notes in publishable form originated during his 2005 course, and was reinvigorated by conversations with Markus Rost at the Abel Symposium 2007 in Oslo. As usual, all mistakes in this paper are the responsibility of the authors.

Rost varieties. In the rest of this introduction, we explain how 0.1 and 0.3 imply the problematic Theorem 0.7, and hence complete the proof of the Bloch-Kato conjecture. We first recall the notions of a ν_i -variety and a Rost variety.

Let X be a smooth projective variety of dimension $d > 0$. Recall from [4, §16] that there is a characteristic class $s_d : K_0(X) \rightarrow \mathbb{Z}$ corresponding to the symmetric polynomial $\sum t_j^d$ in the Chern roots t_j of a bundle; we write $s_d(X)$ for s_d of the tangent bundle T_X . When $d = p^\nu - 1$, we know that $s_d(X) \equiv 0 \pmod{p}$; see [4, 16.6 and 16-E] and [9, pp. 128–9] or [1, II.7].

Definition 0.4. (see [10, 1.20]) A ν_{n-1} -variety over a field k is a smooth projective variety X of dimension $d = p^{n-1} - 1$, with $s_d(X) \not\equiv 0 \pmod{p^2}$.

For example, $s_d(\mathbb{P}^d) = d + 1$ by [4, 16.6]. Thus the projective space \mathbb{P}^{p-1} is a ν_1 -variety, and so is any Brauer-Severi variety of dimension $p - 1$. In Section 8, we will show that the bundle $\mathbb{P}(\mathcal{A})$ over S is a ν_n -variety.

Definition 0.5. A *Rost variety* for a sequence $\{\underline{a}\} = (a_1, \dots, a_n)$ of units in k is a ν_{n-1} -variety such that: $\{a_1, \dots, a_n\}$ vanishes in $K_n^M(k(X))/p$; for each $i < n$ there is a ν_i -variety mapping to X ; and the motivic homology sequence

$$(0.6) \quad H_{-1, -1}(X \times X) \xrightarrow{\pi_0^* - \pi_1^*}, H_{-1, -1}(X) \rightarrow H_{-1, -1}(k) \quad (= k^\times).$$

is exact. Part of Theorem 0.7 states that Rost varieties exist for every $\{\underline{a}\}$.

Remark 0.6.1. Rost originally defined a *Norm Variety* for $\{\underline{a}\}$ to be a projective splitting variety of dimension p^{n-1} which is a ν_{n-1} -variety. (See [Rost, 10/20/99].) Theorem 0.7(2) says that our definition agrees with Rost's when k is p -special.

Here is the statement of Theorem 0.7, quoted from [10, 1.21]. It assumes that the Bloch-Kato conjecture holds for $n - 1$.

Theorem 0.7. *Let $n \geq 2$ and $0 \neq \{\underline{a}\} = \{a_1, \dots, a_n\} \in K_n^M(k)/p$. Then:*

0) There exists a geometrically irreducible Norm variety for $\{\underline{a}\}$.

Assume further that k is p -special. If X is a Norm variety for $\{\underline{a}\}$, then:

1) X is geometrically irreducible.

2) X is a ν_{n-1} -variety.

3) each element of $\overline{A}_0(X, \mathcal{K}_1)$ is of the form $[x, \alpha]$, where $x \in X$ is a closed point of degree p and $\alpha \in k(x)^\times$.

The construction of geometrically irreducible Norm varieties was carried out in [10, pp. 254–256]; this proves part (0) of Theorem 0.7. Part (1) was proven in [10, 5.4]. Part (2) was proven in [10, 5.2], assuming Rost’s Chain Lemma (see 0.1), and part (3) was proven in [10, p. 271], assuming not only the Chain Lemma but also the Norm Principle (see 0.3 below).

As stated in the introduction of [10], the construction of Norm varieties and the proof of Theorem 0.7 are part of an inductive proof of the Bloch-Kato conjecture. We point out that in the present paper, the inductive assumption (that the Bloch-Kato conjecture for $n - 1$ holds) is never used. It only appears in [10] to prove that the candidates for norm varieties constructed there are p -generic splitting varieties. (However, the Norm Principle 0.3 is itself a statement about norm varieties.) In particular, the Chain Lemma 0.1 holds in all degrees independently of the Bloch-Kato conjecture.

1. FORMS ON VECTOR BUNDLES

We begin with a presentation of some well known facts about p -forms.

If V is a vector space over a field k , a p -form on V is a symmetric p -linear function on V , *i.e.*, a linear map $\phi : \text{Sym}^p(V) \rightarrow k$. It determines a p -ary form, *i.e.*, a function $\varphi : V \rightarrow k$ satisfying $\varphi(\lambda v) = \lambda^p \varphi(v)$, by $\varphi(v) = \phi(v, v, \dots, v)$. If $p!$ is invertible in k , p -linear forms are in 1–1 correspondence with p -ary forms.

If $V = k$ then every p -form may be written as $\varphi(\lambda) = a\lambda^p$ or $\phi(\lambda_1, \dots) = a \prod \lambda_i$ for some $a \in k$. Up to isometry, non-zero 1-dimensional p -forms are in 1–1 correspondence with elements of $k^\times/k^{\times p}$. Therefore an n -tuple of forms φ_i determine a well-defined element of $K_n^M(k)/p$ which we write as $\{\varphi_1, \dots, \varphi_n\}$.

Of course the notion of a p -form on a projective module over a commutative ring makes sense, but it is a special case of p -forms on locally free modules (algebraic vector bundles), which we now define.

Definition 1.1. If \mathcal{E} is a locally free \mathcal{O}_X -module over a scheme X then a p -form on \mathcal{E} is a symmetric p -linear function on \mathcal{E} , *i.e.*, a linear map $\phi : \text{Sym}^p(\mathcal{E}) \rightarrow \mathcal{O}_X$. If \mathcal{E} is invertible, we will sometimes identify the p -form with the diagonal p -ary form $\varphi = \phi \circ \Delta : \mathcal{E} \rightarrow \mathcal{O}_X$; locally, if v is a section generating \mathcal{E} then the form is determined by $a = \varphi(v)$: $\varphi(tv) = a t^p$.

Remark 1.1.1. The geometric vector bundle over a scheme X whose sheaf of sections is \mathcal{E} is $\mathbb{V} = \mathbf{Spec}(S^*(\mathcal{E}^\vee))$, where \mathcal{E}^\vee is the dual \mathcal{O}_X -module of \mathcal{E} . We will sometimes describe p -forms in terms of \mathbb{V} .

The projective space bundle associated to \mathcal{E} is $\pi : \mathbb{P}(\mathcal{E}) = \mathbf{Proj}(S^*) \rightarrow X$, $S^* = S^*(\mathcal{E}^\vee)$. The tautological line bundle on $\mathbb{P}(\mathcal{E})$ is $\mathbb{L} = \mathbf{Spec}(\text{Sym } \mathcal{O}(1))$, and its sheaf of sections is $\mathcal{O}(-1)$. The multiplication $S^* \otimes \mathcal{E}^\vee \rightarrow S^*(1)$ in the symmetric algebra induces a surjection of locally free sheaves $\pi^*(\mathcal{E}^\vee) \rightarrow \mathcal{O}(1)$ and hence an

injection $\mathcal{O}(-1) \rightarrow \pi^*(\mathcal{E})$; this yields a canonical morphism $\mathbb{L} \rightarrow \pi^*(\mathbb{V})$ of the associated geometric vector bundles.

Definition 1.2. Any p -form $\psi : \mathrm{Sym}^p(\mathcal{E}) \rightarrow \mathcal{O}_X$ on \mathcal{E} induces a canonical p -form ϵ on the tautological line bundle \mathbb{L} :

$$\epsilon : \mathcal{O}(-p) = \mathrm{Sym}^p(\mathcal{O}(-1)) \rightarrow \mathrm{Sym}^p(\pi^*\mathcal{E}) = \pi^*\mathrm{Sym}^p(\mathcal{E}) \xrightarrow{\psi} \pi^*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}(\mathcal{E})}.$$

We will use the following notational shorthand. For a scheme Z , a point q on some Z -scheme and a vector bundle V on Z we write $V|_q$ for the fiber of V at q , i.e., the $k(q)$ vector space $q^*(V)$ for $q \rightarrow Z$. If φ is a p -form on a line bundle L , $0 \neq u \in L|_q$ and $a = \varphi|_q(u^p)$, then $\varphi|_q : (L|_q)^p \rightarrow k(q)$ is the p -form $\varphi|_q(tu^p) = at^p$.

Example 1.3. Given an invertible sheaf L on X , and a p -form φ on L , the bundle $V = \mathcal{O} \oplus L$ has the p -form $\psi(t, u) = t^p - \varphi(u)$. Then $\mathbb{P}(V) \rightarrow X$ is a \mathbb{P}^1 -bundle, and its tautological line bundle \mathbb{L} has the p -form ϵ described in 1.2.

Over a point in $\mathbb{P}(V)$ of the form $\infty = (0 : u)$, the p -form on $\mathbb{L}|_\infty$ is $\epsilon(0, \lambda u) = -\lambda^p \varphi(u)$. If $q = (1 : u)$ is any other point on $\mathbb{P}(V)$ then the 1-dimensional subspace $\mathbb{L}|_q$ of the vector space $V|_q$ is generated by $v = (1, u)$ and the p -form $\epsilon|_q$ on $\mathbb{L}|_q$ is determined by $\epsilon(v) = \psi(1, u) = 1 - \varphi(u)$ in the sense that $\epsilon(\lambda v) = \lambda^p(1 - \varphi(u))$.

One application of these ideas is the formation of the sheaf of Kummer algebras associated to a p -form. Recall that if L is a line bundle then the $(p-1)$ st symmetric power of $\mathbb{P}(\mathcal{O} \oplus L)$ is $\mathrm{Sym}^{p-1}\mathbb{P}(\mathcal{O} \oplus L) = \mathbb{P}(\mathcal{A}(L))$, where $\mathcal{A}(L) = \bigoplus_{i=0}^{p-1} L^{\otimes i}$.

Definition 1.4. If L is a line bundle on X , equipped with a p -form ϕ , the *Kummer algebra* $\mathcal{A}_\phi(L)$ is the vector bundle $\mathcal{A}(L) = \bigoplus_{i=0}^{p-1} L^{\otimes i}$ regarded as a bundle of algebras as in [10, 3.11]; locally, if u is a section generating L then $\mathcal{A}(L) \cong \mathcal{O}[u]/(u^p - \phi(u))$. If $x \in X$ and $a = \phi|_x(u)$ then the $k(x)$ -algebra $\mathcal{A}|_x$ is the Kummer algebra $k(x)(\sqrt[p]{a})$, which is a field if $a \notin k(x)^p$ and $\prod k(x)$ otherwise.

Since the norm on $\mathcal{A}_\phi(L)$ is given by a homogeneous polynomial of degree p , we may regard the norm as a map from $\mathrm{Sym}^p \mathcal{A}_\phi(L)$ to \mathcal{O} . The canonical p -form ϵ on the tautological line bundle \mathbb{L} on the projective bundle $\mathbb{P} = \mathbb{P}(\mathcal{A}(L))$, given in 1.2, agrees with the natural p -form:

$$\mathbb{L}^{\otimes p} \rightarrow \mathrm{Sym}^p \pi^* \mathcal{A}(L) \xrightarrow{N} \mathcal{O}_{\mathbb{P}},$$

where $\pi : \mathbb{P} \rightarrow X$ is the structure map and the canonical inclusion of \mathbb{L} into $\pi^*(\mathcal{A}(L)) = \bigoplus_0^{p-1} \pi^* L^{\otimes i}$ induces the first map.

Recall from 1.2 and 1.4 that ϕ is a p -form on L , $\psi = (1, -\varphi)$ is a p -form on $\mathcal{O} \oplus L$ and ϵ is the canonical p -form on \mathbb{L} induced from ψ .

Lemma 1.5. *Suppose that $x \in X$ has $\phi|_x \neq 0$ and that $0 \neq u \in L|_x$. Then $\epsilon|_{(0:u)} \neq 0$. Moreover, $\phi(u) \in k(x)^{\times p}$ iff there is a point $\ell \in \mathbb{P}(\mathcal{O} \oplus L)$ over x so that $\epsilon|_\ell = 0$.*

Proof. Let $w = (t, su)$ be a point of $\mathbb{L}|_x$ over $\ell = (t : su) \in \mathbb{P}(\mathcal{O} \oplus L)|_x$. If $t = 0$ then $\ell = (0 : u)$ and $\epsilon(w) = -s^p \phi(u)$, which is nonzero for $s \neq 0$. If $t \neq 0$ then $\epsilon|_\ell$ is determined by the scalar $\epsilon(w) = \psi(t, su) = t^p - s^p \phi(u)$. Thus $\epsilon|_\ell = 0$ iff $\phi(u) = (t/s)^p$. \square

Remark 1.5.1. Here is an alternative proof, using the Kummer algebra $K = k(x)(a)$, $a = \sqrt[p]{\phi(u)}$. Since $\epsilon(w) = \psi(t, su)$ is the norm of the nonzero element $t - sa$ in K , the norm $\epsilon(w)$ is zero iff the Kummer algebra is split, i.e., $\phi(u) = a^p \in k(x)^{\times p}$.

Finally, the notation $\{\gamma, \dots, \gamma'_{n-1}\}$ in the Chain Lemma 0.1 is a special case of the notation in the following definition.

Definition 1.6. Given line bundles H_1, \dots, H_n on X , p -forms α_i on H_i , and a point $x \in X$ at which each form $\alpha_i|_x$ is nonzero, we write $\{\alpha_1, \dots, \alpha_n\}|_x$ for the element $\{\alpha_1|_x, \dots, \alpha_n|_x\}$ of $K_n^M(k(x))/p$ described before 1.1: if u_i is a generator of $H_i|_x$ and $\alpha_i|_x(u_i) = a_i$ then $\{\alpha_1, \dots, \alpha_n\}|_x = \{a_1, \dots, a_n\}$.

We record the following useful consequence of this construction.

Lemma 1.7. *Suppose that the p -forms α_i are all nonzero at the generic point η of a smooth X . On the open subset U of X of points x on which each $\alpha_i|_x \neq 0$, the symbol $\{\alpha_1|_x, \dots, \alpha_n|_x\}$ in $K_n^M(k(x))/p$ is obtained by specialization from the symbol in $K_n^M(k(X))/p$.*

2. THE CHAIN LEMMA WHEN $n = 2$.

The goal of this section is to construct certain iterated projective bundles together with line bundles and p -forms on them as needed in the case $n = 2$ of the Chain Lemma 0.1. Our presentation is based upon Rost's lectures [Rost].

We begin with a generic construction, which starts with a pair K_0, K_{-1} of line bundles on a variety $X_0 = X_{-1}$ and produces a tower of varieties X_r , equipped with distinguished line bundles K_r . Each X_r is a product of $p - 1$ projective line bundles over X_{r-1} , so X_r has relative dimension $r(p - 1)$ over X_0 .

Definition 2.1. Given a morphism $f_{r-1} : X_{r-1} \rightarrow X_{r-2}$ and line bundles K_{r-1} on X_{r-1} , K_{r-2} on X_{r-2} , we form the projective line bundle $\mathbb{P}(\mathcal{O} \oplus K_{r-1})$ over X_{r-1} and its tautological line bundle \mathbb{L} . By definition, X_r is the product $\prod_1^{p-1} \mathbb{P}(\mathcal{O} \oplus K_{r-1})$ over X_{r-1} . Writing f_r for the projection $X_r \rightarrow X_{r-1}$, and \mathbb{L}_r for the exterior product $\mathbb{L} \boxtimes \dots \boxtimes \mathbb{L}$ on X_r , we define the line bundle K_r on X_r to be $K_r = (f_r \circ f_{r-1})^*(K_{r-2}) \otimes \mathbb{L}_r$.

$$X_r \xrightarrow{f_r} X_{r-1} \xrightarrow{f_{r-1}} X_{r-2} \cdots X_1 \xrightarrow{f_1} X_0 = X_{-1}.$$

Example 2.2 (k-tower). The k -tower is the tower obtained when we start with $X_0 = \text{Spec}(k)$, using the trivial line bundles K_{-1}, K_0 . Note that $X_1 = \prod \mathbb{P}^1$ and $K_1 = \mathbb{L}_1$, while X_2 is a product of projective line bundles over $\prod \mathbb{P}^1$, and $K_2 = \mathbb{L}_2$.

In the Chain Lemma (Theorem 0.1) for $n = 2$ we have $S = X_p$ in the k -tower, and the line bundles are $J = J_1 = K_p$, $J'_1 = f_p^*(K_{p-1})$. Before defining the p -forms γ_1 and γ'_1 in 2.7, we quickly establish 2.6; this verifies part (6) of Theorem 0.1, that the degree of $c_1(K_p)^{p^2-p}$ is prime to p .

If L is a line bundle over X , and $\lambda = c_1(L)$, the Chow ring of $\mathbb{P} = \mathbb{P}(\mathcal{O} \oplus L)$ is $CH(\mathbb{P}) = CH(X)[z]/(z^2 - \lambda z)$, where $z = c_1(\mathbb{L})$. If $\pi : \mathbb{P} \rightarrow X$ then $\pi_*(z) = -1$ in $CH(X)$. Applying this observation to the construction of X_r out of $X = X_{r-1}$ with $\lambda_{r-1} = c_1(K_{r-1})$, we have

$$CH(X_r) = CH(X_{r-1})[z_{r,1}, \dots, z_{r,p-1}]/(\{z_{r,j}^2 - \lambda_{r-1}z_{r,j} \mid j = 1, \dots, p-1\}),$$

where $z_{r,j}$ is the first Chern class of the j th tautological line bundle \mathbb{L} . (Formally, $CH(X_{r-1})$ is identified with a subring of $CH(X_r)$ via the pullback of cycles.) By induction on r , this yields the following result:

Lemma 2.3. $CH^*(X_r)$ is a free $CH^*(X_0)$ -module. A basis consists of the monomials $\prod z_{i,j}^{e_{i,j}}$ for $e_{i,j} \in \{0, 1\}$, $0 < i \leq r$ and $0 < j < p$. As a graded algebra, $CH^*(X_r)/p \cong CH^*(X_0)/p \otimes_{R_0} R_r$, where $R_0 = \mathbb{F}_p[\lambda_0, \lambda_{-1}]$ and

$$R_r = \mathbb{F}_p[\lambda_{-1}, \lambda_0, \dots, \lambda_r, z_{1,1}, \dots, z_{r,p-1}]/I_r,$$

$$I_r = (\{z_{i,j}^2 - \lambda_{i-1}z_{i,j} \mid 1 \leq i \leq r, 0 < j < p\}, \{\lambda_i - \lambda_{i-2} - \sum_{j=1}^{p-1} z_{i,j} \mid 1 \leq i \leq r\}).$$

Definition 2.4. For $r = 1, \dots, p$, set $z_r = \sum_{j=1}^{p-1} z_{r,j}$ and $\zeta_r = \prod z_{r,j}$. It follows from Lemma 2.3 that $\lambda_i = \lambda_{i-2} + z_i$ and $z_i^p = \sum z_{r,j}^p = \sum z_{r,j} \lambda_{r-1}^{p-1} = z_i \lambda_{i-1}^{p-1}$ in R_r and hence in $CH(X_r)/p$.

By Lemma 2.3, if $1 \leq r \leq p$ then multiplication by $\prod \zeta_i \in CH^{r(p-1)}(X_r)$ is an isomorphism $CH_0(X_0)/p \xrightarrow{\sim} CH_0(X_r)/p$. If $X_0 = \text{Spec}(k)$ then $CH_0(X_r)/p \cong \mathbb{F}_p$, and is generated by $\prod \zeta_i$.

Lemma 2.5. If $y \in CH_0(X_0)$, the degree of $y \cdot \zeta_1 \cdots \zeta_r$ is $(-1)^{r(p-1)} \deg(y)$.

Proof. The degree on X_r is the composition of the $(f_i)_*$. The projection formula implies that $(f_r)_*(\zeta_r) = (-1)^{p-1}$, and

$$(f_r)_*(y \cdot \zeta_1 \cdots \zeta_r) = (y \cdot \zeta_1 \cdots \zeta_{r-1}) \cdot (f_r)_*(\zeta_r) = (-1)^{p-1} y \cdot \zeta_1 \cdots \zeta_{r-1}.$$

Hence the result follows by induction on r . \square

Proposition 2.6. For every 0-cycle y on X_0 and $1 \leq r \leq p$, $\lambda_r = c_1(K_r)$ satisfies $y \lambda_r^{r(p-1)} \equiv y \zeta_1 \cdots \zeta_r$ in $CH_0(X_r)/p$, and $\deg(y \lambda_r^{r(p-1)}) \equiv \deg(y) \pmod{p}$.

For the k -tower 2.2 (with $y = 1$), we have $\deg(\lambda_p^{p^2-p}) \equiv 1 \pmod{p}$.

Proof. If $r = 1$ this follows from $y \lambda_{-1} = y \lambda_0 = 0$ in $CH(X_0)$: $\lambda_1 = z_1 + \lambda_{-1}$ and $y \cdot \zeta_1 \equiv y \lambda_1^{p-1}$. For $r \geq 2$, we have $\lambda_r = z_r + \lambda_{r-2}$ and $z_r^p = z_r \lambda_{r-1}^{p-1}$ by 2.4. Because $p - r \geq 0$, we have

$$\begin{aligned} \lambda_r^{r(p-1)} &= (z_r + \lambda_{r-2})^{p(r-1)+(p-r)} \equiv (z_r^p + \lambda_{r-2}^p)^{r-1} \cdot (z_r + \lambda_{r-2})^{p-r} \pmod{p} \\ &= (z_r \lambda_{r-1}^{p-1} + \lambda_{r-2}^p)^{r-1} (z_r + \lambda_{r-2})^{p-r} \equiv \zeta_r \lambda_{r-1}^{(r-1)(p-1)} + T \pmod{p}, \end{aligned}$$

where $T \in CH(X_{r-1})[z_r]$ is a homogeneous polynomial of total degree $< p-1$ in z_r .

By 2.3, the coefficients of yT are elements of $CH(X_{r-1})$ of degree $> \dim(X_{r-1})$, so yT must be zero. Then by the inductive hypothesis,

$$y \lambda_{r-1}^{r(p-1)} \equiv y \zeta_r \lambda_{r-1}^{(r-1)(p-1)} \equiv y \zeta_r \cdot (\zeta_1 \cdots \zeta_{r-1})$$

in $CH^*(X_r)/p$, as claimed. Now the degree assertion follows from Lemma 2.5. \square

The p -forms. We now turn to the p -forms in the Chain Lemma 0.1, using the k -tower 2.2. We will inductively equip the line bundles \mathbb{L}_r and K_r of 2.2 with p -forms Ψ_r and φ_r ; the γ_1 and γ'_1 of the Chain Lemma 0.1 will be φ_p and φ_{p-1} .

When $r = 0$, we equip the trivial line bundles K_{-1}, K_0 on $X_0 = \text{Spec}(k)$ with the p -forms $\varphi_{-1}(t) = a_1 t^p$ and $\varphi_0(t) = a_2 t^p$. The p -form φ_{r-1} on K_{r-1} induces a p -form $\psi(t, u) = t^p - \varphi_{r-1}(u)$ on $\mathcal{O} \oplus K_{r-1}$ and a p -form ϵ on the tautological line bundle \mathbb{L} , as in Example 1.3. As observed in Example 1.3, at the point $q = (1 : x)$ of $\mathbb{P}(\mathcal{O} \oplus K_{r-1})$ we have $\epsilon(y) = \psi(1, x) = 1 - \varphi_{r-1}(x)$.

Definition 2.7. The p -form Ψ_r on \mathbb{L}_r is the product form $\prod \psi$:

$$\Psi_r(y_1 \boxtimes \cdots \boxtimes y_{p-1}) = \prod \psi(y_i).$$

The p -form φ_r on $K_r = (f_{r-1} \circ f_r)^*(K_{r-2}) \otimes \mathbb{L}_r$ is defined to be

$$\varphi_r = (f_{r-1} \circ f_r)^*(\varphi_{r-2}) \otimes \Psi_r.$$

Proposition 2.8. *Let $x = (x_1, \dots, x_{p-1}) \in X_r$ be a point with residue field $E = k(x)$. For $-1 \leq i \leq r$, choose generators u_i and v_i for the one-dimensional E vector spaces $K_i|_x$ and $\mathbb{L}_i|_x$ respectively, in such a way that $u_i = u_{i-2} \otimes v_i$.*

- (1) *If $\varphi_i|_x = 0$ for some $1 \leq i \leq r$ then $\{a_1, a_2\}_E = 0 \in K_2(E)/p$.*
- (2) *If $\varphi_i|_x \neq 0$ for all i , $1 \leq i \leq r$, then*

$$\{a_1, a_2\}_E = (-1)^r \{\varphi_{r-1}(u_{r-1}), \varphi_r(u_r)\} \in K_2(E)/p.$$

Proof. By induction on r . Both parts are obvious if $r = 0$. To prove the first part, we may assume that $\varphi_i|_x \neq 0$ for $1 \leq i \leq r-1$, but $\varphi_r|_x = 0$. We have $u_r = u_{r-2} \otimes v_r$ and by the definition of φ_r , we conclude that

$$0 = \varphi_r(u_r) = \varphi_{r-2}(u_{r-2})\Psi_r(v_r),$$

whence $\Psi_r(v_r) = 0$. Now the element $v_r \neq 0$ is a tensor product of sections w_j and $\Psi_r(v_r) = \prod \psi(w_j)$ so $\psi(w_j) = 0$ for a nonzero section w_j of $\mathbb{L}|_{x_j}$. By Lemma 1.5, $\varphi_{r-1}(u_{r-1})$ is a p th power in E . Consequently, $\{\varphi_{r-2}(u_{r-2}), \varphi_{r-1}(u_{r-1})\}_E = 0$ in $K_2(E)/p$. This symbol equals $\pm\{a_1, a_2\}_E$ in $K_2(E)/p$, by (2) and induction. This finishes the proof of the first assertion.

For the second claim, we can assume by induction that

$$\{a_1, a_2\}_E = \pm\{\varphi_{r-2}(u_{r-2}), \varphi_{r-1}(u_{r-1})\}_E.$$

Now $\varphi_r(u_r) = \varphi_{r-2}(u_{r-2})\Psi_r(v_r)$. But $\{\varphi_{r-1}(u_{r-1}), N_{\varphi_{r-1}}(v_{r-1})\} = 0$ by Lemma 2.9 below. We conclude that

$$\{\varphi_{r-2}(u_{r-2}), \varphi_{r-1}(u_{r-1})\}_E \equiv -\{\varphi_{r-1}(u_{r-1}), \varphi_r(u_r)\}_E \pmod{p};$$

this concludes the proof of the second assertion. \square

Lemma 2.9. *For any field k any $a \in k^\times$ and any b in $K_a = k[\sqrt[p]{a}]$, the symbol $\{a, N_{K_a/k}(b)\}$ is trivial in $K_2(k)/p$.*

Proof. Because $\{a, b\} = p\{\sqrt[p]{a}, b\}$ vanishes in $K_2(K_a)/p$, we have $\{a, N(b)\} = N\{a, b\} = pN(\{\sqrt[p]{a}, b\}) = 0$. \square

Proof of the Chain Lemma 0.1 for $n = 2$. We verify the conditions for the variety $S = X_p$ in the k -tower 2.2; the line bundles $J = J_1 = K_p$, $J'_1 = f_p^*(K_{p-1})$; the p -forms γ_1 and γ'_1 in 0.1 are the forms φ_p and φ_{p-1} of 2.7. Part (1) of Theorem 0.1 is immediate from the construction of $S = X_p$; parts (2) and (4) were proven in Proposition 2.8; parts (3) and (5) follow from (2) and (4); and part (6) is Proposition 2.6 with $y = 1$. \square

NORM PRINCIPLE FOR $n = 2$

The Norm Principle for $n = 2$ was implicit in the Merkurjev-Suslin paper [3, 4.3]. We reproduce their short proof, which uses the Severi-Brauer variety X of the cyclic division algebra $D = A_\zeta(a, b)$ attached to a nontrivial symbol $\{a, b\}$ in $K_2(k)/p$ and a p th root of unity ζ ; X is a Norm variety for the symbol $\{a, b\}$.

Theorem 2.10 (Norm Principle for $n = 2$). *If $x \in X$ and $[k(x) : k] = p^m$ for $m > 1$ then for all $\lambda \in k(x)$ there exists $x' \in X$ and $\lambda' \in k(x')$ so that $[k(x') : k] \leq p$ and $[x, \lambda] = [x', \lambda']$ in $\overline{A}_0(X, \mathcal{K}_1)$.*

Proof. By Merkurjev-Suslin [3, 8.7.2], $N : \overline{A}_0(X, \mathcal{K}_1) \rightarrow k^\times$ is an injection with image $\text{Nrd}(D) \subseteq k^\times$. Therefore the unit $N([x, \lambda])$ of k can be written as the reduced norm of an element $\lambda' \in D$. The subfield $E = k(\lambda')$ of D has degree $\leq p$, and corresponds to a point $x' \in X$. Since $N([x', \lambda']) = \text{Nrd}(\lambda') = N([x, \lambda])$, we have $[x, \lambda] = [x', \lambda']$ in $\overline{A}_0(X, \mathcal{K}_1)$. \square

3. THE SYMBOL CHAIN

Here is the pattern of the chain lemma in all weights.

We start with a sequence a_1, a_2, \dots of units of k , and the function $\Phi_0(t) = t^p$. For $r \geq 1$, we inductively define functions Φ_r in p^r variables and Ψ_r in $p^r - p^{r-1}$ variables, taking values in k , and prove (in 3.4) that $\{a_1, \dots, a_r, \Phi_r(\mathbf{x})\} \equiv 0 \pmod{p}$. Note that Φ_r and Ψ_r depend only upon the units a_1, \dots, a_r . We write \mathbf{x}_i for a sequence of p^i variables x_{ij} (where $j = (j_1, \dots, j_r)$ and $0 \leq j_t < p$), and we inductively define

$$(3.1) \quad \Psi_{r+1}(\mathbf{x}_1, \dots, \mathbf{x}_{p-1}) = \prod_{i=1}^{p-1} [1 - a_{r+1} \Phi_r(\mathbf{x}_i)],$$

$$(3.2) \quad \Phi_{r+1}(\mathbf{x}_0, \dots, \mathbf{x}_{p-1}) = \Phi_r(\mathbf{x}_0) \Psi_{r+1}(\mathbf{x}_1, \dots, \mathbf{x}_{p-1}).$$

We say that two rational functions are *birationally equivalent* if they can be transformed into one another by an automorphism (over the base field k) of the field of rational functions.

Example 3.3. $\Psi_1(x_1, \dots, x_{p-1}) = \prod(1 - a_1 x_i^p)$ and $\Phi_1(x_0, \dots, x_{p-1})$ is $x_0^p \prod(1 - a_1 x_i^p)$, the norm of the element $x_0 \prod(1 - x_i \alpha_1)$ in the Kummer extension $k(\mathbf{x})(\alpha_1)$, $\alpha_1 = \sqrt[p]{a_1}$. Thus Φ_1 is birationally equivalent to symmetrizing in the x_i , followed by the norm from $k[\sqrt[p]{a_1}]$ to k . More generally, $\Psi_r(\mathbf{x}_1, \dots, \mathbf{x}_{p-1})$ is the norm of an element in $k(\mathbf{x}_1, \dots, \mathbf{x}_{p-1})(\sqrt[p]{a_r})$.

Example 3.3.1. It is useful to interpret the map Φ_1 geometrically. Let $R_{k(\alpha)/k} \mathbb{A}^1$ denote the variety, isomorphic to \mathbb{A}^p , which is the Weil restriction ([16]) of the affine line over $k(\alpha)$, so that there is a morphism $N : R_{k(\alpha)/k} \mathbb{A}^1 \rightarrow \mathbb{A}^1$ corresponding to the norm map. The function $k^p \rightarrow k(\alpha)$ defined by

$$(x_0, s_1, \dots, s_{p-1}) \mapsto x_0(1 - s_1 \alpha + s_2 \alpha^2 - \dots \pm s_{p-1} \alpha^{p-1})$$

induces a birational map $\mathbb{A}^p \xrightarrow{m} R_{k(\alpha)/k}\mathbb{A}^1$. Finally, let $q : \mathbb{A}^{p-1} \rightarrow \mathbb{A}^{p-1}/\Sigma_{p-1} \cong \mathbb{A}^{p-1}$ be the symmetrizing map sending (x_1, \dots) to the elementary symmetric functions (s_1, \dots) . Then the following diagram commutes:

$$\begin{array}{ccc} \mathbb{A}^p = \mathbb{A}^1 \times \mathbb{A}^{p-1} & \xrightarrow{1 \times q} & \mathbb{A}^1 \times \mathbb{A}^{p-1} \xrightarrow{\text{birat.}} R_{k(\alpha)/k}\mathbb{A}^1 = \mathbb{A}^p \\ & \searrow \Phi_1 & \downarrow N \\ & & \mathbb{A}^1. \end{array}$$

Remark 3.3.2. If $p = 2$, $\Phi_1(x_0, x_1) = x_0^2(1 - a_1x_1^2)$ is birationally equivalent to the norm form $u^2 - a_1v^2$ for $k(\sqrt{a_1})/k$, and $\Phi_2 = \Phi_1(\mathbf{x}_0)[1 - a_2\Phi_1(\mathbf{x}_1)]$ is birationally equivalent to the norm form $\langle\langle a_1, a_2 \rangle\rangle = (u^2 - a_1v^2)[1 - a_2(w^2 - a_1t^2)]$ for the quaternionic algebra $A_{-1}(a_1, a_2)$.

More generally, Φ_n is birationally equivalent to the Pfister form

$$\langle\langle a_1, \dots, a_r \rangle\rangle = \langle\langle a_1, \dots, a_{r-1} \rangle\rangle \perp a_r \langle\langle a_1, \dots, a_{r-1} \rangle\rangle$$

and Ψ_r is equivalent to the restriction of the Pfister form to the subspace defined by the equations $\mathbf{x}_0 = (1, \dots, 1)$.

Remark 3.3.3 (Rost). Suppose that $p = 3$. Then Φ_2 is birationally equivalent to (symmetrizing, followed by) the reduced norm of the algebra $A_\zeta(a_1, a_2)$ and Φ_3 is equivalent to the norm form of the exceptional Jordan algebra $J(a_1, a_2, a_3)$. When $r = 4$, Rost showed that the set of nonzero values of Φ_4 is a subgroup of k^\times .

For the next lemma, it is useful to introduce the function field F_r in the p^r variables x_{j_1, \dots, j_r} , $0 \leq j_t < p$. Note that F_r is isomorphic to the tensor product of p copies of F_{r-1} .

Lemma 3.4. $\{a_1, \dots, a_r, \Phi_r(\mathbf{x})\} = \{a_1, \dots, a_r, \Psi_r(\mathbf{x})\} = 0 \in K_{r+1}^M(F_r)/p$.

If $b \in k$ is a nonzero value of Φ_r , then $\{a_1, \dots, a_r, b\} = 0 \in K_{r+1}^M(k)/p$.

Proof. By Lemma 2.9, $\{a_r, \Psi_r(\mathbf{x})\} = 0$ because $\Psi_r(\mathbf{x})$ is a norm of an element of $k(\mathbf{x})(\alpha_r)$ by 3.3. If $r = 1$ then $\{a_1, \Phi_1(\mathbf{x})\} = \{a_1, x_0^p\} \equiv 0$ as well. The result for F_r follows by induction:

$$\{a_1, \dots, a_{r+1}, \Phi_{r+1}(\mathbf{x})\} = \{a_1, \dots, a_{r+1}, \Phi_r(\mathbf{x}_0)\}\{a_1, \dots, a_{r+1}, \Psi_{r+1}(\mathbf{x})\} = 0.$$

The result for b follows from the first assertion, and specialization from F_r to k . \square

Remark 3.5. For any value $b \in k^\times$ of Φ_n , any desingularization X of the projective closure of the affine hypersurface $X_b = \{\mathbf{x} : \Phi_n(\mathbf{x}) = b\}$ will be a Norm variety for the symbol $\{a_1, \dots, a_n, b\}$ in $K_{n+1}^M(k)/p$.

Indeed, since $\dim(X_b) = p^n - 1$, we see from Lemma 3.4 that every affine point of X_b splits the symbol. In particular, the generic point of X_b is a splitting field for this symbol. By specialization, every point of X_b and X splits the symbol.

The symmetric group Σ_{p-1} acts on $\{\mathbf{x}_1, \dots, \mathbf{x}_{p-1}\}$ and fixes Φ_n , so it acts on X_b . It is easy to see that X_b/Σ_{p-1} is birationally isomorphic to the Norm variety constructed in [10, §2] using the hypersurface W defined by $N = b$ in the vector bundle of *loc. cit.* By [10, 1.19], X is also a Norm variety.

Definition 3.6. A move of type C_n on a sequence a_1, \dots, a_n in k^\times is a transformation of the kind:

$$\text{Type } C_n : \quad (a_1, \dots, a_n) \mapsto (a_1, \dots, a_{n-2}, a_n \Psi_{n-1}(\mathbf{x}), a_{n-1}^{-1}).$$

Here Ψ_{n-1} is a function of $p^{n-1} - p^{n-2}$ new variables $\mathbf{x}_i = \{\mathbf{x}_{1,1}, \dots, \mathbf{x}_{1,p-1}\}$.

By Lemma 3.4, $\{a_1, \dots, a_n\} = \{a_1, \dots, a_{n-2}, a_n \Psi_{n-1}(\mathbf{x}), a_{n-1}^{-1}\}$, so the move does not change the symbol in $K_n^M(k)$. If we do this move p times, always with a new set of variables \mathbf{x}_i , we obtain a move $(a_1, \dots, a_n) \mapsto (a_1, \dots, a_{n-2}, \gamma_{n-1}, \gamma'_{n-1})$ in which $\gamma_{n-1}, \gamma'_{n-1}$ are functions of $p^n - p^{n-1}$ variables $\mathbf{x}_{i,j}$, $1 \leq i < p$, $1 \leq j \leq p$.

Since these moves do not change the symbol, we have

$$(3.7) \quad \{a_1, \dots, a_n\} = \{a_1, \dots, a_{n-2}, \gamma_{n-1}, \gamma'_{n-1}\}$$

in $K_n^M(k)$. The functions γ_{n-1} and γ'_{n-1} in (3.7) are the ones appearing in the Chain Lemma 0.1.

Formally, if $k(\mathbf{x}_1)$ is the function field of the move of type C_n , then the function field F'_n of the move (3.7) is the tensor product $k(\mathbf{x}_1) \otimes \dots \otimes k(\mathbf{x}_p)$. We will define a variety S_{n-1} with function field F'_n .

Using $p^{n-1} - p^{n-2}$ more variables $\mathbf{x}'_{i,j}$ ($1 \leq i < p$, $1 \leq j \leq p$) we do p moves of type C_{n-1} on $(a_1, \dots, a_{n-2}, \gamma_{n-1})$ to get the sequence $(a_1, \dots, a_{n-3}, \gamma_{n-2}, \gamma'_{n-2}, \gamma'_{n-1})$. The function field of this move is $F'_{n-1} \otimes F'_n$, and we will define a variety S_{n-2} with this function field, together with a morphism $S_{n-2} \rightarrow S_{n-1}$.

Next, apply p moves of type C_{n-2} , then p moves of type C_{n-3} , and so on, ending with p moves of type C_2 . We have the sequence $(\gamma_1, \gamma'_1, \gamma'_2, \dots, \gamma'_{n-1})$ in $p^n - p$ variables $\mathbf{x}_1, \dots, \mathbf{x}_{p-1}$. Moreover, we see from Lemma 3.4 that

$$(3.8) \quad \{a_1, \dots, a_n\} = \{\gamma_1, \gamma'_1, \gamma'_2, \dots, \gamma'_{n-1}\} \quad \text{in } K_n^M(k).$$

The net effect will be to construct a tower

$$(3.9) \quad S = S_1 \xrightarrow{f_r} S_2 \longrightarrow \dots \longrightarrow S_{n-2} \longrightarrow S_{n-1} \longrightarrow S_n = \text{Spec}(k).$$

Let S be any variety containing $U = \mathbb{A}^{p^n - p}$ as an affine open, so that $k(S) = k(\mathbf{x}_1, \dots, \mathbf{x}_{p-1})$, each \mathbf{x}_i is p^{n-1} variables $x_{i,j}$ and all line bundles on U are trivial. Then parts (1) and (2) of the Chain Lemma 0.1 are immediate from (3.7) and (3.8).

Now the only thing to do is to construct $S = S_1$, extend the line bundles (and forms) from U to S , and prove parts (4) and (6) of 0.1.

4. MODEL P_{n-1} FOR MOVES OF TYPE C_n

In this section, we construct a tower of varieties P_r and Q_r over S' , with p -forms on lines bundles over them, which will produce a model of the forms Ψ_r and Φ_r in (3.1) and (3.2). This tower, depicted in (4.0), is defined in 4.2 below.

$$(4.0) \quad P_{n-1} \rightarrow \dots \rightarrow P_r \longrightarrow Q_{r-1} \rightarrow P_{r-1} \longrightarrow \dots \rightarrow Q_1 \rightarrow P_1 \longrightarrow Q_0 = S'$$

The passage from S' to the variety P_{n-1} is a model for the moves of type C_n defined in 3.6.

Definition 4.1. Let X be a variety over some fixed base S' . Given line bundles K, L on X , we can form the vector bundle $V = \mathcal{O} \oplus L$, the \mathbb{P}^1 -bundle $\mathbb{P}(V)$ over X , and \mathbb{L} . Taking products over S' , set

$$P = \prod_1^{p-1} \mathbb{P}(\mathcal{O} \oplus L); \quad Q = X \times_{S'} P$$

On P and Q , we have the exterior products of the tautological line bundles:

$$\mathbb{L}(1, \dots, 1) = \mathbb{L} \boxtimes \mathbb{L} \boxtimes \dots \boxtimes \mathbb{L} \text{ on } P, \quad K \boxtimes \mathbb{L}(1, \dots, 1) \text{ on } Q.$$

Given p -forms φ and ϕ on K and L , respectively, the line bundle \mathbb{L} has the p -form ϵ , as in Example 1.3, and the line bundles $\mathbb{L}(1, \dots, 1)$ and $K \boxtimes \mathbb{L}(1, \dots, 1)$ are equipped with the product p -forms $\Psi = \prod \epsilon$ and $\Phi = \varphi \otimes \Psi$.

Remark 4.1.1. Let $x = (x_1, \dots, x_{p-1})$ denote the generic point of X^{p-1} . The function fields of P and Q are $k(P) = k(x)(y_1, \dots, y_{p-1})$ and $k(Q) = k(x_0) \otimes k(P)$. We may represent their generic points in coordinate form as a $(p-1)$ -tuple $\{(1 : y_i)\}$, where the y_i generate L over x_i . Then $y = \{(1, y_i)\}$ is a generator of $\mathbb{L}(1, \dots, 1)$ at the generic point, and $\Psi(y) = \prod(1 - \phi(y_i))$, $\Phi(y) = \varphi(x_0)\Psi(y)$.

Example 4.1.2. An important special case arises when we begin with two line bundles H on S' , K on X , with p -forms α and φ . In this case, we set $L = H \otimes K$ and equip it with the product form $\phi(u \otimes v) = \alpha(u)\varphi(v)$. At the generic point q of Q we can pick a generator $u \in H|_q$ and set $y_i = u \otimes v_i$; the forms resemble the forms of (3.1) and (3.2):

$$\Psi(y) = \prod(1 - \alpha(u)\varphi(v_i)), \quad \Phi(y) = \varphi(v_0)\Psi(y).$$

Remark 4.1.3. Suppose a group G acts on S' , X , K and L , and K_0, L_0 are nontrivial 1-dimensional representations so that at every fixed point x of X (a) $k(x) = k$, (b) $L_x \cong L_0$. Then G acts on P (resp., Q) with 2^{p-1} fixed points y over each fixed point of X^{p-1} (resp., of X^p), each with $k(y) = k$, and each fiber of $\mathbb{L} = \mathbb{L}(1, \dots, 1)$ (resp., $K \boxtimes \mathbb{L}$) is the representation L_0^j (resp., $K_0 \otimes L_0^j$) for some j ($0 \leq j < p$). Indeed, G acts nontrivially on each term \mathbb{P}^1 of the fiber $\prod \mathbb{P}^1$, so that the fixed points in the fiber are the points (y_1, \dots, y_{p-1}) with each y_i either $(0 : 1)$ or $(1 : 0)$.

We now define the tower (4.0) of P_r and Q_r over a fixed base S' , by induction on r . We start with line bundles H_1, \dots, H_r , and $K_0 = \mathcal{O}_{S'}$ on S' , and set $Q_0 = S'$.

Definition 4.2. Given a variety Q_{r-1} and a line bundle K_{r-1} on Q_{r-1} , we form the varieties $P_r = P$ and $Q_r = Q$ using the construction in Definition 4.1, with $X = Q_{r-1}$, $K = K_{r-1}$ and $L = H_r \otimes K_{r-1}$ as in 4.1.2. To emphasize that P_r only depends upon S' and H_1, \dots, H_r , we will sometimes write $P_r(S'; H_1, \dots, H_r)$. As in 4.1, P_r has the line bundle $\mathbb{L}(1, \dots, 1)$, and Q_r has the line bundle $K_r = K_{r-1} \boxtimes \mathbb{L}(1, \dots, 1)$.

Suppose that we are given p -forms $\alpha_i \neq 0$ on H_i , and we set $\Phi_0(t) = t^p$ on K_0 . Inductively, the line bundle K_{r-1} on Q_{r-1} is equipped with a p -form Φ_{r-1} . As described in 4.1 and 4.1.2, the line bundle $\mathbb{L}(1, \dots, 1)$ on P_r obtains a p -form Ψ_r from the p -form $\alpha_r \otimes \Phi_{r-1}$ on $L = H_r \otimes K_{r-1}$, and K_r obtains a p -form $\Phi_r = \Phi_{r-1} \otimes \Psi_r$.

Example 4.2.1. $Q_1 = P_1$ is $\prod_1^{p-1} \mathbb{P}^1(\mathcal{O} \oplus H_1)$ over S' , equipped with the line bundle $K_1 = \mathbb{L}(1, \dots, 1)$. If H_1 is a trivial bundle with p -form $\alpha_1(t) = a_1 t^p$ then Φ_1 is the p -form Φ_1 of Example 3.3.

P_2 is $\prod_1^{p-1} \mathbb{P}^1(\mathcal{O} \oplus H_2 \otimes K_1)$ over Q_1^{p-1} , and $K_2 = K_1 \boxtimes \mathbb{L}(1, \dots, 1)$.

Lemma 4.3. *If $r > 0$ then $\dim(P_r/S') = (p^r - p^{r-1})$ and $\dim(Q_r/S') = p^r - 1$.*

Proof. Set $d_r = \dim(Q_r/S')$. This follows easily by induction from the formulas $\dim(P_{r+1}/S') = (p-1)(d_r + 1)$, $\dim(Q_{r+1}/S') = p(d_r + 1) - 1$. \square

Choosing generators u_i for H_i at the generic point of S' , we get units $a_i = \alpha_i(u_i)$.

Lemma 4.4. *At the generic points of P_r and Q_r , the p -forms Ψ_n and Φ_n of 4.2 agree with the forms defined in (3.1) and (3.2).*

Proof. This follows by induction on r , using the analysis of 4.1.2. Given a point $q = (q_1, \dots, q_p)$ of Q_{r-1}^{p-1} and a point $\{(1 : y_i)\}$ on P_r over it, $y = \{(1, y_i)\}$ is a nonzero point on $\mathbb{L}(1, \dots, 1)$ and $y_i = 1 \otimes v_i$ for a section v_i of K_{r-1} . Since $\epsilon(1, y_i) = 1 - a_r \Phi_{r-1}(v_i)$ and $\Psi_r(y) = \prod \epsilon(1, y_i)$, the forms Ψ_r agree. Similarly, if v_0 is the generator of K_{r-1} over the generic point q_0 then $y' = v_0 \otimes y$ is a generator of K_r and

$$\Phi_r(y') = \Phi_{r-1}(v_0) \Psi_r(y),$$

which is also in agreement with the formula in (3.2). \square

Recall that K_0 is the trivial line bundle, and that Φ_0 is the standard p -form $\Phi_0(v) = v^p$ on K_0 . Every point of $P_r = \prod \mathbb{P}(\mathcal{O} \oplus L)$ has the form $w = (w_1, \dots, w_{p-1})$, and the projection $P_r \rightarrow \prod Q_{r-1}$ sends $w \in P_r$ to a point $x = (x_1, \dots, x_{p-1})$.

Proposition 4.5. *Let $s \in S'$ be a point such that $a_1|_s, \dots, a_r|_s \neq 0$.*

1. *If $\Psi_r|_w = 0$ for some $w \in P_r$, then $\{a_1, \dots, a_r\}$ vanishes in $K_r^M(k(w))/p$.*
2. *If $\Phi_r|_q = 0$ for some $q = (x_0, w) \in Q_r$, $\{a_1, \dots, a_r\}$ vanishes in $K_r^M(k(q))/p$.*

Proof. Since $\Phi_r = \Phi_{r-1} \otimes \Psi_r$, the assumption that $\Psi_r|_w = 0$ implies that $\Phi_r|_q = 0$ for any $x_0 \in Q_{r-1}$ over s . Conversely, if $\Phi_r|_q = 0$ then either $\Psi_r|_w = 0$ or $\Phi_{r-1}|_{x_0} = 0$. Since $\Phi_0 \neq 0$, we may proceed by induction on r and assume that $\Phi_{r-1}|_{x_j} \neq 0$ for each j , so that $\Phi_r|_q = 0$ is equivalent to $\Psi_r|_w = 0$.

By construction, the p -form on $L = H_r \otimes K_{r-1}$ is $\phi(u_r \otimes v) = a_r \Phi_{r-1}(v)$, where u_r generates the vector space $H_r|_s$ and v is a section of K_{r-1} . Since $\Psi_r|_w$ is the product of the forms $\epsilon|_{w_j}$, some $\epsilon|_{w_j} = 0$. Lemma 1.5 implies that $a_r \Phi_{r-1}(v)$ is a p th power in $k(x_j)$, and hence in $k(w)$, for any generator v of $K_{r-1}|_{x_j}$. By Lemma 3.4, $\{a_1, \dots, a_{r-1}, \Phi_{r-1}\} = 0$ and hence

$$\{a_1, \dots, a_r\} = \{a_1, \dots, a_{r-1}, a_r \Phi_{r-1}\} = 0$$

in $K_r^M(k(w))/p$, as claimed. \square

We conclude this section with some identities in $CH(P_n)/pCH(P_n)$, given in 4.8. To simplify the statements and proofs below, we write $\text{ch}(X)$ for $CH(X)/pCH(X)$, and adopt the following notation.

Definition 4.6. Set $\eta = c_1(H_n) \in \text{ch}^1(S')$, and $\gamma = c_1(\mathbb{L}(1, \dots, 1)) \in \text{ch}^1(P_n)$. Writing \mathbb{P} for the bundle $\mathbb{P}(\mathcal{O} \oplus H_n \otimes K_{n-1})$ over Q_{n-1} , let $c \in \text{ch}(\mathbb{P})$ denote $c_1(\mathbb{L})$ and let $\kappa \in \text{ch}(Q_{n-1})$ denote $c_1(K_{n-1})$. We write $c_j, \kappa_j \in \text{ch}(P_n)$ for the images of c and κ under the j th coordinate pullbacks $\text{ch}(Q_{n-1}) \rightarrow \text{ch}(\mathbb{P}) \rightarrow \text{ch}(P_n)$.

Lemma 4.7. *Suppose that H_1, \dots, H_{n-1} are trivial. Then*

- (a) $\gamma^{p^n} = \gamma^{p^{n-1}} \eta^d$ in $\text{ch}(P_n)$, where $d = p^n - p^{n-1}$;
- (b) *If in addition H_n is trivial, then $\gamma^d = -\prod c_j \kappa_j^e$, where $e = p^{n-1} - 1$.*
- (c) *If $S' = \text{Spec } k$ then the zero-cycles $\kappa^e \in \text{ch}_0(Q_{n-1})$ and $\gamma^d \in \text{ch}_0(P_n)$ have*

$$\deg(\kappa^e) \equiv (-1)^{n-1} \quad \text{and} \quad \deg(\gamma^d) \equiv -1 \quad \text{modulo } p.$$

Proof. First note that because K_{n-1} is defined over the e -dimensional variety $Q_{n-1}(\text{Spec } k; H_1, \dots, H_{n-1})$, the element $\kappa = c_1(K_{n-1})$ satisfies $\kappa^{p^{n-1}} = 0$. Thus $(\eta + \kappa)^{p^{n-1}} = \eta^{p^{n-1}}$ and hence $(\eta + \kappa)^d = \eta^d$. Now the element $c = c_1(\mathbb{L})$ satisfies the relation $c^2 = c(\eta + \kappa)$ in $\text{ch}(\mathbb{P})$ and hence

$$c^{p^n} = c^{p^{n-1}}(\eta + \kappa)^d = c^{p^{n-1}}\eta^d$$

in $\text{ch}^{p^n}(\mathbb{P})$. Now recall that $P_n = \prod \mathbb{P}$. Then $\gamma = \sum c_j$ and

$$\gamma^{p^n} = \sum c_j^{p^n} = \sum c_j^{p^{n-1}}\eta^d = \gamma^{p^{n-1}}\eta^d.$$

When H_n is trivial we have $\eta = 0$ and hence $c^2 = c\kappa$. Setting $b_j = c_j^{p^{n-1}} = c_j\kappa_j^e$, we have $\gamma^d = \gamma^{p^{n-1}(p-1)} = (\sum b_j)^{p-1}$. To evaluate this, we use the algebra trick that since $b_j^2 = 0$ for all j and $p = 0$ we have $(\sum b_j)^{p-1} = (p-1)! \prod b_j = -\prod b_j$.

For (c), note that if $S' = \text{Spec } k$ then $\eta = 0$ and γ^d is a zero-cycle on P_n . By the projection formula for $\pi : P_n \rightarrow \prod Q_{n-1}$, part (b) yields $\pi_*\gamma^d = (-1)^p \prod \kappa_j^e$. Since each Q_{n-1} is an iterated projective space bundle, $CH(\prod Q_{n-1}) = \otimes_1^{p-1} CH(Q_{n-1})$, and the degree of $\prod \kappa_j^e$ is the product of the degrees of the κ_j^e . By induction on n , these degrees are all the same, and nonzero, so $\deg(\prod \kappa_j^e) \equiv 1 \pmod{p}$.

It remains to establish the inductive formula for $\deg(\kappa^e)$. Since it is clear for $n = 0$, and the Q_i are projective space bundles, it suffices to compute that $c_1(K_n)^{p^n-1} = \kappa^e\gamma^d$ in $\text{ch}(Q_n) = \text{ch}(Q_{n-1}) \otimes \text{ch}(P_n)$. Since $\kappa^{e+1} = 0$ and $c_1(K_n) = \kappa + \gamma$ we have

$$c_1(K_n)^{p^n-1} = \kappa^{e+1} + \gamma^{p^n-1} = \gamma^{p^n-1},$$

and hence $c_1(K_n)^d = \gamma^d$. Since $\gamma^{d+1} = 0$, this yields the desired calculation:

$$c_1(K_n)^{p^n-1} = c_1(K_n)^e c_1(K_n)^d = (\kappa + \gamma)^e \gamma^d = \kappa^e \gamma^d. \quad \square$$

Corollary 4.8. *There is a ring homomorphism $\mathbb{F}_p[\lambda, z]/(z^p - \lambda^{p-1}z) \rightarrow \text{ch}(P_n)$, sending λ to $\eta^{p^{n-1}}$ and z to $\gamma^{p^{n-1}}$.*

5. MODEL FOR p MOVES

In this section we construct maps $S_{n-1} \rightarrow S_n$ which model the p moves of type C_n defined in 3.6. Each such move introduces $p^{n-1} - p^{n-2}$ new variables, and will be modelled by a map $Y_r \rightarrow Y_{r-1}$ of relative dimension $p^{n-1} - p^{n-2}$, using the P_{n-1} construction in 4.2. The result (Definition 5.1) will be a tower of the form:

$$\begin{array}{ccccccc} J_{n-1} = L_p & & L_{p-1} & & L_2 & & L_1 & & L_0 = J_n \\ S_{n-1} = Y_p & \xrightarrow{f_p} & Y_{p-1} & \longrightarrow & \cdots & \longrightarrow & Y_2 & \xrightarrow{f_2} & Y_1 & \xrightarrow{f_1} & Y_0 = S_n. \end{array}$$

Fix $n \geq 2$, a variety S_n , and line bundles H_1, \dots, H_{n-2}, H_n and J_n on S_n . The first step in the tower is to form $Y_0 = S_n$ and $Y_1 = P_{n-1}(S_n; H_1, \dots, H_{n-2}, J_n)$, with line bundles $L_0 = J_n$ and $L_1 = H_n \otimes \mathbb{L}(1, \dots, 1)$ as in 4.2. In forming the other Y_r , the base in the P_{n-1} construction 4.2 will become Y_{r-1} and only the final line bundle will change (from J_n to L_{r-1}). Here is the formal definition.

Definition 5.1. For $r > 1$, we define morphisms $f_r : Y_r \rightarrow Y_{r-1}$ and line bundles \mathbb{L}_r^{\boxtimes} and L_r on Y_r as follows. Inductively, we are given a morphism $f_{r-1} : Y_{r-1} \rightarrow Y_{r-2}$ and line bundles L_{r-1} on Y_{r-1} , L_{r-2} on Y_{r-2} . Set $\mathbb{L}_r^{\boxtimes} = \mathbb{L}(1, \dots, 1)$,

$$Y_r = P_{n-1}(Y_{r-1}; H_1, \dots, H_{n-2}, L_{r-1}) \xrightarrow{f_r} Y_{r-1}, \quad L_r = f_r^* f_{r-1}^*(L_{r-2}) \otimes \mathbb{L}_r^{\boxtimes}.$$

Finally, we write S_{n-1} for Y_p and set $J_{n-1} = L_p$, $J'_{n-1} = f_p^*(L_{p-1})$. By Lemma 4.3, $\dim(Y_r/Y_{r-1}) = p^{n-1} - p^{n-2}$ and hence $\dim(S_{n-1}/S_n) = p^n - p^{n-1}$.

For example, when $n = 2$ and H_1 is trivial, this tower is exactly the tower of 2.1: we have $Y_r = P_1(Y_{r-1}; L_{r-1}) = \prod \mathbb{P}^1(\mathcal{O} \oplus L_{r-1})$.

Remark 5.1.1. The line bundles J_{n-1} and J'_{n-1} will be the line bundles of the Chain Lemma 0.1. The rest of tower (3.9) will be obtained in Definition 5.8 by repeating this construction and setting $S = S_1$.

The rest of this section, culminating in Theorem 5.9, is devoted to proving part (6) of the Chain Lemma, that the degree of the zero-cycle $c_1(J_1)^{\dim S}$ is relatively prime to p . In preparation, we need to compare the degrees of the zero-cycles $c_1(J_{n-1})^{\dim S_{n-1}}$ on S_{n-1} and $c_1(J_n)^{\dim S_n}$ on S_n . In order to do so, we introduce the following algebra.

Definition 5.2. We define the graded \mathbb{F}_p -algebra A_r and \bar{A}_r by $\bar{A}_r = A_r/\lambda_{-1}A$ and:

$$A_r = \mathbb{F}_p[\lambda_{-1}, \lambda_0, \dots, \lambda_r, z_1, \dots, z_r]/(\{z_i^p - \lambda_{i-1}^{p-1}z_i, \lambda_i - \lambda_{i-2} - z_i \mid i = 1, \dots, r\}).$$

Remark 5.2.1. By Corollary 4.8, there is a homomorphism $A_p \xrightarrow{\rho} \text{ch}(Y_p)$, sending λ_r to $c_1(L_r)^{p^{n-2}}$ and z_r to $c_1(\mathbb{L}_r^{\boxtimes})^{p^{n-2}}$. When H_{n-1} is trivial, ρ factors through \bar{A}_p .

Lemma 5.3. In \bar{A}_r , every element u of degree 1 satisfies $u^{p^2} = u^p \lambda_0^{p^2-p}$.

Proof. We will show that \bar{A}_r embeds into a product of graded rings of the form $\Lambda_k = \mathbb{F}_p[\lambda_0][v_1, \dots, v_k]/(v_1^p, \dots, v_k^p)$. In each entry, $u = a\lambda_0 + v$ with $v^p = 0$ and $a \in \mathbb{F}_p$, so $u^p = a\lambda_0^p$ and $u^{p^2} = a\lambda_0^{p^2}$, whence the result.

Since $\bar{A}_{r+1} = \bar{A}_r[z]/(z^p - \lambda_r^{p-1}z)$ is flat over \bar{A}_r , it embeds by induction into a product of graded rings of the form $\Lambda' = \Lambda_k[z]/(z^p - u^{p-1}z)$, $u \in \Lambda_k$. If $u \neq 0$, there is an embedding of Λ' into $\prod_{i=0}^{p-1} \Lambda_k$ whose i th component sends z to iu . If $u = 0$, then $\Lambda' \cong \Lambda_{k+1}$. \square

Remark 5.3.1. It follows that if $m > 0$ and $(p^2 - p) \mid m$ then $u^{kp+m} = \lambda_0^m u^{kp}$.

Proposition 5.4. In \bar{A}_r , $\lambda_p^{p^N-p} = \lambda_0^{p^N-p^2} (\prod z_i^{p-1} + T\lambda_0)$, where $\deg(T) = p^2 - p - 1$.

Proof. By Definition 5.2, \bar{A}_p is free over $\mathbb{F}_p[\lambda_0]$, with the elements $\prod z_i^{m_i}$ ($0 \leq m_i < p$) forming a basis. Thus any term of degree $p^N - p$ is a linear combination of $F = \lambda_0^{p^N-p^2} \prod z_i^{p-1}$ and terms of the form $\lambda_0^{m_0} \prod z_i^{m_i}$ where $\sum m_i = p^N - p^2$ and $m_0 > p^N - p^2$. It suffices to determine the coefficient of F in $\lambda_p^{p^N-p}$. Since $\lambda_p^{p^N-p} = \lambda_0^{p^N-p^2} \lambda_p^{p^2-p}$ by Remark 5.3.1, it suffices to consider $N = 2$, when $F = \prod z_i^{p-1}$.

As in the proof of Proposition 2.6, if $p \geq r \geq 2$ we compute in the ring \bar{A}_r that

$$\begin{aligned} \lambda_r^{r(p-1)} &= (z_r + \lambda_{r-2})^{p(r-1)+(p-r)} = (z_r^p + \lambda_{r-2}^p)^{r-1} \cdot (z_r + \lambda_{r-2})^{p-r} \\ &= (z_r \lambda_{r-1}^{p-1} + \lambda_{r-2}^p)^{r-1} (z_r + \lambda_{r-2})^{p-r} = z_r^{p-1} \lambda_{r-1}^{(r-1)(p-1)} + T, \end{aligned}$$

where $T \in \bar{A}_{r-1}[z_r]$ is a homogeneous polynomial of total degree $< p-1$ in z_r . By induction on r , the coefficient of $(z_1 \cdots z_r)^{p-1}$ in $\lambda_r^{r(p-1)}$ is 1 for all r . \square

Lemma 5.5. *If $S_n = \text{Spec}(k)$ and $c = c_1(J_{n-1}) \in CH^1(S_{n-1})$, then*

$$\deg(c^{\dim S_{n-1}}) \equiv 1 \pmod{p}.$$

Proof. Set $d = \dim(S_{n-1}) = p^n - p^{n-1}$; under the map $A_p \xrightarrow{\rho} \text{ch}(S_{n-1})$ of 5.2.1, the degree $p^2 - p$ part of A_p maps to $CH^d(S_{n-1})$. In particular, the zero-cycle $c^d = \rho(\lambda_p)^{p^2-p}$ equals the product of the $\rho(z_i)^{p-1} = c_1(\mathbb{L}_i^{\boxtimes})^{d/p}$ by Proposition 5.4 (the $T\lambda_0^*$ term maps to zero for dimensional reasons). Because $S_{n-1} = Y_p$ is a product of iterated projective space bundles, $CH_0(Y_p)$ is the tensor product of their CH_0 groups, and the degree of c^d is the product of the degrees of the $c_1(\mathbb{L}_i^{\boxtimes})^{d/p}$, each of which is -1 by Lemma 4.7. It follows that $\deg(c^d) \equiv 1 \pmod{p}$. \square

Theorem 5.6. *If S_n has dimension $p^M - p^n$ and H_1, \dots, H_{n-1} are trivial then the zero-cycles $c_1(J_{n-1})^{\dim S_{n-1}} \in CH_0(S_{n-1})$ and $c_1(J_n)^{\dim S_n} \in CH_0(S_n)$ have the same degree modulo p :*

$$\deg(c_1(J_{n-1})^{\dim S_{n-1}}) = \deg(c_1(J_n)^{\dim S_n}) \pmod{p}.$$

Proof. By 5.2.1, there is a homomorphism $A_p \xrightarrow{\rho} \text{ch}(S_{n-1})$, sending λ_r to $c_1(L_r)^{p^{n-2}}$ and z_r to $c_1(\mathbb{L}_r^{\boxtimes})^{p^{n-2}}$. Because H_{n-1} is trivial, ρ factors through \bar{A}_p .

Set $N = M - n + 2$ and $y = \lambda_p^{p^N - p^2}$, so $\rho(y) = c_1(J_n)^{\dim S_n} \in \text{ch}_0(S_n)$. From Proposition 5.4 we have $\lambda_p^{p^N - p^2} \equiv y \prod z_i^{p-1}$ modulo $\ker(\rho)$. From Lemma 2.5, the degree of this element equals the degree of y modulo p . \square

The p -forms. We now define the p -forms on the line bundles J_{n-1} and J'_{n-1} using the tower (5.1). Suppose that the line bundles $L_{-1} = H_n$ and $L_0 = J_n$ on S_n are equipped with the p -forms β_{-1} and β_0 . We endow the line bundle L_1 in Definition 5.1 with the p -form $\beta_1 = f^*(\beta_{-1}) \otimes \Psi_{n-1}(\beta_0)$; inductively, we endow the line bundle L_r with the p -form

$$\beta_r = f^*(\beta_{r-2}) \otimes \Psi_{n-1}(\beta_{r-1}).$$

Example. When $n = 2$ and H_1 is trivial saw that the tower 5.1 is exactly the tower of 2.1. In addition, the p -form $\beta_r = \Psi_1(\beta_{r-1})$ agrees with the p -form $\varphi_r = f^*(\varphi_{r-2}) \otimes \Psi_r$ of 2.7.

Lemma 5.7. *If $\beta_0 = \alpha_{n-1}$ and $\beta_{-1} = \alpha_n$, then (at the generic point of Y_1) the p -form β_p agrees with the form $\alpha_n \Psi_{n-1}$ in (3.6).*

Proof. By Lemma 4.4, the form agrees with the form of (3.1). \square

Definition 5.8. The tower (3.9) of varieties S_i is obtained by downward induction, starting with $S_n = \text{Spec}(k)$ and $J_n = H_{n-1}$. Construction 5.1 yields S_{n-1} , J_{n-1} and J'_{n-1} . Inductively, we repeat construction 5.1 for i , starting with the output S_{i+1} and J_{i+1} of the previous step, to produce S_i , J_i and J'_i .

By downward induction in the tower (3.9), each J_i and J'_i carries a p -form, which we call γ_i and γ'_i , respectively. By 5.7, these forms agree with the forms γ_i and γ'_i of (3.7) and (3.8).

Since $\dim(S_i/S_{i-1}) = p^{i+1} - p^i$ we have $\dim(S_i/S_n) = p^n - p^i$. Thus if we combine Lemma 5.5 and Theorem 5.6, we obtain the following result.

Theorem 5.9. *For each $i < n$, $\deg(c_1(J_i)^{\dim S_i}) \equiv -1 \pmod{p}$.*

Theorem 5.9 establishes part (6) of the Chain Lemma 0.1, that $\deg(c_1(J_1)^{\dim S_1})$.

Proof of the Chain Lemma 0.1. We verify the conditions for the variety $S = S_1$ in the tower (3.9); the line bundles J_i and J'_i and their p -forms are obtained by pulling back from the bundles and forms defined in 5.8. Part (1) of Theorem 0.1 is immediate from the construction of S ; part (6) is Theorem 5.9, combined with Lemma 5.5. Part (2) was just established, and part (4) was proven in Proposition 4.5; parts (3) and (5) follow from (2) and (4). This completes the proof of the Chain Lemma. \square

6. NICE G -ACTIONS

We will extend the Chain Lemma to include an action by $G = \mu_p^n$ on S , J_i , J'_i leaving γ_i and γ'_i invariant, such that the action is admissible in the following sense.

Definition 6.1. (Rost, cf. [7, p.2]) Let G be a group acting on a k -variety X . We say that the action is *nice* if $\text{Fix}_G(X)$ is 0-dimensional, and consists of k -points.

When G also acts on a line bundle L over X , the action on the geometric bundle L is *nice* exactly when G acts nontrivially on $L|_x$ for every fixed point $x \in X$, and in this case $\text{Fix}_G(L)$ is the zero-section over $\text{Fix}_G(X)$.

Suppose that G acts nicely on each of several line bundles L_i over X . We say that G acts *nicely* on $\{L_1, \dots, L_r\}$ if for each fixed point $x \in X$ the image of the canonical representation $G \rightarrow \prod \text{Aut}(L_i|_x) = \prod k(x)^\times$ is $\prod G_i$, with each G_i nontrivial.

Remark 6.1.1. If $X_i \rightarrow S$ are equivariant maps and the X_i are nice, then G also acts nicely on $X_1 \times_S X_2$. However, even if G acts nicely on line bundles L_i it may not act nicely on $L_1 \boxtimes L_2$, because the representation over (x_1, x_2) is the product representation $L_1|_{x_1} \otimes L_2|_{x_2}$.

Example 6.2. Suppose that G acts nicely on a line bundle L over X . Then the induced G -action on $\mathbb{P} = \mathbb{P}(\mathcal{O} \oplus L)$ and its canonical line bundle \mathbb{L} is nice. Indeed, if $x \in X$ is a fixed point then the fixed points of $\mathbb{P}|_x$ consist of the two k -points $\{[\mathcal{O}], [L]\}$, and if $L|_x$ is the representation ρ then G acts on \mathbb{L} at these fixed points as ρ and ρ^{-1} , respectively.

By 6.1.1, G also acts nicely on the products $P = \prod \mathbb{P}(\mathcal{O} \oplus L)$ and $Q = X \times_{S'} P$ of Definition 4.1, but it does not act nicely on $\mathbb{L}(1, \dots, 1)$.

Example 6.3. The group G also acts nicely on the Kummer algebra bundle $\mathcal{A} = \mathcal{A}(L)$ of 1.4, and on its projective space $\mathbb{P}(\mathcal{A})$. Indeed, an elementary calculation shows that $\text{Fix}_G \mathbb{P}(\mathcal{A})$ consists of the p sections $[L^i]$, $0 \leq i < p$ over $\text{Fix}_G(X)$. In each fiber, the (vertical) tangent space at each fixed point is the representation $\rho \oplus \dots \oplus \rho^{p-1}$. If $G = \mu_p$, this is the reduced regular representation.

Over any fixed point $x \in X$, $L|_x$ is trivial, and the symmetric group Σ_p acts on the bundle $\mathcal{A}|_x$, permuting the fixed points. This induces isomorphisms between the tangent spaces at these points.

Example 6.3.1. The action of G on $Y = \mathbb{P}(\mathcal{O} \oplus \mathcal{A})$ is not nice. In this case, an elementary calculation shows that $\text{Fix}_G(Y)$ consists of the points $[L^i]$ of $\mathbb{P}(\mathcal{A})$, $0 < i < p$, together with the projective line $\mathbb{P}(\mathcal{O} \oplus \mathcal{O})$ over every fixed point x of X . For each x , the (vertical) tangent space at $[L^i]$ is $1 \oplus \rho \oplus \dots \oplus \rho^{p-1}$; if $G = \mu_p$, this is the regular representation.

When $G = \mu_p^n$, the following lemma allows us to assume that the action on $L|_x$ is induced by the standard representation $\mu_p \subset k^\times$, via a projection $G \rightarrow \mu_p$.

Lemma 6.4. *Any nontrivial 1-dimensional representation ρ of $G = \mu_p^n$ factors as the composition of a projection $G \rightarrow \mu_p$ with the standard representation of μ_p .*

Proof. The representation ρ is a nonzero element of $(\mathbb{Z}/p)^n = G^* = \text{Hom}(\mu_p^n, \mathbb{G}_m)$, and π is the Pontryagin dual of the induced map $\mathbb{Z}/p \rightarrow G^*$ sending 1 to ρ . \square

The construction of the P_r and Q_r in 4.2 is natural in the given line bundles H_1, \dots, H_n over S' , and so is the construction of the Y_r , S_r and S in 5.1 and 5.8. Since $\prod_{i=1}^n \text{Aut}(H_i)$ acts on the H_i , this group (and any subgroup) will act on the variety S of the Chain Lemma. We will show that it acts nicely on S .

Recall from Definition 4.2 that P_r and Q_r are defined by the construction 4.1 using the line bundle $L_r = H_r \otimes K_{r-1}$ over Q_{r-1} .

Lemma 6.5. *If $S' = \text{Spec}(k)$, then $G = \mu_p^r$ acts nicely on L_r , P_r and Q_r .*

This implies that any subgroup of $\prod_{i=1}^r \text{Aut}(H_i)$ containing μ_p^r also acts nicely.

Proof. We proceed by induction on r , the case $r = 1$ being 6.2, so we may assume that μ_p^{r-1} acts nicely on Q_{r-1} . By 6.1.1, it suffices to show that $G = \mu_p^r$ acts nicely on $\mathbb{P}(\mathcal{O} \oplus L_r)$, where $L_r = H_r \otimes K_{r-1}$. Since the final component μ_p of G acts trivially on K_{r-1} and Q_{r-1} and nontrivially on H_r , $G = \mu_p^{r-1} \times \mu_p$ acts nicely on L_r . By Example 6.2, G acts nicely on $\mathbb{P}(\mathcal{O} \oplus L_r)$. \square

The proof of Lemma 6.5 goes through in slightly greater generality.

Corollary 6.6. *Suppose that $G = \mu_p^n$ acts nicely on S' and on the line bundles $\{H_1, \dots, H_r\}$ over it. Then G acts nicely on L_r , P_r and Q_r .*

Proof. Without loss of generality, we may replace S' by a fixed point $s \in S'$, in which case G acts nicely on $\{H_1, \dots, H_r\}$ through the surjection $\mu_p^n \rightarrow \mu_p^r$. Now we are in the situation of Lemma 6.5. \square

Example 6.6.1. Since μ_p^{n-1} acts nicely on $Y = P_{n-1}(S'; H_1, \dots, H_{n-1})$ and on the bundle K_{n-1} , while μ_p of $G = \mu_p^n$ acts solely on H_n , it follows that the group $\mu_p^n = \mu_p^{n-1} \times \mu_p$ acts nicely on $\{H_1, \dots, H_{n-1}, H_n \otimes \mathbb{L}(1, \dots, 1)\}$ over Y .

We can now process the tower of varieties Y_r defined in 5.1. For notational convenience, we write H_{n-1} for J_n . The case $r = 0$ of the following assertion uses the convention that $L_0 = H_{n-1}$ and $L_{-1} = H_n$.

Proposition 6.7. *Suppose that $G = G_0 \times \mu_p^n$ acts nicely on S_n and (via $G \rightarrow \mu_p^n$) on $\{H_1, \dots, H_n\}$. Then G acts nicely on each Y_r , and on its line bundles $\{H_1, \dots, H_{n-2}, L_r, L_{r-1}\}$.*

Proof. The question being local, we may replace S' by a fixed point $s \in S'$, and G by μ_p^n . We proceed by induction on r , the case $r = 1$ being Example 6.6.1, since $L_1 = H_n \otimes \mathbb{L}(1, \dots, 1)$. Inductively, suppose that G acts nicely on Y_r and on $\{H_1, \dots, H_{n-2}, L_r, L_{r-1}\}$. Thus there is a factor of G isomorphic to μ_p which acts nontrivially on L_r but acts trivially on $\{H_1, \dots, H_{n-2}, L_r\}$. Hence this factor acts trivially on $Y_{r+1} = P_{n-1}(Y_r; H_1, \dots, H_{n-2}, L_r)$ and its line bundle \mathbb{L}^{\boxtimes} , and nontrivially on $L_{r+1} = L_{r-1} \otimes \mathbb{L}^{\boxtimes}$. The assertion follows. \square

Corollary 6.8. *$G = \mu_p^n$ acts nicely on (S, J) .*

Proof. By Definition 5.1, $S_{n-1} = Y_p$, $J_{n-1} = L_p$ and $J'_{n-1} = L_{p-1}$. By 6.7 with $r = p$, G acts nicely on S_{n-1} and on $\{H_1, \dots, H_{n-2}, J_{n-1}, J'_{n-1}\}$. By downward induction, $G = \mu_p^{n-i} \times \mu_p^i$ acts nicely on S_i and $\{H_1, \dots, H_{i-1}, J_i, J'_i\}$ for all $i \leq n$. The case $i = 1$ is the conclusion, since $(S, J) = (S_1, J_1)$. \square

Remark 6.8.1. If $G = \mu_p^n$ acts nicely on S' , Rost [7, p.2] would say that a fixed point $s \in S'$ is *twisting* for $\{H_1, \dots, H_r\}$ if the map $G \rightarrow \mu_p^r \subset \prod k(s)^\times = \prod \text{Aut}(H_i|_s)$ is a surjection.

7. G -FIXED POINT EQUIVALENCES

Let $\mathcal{A} = \mathcal{A}(J)$ be the Kummer algebra over the variety S of the Chain Lemma 0.1, as in 1.4. The group $G = \mu_p^n$ acts nicely on S and J by 6.8, and on \mathcal{A} and $\mathbb{P}(\mathcal{A})$ by 6.3. In this section, we introduce two G -varieties \bar{Y} and Q , parametrized by norm conditions, and show that they are G -fixed point equivalent to $\mathbb{P}(\mathcal{A})$ and $\mathbb{P}(\mathcal{A})^p$, respectively. This will be used in the next section to show that \bar{Y} is G -fixed point equivalent to the Weil restriction of Q_E for any Kummer extension E of k .

We begin by defining fixed point equivalence and the variety Q .

Definition 7.1. Let G be an algebraic group. We say that two G -varieties X and Y are *G -fixed point equivalent* if $\text{Fix}_G X$ and $\text{Fix}_G Y$ are 0-dimensional, lie in the smooth locus of X and Y , and there is a separable extension K of k and a bijection $\text{Fix}_G(X_K) \rightarrow \text{Fix}_G(Y_K)$ under which the families of tangent spaces at the fixed points are isomorphic as G -representations over K .

Definition 7.2. Recall from 1.4 that the norm $\mathcal{A} \xrightarrow{N} \mathcal{O}_S$ is equivariant, and homogeneous of degree p . We define the G -variety Q over $S \times \mathbb{A}^1$, and its fiber Q_w over $w \in k$, by the equation $N(\beta) = w$:

$$Q = \{[\beta, t] \in \mathbb{P}(\mathcal{A} \oplus \mathcal{O}) \times \mathbb{A}^1 : N(\beta) = t^p w\},$$

$$Q_w = \{[\beta, t] \in \mathbb{P}(\mathcal{A} \oplus \mathcal{O}) : N(\beta) = t^p w\}, \quad \text{for } w \in k.$$

Since $\dim(S) = p^n - p$ we have $\dim(Q_w) = p^n - 1$. If $w \neq 0$, then it is proved in [10, §2] that Q_w is geometrically irreducible and that the open subscheme where $t \neq 0$ is smooth.

If $w \neq 0$, Q_w is disjoint from the section $\sigma : S \cong \mathbb{P}(\mathcal{O}) \rightarrow \mathbb{P}(\mathcal{A} \oplus \mathcal{O})$; over each point of S , the point $(0 : 1)$ is disjoint from Q_w . Hence the projection $\mathbb{P}(\mathcal{A} \oplus \mathcal{O}) - \sigma(S) \rightarrow \mathbb{P}(\mathcal{A})$ from these points induces an equivariant morphism $\pi : Q_w \rightarrow Y = \mathbb{P}(\mathcal{A})$, $\pi(\beta, t) = \beta$. This is a cover of degree p over its image, since $\pi(\beta, t) = \pi(\beta, \zeta t)$ for all $\zeta \in \mu_p$.

Theorem 7.3. *If $w \neq 0$, G acts nicely on Q_w and $\text{Fix}_G Q_w \cap (Q_w)_{\text{sing}} = \emptyset$. Moreover, Q_w and $Y = \mathbb{P}(\mathcal{A})$ are G -fixed point equivalent over the field $\ell = k(\sqrt[p]{b})$.*

Proof. Since the maps $Q_w \xrightarrow{\pi} Y \rightarrow S$ are equivariant, π maps $\text{Fix}_G Q_w$ to $\text{Fix}_G Y$, and both lie over the finite set $\text{Fix}_G S$ of k -rational points. Since the tangent space T_y is the product of $T_s S$ and the tangent space of the fiber Y_s , and similarly for Q_w , it suffices to consider a G -fixed point $s \in S$.

By 6.7 and Lemma 6.4, G acts nontrivially on $L = J|_s$ via a projection $G \rightarrow \mu_p$. By Example 6.3, G acts nicely on $\mathbb{P}(\mathcal{A})$. Thus there is no harm in assuming that $G = \mu_p$ and that L is the standard 1-dimensional representation.

Let $y \in Y$ be a G -fixed point lying over s . By 6.2, the tangent space of $Y|_s$ at y is the reduced regular representation, and y is one of $[1], [L], \dots [L^{p-1}]$.

We saw in Example 6.3.1 that a fixed point $[a_0 : a_1 : \dots : a_{p-1} : t]$ of G in $\mathbb{P}(\mathcal{A} \oplus \mathcal{O})|_s$ is either one of the points $e_i = [\dots : 0 : a_i : 0 \dots : 0]$, which do not lie on Q_w , or a point on the projective line $\{[a_0 : 0 : t]\}$. By inspection, $Q_w \otimes_k \ell$ meets the projective line in the ℓ -points $[\zeta \sqrt[p]{b} : 0 : \dots : 0 : 1]$, $\zeta \in \mu_p$. Each of these p points is smooth on Q_w , and the tangent space (over s) is the reduced regular representation of G . \square

Remark 7.3.1. Since $\pi([\zeta \sqrt[p]{b} : 0 : \dots : 0 : 1]) = [1]$ for all ζ , $\text{Fix}_G(Q_w) \xrightarrow{\pi} \text{Fix}_G(Y)$ is *not* a scheme isomorphism over ℓ .

Remark 7.4. For any $w \in k^\times$ of N , any desingularization Q' of Q_w is a smooth, geometrically irreducible splitting variety for the symbol $\{a_1, \dots, a_n, w\}$ in $K_{n+1}^M(k)/p$.

Assuming the Bloch-Kato conjecture for n , Suslin and Joukhovitski show it is a norm variety in [10, §2]. Note that the variety X_w of 3.5 is birationally a cover of Q_w .

To construct \bar{Y} , we fix a Kummer extension $E = k(\epsilon)$ of k . Let \mathcal{B} be the \mathcal{O}_S -subbundle $(\mathcal{A} \otimes 1) \oplus (\mathcal{O}_S \otimes \epsilon)$ of $\mathcal{A}_E = \mathcal{A} \otimes_k E$ and let $N_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{O}_S \otimes_k E$ be the map induced by the norm on \mathcal{A}_E .

Definition 7.5. Let U be the variety $\mathbb{P}(\mathcal{A}) \times \mathbb{P}(\mathcal{B})^{\times(p-1)}$ over $S^{\times p}$, and let L be the line bundle $\mathbb{L}(\mathcal{A}) \boxtimes \mathbb{L}(\mathcal{B})^{\boxtimes(p-1)}$ over U , given as the exterior product of the tautological bundles. The product of the various norms defines an algebraic morphism $N : L \rightarrow \mathcal{O}_S \otimes E$.

Lemma 7.6. *Let $u \in U$ be a point over $(s_0, s_1, \dots, s_{p-1})$, and write A_i for the $k(s_i)$ -algebra $\mathcal{A}|_{s_i}$. Then the following hold.*

- (1) *If $\{a\}$ doesn't split at any of the points s_0, \dots, s_{p-1} , then the norm map $N : L_u \rightarrow k(u) \otimes E$ is non-zero.*
- (2) *If $\{a\}|_{s_0} \neq 0$ in $K_n^M(k(s_0))/p$, then A_0 is a field.*
- (3) *For $i \geq 1$, if $\{a\}|_{E(s_i)} \neq 0$ in $K_n^M(E(s_i))/p$ then $A_i \otimes E$ is a field.*

Proof. The first assertion follows from part (4) of the Chain Lemma 0.1, since by 1.4 the norm on L is induced from the p -form γ_1 on J . Assertions (2-3) follow from part (2) of the Chain Lemma, since $\{a\} \neq 0$ implies that γ is nontrivial. \square

Definition 7.7. Let \mathbb{A}^E denote the Weil restriction $\text{Res}_{E/k} \mathbb{A}^1$, characterized by $\mathbb{A}^E(F) = F \otimes_k E$ ([16]). Let \bar{Y} denote the subvariety of $\mathbb{P}(L \oplus \mathcal{O}) \times \mathbb{A}^E$ consisting of all points $([\alpha : t], w)$ such that $N(\alpha) = t^p w$ in E . We write \bar{Y}_w for the fiber over a point $w \in \mathbb{A}^E$. Note that $\dim(\bar{Y}_w) = p^{n+1} - p = p \dim(Q_w)$.

Notation 7.8. Let $([\alpha : t], w)$ be a k -rational point on \bar{Y} , so that $w \in \mathbb{A}^E(k) = E$. We may regard $[\alpha : t] \in \mathbb{P}(L \oplus \mathcal{O})(k)$ as being given by a point $u \in U(k)$, lying over a point $(s_0, \dots, s_{p-1}) \in S(k)^{\times p}$, and a nonzero pair $(\alpha, t) \in L_u \times k$ (up to scalars). From the definition of L , we see that (up to scalars) α determines a p -tuple $(b_0, b_1 + t_1 \epsilon, \dots, b_{p-1} + t_{p-1} \epsilon)$, where $b_i \in \mathcal{A}|_{s_i}$ and $t_i \in k$. When $\alpha \neq 0$, $b_0 \neq 0$ and for all $i > 0$, $b_i \neq 0$ or $t_i \neq 0$. Finally, writing A_i for $\mathcal{A}|_{s_i}$, the norm condition says that in E :

$$N_{A_0/k}(b_0) \prod_{i=1}^{p-1} N_{A_i \otimes E/E}(b_i + t_i \epsilon) = t^p w.$$

If $k \subseteq F$ is a field extension, then an F -point of \bar{Y} is described as above, replacing k by F and E by $E \otimes_k F$ everywhere.

Remark 7.8.1. If $w \neq 0$, then $\alpha \neq 0$, because $N(\alpha) = t^p w$ and $(\alpha, t) \neq (0, 0)$.

Lemma 7.9. *If \bar{Y} has a k -point with $t = 0$ then $\{\underline{a}\}|_E = 0$ in $K_n^M(E)/p$.*

Proof. We use the description of a k -point of \bar{Y} from 7.8. If $t = 0$, then $\alpha \neq 0$, therefore $b_0 \neq 0 \in A_0$ and $b_i + t_i \epsilon \neq 0 \in A_i \otimes E$. By Lemma 7.6, if $\{\underline{a}\}|_E \neq 0$ in $K_n^M(E)/p$ then A_0 and all the algebras $A_i \otimes E$ are fields, so that $N(\alpha) = N_{A_0/k}(b_0) \prod_{i=1}^{p-1} N_{A_i \otimes E/E}(b_i + t_i \epsilon) \neq 0$, a contradiction to $t^p w = 0$. \square

Consider the projection $\bar{Y} \rightarrow \mathbb{A}^E$ onto the second factor, and write \bar{Y}_w for the (scheme-theoretic) fiber over $w \in \mathbb{A}^E$. Combining 7.6 with 7.9 we obtain the following consequence (in the notation of 7.8):

Corollary 7.10. *If $\{\underline{a}\} \neq 0$ in $K_n^M(E)/p$ and $w \neq 0$ is such that \bar{Y}_w has a k -point, then A_0 and the $A_i \otimes E$ are fields and w is a product of norms of an element of A_0 and elements in the subsets $A_i + \epsilon$ of $A_i \otimes_k E$.*

Remark 7.10.1. In Theorem 7.13 we will see that if w is a generic element of E then such a k -point exists.

The group $G = \mu_p^n$ acts nicely on S and J by 6.8, and on \mathcal{A} and $\mathbb{P}(\mathcal{A})$ by 6.3. It acts trivially on \mathbb{A}^E , so G acts on B , U and \bar{Y} (but not nicely; see 6.1.1).

In the notation of 7.8, if $([\alpha : t], w)$ is a fixed point of the G -action on \bar{Y} then the points $u_0 \in \mathbb{P}(\mathcal{A})$ and $s_i \in S$ are fixed, and therefore are k -rational (see 6.1). If u is defined over F , each point $(b_i : t_i)$ is fixed in $\mathcal{B}|_{s_i}$. Since S acts nicely on J , Example 6.3.1 shows that if $t = 0$ then either $t_i \neq 0$ (and $b_i \in F \subset A_i \otimes F$) or else $t_i = 0$ and $0 \neq b_i \in J|_{s_i}^{\otimes r_i} \otimes F \subseteq A_i \otimes F$ for some r_i , $0 \leq r_i < p$.

Lemma 7.11. *For all w , $\text{Fix}_G \bar{Y}_w$ is disjoint from the locus where $t = 0$.*

Proof. Suppose $([\alpha : 0], w)$ is a fixed point defined over a field F containing k . As explained above, $b_0 \neq 0$ and (for each $i > 0$) $b_i + t_i \epsilon \neq 0$ and either $t_i \neq 0$ or there is an r_i so that $b_i \in J|_{s_i}^{r_i} \otimes F$. Let I be the set of indices such that $t_i \neq 0$.

By Example 6.3, $b_0 \in J|_{s_0}^{\otimes r_0}$ for some r_0 , and hence $N_{A_0}(b_0)$ is a unit in k , because the p -form γ is nontrivial on $J|_{s_0}$. Likewise, if $i \notin I$, then $N_{A_i \otimes F/F}(b_i)$ is a unit in F .

Now suppose $i \in I$, i.e., $t_i \neq 0$, and recall that in this case $b_i \in F \subset A_i \otimes F$. If we write EF for the algebra $E \otimes F \cong F[\epsilon]/(\epsilon^p - e)$, then the norm from $A_i \otimes EF$ to EF is simply the p -th power on elements in EF , so $N_{A_i \otimes EF/EF}(b_i + t_i \epsilon) = (b_i + t_i \epsilon)^p$ as an element in the algebra EF . Taking the product, and keeping in mind $t = 0$, we get the equation

$$\prod_{i \in I} N_{A_i \otimes EF/EF}(b_i + t_i \epsilon) = \prod_{i \in I} (b_i + t_i \epsilon)^p = 0.$$

Because EF is a separable F -algebra, it has no nilpotent elements. We conclude that

$$\prod_{i \in I} (b_i + t_i \epsilon) = 0.$$

The left hand side of this equation is a polynomial of degree at most $p - 1$ in ϵ ; since $\{1, \epsilon, \dots, \epsilon^{p-1}\}$ is a basis of $F \otimes E$ over F , that polynomial must be zero. This implies that $b_i = t_i = 0$ for some i , a contradiction. \square

Proposition 7.12. *If $w \in \mathbb{A}^E$ is generic then $\text{Fix}_G \bar{Y}_w$ lies in the open subvariety where $t \prod_{i=1}^p t_i \neq 0$.*

Remark 7.12.1. The open subvariety in 7.12 is G -isomorphic (by setting t and all t_i to 1) to a closed subvariety of $\mathbb{A}(\mathcal{A})^p$, namely the fiber over w of the map $N_{\mathcal{A} \otimes E/E} : \mathbb{A}(\mathcal{A})^p \rightarrow \mathbb{A}^E$ defined by

$$N(b_0, \dots, b_{p-1}) = N_{A_0/k}(b_0) \prod_{i=1}^{p-1} N_{A_i \otimes E/E}(b_i + \epsilon).$$

Indeed, $\mathbb{A}(\mathcal{A})^p$ is G -isomorphic to an open subvariety of \bar{Y} and $N_{A_i \otimes E/E}$ is the restriction of $\alpha \mapsto N(\alpha)$.

Proof. By Lemma 7.11, $\text{Fix}_G \bar{Y}_w$ is disjoint from the locus where $t = 0$, so we may assume that $t = 1$. Since w is generic, we may also take $w \neq 0$. So let $([\alpha : 1], w)$ be a fixed point defined over $F \supseteq k$ for which $t_j = 0$. As in the proof of the previous lemma, we collect those indices i such that $t_i \neq 0$ into a set I , and write EF for $E \otimes_k F$. Recall that for $i \in I$, we have $b_i \in F$. Since $j \notin I$, we have that $|I| \leq p-2$. For $i \notin I$,

$$N_{A_i \otimes EF/EF}(b_i + t_i \epsilon) = N_{A_i \otimes F/F}(b_i) \in F^\times$$

(the norm cannot be 0 as $t^p w = w \neq 0$ by assumption). So we get that

$$\prod_{i \in I} (b_i + t_i \epsilon)^p = \xi w$$

for some $\xi \in F^\times$. If we view ξw as a point in $\mathbb{P}(E)(F) = (EF - \{0\})/F^\times$, then we get an equation of the form

$$\left[\prod_{i \in I} (b_i + t_i \epsilon)^p \right] = [w].$$

But the left-hand side lies in the image of the morphism $\prod_{i \in I} \mathbb{P}^1 \rightarrow \mathbb{P}(E)$ which sends $[b_i : t_i] \in \mathbb{P}^1(F)$ to $[\prod (b_i + t_i \epsilon)^p] \in \mathbb{P}(E)(F)$. Since $|I| \leq p-2$, this image is a proper closed subvariety, proving the assertion for generic w . \square

Theorem 7.13. *For a generic closed point $w \in \mathbb{A}^E$, \bar{Y}_w is G -fixed point equivalent to the disjoint union of $(p-1)!$ copies of $\mathbb{P}(\mathcal{A})^p$*

Proof. Since both lie over S , it suffices to consider a G -fixed point $s = (s_0, \dots, s_{p-1})$ in $S(k)^p$ and prove the assertion for the fixed points over s . Because G acts nicely on S and J , $k(s) = k$ and (by Lemma 6.4) G acts on J_s via a projection $G \rightarrow \mu_p$ as the standard representation of μ_p . Note that $J_s = J_{s_i}$ for all i .

By Example 6.3, there are precisely p fixed points on $\mathbb{P}(\mathcal{A})$ lying over a given fixed point $s_i \in S(k)$, and at each of these points the (vertical) tangent space is the reduced regular representation of μ_p . Thus each fixed point in $\mathbb{P}(\mathcal{A})^p$ is k -rational, the number of fixed points over s is p^p , and each of their tangent spaces is the sum of p copies of the reduced regular representation.

Since w is generic, we saw in 7.12 that all the fixed points of \bar{Y}_w satisfy $t \neq 0$ and $t_i \neq 0$ for $1 \leq i \leq p-1$. By Remark 7.12.1, they lie in the affine open $\mathbb{A}(\mathcal{A})^p$ of $\mathbb{P}(L \oplus \mathcal{O})$. Because μ_p acts nicely on J_s , an F -point $b = (b_0, \dots, b_{p-1})$ of $\mathbb{A}(\mathcal{A})^p$ is fixed if and only if each $b_i \in F$. That is, $\text{Fix}_G(\mathbb{A}(\mathcal{A})^p) = \mathbb{A}^p$. Now the norm map restricted to the fixed-point set is just the map $\mathbb{A}^p \rightarrow \mathbb{A}^E$ sending b to $b_0^p \prod_{i=1}^{p-1} (b_i + \epsilon)^p$. This map is finite of degree $p^p(p-1)!$, and étale for generic w , so $\text{Fix}_G(\bar{Y}_w)$ has $p^p(p-1)!$ geometric points for generic w . This is the same number as the fixed points in $(p-1)!$ copies of $\mathbb{P}(\mathcal{A})$ over s , so it suffices to check their tangent space representations.

At each fixed point b , the tangent space of $\mathbb{A}(\mathcal{A})^p$ (or \bar{Y}) is the sum of p copies of the regular representation of μ_p . Since this tangent space is also the sum of the tangent space of \mathbb{A}^p (a trivial representation of G) and the normal bundle of \mathbb{A}^p in \bar{Y} , the normal bundle must then be p copies of the reduced regular representation of μ_p . Since the tangent space of \mathbb{A}^p maps isomorphically onto the tangent space of \mathbb{A}^E at w , the tangent space of \bar{Y}_w is the same as the normal bundle of \mathbb{A}^p in \bar{Y} , as required. \square

Remark 7.13.1. The fixed points in \bar{Y}_w are not necessarily rational points, and we only know that the isomorphism of the tangent spaces at the fixed points holds on a separable extension of k . This is parallel to the situation with the fixed points in Q_w described in Theorem 7.3.

8. A ν_n -VARIETY.

The following result will be needed in the proof of the norm principle.

Theorem 8.1. *Let S be the variety of the chain lemma for some symbol $\{\underline{a}\} \in K_n^M(k)/p$ and $\mathcal{A} = \bigoplus_{i=0}^{p-1} J^{\otimes i}$ the sheaf of Kummer algebras over S . Then the projective bundle $\mathbb{P}(\mathcal{A})$ has dimension $d = p^n - 1$ and $p^2 \nmid s_d(\mathbb{P}(\mathcal{A}))$.*

Proof. Let $\pi : \mathbb{P}(\mathcal{A}) \rightarrow S$ be the projection. The statement about the dimension is trivial. In the Grothendieck group $K_0(\mathbb{P}(\mathcal{A}))$, we have that

$$[T_{\mathbb{P}(\mathcal{A})}] = \pi^*([T_S]) + [T_{\mathbb{P}(\mathcal{A})/S}]$$

where $T_{\mathbb{P}(\mathcal{A})/S}$ is the relative tangent bundle. The class s_d is additive, and the dimension of S is less than d , so we conclude that $s_d(\mathbb{P}(\mathcal{A})) = s_d(T_{\mathbb{P}(\mathcal{A})/S})$. Now $[T_{\mathbb{P}(\mathcal{A})/S}] = [\pi^*(\mathcal{A}) \otimes \mathcal{O}(1)_{\mathbb{P}(\mathcal{A})/S}] - 1$; applying additivity again, together with the definition of s_d and the decomposition of \mathcal{A} and hence $\pi^*(\mathcal{A})$ into line bundles, we obtain

$$s_d(\mathbb{P}(\mathcal{A})) = \deg \sum_{i=0}^{p-1} c_1(\pi^* J^{\otimes i} \otimes \mathcal{O}(1))^d.$$

The projective bundle formula presents the Chow ring $CH^*(\mathbb{P}(\mathcal{A}))$ as:

$$CH^*(\mathbb{P}(\mathcal{A})) = CH^*(S)[y] / \left(\prod_{i=0}^{p-1} (y - ix) \right)$$

where $x = -c_1(J) \in CH^1(S)$ and $y = c_1(\mathcal{O}(1)) \in CH^1(\mathbb{P}(\mathcal{A}))$. Then $s_d(\mathbb{P}(\mathcal{A}))$ is the degree of the following element of the ring $CH^*(\mathbb{P}(\mathcal{A}))$:

$$s'_d(\mathbb{P}(\mathcal{A})) = \sum_{i=0}^{p-1} (y - ix)^d = \sum_{i=0}^{p-1} a_i y^i x^{d-i}$$

for some integer coefficients a_i . Since $x \in CH^1(S)$, we have $x^r = 0$ for any $r > \dim(S) = p^n - p$. It follows that $s'_d(\mathbb{P}(\mathcal{A})) = a_{p-1} y^{p-1} x^{\dim(S)}$. By part (6) of the Chain Lemma 0.1, the degree of $x^{\dim(S)} = (-1)^{\dim(S)} c_1(J)^{\dim(S)}$ is prime to p . In addition, $\pi_*(y^{p-1}) = \pi_*(c_1(\mathcal{O}(1))^{p-1}) = [S] \in CH^0(S)$. By the projection formula $s_d(\mathbb{P}(\mathcal{A})) = a_{p-1} \deg x^{\dim(S)}$. Thus to prove the theorem, it suffices to show that $a_{p-1} \equiv p \pmod{p^2}$; this algebraic calculation is achieved in Lemma 8.2 below. \square

Lemma 8.2. *In the ring $R = \mathbb{Z}/p^2[x, y] / (\prod_{i=0}^{p-1} (y - ix))$, the coefficient of y^{p-1} in $u_m = \sum_{i=0}^{p-1} (y - ix)^{m-1}$ is px^b , with $b = p^m - p$.*

Proof. Since u_m is homogeneous of degree $p^m - 1$, it suffices to determine the coefficient of y^{p-1} in u_m in the ring

$$R/(x-1) = \mathbb{Z}/p^2[y]/(\prod_{i=0}^{p-1}(y-i)) \cong \prod_{i=0}^{p-1} \mathbb{Z}/p^2.$$

If $m = 1$, then $u_1 = \sum_{i=0}^{p-1} (y-i)^{p-1}$ is a polynomial of degree $p-1$ with leading term py^{p-1} . Inductively, we use the fact that for all $a \in \mathbb{Z}/p^2$, we have

$$a^{p^2-p} = \begin{cases} 0, & \text{if } p \mid a \\ 1, & \text{else.} \end{cases}$$

Thus for $m \geq 2$, if we set $k = (p^{m-1} - 1)/(p-1)$, then $a^{p^m-1} = a^{(p-1)+k(p^2-p)} = a^{p-1} \in \mathbb{Z}/p^2$, and therefore

$$u_m = \sum_{i=0}^{p-1} (y-i)^{p^m-1} = \sum_{i=0}^{p-1} (y-i)^{p-1} = u_1$$

holds in $R/(x-1)$; the result follows. \square

9. THE NORM PRINCIPLE

We now turn to the Norm Principle, which concerns the group $A_0(X, \mathcal{K}_1)$ associated to a variety X . In the literature, this group is also known as $H_{-1,-1}(X)$ and $H^d(X, \mathcal{K}_{d+1})$, where $d = \dim(X)$. We recall the definition from 0.2.

Definition 9.1. If X is a regular scheme then $A_0(X, \mathcal{K}_1)$ is the cokernel of the map $\bigoplus_y K_2(k(y)) \xrightarrow{(\partial_{xy})} \bigoplus_x k(x)^\times$. In this expression, the first sum is taken over all points $y \in X$ of dimension 1, and the second sum is over all closed points $x \in X$. The map $\partial_{xy} : K_2(k(y)) \rightarrow k(x)^\times$ is the tame symbol associated to the discrete valuation on $k(y)$ associated to x ; if x is not a specialization of y then $\partial_{xy} = 0$. If $x \in X$ is closed and $\alpha \in k(x)^\times$ we write $[x, \alpha]$ for the image of α in $A_0(X, \mathcal{K}_1)$.

The group $A_0(X, \mathcal{K}_1)$ is covariant for proper morphisms $X \rightarrow Y$, and clearly $A_0(\text{Spec } k, \mathcal{K}_1) = k^\times$ for every field k . Thus if $X \rightarrow \text{Spec}(k)$ is proper then there is a morphism $N : A_0(X, \mathcal{K}_1) \rightarrow k^\times$, whose restriction to the group of units of a closed point x is the norm map $k(x)^\times \rightarrow k^\times$. That is, $N[x, \alpha] = N_{k(x)/k}(\alpha)$.

Definition 9.2. When X is smooth and proper over k , we write $\overline{A}_0(X, \mathcal{K}_1)$ for the quotient of $A_0(X, \mathcal{K}_1)$ by the relation that $[x_1, N_{x/x_1}(\alpha)] = [x_2, N_{x/x_2}(\alpha)]$ for every closed point $x = (x_1, x_2)$ of $X \times_k X$ and every $\alpha \in k(x)^\times$.

It is proven in [10, 1.5–1.7] that if X has a k -rational point then $\overline{A}_0(X, \mathcal{K}_1) = k^\times$; if $X(k) = \emptyset$, then both the kernel and cokernel of $N : \overline{A}_0(X, \mathcal{K}_1) \rightarrow k^\times$ have exponent n , where n is the gcd of the degrees $[k(x) : k]$ for closed $x \in X$. In addition, if x, x' are two points of X then for any field map $k(x') \rightarrow k(x)$ over k and any $\alpha \in k(x)^\times$ we have $[x, \alpha] = [x', N_{x/x'}\alpha]$ in $\overline{A}_0(X, \mathcal{K}_1)$.

To illustrate the advantage of passing to \overline{A}_0 , consider a cyclic field extension E/k . Then $A_0(\text{Spec } E, \mathcal{K}_1) = E^\times$ and by Hilbert 90, there is an exact sequence

$$0 \rightarrow \overline{A}_0(\text{Spec } E, \mathcal{K}_1) \rightarrow k^\times \rightarrow \text{Br}(K/k) \rightarrow 0.$$

We now suppose that k is a p -special field, so that the kernel and cokernel of $N : \overline{A}_0(X, \mathcal{K}_1) \rightarrow k^\times$ are p -groups, and that X is a Norm variety (a p -generic splitting variety of dimension $p^n - 1$). The Norm Principle is concerned with reducing the

degrees of the field extensions $k(x)$ used to represent elements of $\overline{A}_0(X, \mathcal{K}_1)$. For this, the following definition is useful.

Definition 9.3. Let $\widetilde{A}_0(k)$ denote the subset of elements θ of $\overline{A}_0(X, \mathcal{K}_1)$ represented by $[x, \alpha]$ where $k(x) = k$ or $[k(x) : k] = p$. If E/k is a field extension, $\widetilde{A}_0(E)$ denotes the corresponding subset of $\overline{A}_0(X_E, \mathcal{K}_1)$.

Lemma 9.4. *If k is p -special and X is a Norm variety, then $\widetilde{A}_0(k)$ is a subgroup of $\overline{A}_0(X, \mathcal{K}_1)$.*

Proof. By the Multiplication Principle [10, 5.7], which depends upon the Chain Lemma 0.1, we know that for each $[x, \alpha], [x', \alpha']$ in $\widetilde{A}_0(k)$, there is a $[x'', \alpha''] \in \widetilde{A}_0(k)$ so that $[x, \alpha] + [x', \alpha'] = [x'', \alpha'']$ in $\overline{A}_0(X, \mathcal{K}_1)$. Hence $\widetilde{A}_0(k)$ is closed under addition. It is nonempty because $E = k[\sqrt[p]{a_1}]$ splits the symbol and therefore $X(E) \neq \emptyset$. It is a subgroup because $[x, \alpha] + [x, \alpha^{-1}] = [x, 1] = 0$. \square

Lemma 9.5 ([10, 1.24]). *If k is p -special and X is a Norm variety, the restriction of $\overline{A}_0(X, \mathcal{K}_1) \xrightarrow{N} k^\times$ to $\widetilde{A}_0(k)$ is an injection.*

Proof. Let $[x, \alpha]$ represent $\theta \in \widetilde{A}_0(k)$. If $N(\theta) = N_{k(x)/k}(\alpha) = 1$ then $\alpha = \sigma(\beta)/\beta$ for some β by Hilbert's Theorem 90. But $[x, \sigma(\beta)] = [x, \beta]$ in $\widetilde{A}_0(k)$; see [10, 1.5]. \square

Example 9.5.1. If X has a k -point z , then the norm map N of 0.2 is an isomorphism $\widetilde{A}_0(k) \cong \overline{A}_0(X, \mathcal{K}_1) \xrightarrow{\cong} k^\times$, split by $\alpha \mapsto [z, \alpha]$. Indeed, for every closed point x of X we have $[x, \alpha] = [z, N_{k(x)/k}\alpha]$ in $\overline{A}_0(X, \mathcal{K}_1)$, by [10, 1.5].

Our goal in the next section is to prove the following theorem. Let E/k be a field extension with $[E : k] = p$. Since k has p th roots of unity, we can write $E = k(\epsilon)$ with $\epsilon^p \in k$.

Theorem 9.6. *Suppose that k is p -special, $\{\underline{a}\}_E \neq 0$ and that X is a Norm variety for $\{\underline{a}\}$. For $[z, \alpha] \in \widetilde{A}_0(E)$, there exist points $x_i \in X$ of degree p over k , $t_i \in k$ and $b_i \in k(x_i)$ such that $N_{E(z)/E}(\alpha) = \prod N_{E(x_i)/E}(b_i + t_i\epsilon)$.*

Theorem 9.6 is the key ingredient in the proof of Theorem 9.7.

Theorem 9.7. *If k is p -special and $[E : k] = p$ then $\overline{A}_0(X_E, \mathcal{K}_1) \xrightarrow{N_{E/k}} \overline{A}_0(X, \mathcal{K}_1)$ sends $\widetilde{A}_0(E)$ to $\widetilde{A}_0(k)$.*

Proof. If $\{\underline{a}\}_E = 0$ then the generic splitting variety X has an E -point x , and Theorem 9.7 is immediate from Example 9.5.1. Indeed, in this case X_E has an E -point x' over x , every element of $\widetilde{A}_0(E) \cong E^\times$ has the form $[x', \alpha]$, and $N_{E/k}[x', \alpha] = [x, \alpha]$. Hence we may assume that $\{\underline{a}\}_E \neq 0$. This has the advantage that $E(x_i) = E \otimes_k k(x_i)$ is a field for every $x_i \in X$.

Choose $\theta = [z, \alpha] \in \widetilde{A}_0(E)$ and let $x_i \in X$, t_i and b_i be the data given by Theorem 9.6. Each x_i lifts to an $E(x_i)$ -point $x_i \otimes E$ of X_E so we may consider the element

$$\theta' = \theta - \sum [x_i \otimes E, b_i + t_i\epsilon] \in \overline{A}_0(X_E, \mathcal{K}_1).$$

By 9.4 over E , θ' belongs to the subgroup $\widetilde{A}_0(E)$. By Theorem 9.6, its norm is

$$N(\theta') = N_{E(z)/E}(\alpha) / \prod N_{E(x_i)/E}(b_i + t_i\epsilon) = 1.$$

By Lemma 9.5, $\theta' = 0$. Hence $N_{E/k}(\theta) = \sum [x_i, N_{E(x_i)/k(x_i)}(b_i + t_i\epsilon)]$ in $\overline{A}_0(X, \mathcal{K}_1)$. Since $\tilde{A}_0(k)$ is a group by 9.4, this is an element of $\tilde{A}_0(k)$. \square

Corollary 9.8 (Theorem 0.7(3)). *If k is p -special then $\tilde{A}_0(k) = \overline{A}_0(X, \mathcal{K}_1)$, and $N : \overline{A}_0(X, \mathcal{K}_1) \rightarrow k^\times$ is an injection.*

Proof. We may suppose that $X(k) = \emptyset$. For every closed $z \in X$ there is an intermediate subfield E with $[k(z) : E] = p$ and a $k(z)$ -point z' in X_E over z . Since $[z', \alpha] \in \tilde{A}_0(E)$, Theorem 9.7 implies that $[z, \alpha] = N[z', \alpha]$ is in $\tilde{A}_0(k)$. This proves the first assertion. The second follows from this and Lemma 9.5. \square

The Norm Principle of the Introduction follows from Theorem 9.7.

Proof of the Norm Principle (Theorem 0.3). We consider a generator $[z, \alpha]$ of $\overline{A}_0(X, \mathcal{K}_1)$. Since $[k(z) : k] = p^\nu$ for $\nu > 0$, there is a subfield E of $k(z)$ with $[k(z) : E] = p$, and z lifts to a $k(z)$ -point z' of X_E . By construction, $[z', \alpha] \in \tilde{A}_0(E)$ and $\overline{A}_0(X_E, \mathcal{K}_1) \rightarrow \overline{A}_0(X, \mathcal{K}_1)$ sends $[z', \alpha]$ to $[z, \alpha]$. By Theorem 9.7, $[z, \alpha]$ is in $\tilde{A}_0(k)$, i.e., is represented by an element $[x, \alpha]$ with $[k(x) : k] = p$. \square

10. EXPRESSING NORMS

Recall that $E = k(\epsilon)$ is a fixed Kummer extension of a p -special field k , and X is a Norm variety over k for the symbol $\{\underline{a}\}$. The purpose of this section is to prove Theorem 9.6, that if an element $w \in E$ is a norm for a Kummer point of X_E then w is a product of norms of the form specified in Theorem 9.6.

Recall from 7.2 that $Q \subseteq \mathbb{P}(\mathcal{A} \oplus \mathcal{O}) \times \mathbb{A}_k^1$ is the variety of all points $([\beta, t], w)$ such that $N(\beta) = t^p w$, and let $q : Q \rightarrow \mathbb{A}_k^1$ be the projection. Extending the base field to E and applying the Weil restriction functor, we obtain a morphism

$$Rq = \text{Res}_{E/k}(q_E) : RQ = \text{Res}_{E/k}(Q_E) \rightarrow \mathbb{A}^E.$$

Moreover, choose once and for all a resolution of singularities $\tilde{Q} \rightarrow Q$, which is an isomorphism where $t \neq 0$. This is possible since Q is smooth where $t \neq 0$, see 7.2.

Remark 10.1. Since k is p -special, so is E . As stated in Lemma 9.5, the norm map $\tilde{A}_0(E) \rightarrow E^\times$ is injective; we identify $\tilde{A}_0(E)$ with its image. Thus $[z, \alpha] \in \tilde{A}_0(E)$ is identified with $N_{E(z)/E}(\alpha) \in E^\times$. By [10, Theorem 5.5], there is a point $s \in S$ such that $E(z) = \mathcal{A}_s \otimes E$; Under the correspondence $E(z) \cong \mathbb{A}(\mathcal{A})_s(E)$, we identify α with a point of $\mathbb{A}(\mathcal{A})(E)$, lying over $s \in S$. Then $N_{E(z)/E}(\alpha) = Rq([\alpha, 1], N(\alpha))$. In other words, $\tilde{A}_0(E) \subseteq E^\times$ is equal to $q(Q(E)) - \{0\}$.

To prove Theorem 9.6 it therefore suffices to show that $\tilde{Y}_w(k)$ is non-empty when $w = Rq([\beta, 1], w)$. To do this, we will produce a correspondence $Z \rightarrow \tilde{Y} \times_{\mathbb{A}^E} RQ$ that is dominant and of degree prime to p over RQ . We construct the correspondence Z using the Multiplication Principle of [10, 5.7] in the following form.

Lemma 10.2 (Multiplication Principle). *Let k be a p -special field. Then the set of values of the map $N : \mathbb{A}(\mathcal{A})(k) \rightarrow k$ is a multiplicative subset of k^\times .*

Proof. Given Remark 10.1, this is a consequence of Lemma 9.4. \square

Lemma 10.3. *Let $F = k(\tilde{Y})$ be the function field. Then there exists a finite extension L/F , of degree prime to p , and a point $\xi \in RQ(L)$ lying over the generic point of \mathbb{A}^E .*

Proof. Let F' be the maximal prime-to- p extension of F ; then the field $EF' = E \otimes_k F'$ is p -special. We may regard the generic point of \bar{Y} as an element in $\bar{Y}(F)$. Applying the inclusion $F \subset F'$ to this element, followed by the projection $\bar{Y} \rightarrow \mathbb{A}^E$, we obtain an element ω of $\mathbb{A}^E(F') = EF'$. By 7.8, ω is a product of norms from $\mathbb{A}(\mathcal{A})(EF')$. By the Multiplication Principle 10.2, there exists $\beta \in \mathbb{A}(\mathcal{A})(EF')$ such that $N(\beta) = \omega$. Now let ξ be the point $([\beta, 1], \omega) \in RQ(F')$. Then $Rq(\xi) = \omega$ and ξ is defined over some finite intermediate extension $F \subseteq L \subseteq F'$, with $[L : F]$ prime to p . \square

Write η_L for the point of $\bar{Y}(L)$ defined by the inclusion $F \subseteq L$. We can now define $\bar{Y} \xleftarrow{f} Z \xrightarrow{g} RQ$ to be a (smooth, projective) model of $(\eta_L, \xi) \in (\bar{Y} \times_{\mathbb{A}^E} RQ)(L)$.

Theorem 10.4. *The morphism $g : Z \rightarrow RQ$ is proper and dominant (hence onto) and of degree prime to p .*

Proof. Let $\omega \in \mathbb{A}^E$ be the generic point, $k(\omega)$ the function field and $E(\omega) = E \otimes k(\omega)$. As degree is a generic notion and invariant under extension of the base field, we may replace $\bar{Y} \leftarrow Z \rightarrow RQ$ by its basechange along the morphism

$$\mathrm{Spec}(E(\omega)) \rightarrow \mathrm{Spec}(k(\omega)) \xrightarrow{\omega} \mathbb{A}^E,$$

to obtain morphisms $f : Z_{E(\omega)} \rightarrow \bar{Y}_{E(\omega)}$ and $g : Z_{E(\omega)} \rightarrow RQ_{E(\omega)}$. Using the normal basis theorem, we can write $E(\omega) = E(\omega_1, \dots, \omega_p)$ for transcendentals ω_i that are permuted under the action of the cyclic group $\mathrm{Gal}(E/k)$.

We will apply the DN Theorem A.1 with base field $k' = E(\omega)$. In the notation of Theorem A.1, we let $r = p$; we write Y for some desingularization of $\bar{Y}_{E(\omega)}$; we let X be $R\tilde{Q}_{E(\omega)}$, and we let W be a model for $Z_{E(\omega)}$ mapping to Y and X . Finally, we let $u_i = \{a_1, \dots, a_n, \omega_i\} \in K_{n+1}^M(k')/p$.

Observe that our base field contains E , so $R\tilde{Q}_{E(\omega)} = \mathrm{Res}_{E/k}(\tilde{Q}_E) \times_{\mathbb{A}^E} E(\omega)$ splits as a product $R\tilde{Q}_{E(\omega)} = \prod_{i=1}^p \tilde{Q}_{\omega_i}$, where Q_{ω_i} is the fiber of $Q \rightarrow \mathbb{A}^1$ over the point $\omega_i \in \mathbb{A}^1(E(\omega)) = E(\omega)$. Therefore we have $X = \prod_{i=1}^p X_i$ where X_i is \tilde{Q}_{ω_i} , the resolution of singularities of Q_{ω_i} . By Remark 7.4, X_i is a smooth, geometrically irreducible splitting variety for the symbol u_i of dimension $p^n - 1$. Thus, hypothesis (1) of the DN Theorem A.1 is satisfied.

By Theorem A.10, $t_{d,1}(X_i) = t_{d,1}(\mathbb{P}(\mathcal{A}))$; by Lemma A.6, we conclude that $s_d(X_i) \equiv vs_d(\mathbb{P}(\mathcal{A})) \pmod{p^2}$ for some unit $v \in \mathbb{Z}/p$. Since $s_d(\mathbb{P}(\mathcal{A})) \not\equiv 0$ by Theorem 8.1, we conclude that hypothesis (3) of the DN Theorem A.1 is satisfied.

Furthermore, $K = k'(X_1 \times \dots \times X_{i-1})$ is contained in a rational function field over E ; in fact, the field $E(\omega_j)(Q_{\omega_j})$ becomes a rational function field once we adjoin $\sqrt[p]{\gamma}$. Since E does not split $\{a\}$, K does not split $\{a\}$ either. It follows that K does not split $u_i = \{a\} \cup \{\omega_i\}$, verifying hypothesis (2) of Theorem A.1.

We have now checked the hypotheses (1–3) of Theorem A.1. It remains to check that X and Y are G -fixed point equivalent up to a prime-to- p factor. In fact, we proved in Theorem 7.13 that $\bar{Y}_{E(\omega)}$ is G -fixed point equivalent to $(p-1)!$ copies of $\mathbb{P}(\mathcal{A})^p$, hence so is Y (since the fixed points lie in the smooth locus), and in Theorem 7.3 that X_i is G -fixed point equivalent to $\mathbb{P}(\mathcal{A})$. That is, Y is G -fixed point equivalent to $(p-1)!$ copies of X . Therefore the DN Theorem applies to show that g is dominant and of degree prime to p , as asserted. \square

Proof of Theorem 9.6. We have proved that there is a diagram $\bar{Y} \xleftarrow{f} Z \xrightarrow{g} RQ$ such that the degree of g is prime to p . By blowing up if necessary we may assume that $g : Z \rightarrow RQ$ factors through $\tilde{g} : Z \rightarrow R\tilde{Q}$, with $\deg(\tilde{g})$ prime to p .

Let $[z, \alpha] \in \tilde{A}_0(E)$, and set $w = N_{E(z)/E}(\alpha)$. By Remark 10.1, there exists a point $([\beta, 1], w) \in RQ(k)$. Lift this to a point in $R\tilde{Q}(k)$ (recall that $R\tilde{Q} \rightarrow RQ$ is an isomorphism where $t \neq 0$). Since $Z \rightarrow R\tilde{Q}$ is a morphism of smooth projective varieties of degree prime to p and k is p -special, we can lift $([\beta, 1], w)$ to a k -point of Z , and then apply $f : Z \rightarrow \bar{Y}$ to get a k -point in \bar{Y}_w . By the definition of \bar{Y} and Corollary 7.10, this means that we can find Kummer extensions $k(x_i)/k$ (corresponding to points $s_i \in S$, and determining points $x_i \in X$ because X is a p -generic splitting variety), elements $b_i \in k(x_i)$ and $t_i \in k$ such that $w = \prod_i N_{E(x_i)/E}(b_i + t_i\epsilon)$, as asserted. \square

A. APPENDIX: THE DN THEOREM

In this appendix, we give a proof of the following Degree theorem, which is used in the proof of the Norm Principle. Throughout, k will be a fixed field of characteristic 0, $p > 2$ will be a prime, $n \geq 1$ will be an integer and we fix $d = p^n - 1$.

Recall from Definition 7.1 that if X and Y are G -fixed point equivalent then $\dim(X) = \dim(Y)$, the fixed points are 0-dimensional and their tangent space representations are isomorphic (over \bar{k}).

Theorem A.1 (DN Theorem). *For $r \geq 1$, let u_1, \dots, u_r be symbols in $K_{n+1}^M(k)/p$ and let $X = \prod_1^r X_i$, where the X_i are irreducible smooth projective G -varieties of dimension $d = p^n - 1$ such that:*

- (1) $k(X_i)$ splits u_i ;
- (2) u_i is non-zero over $k(X_1 \times \dots \times X_{i-1})$; and
- (3) $p^2 \nmid s_d(X_i)$

Let Y be a smooth irreducible projective G -variety which is G -fixed point equivalent to the disjoint union of m copies of X , where $p \nmid m$. Let F be a finite extension of $k(Y)$ of degree prime to p , and $\text{Spec}(F) \rightarrow X$ a point, with model $f : W \rightarrow X$. Then f is dominant and of degree prime to p .

$$\begin{array}{ccc}
 \text{Spec}(F) & \longrightarrow & W \\
 \text{finite} \downarrow & & \downarrow g \\
 \text{Spec}(k(Y)) & \longrightarrow & Y
 \end{array}
 \begin{array}{c}
 \\
 \\
 \searrow f \text{ (dominant)} \\
 X
 \end{array}$$

The proof will use two ingredients: the degree formulas A.2 and A.5 below, due to Levine and Morel; and a standard localization result A.10 in (complex) cobordism theory. The former concern the algebraic cobordism ring $\Omega_*(k)$, and the latter concern the complex bordism ring MU_* . These are related via the Lazard ring \mathbb{L}_* ; combining Quillen's theorem [1, II.8] and the Morel-Levine theorem [2, 4.3.7], we have graded ring isomorphisms:

$$\Omega_*(k) \cong \mathbb{L}_* \cong MU_{2*}.$$

Here is the Levine-Morel generalized degree formula for an irreducible projective variety X , taken from [2, Theorem 4.4.15]. It concerns the ideal $M(X)$ of $\Omega_*(k)$ generated by the classes $[Z]$ of smooth projective varieties Z such that there is a k -morphism $Z \rightarrow X$, and $\dim(Z) < \dim(X)$.

Theorem A.2 (Generalized Degree Formula). *Let $f : Y \rightarrow X$ be a morphism of smooth projective k -varieties. If $\dim(X) = \dim(Y)$ then $[Y] - \deg(f)[X] \in M(X)$.*

Trivially, if $[Z] \in M(X)$ then $M(Z) \subseteq M(X)$. We also have:

Lemma A.3. *Let X be a smooth projective k -variety. If Z and Z' are birationally equivalent, then $[Z] \in M(X)$ holds if and only if $[Z'] \in M(X)$.*

Proof. By [2, 4.4.17], the class of Z modulo $M(Z)$ is a birational invariant. Thus $[Z'] - [Z] \in M(Z)$. Because $M(Z) \subseteq M(X)$, the result follows. \square

We shall also need the Levine-Morel ‘‘higher degree formula’’ A.5, which is taken from [2, Theorem 4.4.24], and concerns the mod p characteristic numbers $t_{d,r}(X)$ of [2, 4.4], where p is prime, $n \geq 1$ and $d = p^n - 1$.

Choose a graded ring homomorphism $\psi : \mathbb{L}_* \rightarrow \mathbb{F}_p[v_n]$ corresponding to some height n formal group law, where v_n has degree d ; many such group laws exist, and the class $t_{d,r}$ will depend on this choice, but only up to a unit.

Definition A.4. For $r > 0$, the homomorphism $t_{d,r} : \Omega_{rd}(k) \cong \mathbb{L}_{rd} \rightarrow \mathbb{F}_p$ sends x to the coefficient of v_n^r in $\psi(x)$. If X is a smooth projective variety over k , of dimension rd , then X determines a class $[X]$ in $\Omega_{rd}(k)$, and $t_{d,r}(X)$ is $t_{d,r}([X])$.

Theorem A.5 (higher degree formula). *Let $f : W \rightarrow X$ be a morphism of smooth projective varieties of dimension rd and suppose that X admits a sequence of surjective morphisms*

$$X = X^{(r)} \rightarrow X^{(r-1)} \rightarrow \dots \rightarrow X^{(0)} = \text{Spec}(k)$$

such that

- (1) $\dim(X^{(i)}) = id$.
- (2) If η is a zero-cycle on $X^{(i)} \times_{X^{(i-1)}} k(X^{(i-1)})$, then p divides the degree of η .

Then $t_{d,r}(W) = \deg(f) t_{d,r}(X)$.

Here are some properties of this characteristic number that we shall need. Recall that if $\dim(X) = d$ then p divides $s_d(X)$, so that $s_d(X)/p$ is an integer.

Lemma A.6. *Let X/k be a smooth projective variety, and $k \subseteq \mathbb{C}$ and embedding.*

- (1) For $r = 1$, there is a unit $u \in \mathbb{F}_p$ such that $t_{d,1}(X) \equiv u s_d(X)/p$.
- (2) If $X = \prod_{i=1}^r X_i$ and $\dim(X_i) = d$, then $t_{d,r}(X) = \prod_{i=1}^r t_{d,1}(X_i)$.
- (3) $t_{d,r}(X)$ depends only on the class of $(X \times_k \mathbb{C})^{an}$ in the complex cobordism ring.

Proof. Part (1) is [2, Proposition 4.4.22]. Part (2) is immediate from the definition of $t_{d,r}$ and the graded multiplicative structure on $\Omega_*(k)$. Finally, part (3) is a consequence of the fact that the natural homomorphism $\Omega_*(k) \rightarrow MU_{2*}$ is an isomorphism (since both rings are isomorphic to the Lazard ring). \square

Remark A.6.1. The class called s_d in this article is the S_d in [2]; the class called $s_d(X)$ in [2] is our class $s_d(X)/p$.

The next lemma is a variant of Theorem A.5. It uses the same hypotheses.

Lemma A.7. *Let X be as in Theorem A.5. Then $\psi(M(X)) = 0$.*

Proof. Consider Z with $[Z] \in M(X)$. If d does not divide $\dim(Z)$, then $\psi([Z]) = 0$ for degree reasons. If $\dim(Z) = 0$, then the image of Z is a closed point of X ; since the degree of such a closed point is divisible by p , we have $\psi([Z]) = 0$. Hence we may assume that $\dim(Z) = sd$ for some $0 < s < r$. The cases $r = 1$ and $s = 0$ are immediate, so we proceed by induction on r and s .

Let $f : Z \rightarrow X$ be a k -morphism with $\dim(Z) = sd$, and let $f_s : Z \rightarrow X^{(s)}$ be the obvious composition. As $\dim(Z) = \dim(X^{(s)})$, the generalized degree formula A.2 applies to show that $[Z] - \deg(f_s)([X^{(s)}]) \in M(X^{(s)})$. By induction on r , $\psi(M(X^{(s)})) = 0$, so $\psi([Z]) = \deg(f_s)\psi([X^{(s)}])$. We claim that $\deg(f_s) \equiv 0 \pmod{p}$, which yields $\psi([Z]) = 0$, as desired.

If f_s is not dominant, then $\deg(f_s) = 0$ by definition. On the other hand, if f_s is dominant, then the generic point of Z maps to a closed point η of $X^{(s+1)} \times_{X^{(s)}} k(X^{(s)})$. By condition (2) of Theorem A.5, p divides $\deg(\eta) = \deg(f_s)$. \square

We will need to show that $\psi(M(Y)) = 0$ for the Y appearing in Theorem A.1. This is accomplished in the next lemma.

Lemma A.8. *Suppose X, Y and W are smooth projective varieties of dimension rd over k , and $f : W \rightarrow X$ and $g : W \rightarrow Y$ are morphisms. Suppose further that $\psi(M(X)) = 0$ and that p does not divide $\deg(g)$. Then $\psi(M(Y)) = 0$.*

Proof. Suppose $[Z] \in M(Y)$. As $g : W \rightarrow Y$ is a proper morphism of smooth varieties, of degree prime to p , we can lift the generic point $\text{Spec}(k(Z)) \rightarrow Y$ to a point $q : \text{Spec}(F) \rightarrow W$ for some field extension $F/k(Z)$ of degree e prime to p . Let \tilde{Z} be a smooth projective model of F possessing a morphism to Z and a morphism to X extending the k -morphism $f \circ q : \text{Spec}(F) \rightarrow X$. Hence $[\tilde{Z}] \in M(X)$. By the degree formula for the map $\tilde{Z} \rightarrow Z$, $e[Z] - [\tilde{Z}] \in M(Z)$. If $\dim(Z) = 0$, then $M(Z) = (0)$. In general, $M(Z)$ is generated by the classes of varieties of dimension less than $\dim(Z)$ that map to Z (hence a fortiori also map to Y) over k . By induction on the dimension of Z , we may assume that $\psi(M(Z)) = 0$. Moreover, $\psi([\tilde{Z}]) = 0$ by assumption; since p does not divide e , we conclude that $\psi([Z]) = 0$ as asserted. \square

Finally, we will use the following standard bordism localization result.

Lemma A.9. *Suppose that the abelian p -group $G = \mu_p^n$ acts without fixed points on an almost complex manifold M . Then $\psi([M]) = 0$ in \mathbb{F}_p .*

Proof. By [11], $[M]$ is in the ideal of MU_* generated by $\{p, [M_1] \dots, [M_{n-1}]\}$, where $\dim_{\mathbb{C}}(M_i) = p^i - 1$. Since p is the only generator of this ideal whose dimension is a multiple of $d = p^n - 1$, ψ is zero on every generator and hence on the ideal. \square

Theorem A.10. *Let G be μ_p^n and let X and Y be compact complex G -manifolds which are G -fixed point equivalent. Then $\psi([X]) = \psi([Y])$.*

Proof. Remove equivariantly isomorphic small balls about the fixed points of X and Y , and let $M = X \cup -Y$ denote the result of joining the rest of X and Y , with the opposite orientation on Y . Then M has a canonical almost complex structure, G acts on M with no fixed points, and $[X] - [Y] = [M]$ in MU_* . By Lemma A.9, $\psi([X]) - \psi([Y]) = \psi([M]) = 0$. \square

We can now prove Theorem A.1. Note that the inclusion $k(Y) \subset F$ induces a dominant rational map $W \rightarrow Y$; we may replace W by a blowup to eliminate the points of indeterminacy and obtain a morphism $g : W \rightarrow Y$, whose degree is prime to p , without affecting the statement of Theorem A.1.

Proof of the DN Theorem A.1. We will apply Theorem A.5 to X and the $X^{(t)} = \prod_{i=1}^t X_i$. We must first check that the hypotheses are satisfied. The first condition is obvious. For the second condition, it is convenient to fix t and set $F = k(X_1 \times \dots \times X_{t-1})$, $X' = X^{(t)} \times_{X^{(t-1)}} F$. By hypotheses (1–2) of Theorem A.1, the symbol u_t is nonzero over F but splits over the generic point of X' ; by specialization, it splits over all closed points. A transfer argument implies that the degree of any closed point η of X' is divisible by p ; this is the second condition. Hence Theorem A.5 applies and we have $t_{d,r}(W) = \deg(f) t_{d,r}(X)$.

By Lemmas A.8 and A.7, we have that $\psi(M(Y)) = 0$; by the generalized degree formula A.2, we conclude that $\psi([W]) = \deg(g) \psi([Y])$, so that $t_{d,r}(W) =$

$\deg(g)t_{d,r}(Y) \neq 0$. Hence

$$\deg(f)t_{d,r}(X) = \deg(g)t_{d,r}(Y).$$

By Theorem A.10 and Lemma A.6(3), $mt_{d,r}(X) = t_{d,r}(Y)$. Condition (3) of Theorem A.1 and Lemma A.6 imply that $t_{d,1}(X_i) \neq 0$ for all i and hence that $t_{d,r}(X) \neq 0$. It follows that $m\deg(g) \equiv \deg(f) \neq 0$ modulo p , as required. \square

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REFERENCES

- [1] J. F. Adams, *stable homotopy and generalized homology*, Univ. Chicago Press, 1974.
- [2] M. Levine and F. Morel. Algebraic Cobordism. *Springer Monographs in Mathematics*, 2007.
- [3] A. Merkurjev and A. Suslin, K -cohomology of Severi-Brauer varieties, *Izv. Akad. Nauk SSSR Ser. Mat.* 46 (1982), 1011–1046 (Russian). English translation *Math. USSR Izvestiya* 21 (1983), 307–340.
- [4] J. Milnor and J. Stasheff, *Characteristic Classes*, Princeton Univ. Press, 1974.
- [5] M. Rost, Chow groups with coefficients, *Doc. Math. J. DMV* 1 (1996), 319–393.
- [6] M. Rost, Chain lemma for splitting fields of symbols, *Preprint*, 1998. Available at <http://www.math.uni-bielefeld.de/fost/chain-lemma.html>
- [7] M. Rost, Construction of Splitting Varieties, *Preprint*, 1998. Available at <http://www.math.uni-bielefeld.de/fost/chain-lemma.html>
- [8] M. Rost, The chain lemma for Kummer elements of degree 3. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(3):185–190, 1999.
- [Rost] M. Rost, *Notes on Lectures given at IAS*, 1999–2000 and Spring Term 2005.
- [9] R. Stong, *Notes on Cobordism Theory*, Princeton Univ. Press, 1968.
- [10] A. Suslin and S. Joukhovitski, Norm Varieties, *J. Pure Appl. Alg.* 206 (2006), 245–276.
- [11] T. tom Dieck, Actions of finite abelian p -groups without stationary points, *Topology* 9 (1970), 359–366.
- [12] V. Voevodsky, Reduced Power operations in Motivic Cohomology, *Publ. IHES* 98 (2003), 1–57.
- [13] V. Voevodsky, On Motivic Cohomology with \mathbb{Z}/l coefficients, *Preprint*, 2003. Available at <http://www.math.uiuc.edu/K-theory/0639/>.
- [14] C. Weibel, Axioms for the Norm Residue Isomorphism, *Preprint*, 2006. To appear in *Proceedings Valladolid Conference*, available at <http://www.math.uiuc.edu/K-theory/0809/>.
- [15] C. Weibel, Patching the Norm Residue Isomorphism Theorem, *Preprint*, 2007. Available at <http://www.math.uiuc.edu/K-theory/0844/>.
- [16] A. Weil, *Adeles and Algebraic Groups*, Birkhäuser, 1982. Based on lectures given 1959–1960.