SOME REMARKS ON ORBIT SETS OF UNIMODULAR ROWS

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Abstract. Let A be a d-dimensional smooth algebra over a perfect field of characteristic not 2. Let $Um_{n+1}(A)/E_{n+1}(A)$ be the set of unimodular rows of length $n+1$ up to elementary transformations. If $n \geq (d+2)/2$, it carries a natural structure of group as discovered by Van der Kallen. If $n = d \geq 3$, we show that this group is isomorphic to a cohomology group $H^d(A, G^{d+1})$. This extends a theorem of Morel, who showed that the set $Um_{d+1}(A)/SL_{d+1}(A)$ is in bijection with $H^d(A, G^{d+1})/SL_{d+1}(A)$. We also extend this theorem to the case $d = 2$. Using this, we compute the groups $Um_{d+1}(A)/E_{d+1}(A)$ when A is a real algebra with trivial canonical bundle and such that $\mathrm{Spec}(A)$ is rational. We then compute the groups $Um_{d+1}(A)/GL_{d+1}(A)$ when d is even, thus obtaining a complete description of stably free modules of rank d on these algebras. We also deduce from our computations that there are no stably free non free modules of top rank over the algebraic real spheres of dimension 3 and 7.

Contents

1. Introduction

Let A be a commutative noetherian ring and P, Q be two projective A-modules which are stably isomorphic, i.e. $P \oplus A^n \simeq Q \oplus A^n$. The question is to know in which

situations this implies $P \simeq Q$. A celebrated theorem of Bass and Schanuel states that this is always the case if P is of rank strictly bigger than the Krull dimension of the ring A (see [2, Theorem 9.3]). If A is an algebra over an algebraically closed field, then Suslin showed that the result can be extended to projective modules whose rank is equal to the dimension of the ring ([16]). In general, this result is wrong as shown by the example of the tangent bundle over the algebraic real two-sphere.

As a special case of the question, the stably free modules were extensively studied. Let d denote the Krull dimension of A. By Bass-Schanuel's cancellation theorem, the study of stably free modules reduces to the case $P \oplus A \simeq A^{d+1}$. Such modules correspond to unimodular rows of length $d+1$. In general, let $Um_{n+1}(A)$ denote the set of unimodular rows of length $n+1$. One sees that $GL_{n+1}(A)$ acts on the right on this set, and so does its subgroup $E_{n+1}(A)$ generated by elementary matrices. It is not hard to see that a unimodular row $(a_1, \ldots, a_{n+1})$ yields a free module if and only if it is the first row of a matrix in $GL_{n+1}(A)$. This observation led to the study of the sets $Um_{n+1}(A)/E_{n+1}(A)$ and $Um_{n+1}(A)/GL_{n+1}(A)$ (which is the same as $Um_{n+1}(A)/SL_{n+1}(A)$). An important step was the discovery by Vaserstein that $Um_3(A)/E_3(A)$ was carrying a natural structure of abelian group under some conditions on A ([19, Theorem 5.2]). These conditions are for example satisfied when A is of Krull dimension 2. Inspired by this case, Van der Kallen put a structure of abelian group on $Um_{n+1}(A)/E_{n+1}(A)$ (under some hypothesis on A) which coincides with the previous one when $n = 2$. This structure comes from the following observation: If $A = C(X)$ is the ring of continous real functions on some nice CW-complex X, then the set of maps from X to $\mathbb{R}^{d+1} \setminus \{0\}$ up to homotopy is the cohomotopy group $\pi^d(X)$. Van der Kallen showed that the group law was in some sense algebraic, thus leading to the group structure on $Um_{n+1}(A)/E_{n+1}(A)$ for any reasonable ring A. The problem is now to actually compute this group and its quotient $Um_{n+1}(A)/SL_{n+1}(A)$.

In his recent preprint [12], Morel showed that the group $Um_{d+1}(A)/SL_{d+1}(A)$ has a cohomological interpretation when A is a d-dimensional smooth algebra over a field k. Indeed, let K^{MW}_{d+1} be the unramified Milnor-Witt sheaf. Then a very easy computation shows that $H^d(\mathbb{A}^{d+1} \setminus \{0\}, K^{MW}_{d+1}) = GW(k)$, the Grothendieck-Witt group of k. A unimodular row $(a_1, \ldots, a_{d+1})$ can be seen as a morphism $f : \mathrm{Spec}(A) \to \mathbb{A}^{d+1} \setminus \{0\}$ and one can consider the pull back $f^*(\langle 1 \rangle)$ in $H^d(A, K^{MW}_{d+1})$, where $\langle 1 \rangle$ denotes the unit in $GW(k)$. Let $\mathcal{H}(k)$ be the $\mathbb{A}^1$-homotopy category of smooth k-schemes. One of the main theorems in [12] states that this map induces a bijection between $\mathrm{Hom}_{\mathcal{H}(k)}(A, \mathbb{A}^{d+1} \setminus \{0\})$ and $H^d(A, K^{MW}_{d+1})$. Furthermore, the natural action of $GL_{d+1}(A)$ on $\mathrm{Hom}_{\mathcal{H}(k)}(\mathrm{Spec}(A), \mathbb{A}^{d+1} \setminus \{0\})$ gives an action on $H^d(A, K^{MW}_{d+1})$, which reduces to an action of $SL_{d+1}(A)$. The quotient $H^d(A, K^{MW}_{d+1})/SL_{d+1}(A)$ is then in bijection with the set of stably free modules of rank d. Thus the above map induces a bijection $Um_{d+1}(A)/SL_{d+1}(A) \to H^d(A, K^{MW}_{d+1})/SL_{d+1}(A)$. For some technical reasons, Morel has to assume that $d \geq 3$ to prove this theorem. Observe also that if the field k is of characteristic different from 2, the group $H^d(A, K^{MW}_{d+1})$ coincide with the group $H^d(A, G^{d+1})$ as defined in [6, Chapter 10] (following the original idea of [1]).

Our first goal in this paper is the following theorem (Theorem 4.9 in the text):

Theorem. *Let A be a smooth k-algebra of dimension d. Suppose that k is perfect. Then the map $\phi : Um_{d+1}(A)/E_{d+1}(A) \to H^d(A, G^{d+1})$ is an isomorphism for $d \geq 3$.*

This result is most probably true if $d = 2$ and the field k is not perfect and of arbitrary characteristic. Our strategy is the following: First we show that $Um_{n+1}(A)/E_{n+1}(A)$ is nothing but the set of morphisms from $\mathrm{Spec}(A)$ to $\mathbb{A}^{n+1}\backslash\{0\}$ up to naive homotopy. Here we say that two morphisms $f, g : \mathrm{Spec}(A) \to \mathbb{A}^{n+1}\backslash\{0\}$ are naively homotopic if there exists a morphism $F : \mathrm{Spec}(A[t]) \to \mathbb{A}^{n+1}\backslash\{0\}$ whose evaluations in 0 and 1 are f and g respectively. Then we show that there is an exact sequence of pointed sets

$$GL_n(A)/E_n(A) \to GL_{n+1}(A)/E_{n+1}(A) \to Um_{n+1}(A)/E_{n+1}(A) \to Um_{n+1}(A)/GL_{n+1}(A) \to 0.$$

which turns out to be an exact sequence of groups in some situations. The next step is to show that the set $GL_n(A)/E_n(A)$ is nothing else than the set $\mathrm{Hom}_{\mathcal{H}(k)}(\mathrm{Spec}(A), Sing^\bullet \mathbb{G}L_n)$ if $n \geq 3$. This is one of the results of [12], but we spend some lines to explain it in Section 4. The theorem is an obvious consequence of this fact.

Our next result extends the theorem of Morel to the case $d = 2$ (Theorem 4.11).

Theorem. *Let A be a smooth k-algebra of dimension 2. The homomorphism ϕ induces an isomorphism*

$$\overline{\phi} : Um_3(A)/SL_3(A) \simeq H^2(A, G^3)/SL_3(A).$$

The idea to prove this result is to use a result of Bhatwadekar and Sridharan relating $Um_3(A)/SL_3(A)$ with the Euler class group $E(A)$ and the weak Euler class group $E_0(A)$ (see [3]). Namely, there is an exact sequence

$$0 \longrightarrow Um_3(A)/SL_3(A) \xrightarrow{\psi} E(A) \longrightarrow E_0(A) \longrightarrow 0.$$

We then use the fact that if A is of regular of dimension 2 then $E(A)$ coincide with the Chow-Witt group $\widetilde{CH}^2(A)$ and $E_0(A)$ is just the Chow group $CH^2(A)$. A comparison of exact sequences then yields the result.

Next we compute the group $H^d(A, G^{d+1})$ where A is a real algebra satisfying some extra conditions:

Theorem. *Let A be a smooth $\mathbb{R}$-algebra of dimension d with trivial canonical bundle. Suppose that $X = \mathrm{Spec}(A)$ is rational. Then*

$$H^d(X, G^{d+j}) \simeq H^d(X, I^{d+j}) \simeq \bigoplus_{C \in \mathcal{C}} \mathbb{Z}$$

where $\mathcal{C}$ is the set of compact connected components of $X(\mathbb{R})$ (endowed with the Euclidian topology).

We also show that when A is even-dimensional, then GL_{d+1} acts trivially on $H^d(A, G^{d+1})$ and we can completely compute the set of stably free modules of rank d in that case.

Theorem. *Let A be a smooth $\mathbb{R}$-algebra of even dimension d with trivial canonical bundle. Suppose that $X = \mathrm{Spec}(A)$ is rational. Then the set of stably free modules of*

rank d is isomorphic to $\bigoplus_{C\in\mathcal{C}}\mathbb{Z}$, where $\mathcal{C}$ is the set of compact connected components of $X(\mathbb{R})$ (endowed with the Euclidian topology).

In odd dimension, things are more complicated. If S^3 and S^7 denote the real algebraic spheres of dimension 3 and 7, we show that all the stably free modules of top rank on these spheres are free.

1.1. **Conventions.** Throughout the article, k will be a commutative field of characteristic different from 2. All k-algebras are commutative and essentially of finite type over k. If A is such an algebra and $\mathfrak{p}$ is any prime ideal in A, we denote by $\kappa(\mathfrak{p})$ the residue field in $\mathfrak{p}$. If $\mathfrak{p}$ is of height n, we denote by $\omega_{\mathfrak{p}}$ the $\kappa(\mathfrak{p})$-vector space $\mathrm{Ext}^n_{A_{\mathfrak{p}}}(\kappa(\mathfrak{p}), A_{\mathfrak{p}})$ (which is of dimension 1 if the ring is regular). When we write $W(\kappa(\mathfrak{p}))$, we always mean the Witt group of $\kappa(\mathfrak{p})$-vector spaces endowed with symmetric isomorphisms for the duality $\mathrm{Hom}_{\kappa(\mathfrak{p})}(\_, \omega_{\mathfrak{p}})$. The Witt group $W(\kappa(\mathfrak{p}))$ is a module over the classical Witt ring of $\kappa(\mathfrak{p})$. If $\langle\alpha\rangle$ denotes the class of $\alpha\in\kappa(\mathfrak{p})^\times$ in the classical Witt group, and ξ is any element of $W(\kappa(\mathfrak{p}))$, we denote by $\langle\alpha\rangle\cdot\xi$ the product of $\langle\alpha\rangle$ and ξ.

2. Unimodular rows and naive homotopies of maps

2.1. **A Vorst-type theorem.** Let A be a k-algebra, where k is a field. For any $m, n\in\mathbb{N}$ such that $m\leq n$, let $Um_{m,n}(A)$ be the set of surjective homomorphisms $A^n\to A^m$. Let $E_n(A)$ be the subgroup of $SL_n(A)$ generated by the elementary matrices. This group acts (on the right) on $Um_{m,n}(A)$ and we denote the set of orbits by $Um_{m,n}(A)/E_n(A)$. In particular, when $m=1$ we get the set of unimodular rows under elementary transformations, and when $m=n$ we get the set $GL_n(A)/E_n(A)$, which is a group when $n\geq 3$.

For any m, n as above, denote by $V(m,n)$ the ideal of $\mathbb{A}^{mn}$ (seen as the set of $m\times n$ matrices) generated by the $m\times m$ minors. Denote by $D(m,n)$ the open subscheme $\mathbb{A}^{mn}\setminus V(m,n)$ of $\mathbb{A}^{mn}$. In particular, $D(1,n)=\mathbb{A}^n\setminus\{0\}$ and $D(n,n)=GL_n(k)$.

Let X, Y be two schemes over k. We say that two homomorphisms $f, g: X\to Y$ are naively homotopic if there exists a morphism $F: X\times\mathbb{A}^1\to Y$ such that $F(0)=f$ and $F(1)=g$ where $F(i)$ denotes the evaluation in $i=0,1$. Being naively homotopic is an equivalence relation and we denote by $\mathrm{Hom}_{\mathbb{A}^1}(X,Y)$ the set of equivalence classes of morphism from X to Y. If $X=\mathrm{Spec}(A)$, remark that $\mathrm{Hom}(X, D(m,n))=Um_{m,n}(A)$ and we can identify the naive homotopy classes as follows:

Theorem 2.1. *Let A be a regular k-algebra and $X=\mathrm{Spec}(A)$. Then*

$$\mathrm{Hom}_{\mathbb{A}^1}(X, D(m,n)) = Um_{m,n}(A)/E_n(A)$$

for any m, n.

Proof. First remark that any elementary matrix is naively homotopic to the identity. Let L and L' be two elements of $Um_{m,n}(A)$. Suppose that there is an element M in $Um_{m,n}(A[t])$ such that $M(0)=L$ and $M(1)=L'$. Consider the exact sequence

$$0\longrightarrow P\longrightarrow A[t]^n\xrightarrow{M}A[t]^m\longrightarrow 0$$

where P is the kernel of M. Remark that P is projective, and therefore it is extended from A by [9], i.e. $P = P(0)[t]$. But $P(0)$ is defined by the following sequence

$$0 \longrightarrow P(0) \longrightarrow A^n \xrightarrow{L} A^m \longrightarrow 0.$$

Comparing the two (split) exact sequences

$$\begin{tikzcd}
0 \arrow[r] & P(0)[t] \arrow[r] \arrow[d, equal] & A[t]^n \arrow[r, "M"] \arrow[d, dashed, "\psi"] & A[t]^m \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P(0)[t] \arrow[r] & A[t]^n \arrow[r, "L"] & A[t]^m \arrow[r] & 0,
\end{tikzcd}$$

we see that there exists an automorphism ψ of $A[t]^n$ such that the diagram commutes. Remark that $\psi(0) = Id$. By [20], $\psi \in E_n(A[t])$. Evaluating at $t = 1$, we get $L' = L\psi(1)$. Thus the result is proved.

□

2.2. **The group structure on $Um_n(A)/E_n(A)$.** The universal weak Mennike symbol on the set $Um_n(A)/E_n(A)$ is the free group $WMS_n(A)$ with generators $wms(v)$ for all $v \in Um_n(A)$ and relations

(i) $wms(v) = wms(vg)$ for any $g \in E_n(A)$.
(ii) If $(q, v_2, \ldots, v_n)$ and $(1+q, v_2, \ldots, v_n)$ are unimodular and $r(q+1) = q$ modulo $(v_2, \ldots, v_n)$, then

$$wms(q, v_2, \ldots, v_n) = wms(r, v_2, \ldots, v_n)wms(1 + q, v_2, \ldots, v_n).$$

By definition, there is a map $wms : Um_n(A)/E_n(A) \to WMS_n(A)$. In [17, Theorem 4.1], it is proven that this map is a bijection under certain conditions. In the same paper, it is shown that $WMS_n(A)$ is abelian in that case ([17, Theorem 3.6]). We condense these informations in the next result:

Theorem 2.2. *Let A be a regular k-algebra of dimension $d \geq 2$. Then the map $wms : Um_n(A)/E_n(A) \to WMS_n(A)$ is a bijection for any $n \geq (d + 4)/2$. Moreover, $WMS_n(A)$ is an abelian group.*

Proof. See [18, Theorem 3.3].

□

2.3. **An exact sequence.** For $n \geq 1$ Consider the morphism of algebraic groups $\mathbb{G}L_n \to \mathbb{G}L_{n+1}$ sending a matrix M to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$. Consider also the morphism $\mathbb{G}L_{n+1} \to \mathbb{A}^{n+1} \setminus \{0\}$ sending a matrix to its first row. We get a sequence

$$\mathbb{G}L_n \longrightarrow \mathbb{G}L_{n+1} \longrightarrow \mathbb{A}^{n+1} \setminus \{0\}.$$

If A is a k-algebra, we get a sequence of pointed sets (where $Um_{n+1}(A)/E_{n+1}(A)$ is pointed by $[1, 0, \ldots, 0]$)

$$GL_n(A)/E_n(A) \longrightarrow GL_{n+1}/E_{n+1}(A) \longrightarrow Um_{n+1}(A)/E_{n+1}(A)$$

which is the sequence of maps up to homotopy obtained from the above sequence (if A is regular).

Proposition 2.3. *Let A be a regular k-algebra of dimension d. For $n \geq 2$, the sequence of pointed sets*

$$GL_n(A)/E_n(A) \longrightarrow GL_{n+1}/E_{n+1}(A) \longrightarrow Um_{n+1}(A)/E_{n+1}(A)$$

is exact. If moreover $n = d$ and $d \geq 3$, then it is an exact sequence of groups.

Proof. We first prove the first assertion. Since the composition

$$\mathbb{G}L_n \longrightarrow \mathbb{G}L_{n+1} \longrightarrow \mathbb{A}^{n+1} \setminus \{0\}$$

send a matrix to the distinguished point of $\mathbb{A}^{n+1} \setminus \{0\}$, then the sequence above is a complex. Let $M \in GL_{n+1}(A)$ be such that there exists $E \in E_{n+1}(A)$ with $ME = \begin{pmatrix} 1 & 0 \\ * & M' \end{pmatrix}$ for some $M' \in GL_n(A)$. There is a matrix $F \in E_{n+1}(A)$ such that $FME = \begin{pmatrix} 1 & 0 \\ 0 & M' \end{pmatrix}$. Now $M^{-1}FM$ is in $E_{n+1}(A)$ since it is homotopic to identity (use [20] again). Therefore $M(M^{-1}FM)E$ comes from $GL_n(A)$ and the sequence is exact.

If $n = d$ and $d \geq 3$ the terms in the sequence are groups. Moreover, the map $GL_{n+1}(A)/E_{n+1}(A) \to Um_{n+1}(A)/E_{n+1}(A)$ is a homomorphism of groups by [17, Theorem 5.3 (ii)].

□

Now the cokernel of the map $GL_{n+1}(A)/E_{n+1}(A) \to Um_{n+1}(A)/E_{n+1}(A)$ is just $Um_{n+1}(A)/GL_{n+1}(A)$ which is the set of isomorphism classes of stably free modules of rank n over A. The following result is an obvious consequence of the above proposition, but we state it for further reference.

Theorem 2.4. *Let A be a regular k-algebra of dimension d. For any $n \geq 2$, there is an exact sequence of pointed sets*

$$GL_n(A)/E_n(A) \to GL_{n+1}(A)/E_{n+1}(A) \to Um_{n+1}(A)/E_{n+1}(A) \to Um_{n+1}(A)/GL_{n+1}(A) \to 0.$$

If $n = d$ and $d \geq 3$, this is an exact sequence of groups.

3. Computations of some cohomology groups

In this section, we compute some cohomology groups of the sheaf G^j, where $j \in N$. The definition and basic properties of this sheaf can be found in [5, §3]. If A is a regular k-algebra of dimension d, then $H^d(A, G^d)$ and $H^d(A, G^{d+1})$ coincide respectively with $H^d(A, K_d^{MW})$ and $H^d(A, K_{d+1}^{MW})$, where K_j^{MW} is the (j-th) unramified Milnor-Witt sheaf as defined in [11]. For the forthcoming computation, the only thing to know is that $H^*(\_, G^*)$ is a contravariant bigraded functor from the category of smooth schemes to the category of abelian groups having two basic properties:

i) $H^*(\_, G^*)$ is a cohomology theory in the sense of Panin and Smirnov ([15]), but it is not oriented.
ii) $H^0(\mathrm{Spec}(k), G^0) = GW(k)$, the Grothendieck-Witt group of k.

3.1. **A preliminary computation.** Let $U = \mathbb{A}^{n+1} - \{0\}$. To compute the groups $H^i(U, G^j)$ for any i, j, it suffices to use the localization sequence

$$\ldots \to H^i_{\{0\}}(\mathbb{A}^{n+1}, G^j) \to H^i(\mathbb{A}^{n+1}, G^j) \to H^i(U, G^j) \xrightarrow{\partial} H^{i+1}_{\{0\}}(\mathbb{A}^{n+1}, G^j) \to \ldots$$

Using that $H^i_{\{0\}}(\mathbb{A}^{n+1}, G^j) = 0$ if $i \neq n+1$ and homotopy invariance, we get $H^0(U, G^j) \simeq H^0(\mathbb{A}^{n+1}, G^j) \simeq H^0(k, G^j) = G^j(k)$, $H^i(U, G^j) = 0$ if $0 < i < n$ and $H^n(U, G^j) \simeq H^{n+1}_{\{0\}}(\mathbb{A}^{n+1}, G^j)$ via ∂. Moreover, we have an isomorphism $H^{n+1}_{\{0\}}(\mathbb{A}^{n+1}, G^j) \simeq H^0(k, G^{j-n-1})$ and finally $H^n(U, G^j) \simeq H^0(k, G^{j-n-1})$. Observe that this isomorphism is $H^0(k, G^0)$-linear. Since we use it in the sequel, we give an explicit description for $j = n+1$.

Observe first that $H^0(k, G^0) = GW(k)$. Let $B = k[x_1, \ldots, x_{n+1}]$ and consider the Koszul complex $Kos(x_2, \ldots, x_{n+1})$ associated to the regular sequence $x_2, \ldots, x_{n+1}$. We get an isomorphism

$$\psi_{x_2,\ldots,x_{n+1}} : B/(x_2, \ldots, x_{n+1}) \simeq \mathrm{Ext}^n_B(B/(x_2, \ldots, x_{n+1}), B)$$

given by $1 \mapsto Kos(x_2, \ldots, x_{n+1})$. Localizing at $\mathfrak{p} = (x_2, \ldots, x_{n+1})$, it becomes an isomorphism $\psi_{x_2,\ldots,x_{n+1}} : \kappa(\mathfrak{p}) \simeq \mathrm{Ext}^n_{B_\mathfrak{p}}(\kappa(\mathfrak{p}), B_\mathfrak{p})$. Observe that $x_1 \in B_\mathfrak{p}^\times$ and consider the couple $(x_1, \langle -\psi_{x_2,\ldots,x_{n+1}}, x_1\psi_{x_2,\ldots,x_{n+1}}\rangle)$ in the fibre product

$$\begin{array}{ccc} G^1(\kappa(\mathfrak{p})) & \longrightarrow & I(\kappa(\mathfrak{p}), \omega_\mathfrak{p}) \\ \downarrow & & \downarrow \\ K^M_1(\kappa(\mathfrak{p})) & \longrightarrow & I(\kappa(\mathfrak{p}))/I^2(\kappa(\mathfrak{p})). \end{array}$$

It defines an element ξ of $H^n(U, G^{n+1})$ which is mapped to $\langle 1\rangle$ (after identification of $H^{n+1}_{\{0\}}(\mathbb{A}^{n+1}, G^{n+1})$ with $GW(k)$) under ∂.

3.2. **An exact sequence.** From the definition of G^j, we have an exact sequence of sheaves for any $j \in \mathbb{N}$

$$0 \longrightarrow I^{j+1} \longrightarrow G^j \longrightarrow K^M_j \longrightarrow 0$$

where I^{j+1} is the unramified sheaf associated to the $(j+1)$-th power of the fundamental ideal and K^M_j is the unramified Milnor K_j-sheaf (see [5, §3] for flasque resolutions of these sheaves).

Let A be a smooth k-algebra of dimension d. Then the above sequence of sheaves gives an exact sequence

$$H^d(A, I^{j+1}) \longrightarrow H^d(A, G^j) \longrightarrow H^d(A, K^M_j) \longrightarrow 0$$

for any $j \in \mathbb{N}$. There is also a natural map of sheaves $G^{j+1} \to I^{j+1}$ which gives a surjective map $H^d(A, G^{j+1}) \to H^d(A, I^{j+1})$ and we get an exact sequence

$$H^d(A, G^{j+1}) \longrightarrow H^d(A, G^j) \longrightarrow H^d(A, K^M_j) \longrightarrow 0$$

for any $j \in \mathbb{N}$. By definition, $H^d(A, G^d)$ is the Chow-Witt group $\widetilde{CH}^d(A)$ and $H^d(A, K^M_d)$ is the Chow group $CH^d(A)$. Putting everything together, we have:

Proposition 3.1. *Let A be a smooth k-algebra of dimension d. There is an exact sequence*

$$H^d(A, G^{d+1}) \longrightarrow \widetilde{CH}^d(A) \longrightarrow CH^d(A) \longrightarrow 0.$$

4. The homomorphism $Um_{n+1}(A)/E_{n+1}(A) \to H^n(A, G^{n+1})$

4.1. **The homomorphism.** Let A be a smooth k-algebra and $X = \mathrm{Spec}(A)$. We define a map

$$\phi : \mathrm{Hom}(X, \mathbb{A}^{n+1} - \{0\}) \to H^n(A, G^{n+1})$$

by $\phi(f) = f^*(\xi)$, where $f^* : H^n(\mathbb{A}^{n+1} - \{0\}, G^{n+1}) \to H^n(A, G^{n+1})$ is the pull-back induced by f ([5, Definition 7.2]). Because of the homotopy invariance of $H^n(A, G^{n+1})$, we get a map

$$\phi : Um_{n+1}(A)/E_{n+1}(A) \to H^n(A, G^{n+1}).$$

Theorem 4.1. *Let A be a smooth k-algebra. Then for any $n \geq 1$ the map $\phi : Um_{n+1}(A)/E_{n+1}(A) \to H^n(A, G^{n+1})$ induces a homomorphism*

$$\Phi : WMS_{n+1}(A) \to H^n(A, G^{n+1}).$$

Proof. Since $H^n(A, G^{n+1})$ is a group and the relation (i) in $WMS_{n+1}(A)$ is clearly satisfied in $H^n(A, G^{n+1})$, it is enough to verify that relation (ii) is also satisfied. We start with a simple computation in $G^1(k(t))$. Using [8, Chapter I, Proposition 5.1], we have $\langle 1, t\rangle = \langle t(t+1), t+1\rangle$ in $I(k(t))$ because both forms represent $t+1$ and they have the same discriminant. Adding $\langle -1, -1\rangle$ on both sides, we get $\langle -1, t\rangle = \langle -1, t(t+1)\rangle + \langle -1, t+1\rangle$ in $I(k(t))$. Therefore we have an equality

$$(t/(t+1), \langle -1, t/(t+1)\rangle) + (t+1, \langle -1, t+1\rangle) = (t, \langle -1, t\rangle) \tag{1}$$

in $G^1(k(t))$. This equality also applies to the generator ξ of $H^n(\mathbb{A}^{n+1} - \{0\}, G^{n+1})$. Suppose now that $(q, v_2, \ldots, v_{n+1})$ is unimodular in A, as well as $(q+1, v_2, \ldots, v_{n+1})$. Remark then that $(q(q+1), v_2, \ldots, v_{n+1})$ is also unimodular. Performing if necessary elementary operations on this unimodular line, we can suppose that the sequence $(v_2, \ldots, v_{n+1})$ is regular in A. Now the pull back of ξ under the map $f : \mathrm{Spec}(A) \to \mathbb{A}^{n+1} - \{0\}$ given by $(q, v_2, \ldots, v_{n+1})$ is precisely the cycle $(q, \langle -1, q\rangle)$ supported on $A/(v_2, \ldots, v_{n+1})$. Since $(q+1, v_2, \ldots, v_{n+1})$ is also unimodular by assumption, we get a cycle $(q+1, \langle -1, q+1\rangle)$ also supported on $A/(v_2, \ldots, v_{n+1})$. Because of relation 1 above, we see that the relation (ii) in $WMS_{n+1}(A)$ is also satisfied in $H^n(A, G^{n+1})$ and the theorem is proved.

□

Applying Theorem 2.2, we get the following corollary:

Corollary 4.2. *Let A be a smooth k-algebra of dimension d. For any $n \geq (d+2)/2$ the map $\phi : Um_{n+1}(A)/E_{n+1}(A) \to H^n(A, G^{n+1})$ is a homomorphism of groups.*

There is an elementary proof of the fact that ϕ is surjective in some non trivial situations. Let $\mathfrak{m}$ be any maximal ideal in A and put $d = \dim(A)$. Then there is a regular sequence $(v_1, \ldots, v_d)$ such that $A/(v_1, \ldots, v_d)$ is a finite length A-module and $A_{\mathfrak{m}}/(v_1, \ldots, v_d)A_{\mathfrak{m}} = A/\mathfrak{m}$ (use [4, Corollary 2.4]). The primary decomposition

of this ideal is $(v_1, \ldots, v_d) = \mathfrak{m} \cap M_1 \cap \ldots \cap M_r$ for some $\mathfrak{m}_i$-primary ideals M_i (where $\mathfrak{m}_i$ are comaximal maximal ideals). Thus

$$A/(v_1, \ldots, v_d) \simeq A/\mathfrak{m} \times A/M_1 \times \ldots \times A/M_r.$$

Let $\alpha \in (A/\mathfrak{m})^\times$. Then there exists an element $a \in A$ such that its class modulo $(v_1, \ldots, v_d)$ is $(\alpha, 1, \ldots, 1)$ under the above isomorphism. Therefore $(a, v_1, \ldots, v_d)$ is unimodular. Consider the Koszul complex $Kos(v_1, \ldots, v_d)$ associated to the regular sequence $(v_1, \ldots, v_d)$. As in section 3, we get an isomorphism

$$\psi_{v_1,\ldots,v_d} : A/(v_1, \ldots, v_d) \to \mathrm{Ext}^d_A(A/(v_1, \ldots, v_d), A)$$

defined by $\psi_{v_1,\ldots,v_d}(1) = Kos(v_1, \ldots, v_d)$. Consider $(a, \langle -\psi_{v_1,\ldots,v_d}, a\psi_{v_1,\ldots,v_d}\rangle)$ in $\bigoplus_{\mathfrak{q} \in \mathrm{Spec}(A)^{(d)}} G^1(A/\mathfrak{q})$. By construction, it vanishes outside $\mathfrak{m}$ and, as α varies, generates $G^1(A/\mathfrak{m})$ because any $(ab, \langle a\psi_{v_1,\ldots,v_d}, b\psi_{v_1,\ldots,v_d}\rangle)$ is equal to

$$(a, \langle -\psi_{v_1,\ldots,v_d}, a\psi_{v_1,\ldots,v_d}\rangle) - (-b, \langle -\psi_{v_1,\ldots,v_d}, -b\psi_{v_1,\ldots,v_d}\rangle)$$

in G^1. We have proven:

Proposition 4.3. *Let A be a regular k-algebra of dimension d. Then the homomorphism $\phi : Um_{d+1}/E_{d+1}(A) \to H^d(A, G^{d+1})$ is surjective.*

Our next goal in the next section is to show that ϕ is in fact an isomorphism when $d \geq 3$, independently of the dimension d of the algebra. The case $d = 2$ will be treated in the sequel.

4.2. **The case $d \geq 3$.** In this section, we will use results of Morel ([12]). We will have to first recall some definitions and results in $\mathbb{A}^1$-homotopy theory. Our reference here will be [13]. Consider the category Sm/k of smooth schemes over k, endowed with the Nisnevich topology. We denote by Sh the category of sheaves on Sm/k (in the Nisnevich topology) and by $\Delta^{op}Sh$ the category of simplicial sheaves over Sm/k. This category is endowed with a model structure ([13, Definition 1.2, Theorem 1.4]), and we denote by $\mathcal{H}_s(k)$ its homotopy category. If F, G are two simplicial sheaves, we denote by $\mathrm{Hom}_{\mathcal{H}_s(k)}(F, G)$ the set of homomorphisms in this category.

Let X be a smooth scheme over k and consider the simplicial sheaf $Sing^\bullet(X)$ defined at the level $n \in \mathbb{N}$ by $U \mapsto (X \times \Delta^n)(U)$ for any smooth scheme U. Here Δ^n denotes the usual n-simplex over k, i.e. $\Delta^n = \mathrm{Spec}(k[x_0, \ldots, x_n]/\sum x_i - 1)$. Observe that there is a canonical map of simplicial presheaves $X \to Sing^\bullet(X)$ (where X is seen as a simplicially constant sheaf). If moreover X is an algebraic group, then the above map is a map of simplicial sheaves of groups.

For any simplicial sheaf F, there exists a fibrant simplicial sheaf RF and a trivial cofibration $F \to RF$. Such an association can be done functorially. If X is a smooth scheme, then $\mathrm{Hom}_{\mathcal{H}_s(k)}(X, F) = \pi_0(RF(X))$ by definition. One of the results of [12] is that the map of simplicial sheaves $\mathbb{G}L_n \to Sing^\bullet\mathbb{G}L_n$ induces an isomorphism $GL_n(A)/E_n(A) \to \mathrm{Hom}_{\mathcal{H}_s(k)}(A, Sing^\bullet\mathbb{G}L_n)$ for $n \geq 3$. The idea is to show that the map $Sing^\bullet\mathbb{G}L_n \to RSing^\bullet\mathbb{G}L_n$ induces for any affine smooth scheme $\mathrm{Spec}(A)$ a weak-equivalence of simplicial sets $(Sing^\bullet\mathbb{G}L_n)(A) \to (RSing^\bullet\mathbb{G}L_n)(A)$ for $n \geq 3$. The explaination of the proof requires first a definition (see [12]).

Definition 4.4. Let F be a presheaf of simplicial sets over Sm/k.

1) We say that F satisfies the affine B.G. property in the Nisnevich topology if for any smooth k-algebra A, any étale A-algebra $A \to B$ and any $f \in A$ such that $A/f \to B/f$ is an isomorphism, the diagram

$$\begin{array}{ccc} F(A) & \longrightarrow & F(B) \\ \downarrow & & \downarrow \\ F(A_f) & \longrightarrow & F(B_f) \end{array}$$

is homotopy cartesian.
2) We say that F satisfies the $\mathbb{A}^1$-invariance property if for any smooth k-algebra A the map $F(A) \to F(A[t])$ induced by the inclusion $A \to A[t]$ is a weak equivalence.

The following theorem is a particular case of a theorem proved by Morel. Its proof is done in [12].

Theorem 4.5. *Let k be a perfect field. Let F be a simplicial sheaf of groups on Sm/k (for the Nisnevich topology). Suppose that F satisfies the affine B.G. property in the Nisnevich topology and the $\mathbb{A}^1$-invariance property. Then for any smooth k-algebra A the map $F(A) \to RF(A)$ is a weak equivalence.*

Corollary 4.6. *Let k be a perfect field and let A be a smooth k-algebra. Then the map of simplicial sheaves $\mathbb{G}L_n \to Sing^\bullet \mathbb{G}L_n$ induces an isomorphism*

$$GL_n(A)/E_n(A) \to \mathrm{Hom}_{\mathcal{H}_s(k)}(A, Sing^\bullet \mathbb{G}L_n)$$

for $n \geq 3$.

Proof. We first prove that $Sing^\bullet \mathbb{G}L_n$ satisfies the properties of Definition 4.4. If F is any sheaf on Sm/k, then it is not hard to see that $Sing^\bullet F$ is $\mathbb{A}^1$-invariant (see [12]). The affine B.G. property is also proven in [12] and requires $n \geq 3$. Theorem 4.5 shows then that $(Sing^\bullet \mathbb{G}L_n)(A)$ is weak-equivalent to $(RSing^\bullet \mathbb{G}L_n)(A)$. Therefore $\pi_0((Sing^\bullet \mathbb{G}L_n)(A)) \simeq \pi_0((RSing^\bullet \mathbb{G}L_n)(A)$. The left-hand term is just $GL_n(A)/E_n(A)$ by Theorem 2.1 and the other term is $\mathrm{Hom}_{\mathcal{H}_s(k)}(A, Sing^\bullet \mathbb{G}L_n)$ by definition.

□

Let now $\mathcal{H}(k)$ be the $\mathbb{A}^1$-homotopy category of smooth schemes over k. It can be seen as the full subcategory of $\mathbb{A}^1$-local objects in $\mathcal{H}_s(k)$ ([13, Theorem 3.2]). It turns out that $Sing^\bullet \mathbb{G}L_n$ is $\mathbb{A}^1$-local for $n \neq 2$. So $\mathrm{Hom}_{\mathcal{H}_s(k)}(A, Sing^\bullet \mathbb{G}L_n) = \mathrm{Hom}_{\mathcal{H}(k)}(A, Sing^\bullet \mathbb{G}L_n)$.

Consider the (pointed) map of simplicial sheaves $Sing^\bullet \mathbb{G}L_n \to Sing^\bullet \mathbb{G}L_{n+1}$ induced by the inclusion $\mathbb{G}L_n \to \mathbb{G}L_{n+1}$ sending M to $\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$. It is a cofibration whose cofiber is $Sing^\bullet \mathbb{G}L_{n+1}/Sing^\bullet \mathbb{G}L_n$, and it is not hard to see that the latter is isomorphic to $Sing^\bullet(\mathbb{G}L_{n+1}/\mathbb{G}L_n)$. Moreover, the map of simplicial sheaves $Sing^\bullet(\mathbb{G}L_{n+1}/\mathbb{G}L_n) \to Sing^\bullet(\mathbb{A}^{n+1} \setminus \{0\})$ is a weak equivalence in $\mathcal{H}(k)$ and the following sequence

$$Sing^\bullet \mathbb{G}L_n \longrightarrow Sing^\bullet \mathbb{G}L_{n+1} \longrightarrow Sing^\bullet(\mathbb{A}^{n+1} \setminus \{0\})$$

is a fibration sequence in $\mathcal{H}(k)$ ([12]). This is one of the ingredients of the proof of the following theorem of Morel ([12] again):

Theorem 4.7. *Let A be a smooth k-algebra and let $n \geq 3$. Suppose that A is of dimension $d \leq n$. Then the natural map $\mathrm{Hom}_{\mathcal{H}(k)}(A, \mathbb{A}^{n+1} \setminus \{0\}) \to H^n(A, G^{n+1})$ is a bijection. This induces a bijection between the set of stably free modules of rank n and $H^n(A, G^{n+1})/GL_{n+1}(A)$.*

Remark 4.8. Remark that if $d < n$ then the set of stably free modules of rank n and $H^n(A, G^{n+1})$ are both trivial.

This allows to prove the following theorem:

Theorem 4.9. *Let A be a smooth k-algebra of dimension d. Suppose that k is perfect. Then the map $\phi : Um_{d+1}(A)/E_{d+1}(A) \to H^d(A, G^{d+1})$ is an isomorphism for $d \geq 3$.*

Proof. By Theorem 2.4, there is an exact sequence of groups

$$GL_d(A)/E_d(A) \to GL_{d+1}(A)/E_{d+1}(A) \to Um_{d+1}(A)/E_{d+1}(A) \to Um_{d+1}(A)/GL_{d+1}(A) \to 0.$$

Because

$$Sing^\bullet \mathbb{G}L_d \longrightarrow Sing^\bullet \mathbb{G}L_{d+1} \longrightarrow Sing^\bullet(\mathbb{A}^{d+1} \setminus \{0\})$$

is a fibration sequence and Theorem 4.7, we have an exact sequence

$$\mathrm{Hom}_{\mathcal{H}(k)}(A, Sing^\bullet \mathbb{G}L_d) \to \mathrm{Hom}_{\mathcal{H}(k)}(A, Sing^\bullet \mathbb{G}L_{d+1}) \to H^d(A, G^{d+1}) \to H^d(A, G^{d+1})/GL_{d+1}(A) \to 0$$

Using the definition of ϕ, as well as Corollary 4.6, we get a commutative diagram

$$\begin{tikzcd}
GL_d(A)/E_d(A) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{H}(k)}(A, Sing^\bullet \mathbb{G}L_d) \arrow[d] \\
GL_{d+1}(A)/E_{d+1}(A) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{H}(k)}(A, Sing^\bullet \mathbb{G}L_{d+1}) \arrow[d] \\
Um_{d+1}(A)/E_{d+1}(A) \arrow[r] \arrow[d] & H^d(A, G^{d+1}) \arrow[d] \\
Um_{d+1}(A)/GL_{d+1}(A) \arrow[r] \arrow[d] & H^d(A, G^{d+1})/GL_{d+1}(A) \arrow[d] \\
0 & 0
\end{tikzcd}$$

We conclude by applying Theorem 4.7 again.

□

Remark 4.10. As in the previous theorem, remark that if $n > d$, then $H^n(A, G^{n+1})$ and $Um_{n+1}(A)/E_{n+1}(A)$ are both trivial.

4.3. **The case $d = 2$.** We first recall some definitions. Let A be a k-algebra of dimension d, where k is of characteristic 0. Then one can define the *Euler class group* $E(A)$ of A ([3, §4]) and the *weak Euler class group* $E_0(A)$ of A ([3, §6]). In short, $E(A)$ is the group generated by pairs (J, ω_J), where $J \subset A$ is an ideal of height d such that J/J^2 is generated by d elements and ω_J is an equivalent class of surjections $(A/J)^d \to J/J^2$, modulo relations similiar to rational equivalence.

The group $E_0(A)$ is generated by elements (J), where J is an ideal of height d as above. There is a natural surjection $E(A) \to E_0(A)$. If d is even, there is an exact sequence ([3, Theorem 7.6])

$$Um_{d+1}(A)/SL_{d+1}(A) \xrightarrow{\psi} E(A) \longrightarrow E_0(A) \longrightarrow 0$$

where ψ is defined as follows:

Let $(a_1, \dots, a_{d+1})$ be a unimodular row. By performing if necessary elementary operations, we can suppose that the ideal $J = (a_2, \dots, a_{d+1})$ is of height d. Let $e_2, \dots, e_{d+1}$ be a basis of $(A/J)^d$ and let $\omega_J : (A/J)^d \to J/J^2$ be the surjection defined by $\omega_J(e_i) = a_i$ for any i. Because $(a_1, \dots, a_{d+1})$ is unimodular and $(a_2, \dots, a_{d+1})$ is of height d, $a_1 \in (A/J)^\times$ and we can define ψ by $\psi(a_1, \dots, a_{d+1}) = (J, a_1\omega_J)$ in $E(A)$. The proof that this is well defined is done in [3, §7] and this is where we need that A contains $\mathbb{Q}$.

Suppose now that A is of dimension 2. Then the above sequence is exact on the left also, i.e. we have a short exact sequence ([3, Proposition 7.3, Proposition 7.5])

$$0 \longrightarrow Um_3(A)/SL_3(A) \xrightarrow{\psi} E(A) \longrightarrow E_0(A) \longrightarrow 0.$$

If A is smooth over k, then $\phi : Um_3(A)/E_3(A) \to H^2(A, G^3)$ gives a homomorphism $SL_3(A)/E_3(A) \to H^2(A, G^3)$ (after composition with the homomorphism $SL_3(A)/E_3(A) \to Um_3(A)/E_3(A)$).

Theorem 4.11. *Let A be a smooth k-algebra of dimension 2, where k is a field of characteristic 0. The homomorphism ϕ induces an isomorphism*

$$\overline{\phi} : Um_3(A)/SL_3(A) \simeq H^2(A, G^3)/SL_3(A).$$

Proof. Observe first that $\overline{\phi}$ is surjective by Proposition 4.3. Now there are surjective homomorphisms $E(A) \to \widetilde{CH}^2(A)$ and $E_0(A) \to CH^2(A)$ ([6, Proposition 17.2.10]) making the following diagram commutative:

$$\begin{array}{ccc} E(A) & \longrightarrow & E_0(A) \\ \downarrow & & \downarrow \\ \widetilde{CH}^2(A) & \longrightarrow & CH^2(A). \end{array}$$

Because $\dim(A) = 2$, the homomorphism $E(A) \to \widetilde{CH}^2(A)$ is an isomorphism ([6, Theorem 15.3.11] and [3, Theorem 7.2]). We then get a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Um_3(A)/SL_3(A) & \xrightarrow{\psi} & E(A) & \longrightarrow & E_0(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{\phi}} & & \downarrow{\scriptstyle \simeq} & & \downarrow & & \\ & & H^2(A, G^3)/SL_3(A) & \longrightarrow & \widetilde{CH}^2(A) & \longrightarrow & CH^2(A) & \longrightarrow & 0 \end{array}$$

Therefore there exists a homomorphism $f : H^2(A, G^3)/SL_3(A) \to Um_3(A)/SL_3(A)$ such that $f\overline{\phi} = Id$. So $\overline{\phi}$ is also injective.

□

5. Computations for real varieties

5.1. **Computation of $H^d(A, G^{d+j})$.** From now on, A is a smooth $\mathbb{R}$-algebra of dimension $d \geq 2$ with trivial orientation, i.e. $\omega_{A/\mathbb{R}} \simeq A$. Put $X = \mathrm{Spec}(A)$. First we compute $H^d(X, I^{d+j})$ for any $j \geq 0$.

Proposition 5.1. *For any $j \geq 0$, we have $H^d(X, I^{d+j}) \simeq \bigoplus_{C \in \mathcal{C}} \mathbb{Z}$ where $\mathcal{C}$ is the set of compact connected components of $X(\mathbb{R})$. More precisely, choose a real point x_C for any C in $\mathcal{C}$ and a generator ξ_{x_C} of $\mathrm{Ext}^d_A(\kappa(x_C), A)$. Then the generators are the classes of the forms $(\langle 1, 1\rangle)^j \cdot \xi_{x_C}$ in $I^j(\kappa(x))$.*

Proof. For $j = 0$, this is [6, Theorem 16.3.8]. We prove the result by induction on j. Consider the form $\langle 1, 1\rangle \in I(\mathbb{R})$. It can be seen as an element of $H^0(\mathbb{R}, I)$. The multiplication by this element yields a homomorphism

$$\cdot\langle 1, 1\rangle : H^d(A, I^{d+j}) \to H^d(A, I^{d+j+1}).$$

Now the homomorphism of sheaves $I^{d+j+1} \to I^{d+j}$ induces a homomorphism $H^d(A, I^{d+j+1}) \to H^d(A, I^{j+d})$. It is easy to check that the composition of these two homomorphism is the multiplication by 2 from $H^d(A, I^{j+d})$ to itself. By induction $H^d(A, I^{j+d})$ is a sum of copies of $\mathbb{Z}$, and therefore the multiplication by 2 is injective. So the homomorphism

$$\cdot\langle 1, 1\rangle : H^d(A, I^{d+j}) \to H^d(A, I^{d+j+1})$$

is injective. But the multiplication by $\langle 1, 1\rangle$ is surjective as a map from $\bigoplus_{x \in X^{(d)}} I^j(\kappa(x))$ to $\bigoplus_{x \in X^{(d)}} I^{j+1}(\kappa(x))$ because all residue fields are $\mathbb{R}$ or $\mathbb{C}$. Therefore the multiplication by $\langle 1, 1\rangle$ is also surjective on cohomology groups.

□

The next result is an obvious consequence of the proposition.

Corollary 5.2. *For any $j \geq 0$, we have $H^d(X, I^{d+j}/I^{d+j+1}) \simeq \bigoplus_{C \in \mathcal{C}} \mathbb{Z}/2\mathbb{Z}$ and an exact sequence of cohomology groups*

$$0 \longrightarrow H^d(A, I^{d+j+1}) \longrightarrow H^d(A, I^{d+j}) \longrightarrow H^d(A, I^{d+j}/I^{d+j+1}) \longrightarrow 0.$$

Next we exhibit some exact sequence which will be useful for the computation of $H^d(X, G^{d+1})$. We first prove a preliminary result. Let $f : X \otimes \mathbb{C} \to X$ be the finite morphism induced by the inclusion $\mathbb{R} \subset \mathbb{C}$. For any $j \geq 0$, it yields a morphism $f_* : H^d(X \otimes \mathbb{C}, K^M_{d+j}) \to H^d(X, K^M_{d+j})$. Moreover, the natural projection gives a homomorphism $H^d(X, K^M_{d+j}) \to H^d(X, K^M_{d+j}/2K^M_{d+j})$.

Proposition 5.3. *For any $j \geq 1$, the sequence*

$$H^d(X \otimes \mathbb{C}, K^M_{d+j}) \xrightarrow{f_*} H^d(X, K^M_{d+j}) \longrightarrow H^d(X, K^M_{d+j}/2K^M_{d+j}) \longrightarrow 0$$

is exact.

Proof. It suffices to show that the sequence of groups

$$\bigoplus_{x\in(X\otimes\mathbb{C})^{(d)}} K_j^M(\kappa(x)) \xrightarrow{f_*} \bigoplus_{y\in X^{(d)}} K_j^M(\kappa(y)) \to \bigoplus_{y\in X^{(d)}} K_j^M(\kappa(y))/2K_j^M(\kappa(y)) \to 0$$

is exact. We have two distinct cases, depending on whether y is a complex point or a real point. Suppose first that y is a complex point. Then there are two points x_1 and x_2 in $(X\otimes\mathbb{C})^{(d)}$ over y and the above sequence becomes

$$K_j^M(\mathbb{C})\oplus K_j^M(\mathbb{C}) \xrightarrow{f_*} K_j^M(\mathbb{C}) \longrightarrow K_j^M(\mathbb{C})/2K_j^M(\mathbb{C}) \longrightarrow 0$$

where f_* is just the sum (which is surjective). Since $j\geq 1$, $K_j^M(\mathbb{C})$ is 2-divisible and therefore $K_j^M(\mathbb{C})/2K_j^M(\mathbb{C})=0$.

Suppose now that y is a real point. There is only a complex point over y and the sequence becomes

$$K_j^M(\mathbb{C}) \xrightarrow{f_*} K_j^M(\mathbb{R}) \longrightarrow K_j^M(\mathbb{R})/2K_j^M(\mathbb{R}) \longrightarrow 0.$$

Here f_* is just the transfer map given by the inclusion $\mathbb{R}\subset\mathbb{C}$. But $K_j^M(\mathbb{R})$ is just the direct sum of a 2-divisible group D generated by symbols $\{a_1,\dots,a_j\}$ with $a_i>0$ and a factor $\mathbb{Z}/2\mathbb{Z}$ generated by $\{-1,\dots,-1\}$. Now f_* is surjective on D (use [10, Proposition 14.64]) and 0 on the subgroup generated by $\{-1,\dots,-1\}$ because $K_j^M(\mathbb{C})$ is 2-divisible. So the sequence is exact.

□

As a corollary, we get:

Proposition 5.4. *Let X be a real smooth affine variety with trivial canonical bundle. Then for any $j\geq 0$, the sequence*

$$H^d(X\otimes\mathbb{C},G^{d+j}) \xrightarrow{f_*} H^d(X,G^{d+j}) \longrightarrow H^d(X,I^{d+j}) \longrightarrow 0$$

is split exact, where the first homomorphism is induced by the finite morphism $f: X\otimes\mathbb{C}\to X$ and the second by the map of sheaves $G^{d+j}\to I^{d+j}$. Moreover, the morphism of sheaves $G^{d+j}\to K_{d+j}^M$ induces an isomorphism $H^d(X\otimes\mathbb{C},G^{d+j})\to H^d(X\otimes\mathbb{C},K_{d+j}^M)$.

Proof. If $j=0$, this is [6, Theorem 16.6.4] and [6, Remark 10.2.16]. We suppose now that $j\geq 1$. First observe that, since $I(\mathbb{C})=0$, we have $G^j(\mathbb{C})=K_j^M(\mathbb{C})$. This proves the last assertion of the theorem. This also proves that the composition

$$H^d(X\otimes\mathbb{C},G^{d+j}) \xrightarrow{f_*} H^d(X,G^{d+j}) \longrightarrow H^d(X,I^{d+j})$$

is zero since the groups $G^j(\kappa(x))$ are the fibre products of $K_j^M(\kappa(x))$ and $I^j(\kappa(x))$ over $I^j(\kappa(x))/I^{j+1}(\kappa(x))$ for any $x\in X^{(d)}$. Now remember that there is a commutative diagram of sheaves whose columns are exact

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
I^{d+j+1} \arrow[r, equal] \arrow[d] & I^{d+j+1} \arrow[d] \\
G^{d+j} \arrow[r] \arrow[d] & I^{d+j} \arrow[d] \\
K^M_{d+j} \arrow[r] \arrow[d] & I^{d+j}/I^{d+j+1} \arrow[d] \\
0 & 0.
\end{tikzcd}$$

This yields the following commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & H^d(X, I^{d+j+1}) \arrow[r, equal] \arrow[d] & H^d(X, I^{d+j+1}) \arrow[d] & & \\
H^d(X \otimes \mathbb{C}, G^{d+j}) \arrow[r] \arrow[d] & H^d(X, G^{d+j}) \arrow[r] \arrow[d] & H^d(X, I^{d+j}) \arrow[r] \arrow[d] & 0 & \\
H^d(X \otimes \mathbb{C}, K^M_{d+j}) \arrow[r] \arrow[d] & H^d(X, K^M_{d+j}) \arrow[r] \arrow[d] & H^d(X, I^{d+j}/I^{d+j+1}) \arrow[r] \arrow[d] & 0 & \\
0 & 0 & 0 & &
\end{tikzcd}$$

where the columns and the last line are exact. An easy chase shows that the line in the middle is also exact.

□

Next we prove that $H^d(X \otimes \mathbb{C}, K^M_{d+j}) = 0$ for some interesting algebras. Recall that a real variety X is said to be rational if $X \otimes \mathbb{C}$ is birational to $\mathbb{P}^d$.

Proposition 5.5. *Let A be a smooth $\mathbb{R}$-algebra of dimension d. Suppose that $X = \mathrm{Spec}(A)$ is rational. Then $H^d(X \otimes \mathbb{C}, K_{d+j}) = 0$ for any $j \geq 0$.*

Proof. Suppose first $j = 0$. Then $H^d(X \otimes \mathbb{C}, K^M_d) = CH^d(X \otimes \mathbb{C}) = 0$ because $X \otimes \mathbb{C}$ is rational. Using [14, Corollary 3.4] and Roitman's theorem, this shows that any maximal ideal $\mathfrak{m}$ in $A \otimes \mathbb{C}$ is complete intersection. Let $\{a_1, \ldots, a_j\}$ be an element of $K^M_j(\mathbb{C}) = K^M_j((A \otimes \mathbb{C})/\mathfrak{m})$. Let $(f_1, \ldots, f_d)$ be a regular sequence generating $\mathfrak{m}$. Consider the symbol $\{f_d, a_1, a_2, \ldots, a_j\}$ defined on the residue fields of the generic points of $(A \otimes \mathbb{C})/(f_1, \ldots, f_{d-1})$. It defines an element of $\oplus_{x \in \mathrm{Spec}(A \otimes \mathbb{C})^{(d-1)}} K^M_{j+1}(\kappa(x))$ whose boundary is $\{a_1, \ldots, a_j\}$.

□

Finally, we get:

Theorem 5.6. *Let A be a smooth $\mathbb{R}$-algebra of dimension d with trivial canonical bundle. Suppose that $X = \mathrm{Spec}(A)$ is rational. Then*

$$H^d(X, G^{d+j}) \simeq H^d(X, I^{d+j}) \simeq \bigoplus_{C \in \mathcal{C}} \mathbb{Z}$$

where $\mathcal{C}$ is the set of compact connected components of $X(\mathbb{R})$ (endowed with the Euclidian topology).

Proof. The first isomorphism is clear in view of Proposition 5.4 and Proposition 5.5. The second isomorphism is just Proposition 5.1.

□

Remark 5.7. If $d \geq 3$ this shows that $\mathrm{Hom}_{\mathbb{A}^1}(X, \mathbb{A}^{d+1} \setminus \{0\}) = Um_{d+1}(A)/E_{d+1}(A)$ (which is isomorphic to $H^d(X, G^{d+1})$) is isomorphic to the cohomotopy group $\pi^d(X(\mathbb{R}))$.

5.2. **Stably free modules.** We first recall a few things. Let A be a regular k-algebra of dimension d and let P be a projective module of rank d over A. To such a module, one can associate an Euler class $\tilde{c}_d(P)$ ([12] or [6, Chapter 13]) which satifies the following property (proven in [12] if $d \geq 4$, in [7] if $d = 3$ and in [6] if $d = 2$): $\tilde{c}_d(P) = 0$ if and only if $P \simeq Q \oplus A$.

We begin with a theorem:

Theorem 5.8. *Let A be a smooth $\mathbb{R}$-algebra of even dimension d with trivial canonical bundle. Suppose that $X = \mathrm{Spec}(A)$ is rational. Then the set of stably free modules of rank d is isomorphic to $\bigoplus_{C \in \mathcal{C}} \mathbb{Z}$, where $\mathcal{C}$ is the set of compact connected components of $X(\mathbb{R})$ (endowed with the Euclidian topology).*

Proof. By Proposition 3.1, there is an exact sequence

$$H^d(X, G^{d+1}) \longrightarrow \widetilde{CH}^d(X) \longrightarrow CH^d(X) \longrightarrow 0.$$

Theorem 5.6, shows that this sequence is exact on the left also.

Suppose that $d \geq 3$. Because of Theorem 4.9, we get a short exact sequence:

$$0 \longrightarrow Um_{d+1}(A)/E_{d+1}(A) \longrightarrow \widetilde{CH}^d(X) \longrightarrow CH^d(X) \longrightarrow 0$$

and $Um_{d+1}(A)/E_{d+1}(A) \simeq \bigoplus_{C \in \mathcal{C}} \mathbb{Z}$ by Theorem 5.6. Using [3, §7], we see that the homomorphism $Um_{d+1}(A)/E_{d+1}(A) \to \widetilde{CH}^d(X)$ associates to a stably free module P (representing a unimodular row) its Euler class. The Euler class of A^d being trivial, a unimodular row coming from GL_{d+1} has therefore image 0 in $\widetilde{CH}^d(X)$. The exact sequence above shows that $GL_{d+1}(A)$ acts trivially on $Um_{d+1}(A)/E_{d+1}(A)$. This proves the result when $d \geq 4$.

Suppose now that $d = 2$. Because of Theorem 4.11, it suffices to compute $H^2(A, G^3)/SL_3(A)$. The same argument as above shows that the action of $SL_3(A)$ on $H^2(A, G^3)$ is trivial. This concludes the proof.

□

In the same flavour, we have:

Theorem 5.9. *Let A be a smooth $\mathbb{R}$-algebra of even dimension d with trivial canonical bundle. Suppose that $X = \mathrm{Spec}(A)$ is rational. Then a stably free module of rank d over A is free if and only if its Euler class is 0.*

Proof. Again, the exact sequence

$$H^d(X, G^{d+1}) \longrightarrow \widetilde{CH}^d(X) \longrightarrow CH^d(X) \longrightarrow 0.$$

is also exact on the left by Theorem 5.6 and the homomorphism $H^d(X, G^{d+1}) \to \widetilde{CH}^d(X)$ sends a stably free module to its Euler class.

□

As an example, let S^d denote the algebraic real sphere of dimension d, i.e. $S^d = \mathrm{Spec}(\mathbb{R}[x_1, \ldots, x_{d+1}]/\sum x_i^2 - 1)$.

Corollary 5.10. *The set of stably free modules of rank $2d$ over S^{2d} is isomorphic to $\mathbb{Z}$. It is generated by the tangent bundle.*

Proof. The first statement is an obvious corollary of the above theorem. The fact that $H^{2d}(S^{2d}, G^{2d+1})$ is generated by the class of the tangent bundle is a consequence of Proposition 5.1.

□

In odd dimension, the situation is a bit more complicated as illustrated by the following result:

Proposition 5.11. *All stably free modules of top rank on S^3 and S^7 are free.*

Proof. We do the proof for S^3, the case of S^7 being similar. Theorem 5.8 shows that $Um_4(S^3)/E_4(S^3) \simeq \mathbb{Z}$ with generator the tangent bundle. It is well known that the tangent bundle over S^3 is free and therefore its associated unimodular row comes from $GL_4(S^3)$. This shows that $Um_4(S^3)/GL_4(S^3) = 0$.

□

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