SOME REMARKS ON MOTIVIC HOMOTOPY THEORY OVER ALGEBRAICALLY CLOSED FIELDS

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1. Introduction

The purpose of this note is to point some aspects of the 2-complete stable motivic homotopy theory over algebraically closed fields of characteristic 0. This means the category of $\mathbb{P}^1$-motivic spectra, Bousfield localized at the pushforward of the Moore spectrum $M\mathbb{Z}/2$. This is, in some sense, the part of motivic stable homotopy theory which is the closest to ordinary topology, although even here, the theory one obtains, as we shall note, contains interesting new phenomena. We shall work in this category throughout this paper, unless specified otherwise. We shall follow the notational convention from [6] (analogous to Real-oriented homotopy theory), which means that for a motivic (generalized) cohomology theory, we write

$$E^{k+\ell\alpha}(X) = E^{k+\ell,\ell}X,$$

and similarly for homology.

One can then define $BPGL$ and a motivic analog of the Adams-Novikov spectral sequence converging to the 2-completed motivic stable homotopy theory π_*^{Mot}. The E_2-term is equal to the topological E_2-term, but with dimensions shifted by twists, and tensored with $\mathbb{Z}[\theta]$ where θ is "Tate twist" (see below). The differentials mimic the topological differentials, but there is a difference in twist. An example is given by the following peculiar result:

Theorem 1. *Over an algebraically closed field k of characteristic 0, inside the α-family, if we denote by $a_{4k} \in \pi^{Mot}_{4k+(4k-1)\alpha}(S^0_{Mot})^\wedge_2$, resp. $a_{4k+1} \in \pi^{Mot}_{4k+1+4k\alpha}(S^0_{Mot})^\wedge_2$ the elements represented by α_{4k}, α_{4k+1}, then*

$$a_{4k}\eta^m \neq 0,\ a_{4k+1}\eta^m \neq 0$$

for any positive integer k, m, while

$$a_{4k}\eta^3\theta = a_{4k+1}\eta^3\theta = \eta^4\theta = 0.$$

Similarly outside the α-family, the element $\alpha_1^2\eta_{3/2} \in Ext^{5,15(1+\alpha)}_{BPGL_*}$ represents a non-zero element

$$x \in \pi^{Mot}_{10+15\alpha}(S^0_{Mot})^\wedge_2$$

which maps to 0 in etale homotopy theory, and satisfies

$$\theta x = 0.$$

Remark: Note that computing Morel's Milnor-Witt ring [13], one easily sees that $\eta^m \neq 0$.

2-adic etale stable homotopy theory over an algebraically closed field turns out to be just topological stable homotopy theory $\otimes\mathbb{Z}[\theta,\theta^{-1}]$. Because of this, Theorem

1 gives an example where motivic stable homotopy groups of spheres substantially deviate from the topological case, by producing an element which "forgets" to 0.

Prompted by a suggestion of Dan Isaksen [7], we also investigate the "complex" motivic J-homomorphism in the present context. We may define F_{Mot} as the direct limit of spaces of self-homotopy equivalences of a fibrant replacement of $S^{n(1+\alpha)}$. Let GL_{Mot} be the direct limit of the group schemes GL_n. Then similarly as in topology, we have a map

$$\mathcal{J} : GL_{Mot} \to F_{Mot} \tag{1}$$

Now let us pass to homotopy groups. The $(m+n\alpha)$'th homotopy group of the right hand side of (1) is the stable homotopy group $\pi_{m+n\alpha}S^0_{Mot}$ for $m > 0$. On the other hand, the $(m+n\alpha)$'th homotopy group of the left hand side is the algebraic K_{m-n-1}-group of the ground field K. Obviously, this can be non-zero only when $m - n > 0$, and additionally we must have $m, n \geq 0$. After reindexing, we get a map for all $k, \ell \geq 0$,

$$J : K_k(K) \to \pi_{(k-1)+\ell(1+\alpha)}S^0. \tag{2}$$

This is (the "complex version") of the motivic J-homomorphism. To put this in our context, we would like a "2-complete version" of the map (2). One complication is that it is at present not known how to realize (1) as a map of ($\mathbb{P}^1$-stable) spectra, since F_{Mot} should be the infinite loop space associated with the multiplicative spectrum of the sphere spectrum, but no infinite loop space machine is known for $\mathbb{P}^1$-stable spectra, so this is not known. For the moment, we recall that there is a canonical pushforward functor PF from the category simplicial sets to the category of Morel-Voevodsky $\mathbb{A}^1$-spaces, which possesses a right adjoint, which we shall denote by FP. Further, PF obviously preserves cofibrations and acyclic cofibrations, so this is a Quillen adjunction. There is a similar Quillen adjoint pair PF_s, FP_s between the category of spectra and $\mathbb{P}^1$-stable motivic spectra. Since PF_s turns shift desuspensions of suspension spectra to shift desuspension of suspension spectra on PF, by commutation of adjoints, we have, for a $\mathbb{P}^1$-stable motivic spectrum E, and an integer n,

$$FP_s(E)_n \simeq FP(E_n). \tag{3}$$

Now we can "2-complete" (1) by applying $FP\Omega^{\ell\alpha}$ to both sides and then 2-complete in the category of spaces (=simplicial sets). We claim that for $k + \ell > 1$, this gives a map

$$\pi_k(K^{alg}(K)^\wedge_2) \to \pi_{(k-1)+\ell(1+\alpha)}(S^0_{Mot})^\wedge_2. \tag{4}$$

Indeed, since the 2-completion of the 0-connected component of the infinite loop space of a spectrum (in the category spaces) is the 0-component of the infinite loop space of the 2-completion of the spectrum, it suffices to prove the following

Lemma 2. *The functor FP_s preserves 2-completion on the level of homotopy categories.*

We will prove this in section 5 below. Now it is also known by a calculation of Suslin [16] that the left hand side of (4) is $\mathbb{Z}_2$ when k is even and 0 otherwise. On the other hand, as we shall see below, the right hand side of (4) only depends on $k + \ell$. In Section 5, we shall prove the following result

Theorem 3. *The image of (4) is isomorphic to the 2-primary component of the image of the (ordinary complex) J-homomorphism in dimension $k+2\ell-1$.*

In order to prove Theorem 1, we need to know the structure of algebraic cobordism over an algebraically closed field. This in turn needs the Adams spectral sequence, which needs the algebra of bistable operations in motivic cohomology (the motivic Steenrod algebra). This calculation was done by Voevodsky in the late 90's, but not published at the time of the writing of this note. [1] To make the present note self-contained, we give the computation here in the case of an algebraically closed field (although the method works in greater generality). Namely, by investigating the structure of symmetric products, we prove

Theorem 4. *(Voevodsky) Let k be an algebraically closed field. Denoting by H^{Mot} the $H\mathbb{Z}/2$-motivic (co)homology spectrum over k, and by θ Tate twist (of cohomological dimension $\alpha-1$), the algebra of bistable operations $H^{Mot*}H^{Mot}$ is generated as a $\mathbb{Z}/2[\theta]$-module by reduced power operations P^s (of dimension $s(1+\alpha)$) and the Bockstein (of dimension 1).*

By [18], we therefore have

Corollary 5. *The dual motivic Steenrod algebra over an algebraically closed field k is given by*

$$H_*^{Mot}H^{Mot}=\mathbb{Z}/2[\theta,\tau_0,\tau_1,\tau_2...\xi_1,\xi_2,...]/(\tau_i^2=\xi_{i+1}\theta)$$

where the dimensions of ξ_i,τ_i,θ are $(2^i-1)(1+\alpha)$, $(2^i-1)(1+\alpha)+1$, $1-\alpha$, respectively.

Let a *positive cell spectrum of finite type* in the $\mathbb{P}^1$-stable motivic category be a spectrum which can be obtained by successively attaching cells (=cones on spheres) in dimensions $m+n\alpha$, $m,n\geq 0$, with only finitely many cells in each dimension.

Theorem 6. *Over an algebraically closed field, the Adams spectral sequence*

$$Ext_{H_*^{Mot}H^{Mot}}(H_*^{Mot}(X),H_*^{Mot})\Rightarrow \pi_*(X_2^\wedge)$$

converges for any positive cell spectrum X of finite type.

The proof will be given in the Appendix (Section 6).

The algebraic cobordism spectrum is a positive cell spectrum of finite type (using Schubert cells). Using this, we can calculate the 2-completed algebraic cobordism groups:

Theorem 7. *Over an algebraically closed field k, the 2-completed algebraic cobordism groups are given by*

$$(MGL_2^\wedge)_*=\mathbb{Z}_2[\theta,v_1,v_2,...]$$

where v_n has dimension $(2^n-1)(1+\alpha)$.

Completed at 2, one can next construct $BPGL$ by mimicking, again, the construction from topology. Using this, we can then conclude

[1] After this note was written, [19] appeared on the K-theory archive. That manuscript was not available when the first version of the present note was written, which is why we were compelled to obtain an independent proof.

Theorem 8. *There exists a convergent Adams-Novikov spectral sequence*

$$Ext_{BPGL_*BPGL}(BPGL_*, BPGL_*) \Rightarrow \pi_*((S^0_{Mot})^\wedge_2).$$

□

The present note is organized as follows: In Section 2, we discuss the main technical tool of our approach to Voevodsky's result, the motivic transfer. In Section 3, we apply this technique to analyzing the motivic cohomology of the symmetric smash-powers of spheres. In Section 4, we prove our main theorems and discuss the motivic Adams-Novikov spectral sequence. The Proof of Theorem 3 will be done in Section 5.

2. The motivic Elmendorf construction

In this paper, we discuss symmetric products of varieties, which are generally not smooth, so we work with the cd-h topology in finite schemes over a field k.

Definition: A 1-*point compactification* X^* of a quasiprojective variety X embedded into a projective variety $\overline{X}$ is $\overline{X}/(\overline{X}-X)$.

Examples: $S^{n(1+\alpha)}$ is a 1-point compactification of $\mathbb{A}^n$. $Quot_{\Sigma_d} \bigwedge_d S^{n(1+\alpha)}$ is a 1-point compactification of $Quot_{\Sigma_d}\mathbb{A}^{nd}$.

We should note that if X is smooth and $\overline{X}-X = D_1 \cup ... \cup D_m$ where D_i are divisors with normal crossings, then denoting by Q the poset of non-empty subsets of $\{1, ..., m\}$ with respect to inclusion, the natural map

$$\underset{S\in Q}{holim} \bigcap_{i\in S} D_i \to \overline{X} - X$$

is an equivalence, which gives a model of the 1-point compactification in the smooth category.

Lemma 9. *Let X^* be a 1-point compactification of a smooth quasiprojective variety X of dimension n. Then, in the $\mathbb{A}^1$-stable category,*

$$X^* \simeq \Sigma^{n(1+\alpha)} D(X^\xi)$$

where D denotes Spanier-Whitehead dual and X^ξ denotes the Thom spectrum of X with respect to its virtual normal bundle of dimension 0*.*

□

Now the main point of this section is to consider "smooth models" of quotients of smooth varieties by finite groups G. In topology, there is the following method, known as the Elmendorf construction: The *orbit category* $\mathfrak{O}$ is the category of G-orbits and G-equivariant maps. Equivalently, it is the category of subgroups and outer subconjugacies (i.e. subconjugacies modulo inner conjugacies in the source group). Then the fixed points $X^?$ of a space X under subgroups of G form a contravariant functor

$$\mathfrak{O} \to Spaces.$$

An example of a covariant functor on the orbit category is $G/?$. Another example of a covariant functor is $E\mathcal{F}_?$ where $E\mathcal{F}_H$ is a G-CW complex characterized up to G-homotopy equivalence by the property that

$$(E\mathcal{F}_H)^K \simeq \begin{array}{ll} * & \text{for } K \text{ subconjugate to } H \\ \emptyset & \text{else.} \end{array} \tag{5}$$

One can make $E\mathcal{F}_?$ into a covariant functor on the orbit category, for example by taking the 2-sided bar construction

$$E\mathcal{F}_? = B(*, \mathfrak{O}, \mathfrak{O}/?). \tag{6}$$

(Here $\mathfrak{O}/H$ denotes the contravariant functor which assigns to K the set of morphisms $K \to H$ in $\mathfrak{O}$.) The notation $E\mathcal{F}_H$ is explained by noting that the set $\mathcal{F}_H$ of subgroups subconjugate to H is an example of a *family*, i.e. set of subgroups of G closed under subconjugacy. Analogously to (5), one can define the classifying space of any family. Then, in topology, the natural map

$$E\mathcal{F}_? \times_{\mathfrak{O}} X^? \to X/G \tag{7}$$

is an equivalence for any G-CW complex X.

In $\mathbb{A}^1$-homotopy, (5) does not characterize $E\mathcal{F}_H$ up to G-$\mathbb{A}^1$-equivalence, and the construction (6) is usually wrong for the purposes of (7). An alternative construction which will work in the case we are interested in (the symmetric products) is obtained as follows: Let G act effectively on $d = \{1, ..., d\}$. Then set

$$E\mathcal{F}_H = \mathbb{A}^{d\infty} - \bigcup_{K \notin \mathcal{F}_H} (\mathbb{A}^{d\infty})^K. \tag{8}$$

In fact, we also want to adapt (7) to take the form

$$E\mathcal{F}_? \times_{(\mathfrak{O})} X^? \to Quot_G(X). \tag{9}$$

Here on the right hand side, $Quot_G$ is the functor which on a G-equivariant scheme X takes X/G in the category of schemes, and commutes with direct limits. Denote

$$N(H, K) = \{g \in G | g^{-1}Hg \subseteq K\}.$$

Then on the left hand side of (9), we mean the obvious coequalizer
(10)

$$\coprod_{(H),(K)} Quot_{W_G(H)} \coprod_{N(H,K)/H} (E\mathcal{F}_H \times X^K) \rightrightarrows \coprod_{(H)} Quot_{W_G(H)}(E\mathcal{F}_H \times X^H)$$

where $W_G(H)$ denotes the Weyl group of H in G, and the coproducts on the right (resp. left) side of (10) are over conjugacy classes of subgroups (resp. subconjugate pairs of conjugacy classes of subgroups of G). The key point is that one can construct inductively the subspace $Y_{\mathcal{F}}$ of the construction (10) on a family $\mathcal{F}$ of subgroups of G: for $H \notin \mathcal{F}$, such that every proper subgroup of H is in $\mathcal{F}$, to construct $Y_{\mathcal{F} \cup (H)}$, one attaches the subspace

$$E\mathcal{F}_H \times (\bigcup X^K)$$

to $Y_{\mathcal{F}}$ according to the prescribed identification given by (10).When $\mathcal{F}$ is the family of all subgroups of G, $Y_{\mathcal{F}}$ is the left hand side of (10).

For any construction of $E\mathcal{F}_H$, one then has (by induction on the strata)

Lemma 10. *Let* $X^{>H} = \bigcup_{K \notin \mathcal{F}_H} X^K$. *If the collapse map*

$$E\mathcal{F}_{H+} \wedge (X^H/X^{>H}) \rightarrow X^H/X^{>H} \tag{11}$$

is an H-$\mathbb{A}^1$*-equivalence, then (9) is an* $\mathbb{A}^1$*-equivalence. In fact, more generally, for any subgroup* $\Gamma \subseteq G$*, the natural map*

$$(\Gamma \backslash G/? \times E\mathcal{F}_?) \times_{(\mathfrak{O})} X^?) \rightarrow Quot_\Gamma(X) \tag{12}$$

is a Γ-$\mathbb{A}^1$*-equivalence.*

□

Corollary 11. *Under the assumptions of Lemma 10, let let* $\mathfrak{H}(X)$ *denote Bloch's Chow chain complex on* X *(resp. its natural extension to* G-$\mathbb{A}^1$*-spaces). Then the transfer maps associated with free group action for each subgroup* H

$$t : \mathfrak{H}(Quot_{W(H)} E\mathcal{F}_? \times X^?) \rightarrow \mathfrak{H}(Quot_{W(H)}(\Gamma \backslash G/? \times E\mathcal{F}_?) \times X^?)$$

multiplied by the multiplicity factors

$$|\pi_H(\Gamma \cap N(H))|$$

where $\pi_H : N(H) \rightarrow W(H) = N(H)/H$ *is the projection fit via (12) together to define a transfer map in the stable category*

$$t : \mathfrak{H}(Quot_G(X)) \rightarrow \mathfrak{H}(Quot_\Gamma(X)).$$

Further, there exists a filtration on $\mathfrak{H}(X/\Gamma)$ *such that the associated graded map of the composition*

$$\mathfrak{H}(Quot_\Gamma X) \rightarrow \mathfrak{H}(Quot_G X) \rightarrow \mathfrak{H}(Quot_\Gamma X)$$

is multiplication by $|G/H|$.

In the last sentence of the Corollary, the filtration could probably be eliminated by more careful consideration, but it does not matter for our purposes.

3. Symmetric products

In this section, we are interested in the following example:

$$X = \bigwedge_d S^{n(1+\alpha)} \tag{13}$$

where Σ_d acts by permutation of coordinates. We claim that condition (11) is satisfied when we set (8) with Σ_d acting on d by the standard permutation representation.

First of all, we note that the strata $X^H/X^{>H}$ are only non-trivial when

$$H = \Sigma_{d_1} \times ... \times \Sigma_{d_k}, \tag{14}$$

$d_1 + ... + d_k = d$ where Σ_{d_i} acts on d_i by the standard permutation representation. Then $X^H/X^{>H}$ is the 1-point compactification of the pure stratum

$$X^H - X^{>H} = \mathbb{A}^{nk} - \Delta \tag{15}$$

where Δ is the big diagonal (the union of elements with 2 or more coordinates coinciding). Using (8) and (15), we can then define an $\mathbb{A}^1$-homotopy inverse

$$\iota : \mathbb{A}^{nk} - \Delta \to E\mathcal{F}_H \times (\mathbb{A}^{nk} - \Delta) \tag{16}$$

of the natural projection

$$p : E\mathcal{F}_H \times (\mathbb{A}^{nk} - \Delta) \to \mathbb{A}^{nk} - \Delta \tag{17}$$

(p collapses the first coordinate to a point). To define ι, use Id for the second coordinate in the target of (16). For the first coordinate, we need a map

$$\mathbb{A}^{nk} - \Delta \to \mathbb{A}^{d\infty} - \bigcup_{K \notin \mathcal{F}_H} (\mathbb{A}^{d\infty})^K.$$

But this is obvious: simply send

$$\begin{array}{l} ((x_{11}, ..., x_{n1}), ..., (x_{1k}, ..., x_{nk})) \\ \mapsto ((x_{11}, ...x_{n1}, 0, 0, ...)^{d_1}, ..., (x_{1k}, ..., x_{nk}, 0, 0, ...)^{d_k}) \in (\mathbb{A}^\infty)^d. \end{array}$$

By definition, (17) is strictly left inverse to (16). To construct an $\mathbb{A}^1$-homotopy

$$ip \simeq Id, \tag{18}$$

first recall that we have the "Milnor trick" homotopy

$$k_t : \mathbb{A}^\infty \to \mathbb{A}^\infty,$$

$$k_t(x_1, x_2, ...) = (1 - t)(x_1, x_2, ...) + t(0, ..., 0, x_1, 0, ..., 0, x_2, ...)$$

(n 0's inserted before each coordinate). Clearly, $(k_t)^d$ restricts to an H-homotopy

$$\ell_t : E\mathcal{F}_H \to E\mathcal{F}_H$$

where $\ell_0 = Id$, $\ell_1 \subseteq (0 \times 0 \times ...0 \times \mathbb{A} \times 0 \times ... \times 0 \times \mathbb{A} \times ...)^d$. (18) then follows if we can construct a homotopy

$$ip \simeq \ell_1.$$

But for this purpose, we may now simply use

$$t(ip) + (1 - t)\ell_1,$$

which completes the proof of our claim.

Now we see that an $\mathbb{A}^1$-homotopy inverse and for the map in the assumption of Lemma 10 is in our case obtained simply by 1-point compactifying (16), (18) in the $\mathbb{A}^{nk} - \Delta$ coordinate. We obtain

Proposition 12. *Let $S \subset \Sigma_d$ be the 2-Sylow subgroup obtained by the standard permutation representation of a product of wreath products of copies of $\mathbb{Z}/2$. Then there exists for X in (13) a transfer map*

$$\tau : \mathfrak{H}(Quot_{\Sigma_d}(X)) \to \mathfrak{H}(X' = Quot_S(\bigwedge_d S^{n(1+\alpha)}))$$

such that if we denote by $p : \mathfrak{H}(X') \to \mathfrak{H}(Quot_{\Sigma_d}(X))$ the natural projection, then τp is a 2-complete equivalence.

Proof: By Corollary 11, there is a filtration such that on the associated graded pieces, τp is the multiplication by the odd number $|\Sigma_d/S|$. □

Because of Proposition 12, it now makes sense to examine the motivic (co)homology of X' in what we call the stable range, i.e. the range

$$i + j\alpha$$

where $(2n \leq) i + j << 4n$.

Proposition 13. *There exists a stratification of X' where the quotientes of strata are 1-point compactifications of the corresponding pure strata, and each pure stratum is of one of the following forms:*

$$(\mathbb{G}_m)^{\times \ell} \times \mathbb{A}^{n+k}, \quad k << 2n, \tag{19}$$

or

$$\text{An } \mathbb{A}^m\text{-bundle } V \text{ on a smooth variety } Y \text{ where } m >> n. \tag{20}$$

Furthermore, strata of the form (19) do not occur unless $d = 2^k$ for some integer k.

Proof: Induction. It suffices to consider $d = 2^k$, since otherwise X' is the smash-product of the cases for 2^{k_i} where k_i are places of the 1's in the binary expansion of k.

Now the case $k = 1$ is obvious, so consider

$$X' = Quot_{\mathbb{Z}/2}(X'' \wedge X'') \tag{21}$$

where X'' is stratified as in the statement of the Proposition. In the rest of the proof, all quotients are taken in the category of schemes, so we dispense with the *Quot* notation accordingly. So first, we take the stratification of $X'' \wedge X''$ by the smash-product of two copies of the stratification of X'', and then take a $\mathbb{Z}/2$-quotient. If the pure strata of X'' are $A_1, ..., A_N$, then this gives pure strata of (21) of the form

$$A_i \times A_j, \ i < j, \tag{22}$$

$$(A_i \times A_i)/(\mathbb{Z}/2) \tag{23}$$

((23) is taken in the scheme theoretical sense, so it is really $Quot_{\mathbb{Z}/2}(A_i \times A_i)$). The strata (22) are clearly of the form (20), the strata (23) may need to be further stratified.

More concretely, if A_i is of the form (20), we may stratify (23) by taking the bundle $Sym^2(V)$ on Y, and then the $\mathbb{Z}/2$-quotient of the bundle induced from $V \times V$ on $(Y \times Y) - Y$. Both are clearly of the form (20).

Suppose, then, that A_i is of the form (19). Once again, then, we have the pure stratum

$$((\mathbb{G}_m)^{\times \ell} \times (\mathbb{G}_m)^{\times \ell}) - (\mathbb{G}_m)^{\times \ell}) \times_{\mathbb{Z}/2} (\mathbb{A}^{n+k} \times \mathbb{A}^{n+k}), \tag{24}$$

which is of the form (20). What we are left with is

$$(\mathbb{G}_m)^{\times \ell} \times Sym^2(\mathbb{A}^{n+k}). \tag{25}$$

We stratify the second coordinate of (25) by taking as the j'th stratum the $\mathbb{Z}/2$-quotient of the subspace of

$$\mathbb{A}^{n+k} \times \mathbb{A}^{n+k}$$

consisting of pairs of elements whose all respective coordinates coincide except the first j. Assuming the characteristic of the ground field is not 2, the bottom pure stratum is

$$\mathbb{A}^{n+k}, \tag{26}$$

the higher strata are

$$\mathbb{G}_m \times \mathbb{A}^{n+k+i}, \tag{27}$$

leading to strata of (25) of type (19), as claimed. □

The relationship between symmetric products and the motivic cohomology spectrum is proved in [17]. Essentially, we have a filtration on the motivic Eilenberg-Mac Lane spaces in dimension $n(1+\alpha)$ such that the associated graded pieces are (13). We derive here one more consequence of the above methods which will be useful later.

Proposition 14. *There motivic Eilenberg-Mac Lane spectrum $H\mathbb{Z}/2^{Mot}$ is cell (i.e. can be obtained by successivelly attaching cones to $k+\ell\alpha$-dimensional homotopy classes, see the Appendix).*

Proof: Since the motivic Eilenberg-Mac Lane spectrum is equivalent to the homotopy direct limit of the motivic (bistable) suspension spectra of $n(1+\alpha)$-dimensional Eilenberg-MacLane spaces, which in turn can be constructed by successively attaching symmetric products of the form (13), it suffices to prove the statement for the suspension spectra of (13). By further filtering with respect to fixed point, we can next pass to 1-point compactifications of the unordered configuration spaces of d points in $\mathbb{A}^n$. Since $H\mathbb{Z}/2^{Mot} = H\mathbb{Z} \wedge M\mathbb{Z}/2$ is 2-complete (note that $[Z, M\mathbb{Z}/2 \wedge X]$ coincides, up to suspension, with $[Z \wedge M\mathbb{Z}/2, X]$), we can additionally work in the 2-complete motivic category, so we can take advantage of transfer in (motivic) stable homotopy and work with the suspension spectra of the 1-point compactification Φ of the quotient of the unordered configuration space of d points in $\mathbb{A}^n$ by an iterated wreath product of copies of $\mathbb{Z}/2$. (Recall that, up to equivalence, a wedge summand of a cell spectrum is cell by the "Eilenberg swindle", although such a claim is false for finite cell objects.) But by the general principle that a finite union is a homotopy direct limit of the diagram of intersections, Φ is further a homotopy direct limit of spaces of the form X' in Proposition 12. For such spaces, in turn, the suspension spectrum is cell by the proof of Proposition 13. □

4. Proofs of the theorems and the motivic Adams-Novikov spectral sequence

Now although the statement of Proposition 13 is precisely as needed for the induction, note that the proof actually precisely accounts for the strata of type (19), showing that the motivic $\mathbb{Z}/2$-cohomology

$$H^*_{Mot}(X', \mathbb{Z}/2),\ X' = Quot_S(\bigwedge_d S^{n(1+\alpha)}), \tag{28}$$

in the stable range is the subspace of $\mathbb{Z}/2$-valued etale cohomology which is the tensor product over $\mathbb{Z}/2$ of $\mathbb{Z}/2[\theta]$ with the $\mathbb{Z}/2$-module with basis

$$Q_k ... Q_1 \alpha \tag{29}$$

where α is the characteristic class of dimension $n(1+\alpha)$, and Q_i is either of the form P^s or βP^s where P^s is of dimension $s(1+\alpha)$, and β is of dimension 1. Now recall Proposition 12. Passing from (29) to the cohomology of X amounts to taking a certain direct summand. Passing to etale cohomology amounts to inverting θ, but also in the etale cohomology one knows that the cohomology is the same as in the topological situation, tensored with $\mathbb{Z}/2[\theta, \theta^{-1}]$. Thus, in etale cohomology, the direct summand is obtained by imposing the Adem relations. But the Adem relations respect twist, and so we see that the summand of (28) corresponding to the cohomology of X (in the stable range) is generated, as a $\mathbb{Z}/2[\theta]$-module, by admissible words of the form (29). Dimensional accounting ([18]) shows that the elements P^s, βP^s we constructed must, in fact, be the reduced power operations and their Bocksteins. Thus, we have proved Theorem 4, and hence Corollary 5.

Let us now turn to Theorem 7. By Theorem 4 and [12], we know that the motivic Adams spectral sequence in our situation converges to the homotopy of MGL (consider finite Thom spectra and pass to direct limit - convergence follows from the fact that the homotopy will eventually stabilize, i.e. remain constant, in dimensions $k + \ell\alpha$ with $k + \ell < N$, N increasing).

Now (recall that we are over an algebraically closed field) one has

$$H\mathbb{Z}/2_*^{Mot} MGL = \mathbb{Z}/2[\xi_1^2, \xi_2^2, ...] \otimes \mathbb{Z}/2[m_i | i \neq 2^k - 1] \tag{30}$$

as a comodule over $H\mathbb{Z}/2_*^{Mot} H\mathbb{Z}/2^{Mot}$ (m_i are primitive, $i \neq 2^k - 1$), by arguments which parallel exactly the topological case. Now using the Adams spectral sequence, Theorem 7 follows.

Therefore, over an algebraically closed field k of characteristic 0, we have formal group law theory for algebraically oriented spectra, which parallels the topological case (see also [9, 10, 11]). In particular, we have the Quillen idempotent, and $MGL_2^\wedge$ is a wedge of suspensions of copies of a spectrum $BPGL$ where

$$BPGL_* = \mathbb{Z}_2[\theta, v_1, v_2, ...], \;\; dim(v_k) = (2^k - 1)(1+\alpha).$$

Following arguments of Adams [1], we then see that the motivic analogue of the Adams-Novikov spectral sequence also converges to 2-completed stable homotopy groups. One also has an isomorphism of Hopf algebroids

$$(BPGL_*, BPGL_* BPGL) \cong (BP_*, BP_* BP) \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[\theta]$$

where the Tate twist is primitive. Therefore, an analogous relation is true for the ANSS E_2-terms:

$$Ext_{BPGL_*}^{s, t(1+\alpha)+n(1-\alpha)} = \begin{array}{ll} Ext_{BP_*}^{s,2t} & \text{for } n \geq 0 \\ 0 & \text{for } n < 0. \end{array} \tag{31}$$

Now to prove Theorem 1, we need a device for comparing differentials to the topological case. In characteristic 0, we know that maps of algebraically closed fields induce isomorphisms of 2-completed (not rational!) stable homotopy theory, and in the case of $k = \mathbb{C}$, we have a topological realization map, sufficient for our purposes.

Now in topology, the first non-trivial differential in the ANSS at $p = 2$ is

$$d_3 : Ext^{2,28}_{BP_*} \to Ext^{5,30}_{BP_*}. \tag{32}$$

By the comparison, then, in the motivic case over an algebraically closed field, we must have a non-trivial

$$d_3 : Ext^{2,14(1+\alpha)}_{BPGL_*} \to Ext^{5,14(1+\alpha)+2}_{BPGL_*}. \tag{33}$$

We see that the target of (33) is θ times the generator of $Ext^{5,15(1+\alpha)}_{BPGL_*}$ corresponding to the target $\alpha_1^2\eta_{3/2}$ of (32). The examples in the α family are treated analogously, using the differentials originating in α_{4k+2}, α_{4k+3}. This concludes the proof of Theorem 1.

5. The 2-complete J-homomorphism

Proof of Lemma 2: First we note that on the level of homotopy categories, PF_s and FP_s both preserve 2-equivalences. To this end, a 2-equivalence is the same thing as an equivalence after smashing with $M\mathbb{Z}/2$, which is the same as an equivalence on the level of cofibers of the map 2. But since PF_s and FP_s preserve cofibrations and 2, they preserve 2-equivalences.

Thus, we know that for a motivic spectrum X, the canonical map

$$FP_s(X) \to FP_s(X^\wedge_2)$$

is a 2-equivalence. On the other hand, if $Y \to Z$ is a 2-equivalence, then

$$PF_sY \to PF_sZ$$

is a 2-equivalence, so the induced map on homotopy classes

$$[PF_sZ, X^\wedge_2] \to [PF_sY, X^\wedge_2] \tag{34}$$

is an isomorphism. But by adjunction, (34) is the same thing as

$$[Z, FP_s(X^\wedge_2)] \to [Y, FP_s(X^\wedge_2)], \tag{35}$$

so $FP_s(X^\wedge_2)$ is 2-complete. □

Now similarly as the J-homomorphism calculation in the topological case, the proof of Theorem 3 has two different parts, detection and vanishing. For both parts we must use a certain ingenuity to mimic the corresponding arguments in topology. To tackle the detection part first, it is tempting to use the version of the ANSS described above, mimicking the Thom spectra approach outlined in Ravenel [15]. The difficulty is however that the element in $ImJ_{\mathbb{C}}$ in dimension $1 \mod 8$ is detected in Ext^3, so the Thom space construction does not work immediately, and a more elaborate argument is needed there. For an inclusion of fields $i^{Op} : K \subset L$, (the reason of the notation is that we are formally thinking of $i : Spec(L) \to Spec(K)$, although this is not in the motivic category), we have the base change functor i^* from $\mathbb{A}^1$-spaces over K to $\mathbb{A}^1$-spaces over L which commutes with pushover. (Similarly for spectra.) Clearly, i^* has a right adjoint, which we will denote by i_*, and preserves cofibrations and cofibration equivalences, so it is a Quillen adjunction, and on the homotopy category, i_* commutes with FP. A similar argument also applies to the $M\mathbb{Z}/2$-localized $\mathbb{P}^1$-stable categories. Now on $M\mathbb{Z}/2$-localized $\mathbb{P}^1$-stable categories, when both K, L are algebraically closed, i^* induces iso in $\mod 2$ motivic homology, and hence by our spectral sequence on $\mod 2$ homotopy of finite homotopy colimits of varieties. Hence, by a colimit argument, i^* induces an iso on homotopy groups in the $M\mathbb{Z}/2$-localized $\mathbb{P}^1$-stable categories, but so does i_* (by the commutation with FP). Thus, i^* and i_* are inverse equivalences of $M\mathbb{Z}/2$-localized $\mathbb{P}^1$-stable categories. This means that we may change fields to work in an algebraically closed field of our choice, say, $\mathbb{C}$.

But then we have the ordinary topological realization functor $|?|$, under which the map (1) becomes just the ordinary complex J-homomorphism. Furthermore, clearly we have a natural equivalence

$$X \xrightarrow{\cong} |PF(X)|. \tag{36}$$

Now consider the diagram

$$\begin{array}{ccc} PF(FP(GL_{Mot})) & \xrightarrow{PF(FP(\mathcal{J}))} & PF(FP(F_{Mot})) \\ \downarrow & & \downarrow \\ GL_{Mot} & \xrightarrow{\mathcal{J}} & F_{Mot} \end{array} \tag{37}$$

where the vertical maps are adjunction counits. Applying topological realization to (37), we then get

$$\begin{array}{ccc} FP(GL_{Mot}) & \xrightarrow{FP(\mathcal{J})} & FP(F_{Mot}) \\ \downarrow & & \downarrow \\ GL & \longrightarrow & F \end{array} \tag{38}$$

where the bottom row is the ordinary topological J-map. Thus, if we know that the left vertical arrow of (38) is an iso in homotopy after 2-completion, we are done by comparison with the ordinary topological J-homomorphism.

The key point is that we analogously have a natural map

$$FP(E) \to |E|$$

for a $\mathbb{P}^1$-spectrum E, and since taking the 0-space of a spectrum obviously commutes with topological realization, the left vertical arrow of (38) coincides with the map obtained by passing to -1-spaces of the natural map

$$FP(K_{Mot}) \to |K_{Mot}|. \tag{39}$$

But this map is just the usual map from algebraic to topological K-theory spectra. In effect, on 0-components of the 0-spaces, if we let

$$BGL_{disc} \to FPBGL_{Mot} \tag{40}$$

be the $+$-construction (GL_{disc} is just the direct limit of the general linear groups of the ground field, considered as a discrete group), it is the adjoint of the obvious map

$$PFBGL_{disc} \to BGL_{Mot}.$$

This follows by reinterpreting the proof of Proposition 3.9 of [14]. Now applying $|PF?|$ to (40), we get a commutative diagram

$$\begin{array}{ccc} BGL_{disc} & \xrightarrow{+} & FPBGL_{Mot} \\ & \searrow & \downarrow \\ & & BGL \end{array}$$

so the vertical map is the right map by universality of the +-construction.

By Suslin [16], (39) is an equivalence after 2-completion.

To prove that the elements not detected in the topological realization actually vanish in the motivic J-homomorphism, we mimic the method of Adams [3] in the $\mathbb{A}^1$-homotopy category. First, let $W = (S^{1+\alpha})^{\times n}$. Let also $V = S^{n(1+\alpha)}$. Then there is a natural projection

$$p : W \to V. \tag{41}$$

Lemma 15. *The suspension of the natural map from V to the cofiber of p is null-homotopic.*

Proof: First, it is a well known fact that when Y has a point (a trivial assumption in spaces, but not in $\mathbb{A}^1$-spaces), we have

$$X * Y \simeq X \wedge \tilde{\Sigma} Y \tag{42}$$

where $\tilde{\Sigma}$ denotes the unreduced suspension and $*$ denotes the join. The obvious inclusion $X * Y \to \Sigma(X \times Y)$ along with (42) then gives a splitting (=right inverse up to homotopy) of the collapse map

$$\Sigma(X \times Y) \to (\Sigma X \wedge Y). \tag{43}$$

Therefore, for n based spaces $X_1,...,X_n$, each of the collapse maps

$$\Sigma(X_1 \times ... \times X_n) \to X_1 \wedge \Sigma(X_2 \times ... \times X_n) \to ... \to X_1 \wedge ... \wedge \Sigma X_n$$

has a right inverse up to homotopy, and hence so does their composition. □

Now consider the diagram

$$\begin{array}{ccc} \pi^s_{n(1+\alpha)-1} S^0_{Mot} & \longrightarrow & [W, BF_{Mot}] \\ \uparrow & & \uparrow \\ K^0_{Mot} S^{n(1+\alpha)} & \longrightarrow & K^0_{Mot} W \end{array} \tag{44}$$

Recall from [14] that in the homotopy motivic category, B is inverse to Ω in group-like monoids. The proof there extends in a standard fashion to delooping E_∞-spaces an arbitrary finite number of times. (What we miss in the $\mathbb{A}^1$-category is a device which would guarantee delooping with respect to copies of S^α.) F_{Mot}, by mimicking the construction from topology, is still a group-like E_∞ space, and hence can be delooped an arbitrary finite number of times. This allows us to rewrite the motivic J-homomorphism on K^0_{Mot} as

$$K^0_{Mot} S^{n(1+\alpha)} \to \pi^s_{n(1+\alpha)} BF_{Mot},$$

which in turn can be rewritten as the left hand vertical arrow of (44). The right hand vertical arrow is the analogous map with $S^{n(1+\alpha)}$ replaced by W.

Lemma 16. *The horizontal arrows of (44) are injective.*

Proof: The arrows in question are induced by the map of $\mathbb{A}^1$-spaces (41). Since however the space we map into in each case can be delooped, the maps in question can be rewritten as induced by the suspension of (41). By Lemma 15, however, this suspension is a homotopy retraction. □

Lemma 17. *For the generator $z \in \tilde{K}^0_{Mot} S^{n(1+\alpha)} \cong \mathbb{Z}$, $(\psi^3 - 1)z$ maps to 0 in the 2-completed motivic J-homomorphism (where ψ denotes the Adams operations).*

Proof: We wish to completely mimic the proof of Adams [3]. By Lemma 16, we may pass to W, where z is a sum of copies of the tautological line bundle ξ over the individual copies of $S^{1+\alpha} = \mathbb{P}^1$. We obviously have a fiberwise degree 3 map from the associated spherical bundle (with fiber $S^{1+\alpha}$) of ξ to the associated sphere bundle of ξ^3, simply by taking fiberwise the completion of the map $\mathbb{P}^1 \to \mathbb{P}^1$ extending the 3'rd power map $\mathbb{G}_m \to \mathbb{G}_m$. On the other hand, there is also a fiberwise degree 3 map from the associated sphere bundle of $\xi+1$ (with fiber $S^{2(1+\alpha)}$) to itself: simply take the identity, and add it to itself three times fiberwise using the suspension coordinate. Thus, we would like to say that there is a fiberwise degree 3 map from the associated sphere bundle of the virtual bundle $\xi - \xi^3$ to the trivial sphere bundle. We can make precise sense of this by passing to an affinization $\mathbb{P}^{1\prime}$ of the base space $\mathbb{P}^1$; then there exists a number N such that $\eta = N + \xi' - \xi'^3$ (where ξ' is the pullback of ξ) is realized by an actual vector bundle, and we then have a fiberwise degree 3 map from the associated sphere bundle (with fiber $S^{N(1+\alpha)}$) to the trivial sphere bundle $\mathbb{P}^{1\prime} \times S^{N(1+\alpha)}$. (Note that the Grothendieck-Witt ring of an algebraically closed field is $\mathbb{Z}$, so we may operate with the degree in the same way as in topology.)

Now the next step is to note that since we have a "cell decomposition" of $\mathbb{P}^{1\prime}$, we can mimic precisely the topological proof of Adams [2, 3] to conclude that there is a natural number M such that the associated sphere bundle of $3^M\eta$ (with fiber $S^{3^M N(1+\alpha)}$) has a fiberwise degree 1 map to the trivial sphere bundle $\mathbb{P}^{1\prime} \times S^{3^M N(1+\alpha)}$.

We are then done if we can mimic enough of the theory of classification of spherical bundles in the $\mathbb{A}^1$-category to conclude that

(45) When τ is a vector bundle on $\mathbb{P}^{1\prime}$ of dimension m such that there exists a fiberwise degree 1 map from the associated sphere bundle of τ to the trivial sphere bundle, then the image of $\tau - m$ under the motivic J-homomorphism vanishes.

This can be done as follows: by gluing over coordinate patches, one can replace a spherical bundle by its "associated principal bundle" π with fiber F_{Mot} (each fiber S is replaced by the $\mathbb{A}^1$-space of stable based equivalences $S \to S^{m(1+\alpha)}$; of course, we must first pass to fibrant replacements). Then having a fiberwise degree 1 map implies that the principal bundle π has a section. But now spherical bundles coming from vector bundles are classified (i.e. pulled back from the universal bundle) by maps into BF_{Mot} (since vector bundles of a fixed dimension are classified by maps into BGL_{Mot}). On the level of principal bundles, this means that we have a map of principal bundles

$$\pi \to u \tag{46}$$

where

$$u : B(F_{Mot}, F_{Mot}, *) \to BF_{Mot}$$

is the universal bundle (the projection). Since $B(F_{Mot}, F_{Mot}, *)$ is contractible, a principal bundle with a section is then necessarily classified by a map homotopic to the constant map, thus proving (45), and the Lemma. □

The Adams operations on $\tilde{K}^0_{Mot} S^{n(1+\alpha)} \cong \mathbb{Z}$ are computed in the same way as in ordinary reduced topological K-theory of S^{2n}, and it is well known that the vanishing of the complex J-homomorphism of $\psi^3 - 1$ in topology is a tight bound for the order of the image of the J-group in all dimensions except $6 \mod 8$ (counting the dimensions as in diagram (44). To translate the argument to the the $\mathbb{A}^1$-category, it suffices to note that over an algebraically closed field, multiplying by the Tate twist induces an isomorphism on homotopy groups of the 2-*completion* of the K_{Mot}-theory spectrum, and thus the bound can be applied also after multiplying by powers of the Tate twist.

Actually, commutation with the Tate twist has to be justified. Unstably, $S^\alpha = \mathbb{G}_m$ represents $H^\alpha(?, \mathbb{Z})$. Thus, we cannot quite conclude that the Tate twist is represented by a map $S^1 \to S^\alpha$ (in fact, that is usually false), but we can conclude that this is true after 2-completion. However, (over an algebraically closed field) we do have

$$FP(\mathbb{G}_m)^\wedge_2 \simeq B\mathbb{Z}_2 = (S^1)^\wedge_2, \tag{47}$$

(by comparison of homotopy groups), so for any spectrum E, we can conclude that the effect of ℓ-fold Tate twist on the $n+m\alpha$'th homotopy groups of the 2-completion of E, where $m, n-\ell \geq 0$, can be computed by the map

$$\begin{aligned} &Map(S^{\ell\alpha}, \Omega^{n+(m-\ell)\alpha} E_0) \to \\ &Map(\bigwedge_\ell FP(S^\alpha)^\wedge_2, FP(\Omega^{n+(m-\ell)\alpha} E_0)^\wedge_2) \simeq \Omega^\ell FP(\Omega^{n+(m-\ell)\alpha} E_0)^\wedge_2. \end{aligned} \tag{48}$$

This is entirely an unstable construction, so it commutes with the motivic J-map.

The dimension in which Lemma 17 does not suffice for a tight bound of the image of J_{Mot} is $n = 3 \mod 4$ (as counted in diagram (44)). In topology, this corresponds to the case of dimension $6 \mod 8$. In this case, Adams [2] simply argues that the image of the J homomorphism vanishes because it factors through the image of the real J-homomorphism, and the corresponding homotopy group of the orthogonal group vanishes.

We can also mimic this argument in $\mathbb{A}^1$-homotopy theory. We may embed

$$GL_n \to O_{2n}. \tag{49}$$

Over an algebraically closed field, we can consider O_{2n} as the automorphism group of the hyperbolic quadratic form

$$x_1 y_1 + ... + x_n y_n,$$

and then GL_n embeds by

$$A \mapsto \left(\begin{array}{cc} A & 0 \\ 0 & (A^T)^{-1}. \end{array} \right)$$

Furthermore, O_{2n} thus described acts on the affine quadric

$$x_1 y_1 + ... + x_n y_n = 1,$$

which is homotopy equivalent to $\mathbb{A}^{n(1+\alpha)} - \{0\} \simeq S^{n(1+\alpha)-1}$ via the projection to the x-coordinates. This is sufficient to show that (1) factors through O_{Mot}, the direct limit of motivic orthogonal groups. The fact that the 2-complete homotopy groups of O_{Mot} vanish in dimensions $5 \mod 8$ now follows from the results of Hornbostel [5], (the definition of the motivic orthogonal spectrum on p. 678, and Definition 4.1 on p. 674), in conjunction with the calculations of Karoubi [8].

6. Appendix: The motivic Adams spectral sequences

The key point is that we focus on cell spectra [4], i.e. homotopy colimits of sequences of (bistable) motivic spectra K_n, where K_{n+1} is obtained from K_n by taking the homotopy cofiber of a map

$$\bigvee_{i \in I_n} S^{m_i} \to K_n \tag{50}$$

where m_i is of the form $k_i + \ell_i \alpha$, $k_i, \ell_i \in \mathbb{Z}$, and S^{m_i} here denotes the m_i'th suspension of the motivic sphere spectrum. We call a motivic cell spectrum of finite type if the k_i's are bounded below and there are only finitely many cells for each k_i. For formal reasons, we know that a map of cell spectra which induces isomorphism in $\pi_{k+\ell\alpha}$ for all $k, \ell \in \mathbb{Z}$, (we will call such map a very weak equivalence), is an equivalence [4]. Also for formal reasons, we have cell approximation, i.e. for every motivic spectrum X there exists a very weak equivalence $X' \to X$ where X' is cell. Now we can define 2-localization in the category of cell spectra. For formal reasons, it is the same as the cell approximation of 2-localization.

Now for a spectrum X, the Adams resolution is the semicosimplicial object (without degeneracies)

$$X \wedge H\mathbb{Z}/2^{Mot} \wedge ... \wedge H\mathbb{Z}/2^{Mot} \tag{51}$$

where cofaces are given by the unit $S \to H\mathbb{Z}/2^{Mot}$. If we are working in a foundational setup where $H\mathbb{Z}/2$ is a rigid ring spectrum, we can make (51) a cosimplicial object where the codegeneracies are given by the multiplication $H\mathbb{Z}/2^{Mot} \wedge H\mathbb{Z}/2^{Mot} \to H\mathbb{Z}/2^{Mot}$. However, that is not important to us. There is an obvious canonical map ϕ from X to the cosimplicial realization of its Adams resolution X^{Ad}. Recall that throughout this paper, we work over an algebraically closed field of characteristic 0.

Lemma 18. *Let X be the cell approximation of the 2-completion of a cell spectrum of finite type. Then the map $\phi : X \to X^{Ad}$ induces isomorphism in π_k, $k \in \mathbb{Z}$.*

Proof: In the present situation, we know by Morel [13] that the homotopy groups $\pi_k(X)$, $(k \in \mathbb{Z})$, are finitely generated $\mathbb{Z}_2$-modules, and are 0 for sufficiently low k. Let k be the lowest for which $A_k = \pi_k(X) \neq 0$. Since for motivic cell spectra we also have the Künneth theorem based on motivic stable homotopy theory, and the universal coefficient theorem, again by the connectivity results of [13], there is a characteristic map

$$X \to \Sigma^k (HA_k)^{Mot}. \tag{52}$$

Further, since we are over an algebraically closed field, and A is a finitely generated $\mathbb{Z}_2$-module, $\pi_i HA^{Mot}$ for $i \in \mathbb{Z}$ is equal to 0 except when $i = k$, in which case it is A. Consequently, for the fiber X^{k+1} of (52), the dimension of the lowest non-trivial homotopy group goes up at least by 1. Note that by Proposition 14, we do not leave the cell spectra category. Because of stability, by iterating this procedure, we obtain cofibration sequences

$$X^m \to X \to X_m \tag{53}$$

$$X_m \to X_{m-1} \to \Sigma A_m \tag{54}$$

where the right hand map (53) induces isomorphism on homotopy groups of integral dimensions $\leq m$, and X_m has only non-trivial homotopy groups in dimensions between k and m. Note that the induced map from X to $\underset{\leftarrow}{holim}\ X_m$ induces iso in homotopy groups of integral dimensions, but we do not know that it is even a very weak equivalence.

On the other hand, by commutation of homotopy inverse limits and the long exact sequence of homotopy groups (of integral dimensions), it now suffices to prove our result for $(HA_k)^{Mot}$ where A_k is a finite-dimensional $\mathbb{Z}_2$-module. This is done by direct computation, which, however, by Corollary 5, is a rehash of the corresponding computation in topology. □

Example: Let us look at the cell approximation of the 2-completion of S^α (which we will denote by $(S^\alpha)^\wedge_2$. By Proposition 14, cell approximation preserves homology. Therefore, by Lemma 18, and Corollary 5, we have a convergent Adams-like spectral sequence

$$Ext_{A_*}(\mathbb{Z}/2, \Sigma\mathbb{Z}/2) \Rightarrow \pi_*((S^\alpha)^\wedge_2). \tag{55}$$

Here the $*$ denotes integral dimensions only (no twist), A_* is the ordinary ("topological") Steenrod algebra, and Σ is shift up by 1. The E_2 term (55) is obtained by reading off the twist 0 part of the α-suspension of the motivic Steenrod algebra (and its tensor products with itself).

Now (55) has the same E_2-term as the classical Adams spectral sequence (suspended by 1), which we cannot solve, but we know one thing, namely the element in the lowest dimension has no possible differential targets, and hence has to be a permanent cycle. We have therefore proved

Lemma 19. *The Tate twist element $\theta \in H_{1-\alpha}$ lifts to a stable homotopy class in $\pi_{1-\alpha}(S^{Mot})^\wedge_2$.*

□

Theorem 6 now follows from the following

Lemma 20. *Under the assumption of Lemma 18, the map $\phi : X \to X^{Ad}$ is a very weak equivalence.*

Proof: We work similarly as in the proof of Lemma 18, but this time with taking into account all twists. In other words, we let

$$A_{k+\ell\alpha} = \pi_{k+\ell\alpha} X,$$

assuming all such groups vanish for lower k or the same k and lower ℓ. Then again by the Künneth theorem and universal coefficient theorem, we obtain a map

$$X \to \Sigma^{k+\ell\alpha}(HA_{k+\ell\alpha})^{Mot} \tag{56}$$

which induces iso in $\pi_{k+\ell\alpha}$. Now we claim that

(57) The fiber $X^{k,\ell+1}$ has no nonzero homotopy groups in dimensions $m+s\alpha$ for $m<k$ or $m=k$ and $s\leq \ell$.

Note carefully that this is no longer automatic, since when taking into account twist,

$$\Sigma^{k+\ell\alpha}(HA_{k+\ell\alpha})^{Mot} \tag{58}$$

no longer only one non-zero homotopy group. However, all the homotopy groups of (58) are multiples of the $k+\ell\alpha$-dimensional homotopy group by powers of the Tate twist (since we are over an algebraically closed field, and $A_{k+\ell\alpha}$ is a finitely generated $\mathbb{Z}_2$-module). By Lemma 19, however, all these elements lift to the homotopy of X, which implies (57).

Now we can complete the proof similarly as the proof of Lemma 18: We obtain a series of successive fibrations

$$X^{m,s} \to X \to X_{m,s},$$

$$X_{m,s} \to X_{m,s-1} \to \Sigma\Sigma^{m+\ell\alpha}(HA_{k+\ell\alpha})^{Mot},$$

$X_{k,\ell-1}=0$. To pass from m to $m+1$, one puts

$$X^m = \underset{\leftarrow}{holim}\; X^{m,s},$$

(and similarly for the X_m's), but we see inductively that for each $q>m$, there exists an s_q such that $\pi_{q,s}X^m=0$ for $s\leq s_q$. Thus, we can put $X^{m+1,s_{m+1}}=X^m$. Further, this time, the induced map from X to the homotopy limit of the X_m's is a very weak equivalence.

By commutation of homotopy inverse limits, again, therefore, it suffices to prove the statement for $HA_{k+\ell\alpha}$, which because of Corollary 5 follows by a rehash of the topological calculation, this time carrying along a polynomial generator θ. $\square$

To prove Theorem 8, we may now consider the MGL-based Adams resolution of S^{Mot}, 2-completing every term. Now replacing every term by its $H\mathbb{Z}/2^{Mot}$ based Adams resolution gives a very weak equivalence by Theorem 6, but at the same time the induced map from $S_2^\wedge$ to this composite resolution is a very weak equivalence by the same theorem.

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