DUALIZATION IN ALGEBRAIC K-THEORY AND INVARIANT e^1 OF QUADRATIC FORMS OVER SCHEMES

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Abstract. Let $(\mathcal{A}, D, \delta)$ be an exact category with a duality functor D. These data define objects:

- a Witt group $W(\mathcal{A}, D, \delta)$ consisting of isomorphism classes of symmetric bilinear forms (A, φ), i.e. of isomorphisms $\varphi : A \to DA$ which are symmetric ($D\varphi \circ \delta_A = \varphi$), modulo metabolic ones,
- and two sequences of groups

$$E^i_+(\mathcal{A}, D) = Ker(K_i(\mathcal{A}) \xrightarrow{1-D} K_i(\mathcal{A}))/Im(K_i(\mathcal{A}) \xrightarrow{1+D} K_i(\mathcal{A}))$$
$$E^i_-(\mathcal{A}, D) = Ker(K_i(\mathcal{A}) \xrightarrow{1+D} K_i(\mathcal{A}))/Im(K_i(\mathcal{A}) \xrightarrow{1-D} K_i(\mathcal{A})).$$

The natural homomorphism $e^0 : W(\mathcal{A}, D, \delta) \to E^0_+(\mathcal{A}, D)$ is a natural generalization of dimension index and has several applications for usual Witt groups of schemes.

Let $I(\mathcal{A}, D, \delta) = Ker(e^0)$. There exists a natural homomorphism $e^1 : I(\mathcal{A}, D, \delta) \to E^1_+(\mathcal{A}, D)/\mathcal{H}$ which coincides with the usual discriminant if $\mathcal{A}$ is a category of finite dimensional vector spaces over a field F with usual dualization.

To define this generalized discriminant map e^1 one needs a self-dual K-theory space, e.g. the bisimplicial set $T(\mathcal{A})$ constructed by A. Nenashev. Roughly speaking, e^1 assigns to a (A, φ) a class of the loop in $T(\mathcal{A})$, corresponding to a double 4-term exact sequence, obtained by glueing with its dual a common resolution of a form Witt equivalent to (A, φ) and a hyperbolic form. This class is well defined modulo subgroup $\mathcal{H}$ of classes assigned to pairs of metabolic forms. There is an epimorphism $E^0_-(\mathcal{A}, D) \twoheadrightarrow \mathcal{H}$. In particular $\mathcal{H} = 0$ if $E^0_-(\mathcal{A}, D) = 0$.

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1. Introduction

To obtain a proper generalization of the classical notion of the discriminant of a quadratic form

$$e^1 : W(F) \to k_1(F)$$

$$e^1(\langle a_1, a_2, \ldots, a_n\rangle) = (-1)^{n(n-1)/2} a_1 \cdot a_2 \cdot \cdots \cdot a_n \bmod F^{*2}$$

for symmetric bilinear forms over schemes, the framework of exact categories with duality seems to be the best one, as it involves K-theory.

Section 2 contains a short review of the needed results from K-theory; details are partially published ([10], [11], [12]); other will be published subsequently ([13]). The group $K_1(\mathcal{M}) = \pi_1(\Omega BQ(\mathcal{M}), 0)$ of an exact category $\mathcal{M}$ is generated by loops corresponding to short double exact sequences and the presentation of $K_1(\mathcal{M})$ by these generators and relations, due to Nenashev, is known (see section 2.3 below). In general a double exact sequence of arbitrary length defines a loop in $\Omega BQ(\mathcal{M})$ (or other K-theory space of $\mathcal{M}$). As K-theory spaces we use G-construction $G(\mathcal{M})$ and its self-dual version, denoted $T(\mathcal{M})$ here and in [13, App. B], or $W(\mathcal{M})$ in [12]. We are interested in exact categories with a duality functor D (section 2.1 below); in such a case there arises an action of the two element group $\{1, D\}$ on $K_1(\mathcal{M})$.

The main example of exact category with duality functor is the category of locally free sheaves of $\mathcal{O}_X$-modules of finite rank on a scheme X with a duality functor $D : V \mapsto V\hat{\ } \otimes L$ (L - a line bundle) with either the canonical isomorphism of V with its double dual (for symmetric L-valued forms) or the opposite of this isomorphism (for skew symmetric L-valued forms).

For each of Witt groups $W^+(X, L) = W(X, L)$, $W^-(X, L)$ of a scheme X of forms with values in a line bundle L there is a natural homomorphism

$$e^0 : W^{\pm}(X, L) \to E^0(X, L)$$

where $E^0(X, L)$ is certain subfactor of $K_0(X)$, a member of the family

$$E^n(X, L) = E^n_+(X, L),\ E^n_-(X, L)$$

of subfactors of $K_n(X)$, namely the Tate cohomology groups of $\{1, D\}$ with values in the group $K_1(X)$. (see definition 3.2 below). The homomorphism e^0 (depending on L) is induced by the forgetful functor and reduces to usual dimension index e^0 in the classical case $X = \mathrm{Spec}(F)$, F - a field of characteristic different from 2. The pull-back

$$\begin{array}{ccc} W_2(X, L) & \longrightarrow & W(X, L) \\ \downarrow & & \downarrow e^0 \\ W(X, L) & \xrightarrow{e^0} & E^0(X, L) \end{array} \quad ,$$

i.e. the set of pairs $([(V, \alpha)], [(W, \beta)])$ of Witt classes with equal values of e^0, may be parametrized by a set $\mathcal{W}(X, L)$ of pairs of exact sequences

$$\begin{cases} 0 \longleftarrow V \xleftarrow{b} B \xleftarrow{a} A \longleftarrow 0, & \alpha \\ 0 \longleftarrow W \xleftarrow[b']{} B \xleftarrow[a']{} A \longleftarrow 0, & \beta \end{cases}$$

with the same objects A, B (we call such a pair *a common resolution* of V, W) and (skew-)selfdual isomorphisms $\alpha : V \to V\hat{}\otimes L$, $\beta : W \to W\hat{}\otimes L$ representing given Witt classes:

$$\begin{cases} 0 \leftarrow V \leftarrow B \leftarrow A \leftarrow 0, & \alpha \\ 0 \leftarrow W \leftarrow B \leftarrow A \leftarrow 0, & \beta \end{cases} \longmapsto ([(V, \alpha)], [(W, \beta)])$$

(corollary 3.2). In this case we define the relative discriminant $\varepsilon^1(\alpha \div \beta) \in E^1(X, L)$ (definition 3.4) by the formula

$$\varepsilon^1(\alpha \div \beta) = \text{ class of d.e.s. } \quad DA \overset{Da}{\underset{Da'}{\Longleftarrow}} DB \overset{Db\circ\alpha\circ b}{\underset{Db'\circ\beta\circ b'}{\Longleftarrow}} B \overset{a}{\underset{a'}{\Longleftarrow}} A$$

(we say that the double exact sequence is obtained by gluing the common resolution with its L-dual along α and β). This relative discriminant map $\varepsilon^1 : \mathcal{W}(X, L) \to E^1(X, L)$ is additive:

$$\varepsilon^1((\alpha \oplus \alpha') \div (\beta \oplus \beta')) = \varepsilon^1(\alpha \div \beta) + \varepsilon^1(\alpha' \div \beta'),$$

it vanishes on pairs of equal forms:

$$\varepsilon^1(\alpha \div \alpha) = 0$$

and its value does not depend on a choice of common resolution (theorem 3.3), but need not to be constant on a class of Witt equivalence. In fact, there exist pairs of hyperbolic forms with the same values of e^0 and nontrivial relative discriminant (example 3.1).

Let $\mathcal{H}(X, L)$ be a subgroup of $E^1(X, L)$ consisting of relative discriminants of pairs of hyperbolic forms with equal values of e^0. Consider the set of common resolutions of pairs of hyperbolic forms and natural map of this set into $E^0_-(X, L)$. The relative discriminant map is constant on each fiber of this map (prop. 3.5) - if a pair of hyperbolic forms with equal values of e^0 defines trivial element of the group $E^0_-(X, L)$, then relative discriminant of such a pair is trivial. It follows that $E^0_-(X, L)$ maps onto $\mathcal{H}(X, L)$. In particular $\mathcal{H}(X, L)$ is trivial provided $E^0_-(X, L)$ is trivial.

Note that even in the case of a projective line over a field the group $\mathcal{H}(X, \mathcal{O}_X)$ is nontrivial (remark 3.1).

We define (definition 3.5 below) the first k-group of X (with respect to the dualization $D : V \mapsto \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(V, L)$) as

$$k_1(X, L) = E^1(X, L)/\mathcal{H}(X, L)$$

and the discriminant map (depending on L)

$$\begin{aligned} e^1 \quad &: \quad I(X, L) \to k_1(X, L) \\ e^1(\varphi) \quad &= \quad \varepsilon^1(\varphi \div 0) \bmod \mathcal{H}(X, L). \end{aligned}$$

where $I(X, L) = \mathrm{Ker}\ e^0$ (definition 3.6 below). In the classical case $X = \mathrm{Spec}(F)$, F - a field of characteristic different from 2, there is no nontrivial line bundles L, $I(X)$ is the fundamental ideal of Witt ring $W(X)$, $E^0_-(X) = 0$, $k_1(X) = E^1(X) = \dot{F}/\dot{F}^2$ and

$$e^1(\langle a_1, a_2, \ldots, a_{2n}\rangle) = \varepsilon^1(\langle a_1, a_2, \ldots, a_{2n}\rangle \div 0) = (-1)^{\frac{2n(2n-1)}{2}} a_1 a_2 \cdot \cdots \cdot a_n$$

is usual discriminant of a quadratic form (example 3.2 below).

The discriminant map is clearly functorial - given morphism $f : X \to Y$ of schemes, the functor f^* induces homomorphisms of all groups involved, and

$$e^1 \circ f^* = f^* \circ e^1.$$

In the particular case of a variety X over a field F such that $K_1(X) = K_1(F) \otimes_{\mathbb{Z}} K_0(X)$ E^1-groups of X are following:

$$\begin{aligned} E^1(X, L) &= k_1(F) \otimes_{\mathbb{Z}} E^0(X, L) \oplus \mu_2(F) \otimes_{\mathbb{Z}} E^0_-(X, L) \\ E^1_-(X, L) &= k_1(F) \otimes_{\mathbb{Z}} E^0_-(X, L) \oplus \mu_2(F) \otimes_{\mathbb{Z}} E^0(X, L) \end{aligned}$$

where $\mu_2(F)$ is the group of square roots of 1 in the field F.

If X is a variety over a field F (so all Witt groups of X are $W(F)$-modules), the map e^1 has following property:

$$I(F)I(X, L) \subset \text{Ker } e^1$$

(theorem 3.9 below). The framework of exact categories with duality provides an uniform notation for the cases of symmetric and skew-symmetric forms, and different line bundles L.

There are some immediate applications of relative discriminant to classes of metabolic spaces in a Grothendieck group. The discriminant map is applied to the Witt group of symmetric bilinear forms over an anisotropic conic with values in a line bundle.

The paper depends on bisimplicial computations done by Sasha Nenashev.

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paper with the map on the level of spectra, studied in [24], should be investigated. Nevertheless, the construction of e^1 given here is designed especially for explicit computations, as is shown in section 4. Moreover, there are some hopes to define higher maps e^n in a similar way.

2. Witt groups and K-theory

2.1. **Dualization and forms.** Exact categories and their higher algebraic K-theory were defined in [17] by D. Quillen as follows.

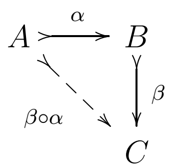
Definition 2.1. *Exact category* $\mathfrak{M} = (\mathfrak{M}, \mathfrak{A}, \mathfrak{E})$ is an additive category $\mathfrak{M}$, having a set of isomorphism classes of objects, embedded as a full subcategory of an abelian category $\mathfrak{A}$, closed under extensions in $\mathfrak{A}$, with a family $\mathfrak{E}$ of exact (in $\mathfrak{A}$) sequences

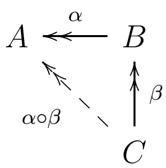
$$0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0 \tag{1}$$

(called *admissible exact sequences*) satisfying the following conditions:

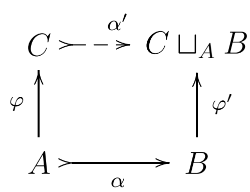
a) all split exact sequences of objects of $\mathfrak{M}$ are in $\mathfrak{E}$; if (1) is in $\mathfrak{E}$, then α is a kernel of β in $\mathfrak{M}$ and β is the cokernel of α in $\mathfrak{M}$;

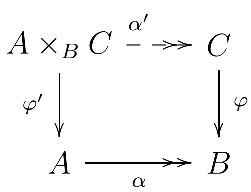
b) a composition of admissible epimorphisms (monomorphisms) is an admissible epimorphism (monomorphism);





a (co)base change of an admissible epimorphism (monomorphism) is an admissible epimorphism (monomorphism);





c) if $M \to M''$ possesses a kernel in $\mathfrak{M}$ and the composition $N \to M \to M''$ is an admissible epimorphism, then $M \to M''$ is an admissible epimorphism; the dual statement for monomorphisms holds true.

For example, given an admissible exact sequence (1) and a map $N \xrightarrow{\varphi} M''$ the sequence

$$0 \to M' \xrightarrow{\overline{\alpha}} M \times_{M''} N \xrightarrow{p_2} N \to 0$$

is an admissible exact sequence.

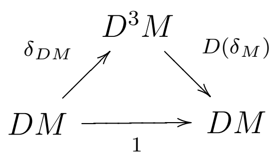
It is known now that condition c) is a consequence of a) - b) (Keller, [4, App. A]).

We will use arrows $\rightarrowtail$ and $\twoheadrightarrow$ to denote admissible monomorphisms and admissible epimorphisms respectively.

An *admissible subobject* is a kernel of an admissible epimorphism.

Recall that an *exact functor* between exact categories is an additive functor, which takes admissible exact sequences to admissible exact sequences.

Definition 2.2. A *duality* or a structure of a *Hermitian category* on $\mathfrak{M}$ (cf. [18, p. 241]) is a pair (D, δ), where the *dualization functor* $D : \mathfrak{M} \to \mathfrak{M}$ is an exact contravariant functor and $\delta : 1_{\mathfrak{M}} \to D^2$ is a natural isomorphism such that $(D\delta) \circ (\delta_D) = 1_D$, i.e. the diagram



commutes for every object M of $\mathfrak{M}$.

Note that by the definition the dual of an admissible exact sequence is an admissible exact sequence.

We focus our attention on the following three examples:

1. Let $\mathfrak{M}$ be the category of finitely generated vector spaces over some fixed field F, $DV = V^* = \mathrm{Hom}_F(V, F)$,

$$D(f)(\phi) = \phi \circ f$$

and $\delta_V : V \to D^2V = V^{**}$ is the canonical isomorphism of V with its double dual:

$$\delta_V(v)(\phi) = \phi(v)$$

for $\phi : V \to F$. The condition $(D\delta) \circ (\delta_D) = 1_D$ means that the map defined by

$$\begin{aligned} \psi &\mapsto \delta_{DV}(\psi) \\ \delta_{DV}(\psi)(\phi) &= \phi(\psi) \end{aligned}$$

composed with the $D(\delta_V)$

$$D(\delta_V)(\chi) = \chi \circ \delta_V$$

produces the map $\psi \mapsto (v \mapsto \psi(v))$ i.e. the map $\psi \mapsto \psi$. If $f : V \times V \to F$ is a symmetric bilinear form, then it is customary to factor out the kernel and deal exclusively with nonsingular symmetric bilinear form. For such a form there is isomorphism - the adjoint linear map $\phi : V \to DV$ - defined by $\phi(v)(w) = f(v, w)$. The symmetry $f(v, w) = f(w, v)$ of f is equivalent to the condition $(D\phi) \circ \delta_V = \phi$.

Changing the δ_V to the opposite to the canonical isomorphism of V with its double dual, yields that the condition $(D\phi) \circ \delta_V = \phi$ is equivalent to $f(v, w) = -f(w, v)$, i.e. gives a formalism for skew-symmetric bilinear forms.

2. Let $\mathfrak{M}$ be the category of vector bundles on a scheme X. We set

$$\begin{aligned} D\mathcal{V} &= \mathcal{V}\hat{} = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{V}, \mathcal{O}_X), \\ D\varphi &= \varphi\hat{} = - \circ \varphi : \mathcal{W}\hat{} \to \mathcal{V}\hat{} \text{ for } \varphi : \mathcal{V} \to \mathcal{W} \end{aligned}$$

and $\delta_{\mathcal{V}} : \mathcal{V} \to D^2\mathcal{V} = \mathcal{V}^{\wedge\wedge}$ is the canonical isomorphism of a $\mathcal{V}$ with its double dual. In this case we define a symmetric bilinear form as its adjoint - an isomorphism $\varphi : \mathcal{V} \to \mathcal{V}^{\wedge}$ such that $\varphi^{\wedge} \circ \delta_{\mathcal{V}} = D\varphi \circ \delta_{\mathcal{V}} = \varphi$. Note that on the fibres there arise a family of usual symmetric bilinear forms, parametrized by points of X.

Changing $\delta_{\mathcal{V}}$ to the opposite to the canonical isomorphism of $\mathcal{V}$ with its double dual yields a formalism for skew-symmetric bilinear forms.

3. Let $\mathfrak{M}$ be the category of vector bundles on a scheme X, and $\mathcal{L}$ - a line bundle on X. We set

$$\begin{aligned} D\mathcal{V} &= \mathcal{V}^{\wedge} \otimes_{\mathcal{O}_X} \mathcal{L} = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(V, \mathcal{L}), \\ D\varphi &= \varphi^{\wedge} \otimes_{\mathcal{O}_X} 1_{\mathcal{L}}, \end{aligned}$$

and

$$\delta_{\mathcal{V}} : \mathcal{V} \to D^2\mathcal{V} = (\mathcal{V}^{\wedge} \otimes_{\mathcal{O}_X} \mathcal{L})^{\wedge} \otimes_{\mathcal{O}_X} \mathcal{L}$$

- a composition of canonical isomorphisms (note that $D^2\mathcal{V} \cong \mathcal{V}^{\wedge\wedge} \otimes (\mathcal{L}^{\wedge} \otimes_{\mathcal{O}_X} \mathcal{L}) \cong \mathcal{V}^{\wedge\wedge}$). An isomorphism $\varphi : \mathcal{V} \to \mathcal{V}^{\wedge} \otimes_{\mathcal{O}_X} \mathcal{L}$ is an $\mathcal{L}$*-valued symmetric bilinear* form iff $(\varphi^{\wedge} \otimes 1_{\mathcal{L}}) \circ \delta_{\mathcal{V}} = D\varphi \circ \delta_{\mathcal{V}} = \varphi$.

Changing $\delta_{\mathcal{V}}$ to the opposite to the canonical isomorphism

$$\mathcal{V} \xrightarrow{\sim} (\mathcal{V}^{\wedge} \otimes_{\mathcal{O}_X} \mathcal{L})^{\wedge} \otimes_{\mathcal{O}_X} \mathcal{L}$$

gives a formalism for $\mathcal{L}$-valued skew-symmetric bilinear forms.

One may easily generalize concepts known for example 1 to the other two.

Definition 2.3. For a category $\mathfrak{M}$ with a duality (D, δ) a morphism $\varphi : V \to DV$ is a *self-dual morphism* iff $D\varphi \circ \delta_V = \varphi$, or the diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & DV \\ {\scriptstyle \delta_V}\downarrow & & \downarrow{\scriptstyle 1} \\ D^2V & \xrightarrow{D\varphi} & DV \end{array}$$

commutes.

A *symmetric bilinear form* (or a *symmetric bilinear space*) is a self-dual isomorphism $\varphi : V \to DV$.

Remark 2.1. If (D, δ) is a duality, then $(D, -\delta)$ is a duality. A morphism is self-dual with respect to $(D, -\delta)$ iff it is skew self-dual with respect to (D, δ).

Example 2.1. $(0, 0)$ is a symmetric bilinear form for arbitrary (D, δ).

There are obvious notions of an isomorphism of symmetric bilinear forms and of a direct sum of symmetric bilinear forms. In all three examples there is a well defined bi-exact tensor product of symmetric bilinear forms.

Definition 2.4. Let $\varphi : V \to DV$ be a symmetric bilinear form. For an admissible monomorphism $\alpha : U \rightarrowtail V$ the *orthogonal complement* $U^\perp$ of $U \overset{\alpha}{\rightarrowtail} V$ is the kernel of composition $V \xrightarrow{\varphi} DV \xrightarrow{D\alpha} DU$, so there is an exact sequence

$$0 \to U^\perp \rightarrowtail V \xrightarrow{(D\alpha)\circ\varphi} DU.$$

The kernel $U^\perp$ exists, since α is admissible.

A direct generalization of above notions to self-dual morphisms which are not isomorphisms is impossible.

Example 2.2. Let $\mathfrak{M}$ be a full subcategory of the category of finitely generated abelian groups, having free abelian groups as objects, and exact sequences of free abelian groups as admissible exact sequences, with the dualization functor

$$DA = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}).$$

The multiplication by 2

$$\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z}$$

is a self-dual map, and even is a bimorphism (a monomorphism and an epimorphism) in $\mathfrak{M}$. This is a symmetric bilinear map, which is not a form. It has a nontrivial cokernel $\mathbb{Z}/2\mathbb{Z}$ in the ambient abelian category, but the functor D is not defined on $\mathbb{Z}/2\mathbb{Z}$.

Example 2.3. Let $\varphi : V \to DV$ be a self-dual morphism. The morphism φ has a kernel in the ambient abelian category, but it needs not to have a kernel in the category $\mathfrak{M}$; even if it has a kernel in $\mathfrak{M}$, this may differ from the kernel in the ambient abelian category (e.g. consider the opposite to the category $\mathfrak{M}$ of the last example, and the same morphism), unless the kernel in the ambient abelian category is an object of $\mathfrak{M}$.

To avoid a situation described in the last example, we define symmetric bilinear maps as follows:

Definition 2.5. A symmetric bilinear map $\varphi : V \to DV$ is a self-dual morphism, which has a decomposition $\varphi = \mu \circ \eta$ with an admissible monomorphism μ and an admissible epimorphism η.

Remark 2.2. This is a terminological novelty: symmetric bilinear form is nonsingular by the definition; symmetric bilinear map may be singular.

A symmetric bilinear map φ has both the kernel and cokernel, and $\operatorname{Coker}\varphi \cong D\operatorname{Ker}\varphi$. A symmetric bilinear map is a symmetric bilinear form iff $\operatorname{Ker}\varphi = 0$. If $\varphi : V \to DV$ is a symmetric bilinear map and $\varphi = \mu \circ \eta$, then there is a commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{\eta} & W & \xrightarrow{\mu} & DV \\ {\scriptstyle \delta_V}\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \psi} & & \| \\ D^2V & \xrightarrow[D\mu]{} & DW & \xrightarrow[D\eta]{} & DV \end{array}$$

with an isomorphism ψ, which must be self-dual. Thus every symmetric bilinear map is of the form

$$\varphi = D\eta \circ \psi \circ \eta$$

for a symmetric bilinear form ψ and an admissible epimorphism η.

If $\varphi : V \to DV$ is a symmetric bilinear map with a decomposition as above, $\alpha : U \to V$ is a morphism, $\operatorname{Ker}\varphi = R \rightarrowtail V$ factors through α,

$$R \rightarrowtail V \quad = \quad R \xrightarrow{\beta} U \xrightarrow{\alpha} V$$

and $\beta : R \to U$ has a cokernel in $\mathfrak{M}$, then β is an admissible monomorphism. If, in addition, $\alpha : U \to V$ is an admissible monomorphism, $\operatorname{Coker}\beta = U \twoheadrightarrow S$, then $S \to W$ is an admissible monomorphism. Thus it has an orthogonal complement $S^\perp$ in the symmetric bilinear space (W, ψ). Then $V \times_W S^\perp$ is an object of $\mathfrak{M}$, $V \times_W S^\perp \to S^\perp$ is an admissible epimorphism, and $V \times_W S^\perp \to V$ is an admissible monomorphism. Moreover, $V \times_W S^\perp \rightarrowtail V$ is the kernel of $D\alpha \circ \varphi$ and $R \to V \times_W S^\perp$ is an admissible monomorphism.

Definition 2.6. Let $\varphi : V \to DV$ be a symmetric bilinear map. If $\alpha : U \rightarrowtail V$ is an admissible subobject such that $\operatorname{Ker}\varphi$ factors through α with cokernel in $\mathfrak{M}$, then the orthogonal complement $U^\perp \rightarrowtail V$ of $\alpha : U \rightarrowtail V$ is the kernel of $D\alpha \circ \varphi$

$$U^\perp \rightarrowtail V \qquad = \qquad \operatorname{Ker}(D\alpha \circ \varphi).$$

An orthogonal complement of an admissible subobject is itself an admissible subobject, since the sequence

$$U^\perp \rightarrowtail V \twoheadrightarrow DS$$

is admissible. Moreover, $\operatorname{Ker}\varphi$ factors through $U^\perp \rightarrowtail V$ with cokernel in $\mathfrak{M}$.

If $\varphi : V \to DV$ is a symmetric bilinear form and $\kappa : V \twoheadrightarrow V/U$ is a cokernel of an admissible suboject $j : U \rightarrowtail V$, then φ induces an isomorphism $U^\perp \cong D(V/U)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D(V/U) & \xrightarrow{D\kappa} & DV & \xrightarrow{Di} & DU & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \overline{\varphi}} & & \uparrow{\scriptstyle \varphi} & & \| & & \\
0 & \longrightarrow & U^\perp & \xrightarrow{j} & V & \xrightarrow{Di\circ\varphi} & DU & \longrightarrow & 0
\end{array} \quad .$$

Lemma 2.1. *If $i : U \to V$ is an admissible subobject such that* $\operatorname{Ker}\varphi$ *factors through U with cokernel in $\mathfrak{M}$, then $U^{\perp\perp} = U$.*

Proof. Dualization of the exact sequence

$$0 \to U^{\perp} \xrightarrow{j} V \xrightarrow{Di\circ\varphi} DU \to 0$$

yields an exact sequence

$$0 \to D^2U \xrightarrow{D\varphi\circ D^2i} DV \xrightarrow{Dj} D(U^{\perp}) \to 0.$$

Since $\operatorname{Ker}\varphi$ factors through U, it follows from the next lemma that sequences

$$\begin{array}{lll} 0 & \to & D^2U \xrightarrow{D^2i} D^2V \xrightarrow{Dj\circ D\varphi} D(U^{\perp}) \\ 0 & \to & U \xrightarrow{D^2i\circ\delta_U} D^2V \xrightarrow{Dj\circ D\varphi} D(U^{\perp}) \\ 0 & \to & U \xrightarrow{\delta_V\circ i} D^2V \xrightarrow{Dj\circ D\varphi} D(U^{\perp}) \\ 0 & \to & U \xrightarrow{i} V \xrightarrow{Dj\circ\varphi} D(U^{\perp}) \end{array}$$

are exact in ambient abelian category. Hence $U \xrightarrow{i} V = \operatorname{Ker}(Dj \circ \varphi)$, i.e. $U = U^{\perp\perp}$. □

Lemma 2.2. *Assume there are morphisms*

$$\alpha : A \to B, \quad \beta : B \to C, \quad \gamma : C \to D$$

in an abelian category. If $\operatorname{Ker}\beta$ *factors through α, then the exactness of*

$$0 \to A \xrightarrow{\beta\circ\alpha} C \xrightarrow{\gamma} D$$

yields the exactness of

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\gamma\circ\beta} D. \qquad \square$$

The subobjects $U, U^{\perp}$ of V may have various mutual positions; two partial cases are important.

Definition 2.7. Let $\varphi : V \to DV$ be a symmetric bilinear map, and $i : U \to V$ be an admissible monomorphism such that $\operatorname{Ker}\varphi$ factors through it.
1. $i : U \to V$ is *nonsingular* iff $U \times_V U^{\perp} = 0$.
2. $i : U \to V$ is *totally isotropic* (or *sublagrangian*) iff it factors through $j : U^{\perp} \to V$ (i.e. there exists an $\alpha' : U \to U^{\perp}$ such that $i = j \circ \alpha'$) and the cokernel $U^{\perp}/U$ is in $\mathfrak{M}$.

Of course, for a symmetric bilinear map $\varphi : V \to DV$ all nontrivial subobjects are nonsingular only if it is a symmetric bilinear form. Nevertheless, there are symmetric forms with singular nontrivial subobjects. If $i : U \to V$ is nonsingular, then $Di \circ \varphi \circ i : U \to DU$ is an isomorphism - an induced symmmetric bilinear form $\varphi \mid_U$.

If $\varphi : V \to DV$ is a symmetric bilinear form and $i : U \to V$ is totally isotropic, then $Di \circ \varphi$ induces an isomorphism $V/U^{\perp} \to DU$ and a morphism $V/U \to DU$. If $U \rightarrowtail V$ is totally isotropic, then $U \to U^{\perp}$ is an admissible monomorphism.

Definition 2.8. Let $\varphi : V \to DV$ be a symmetric bilinear map. An admissible monomorphism $\alpha : U \to V$ is a *Lagrangian* (or a *metabolizer*) iff it is the kernel of $D\alpha \circ \varphi$ ($D\alpha \circ \varphi \circ \alpha = 0$ and given $\beta : T \to V$ such that $D\alpha \circ \varphi \circ \beta = 0$, there exists unique $\overline{\beta} : T \to U$ such that $\alpha \circ \overline{\beta} = \beta$), i.e. the diagram

$$\begin{array}{ccccc} U & \overset{\alpha}{\rightarrowtail} & V & \xrightarrow{D\alpha\circ\varphi} & DU \\ \downarrow{\scriptstyle \delta_U} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle 1} \\ D^2U & \xrightarrow{D\varphi\circ D^2\alpha} & DV & \overset{D\alpha}{\twoheadrightarrow} & DU \end{array}$$

with exact rows commutes. A symmetric bilinear map $\varphi : V \to DV$ is *metabolic* iff it possesses a Lagrangian.

If (V, φ) is a metabolic space (φ is an isomorphism), then in addition the map $D\alpha \circ \varphi$ is an epimorphism and $D\varphi \circ D^2\alpha$ is a monomorphism. In such a case $Di \circ \varphi$ induces an isomorphism $V/U \to DU$.

Example 2.4. A *hyperbolic form*

$$\begin{bmatrix} 0 & 1_{DV} \\ \delta_V & 0 \end{bmatrix} : V \oplus DV \to DV \oplus D^2V \cong V \oplus DV$$

is a metabolic space with the Lagrangian $V \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} V \oplus DV$.

Example 2.5. For an arbitrary symmetric bilinear form (V, φ), the form

$$\begin{bmatrix} \varphi & 0 \\ 0 & -\varphi \end{bmatrix} : V \oplus V \to DV \oplus DV$$

has a Lagrangian $V \xrightarrow{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} V \oplus V$, so it is metabolic.

Example 2.6. For an arbitrary self-dual map $\alpha : U \to DU$ (i.e. $D\alpha \circ \delta_U = \alpha$) the *split metabolic space* $H(U, \alpha) = (U \oplus DU, \begin{bmatrix} \alpha & 1_{DU} \\ \delta_U & 0 \end{bmatrix})$ is metabolic, since the subobject

$$\begin{bmatrix} 0 \\ 1_{DU} \end{bmatrix} : DU \to U \oplus DU$$

is a Lagrangian:

$$\begin{aligned} \operatorname{Ker}\left(D \begin{bmatrix} 0 \\ 1_{DU} \end{bmatrix} \begin{bmatrix} \alpha & 1_{DU} \\ \delta_U & 0 \end{bmatrix} \right) &= \operatorname{Ker}\left([0, 1_{D^2U}] \begin{bmatrix} \alpha & 1_{DU} \\ \delta_U & 0 \end{bmatrix} \right) \\ &= \operatorname{Ker}\,[\delta_U, 0] = \begin{bmatrix} 0 \\ 1_{DU} \end{bmatrix}. \end{aligned}$$

Example 2.7. For a split metabolic space $H(U, \alpha) = (U \oplus DU, \begin{bmatrix} \alpha & 1_{DU} \\ \delta_U & 0 \end{bmatrix})$ and arbitrary map $\gamma : U \to DU$, there is an isomorphism $H(U, \alpha) \cong H(U, \alpha + \gamma + D\gamma \circ \delta_U)$. The isomorphism is given by

$$\begin{bmatrix} 1_U & 0 \\ \gamma & 1_{DU} \end{bmatrix} : U \oplus DU \to U \oplus DU$$

since

$$\begin{aligned}
&\left(D\begin{bmatrix} 1_U & 0 \\ \gamma & 1_{DU} \end{bmatrix}\right)\begin{bmatrix} \alpha & 1_{DU} \\ \delta_U & 0 \end{bmatrix}\begin{bmatrix} 1_U & 0 \\ \gamma & 1_{DU} \end{bmatrix} \\
=& \begin{bmatrix} 1_{DU} & D\gamma \\ 0 & 1_{D^2U} \end{bmatrix}\begin{bmatrix} \alpha & 1_{DU} \\ \delta_U & 0 \end{bmatrix}\begin{bmatrix} 1_U & 0 \\ \gamma & 1_{DU} \end{bmatrix} \\
=& \begin{bmatrix} 1_{DU} & D\gamma \\ 0 & 1_{D^2U} \end{bmatrix}\begin{bmatrix} \alpha+\gamma & 1_{DU} \\ \delta_U & 0 \end{bmatrix} = \begin{bmatrix} \alpha+\gamma+D\gamma\circ\delta_U & 1_{DU} \\ \delta_U & 0 \end{bmatrix}.
\end{aligned}$$

It follows that a split metabolic space need not be hyperbolic; it is so for symmetric bilinear forms over a field of characteristic 2.

Example 2.8. If

$$U \overset{i}{\rightarrowtail} V \overset{j}{\twoheadrightarrow} W$$

is an admissible exact sequence, then

$$\left[\begin{smallmatrix} i & 0 \\ 0 & Dj \end{smallmatrix}\right] : U \oplus DW \to V \oplus DV$$

is a Lagrangian for the hyperbolic space

$$\left(V \oplus DV, \left[\begin{smallmatrix} 0 & 1 \\ \delta_V & 0 \end{smallmatrix}\right]\right).$$

In fact,

$$\begin{aligned}
D\begin{bmatrix} i & 0 \\ 0 & Dj \end{bmatrix}\circ\begin{bmatrix} 0 & 1 \\ \delta_V & 0 \end{bmatrix} &= \begin{bmatrix} Di & 0 \\ 0 & D^2j \end{bmatrix}\circ\begin{bmatrix} 0 & 1 \\ \delta_V & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & Di \\ D^2j\circ\delta_V & 0 \end{bmatrix} = \begin{bmatrix} 0 & Di \\ \delta_W\circ j & 0 \end{bmatrix}
\end{aligned}$$

and the sequence

$$U \oplus DW \xrightarrow{\left[\begin{smallmatrix} i & 0 \\ 0 & Dj \end{smallmatrix}\right]} V \oplus DV \xrightarrow{\left[\begin{smallmatrix} 0 & Di \\ \delta_W \circ j & 0 \end{smallmatrix}\right]} DU \oplus D^2W$$

is exact, since if for any $\begin{bmatrix} f \\ g \end{bmatrix} : K \to V \oplus DV$ the equality

$$\begin{bmatrix} 0 & Di \\ \delta_W \circ j & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} Di \circ g \\ \delta_W \circ j \circ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

holds, then g factors through Ker Di, $g = Dj \circ \overline{g}$. Therefore f factors through Ker $(\delta_W \circ j) =$ Ker j, $f = i \circ \overline{f}$, so

$$\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & Dj \end{bmatrix} \begin{bmatrix} \overline{f} \\ \overline{g} \end{bmatrix}.$$

2.2. **Witt groups.**

Proposition 2.3. *Let $\varphi : V \to DV$ be a symmetric bilinear map. If $i : U \rightarrowtail V$ is totally isotropic,*

$$U \xrightarrow{i} V \qquad = \qquad U \xrightarrow{\bar{i}} U^{\perp} \xrightarrow{j} V$$

and $\kappa : U^{\perp} \to U^{\perp}/U$ is the natural map, then φ induces the unique symmetric bilinear form $\widetilde{\varphi} : U^{\perp}/U \to D(U^{\perp}/U)$ such that

$$D\kappa \circ \widetilde{\varphi} \circ \kappa = Dj \circ \varphi \circ j.$$

Proof. By the assumption there is an exact sequence

$$U^{\perp} \xrightarrow{j} V \xrightarrow{(Di)\circ\varphi} DU.$$

Hence

$$\begin{aligned} 0 &= D(Di \circ \varphi \circ j) = Dj \circ D\varphi \circ D^2 i \\ 0 &= 0 \circ \delta_U = Dj \circ D\varphi \circ D^2 i \circ \delta_U = Dj \circ D\varphi \circ \delta_V \circ i = Dj \circ \varphi \circ i \end{aligned}$$

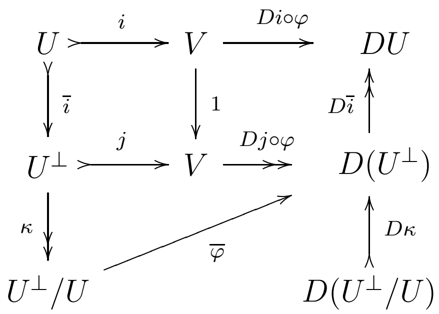
and i factors through $\mathrm{Ker}(Dj \circ \varphi)$. Moreover

$$0 = Dj \circ \varphi \circ i = Dj \circ \varphi \circ j \circ \bar{i},$$

so $\bar{i}$ factors through $\mathrm{Ker}(Dj \circ \varphi \circ j)$ and there is an induced map $\overline{\varphi} : U^\perp/U \to D(U^\perp)$ such that

$$\overline{\varphi} \circ \kappa = Dj \circ \varphi \circ j$$

The commutative diagram



has exact columns, and the induced map $\overline{\varphi} : U^\perp/U \to D(U^\perp)$ has the property

$$D\bar{i} \circ \overline{\varphi} \circ \kappa = D\bar{i} \circ Dj \circ \varphi \circ j = D(j \circ \bar{i}) \circ \varphi \circ j = Di \circ \varphi \circ j = 0,$$

which yields

$$D\bar{i} \circ \overline{\varphi} = 0,$$

since κ is an epimorphism. It follows that $\overline{\varphi}$ factors through $D\kappa : D(U^\perp/U) \to D(U^\perp)$; let $\widetilde{\varphi} : U^\perp/U \to D(U^\perp/U)$ be such that $\overline{\varphi} = D\kappa \circ \widetilde{\varphi}$. This induced map $\widetilde{\varphi}$ is the unique map which has the property that

$$D\kappa \circ \widetilde{\varphi} \circ \kappa = Dj \circ \varphi \circ j.$$

since $D\kappa$ is a monomorphism and κ is an epimorphism. It follows that $\widetilde{\varphi}$ is symmetric, since $D\widetilde{\varphi}\circ\delta_{U^\perp/U}$ has the same property:

$$\begin{aligned}
D\left(D\kappa\circ\widetilde{\varphi}\circ\kappa\right)\circ\delta_{U^\perp} &= D\left(Dj\circ\varphi\circ j\right)\circ\delta_{U^\perp}\\
D\kappa\circ D\widetilde{\varphi}\circ D^2\kappa\circ\delta_{U^\perp} &= Dj\circ D\varphi\circ D^2 j\circ\delta_{U^\perp}\\
D\kappa\circ D\widetilde{\varphi}\circ\delta_{U^\perp/U}\circ\kappa &= Dj\circ D\varphi\circ\delta_V\circ j\\
D\kappa\circ\left(D\widetilde{\varphi}\circ\delta_{U^\perp/U}\right)\circ\kappa &= Dj\circ\varphi\circ j\\
D\widetilde{\varphi}\circ\delta_{U^\perp/U} &= \widetilde{\varphi}.
\end{aligned}$$

It is easy to check that $\widetilde{\varphi}$ is an isomorphism. □

In such a case (V,φ) and $(U^\perp/U,\widetilde{\varphi})$ are said to be *directly Witt equivalent*.

Example 2.9. A metabolic form is directly Witt equivalent to $0=(0,0)$.

Example 2.10. Let $\varphi : V\to DV$ be a symmetric bilinear map with the kernel $k : R\rightarrowtail V$. Then $R^\perp = V$:

$$\begin{aligned}
\delta_V\left(R^\perp\right) &= \delta_V\left(\operatorname{Ker}\left(Dk\circ\varphi\right)\right) = \delta_V\left(\operatorname{Ker}\left(Dk\circ D\varphi\circ\delta_V\right)\right)\\
&= \operatorname{Ker}\left(Dk\circ D\varphi\right) = \operatorname{Ker}\left(D(\varphi\circ k)\right) = \operatorname{Ker} D\left(R\xrightarrow{0} DV\right)\\
&= \operatorname{Ker}\left(D^2V\xrightarrow{0} DR\right) = D^2V
\end{aligned}$$

$$R^\perp = V$$

and (V,φ) is directly Witt equivalent to $(V/R,\widetilde{\varphi})$.

In general the Witt equivalence is a transitive completion of the direct Witt equivalence.

Definition 2.9. Two symmetric bilinear forms $\varphi : V\to DV$ and $\psi : U\to DU$ are *Witt equivalent*, $(V,\varphi)\approx(U,\psi)$, iff there exist metabolic forms $\chi_1 : M_1\to DM_1$, $\chi_2 : M_2\to DM_2$ such that

$$(V,\varphi)\oplus(M_1,\chi_1)\cong(U,\psi)\oplus(M_2,\chi_2).$$

Remark 2.3. In the classical algebraic theory of quadratic forms $(V,\varphi) \cong (W,\psi)$ or $\varphi \cong \psi$ means an isomorphism, $(V,\varphi) = (W,\psi)$ or $\varphi = \psi$ means the Witt equivalence, and $\varphi \approx a$ means that φ represents a. There is no notion of representing elements in this categorial context, so we may reserve $=$ for the identity relation and $\approx$ the for Witt equivalence.

Proposition 2.4. *Two symmetric bilinear maps $\varphi : V \to DV$ and $\psi : U \to DU$ are Witt equivalent, $(V,\varphi) \approx (U,\psi)$ iff the space $(V \oplus U, \varphi \oplus (-\psi))$ is Witt equivalent to* 0.

Proof. It is obvious that if for metabolic (W,μ) the form $(V \oplus U \oplus W, \varphi \oplus (-\psi) \oplus \mu)$ is metabolic, then

$$(V,\varphi) \oplus ((U,\psi) \oplus (U,-\psi) \oplus (W,\mu)) \cong (U,\psi) \oplus (V \oplus U \oplus W, \varphi \oplus (-\psi) \oplus \mu)$$

so (V,φ) and (U,ψ) are Witt equivalent.

Conversely, if $(V,\varphi) \approx (U,\psi)$, and $(V,\varphi)\oplus(W,\mu) \cong (U,\psi)\oplus(W',\mu')$ for some metabolic (W,μ), (W',μ'), then the graph of this isomorphism is a Lagrangian in

$$(V,\varphi) \oplus (W,\mu) \oplus (U,-\psi) \oplus (W',-\mu')$$

so $(V \oplus U, \varphi \oplus (-\psi))$ is Witt equivalent to 0. □

Even for projective modules over a ring, the Witt equivalence of spaces of equal rank is weaker than an isomorphism.

Example 2.11. For $X = \mathbf{Spec}\,\mathbb{Z}$, a well known lattice Γ_8 generated by all $e_i + e_j$, and $\frac{1}{2}\sum_{i=1}^{8} e_i$ in the Euclidean space $\mathbb{R}^8$ with the usual scalar product defines a rank 8 free abelian group together with a selfdual isomorphism $\beta : \Gamma_8 \to \mathrm{Hom}(\Gamma_8, \mathbb{Z})$ which is not isomorphic to $8 \cdot \langle 1 \rangle = \langle 1,1,1,1,1,1,1,1 \rangle$ since the integer $\beta(u)(u)$ is even for arbitrary $u \in \Gamma_8$ (see [9, Chapt. 2].) Nevertheless, Γ_8 and $8 \cdot \langle 1 \rangle$ are Witt equivalent, since (Theorem 4.3 of [9, Chapt. 2]) the space $\Gamma_8 \oplus 8 \cdot \langle -1 \rangle$ is hyperbolic.

Example 2.12. For $X = \mathbf{Spec}\,\mathbb{R}[x,y]$ it is known that $W(\mathbb{R}) \to W(\mathbb{R}[x,y])$ is an isomorphism (the Karoubi theorem). Parimala in [14] produced a sequence of invertible symmetric matrices

$$S_n = \begin{bmatrix} 4+y^{2n}(1+x^2) & xy^n(1+y^{2n}) & 0 & y^n(1+x^2y^{2n}) \\ xy^n(1+y^{2n}) & 1+x^2y^{4n} & -y^n(1+x^2y^{2n}) & 0 \\ 0 & -y^n(1+x^2y^{2n}) & 4+y^{2n}(1+x^2) & xy^n(1+y^{2n}) \\ y^n(1+x^2y^{2n}) & 0 & xy^n(1+y^{2n}) & 1+x^2y^{4n} \end{bmatrix}$$

such that over $\mathbb{R}[x,y]$

(1) S_0 is congruent to the identity matrix;

(2) if $m \neq n$, then S_m is not congruent to S_n,

(3) for $n > 0$ the bilinear space $P_n = (\mathbb{R}[x,y]^4, S_n)$ is not extended from $\mathbb{R}$.

(4) for $n > 0$ the bilinear space $P_n = (\mathbb{R}[x,y]^4, S_n)$ is indecomposable (so it has no orthogonal base).

Thus for $n > 0$ the bilinear space P_n is not isomorphic to any symmetric bilinear space of dimension four extended from $\mathbb{R}$. On the other hand, over the field $\mathbb{R}(x,y)$ the space P_n is isomorphic to a totally positive space

$$\langle 1+x^2y^{4n}, \frac{4+2x^2y^{4n}+y^{2n}+y^{6n}x^4}{1+x^2y^{4n}}, \frac{4+4x^2y^{4n}}{4+2x^2y^{4n}+y^{2n}+y^{6n}x^4}, \frac{16}{4+4x^2y^{4n}}\rangle$$

and the map $W(\mathbb{R}) \cong W(\mathbb{R}[x,y]) \to W(\mathbb{R}(x,y))$ is an injection, so over $\mathbb{R}[x,y]$ this form is Witt equivalent to $4 \cdot \langle 1 \rangle$.

The symmetric bilinear space P_n extends from $\mathbf{Spec}\,\mathbb{R}[x,y] = \mathbb{A}^2_{\mathbb{R}}$ to the whole projective plane $\mathbb{P}^2_{\mathbb{R}}$ (Theorem 4.1 of [6]).

Note that a space Witt equivalent to 0 need not be metabolic, even in the exact category of finitely generated projective modules over a commutative ring.

It is easy to see that the Witt equivalence is transitive.

Proposition 2.5. *Let $\varphi : V \to DV$ be a symmetric bilinear map. If $i : U \to V$ is totally isotropic, has a decomposition*

$$U \xrightarrow{i} V = U \xrightarrow{\bar{i}} U^{\perp} \xrightarrow{j} V$$

and $\kappa : U^{\perp} \twoheadrightarrow U^{\perp}/U$*, then* (V,φ) *and* $(U^{\perp}/U, \widetilde{\varphi})$ *are Witt equivalent.*

Proof. The symmetric bilinear space $(V,\varphi) \oplus (U^{\perp}/U, -\widetilde{\varphi})$ is metabolic, since the map

$$\begin{bmatrix} j \\ \kappa \end{bmatrix} : U^{\perp} \to V \oplus U^{\perp}/U$$

coincides with its orthogonal complement: firstly

$$\begin{bmatrix} Dj \circ \varphi & -D\kappa \circ \widetilde{\varphi} \end{bmatrix} \begin{bmatrix} j \\ \kappa \end{bmatrix} = 0$$

and secondly $U^{\perp\perp} = U$, so there are exact sequences

$$U^{\perp} \overset{j}{\rightarrowtail} V \overset{Di\circ\varphi}{\twoheadrightarrow} DU \quad ;$$

$$U \overset{i}{\rightarrowtail} V \overset{Dj\circ\varphi}{\twoheadrightarrow} DU^{\perp}$$

if $\begin{bmatrix} Dj \circ \varphi & -D\kappa \circ \widetilde{\varphi} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = 0$ for a map $\begin{bmatrix} f \\ g \end{bmatrix} : K \to V \oplus U^{\perp}/U$, then

$$\begin{aligned} Dj \circ \varphi \circ f - D\kappa \circ \widetilde{\varphi} \circ g &= 0 \\ D\overline{i} \circ Dj \circ \varphi \circ f - D\overline{i} \circ D\kappa \circ \widetilde{\varphi} \circ g &= 0 \\ D(j \circ \overline{i}) \circ \varphi \circ f - D(\kappa \circ \overline{i}) \circ \widetilde{\varphi} \circ g &= 0 \\ Di \circ \varphi \circ f &= 0 \end{aligned}$$

f factors through $U^{\perp} \xrightarrow{j} V = \mathrm{Ker}\,(Di \circ \varphi)$, i.e. $f = j \circ \overline{f}$ and

$$\begin{aligned}
Dj \circ \varphi \circ f - D\kappa \circ \widetilde{\varphi} \circ g &= 0 \\
Dj \circ \varphi \circ j \circ \overline{f} - D\kappa \circ \widetilde{\varphi} \circ g &= 0 \\
D\kappa \circ \widetilde{\varphi} \circ \kappa \circ \overline{f} - D\kappa \circ \widetilde{\varphi} \circ g &= 0 \\
D\kappa \circ \widetilde{\varphi} \circ (\kappa \circ \overline{f} - g) &= 0 \\
\widetilde{\varphi} \circ (\kappa \circ \overline{f} - g) &= 0 \\
\kappa \circ \overline{f} - g &= 0 \\
g &= \kappa \circ \overline{f}
\end{aligned}$$

and $\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} j \circ \overline{f} \\ \kappa \circ \overline{f} \end{bmatrix} = \begin{bmatrix} j \\ \kappa \end{bmatrix} \circ \overline{f}$. Thus $\begin{bmatrix} j \\ \kappa \end{bmatrix} : U^{\perp} \to V \oplus U^{\perp}/U$ is the kernel of $\begin{bmatrix} Dj \circ \varphi & -D\kappa \circ \widetilde{\varphi} \end{bmatrix}$, which is the orthogonal complement of $\begin{bmatrix} j \\ \kappa \end{bmatrix} : U^{\perp} \to V \oplus U^{\perp}/U$. □

Definition 2.10. The *Witt group* $W(\mathfrak{M}, D, \delta)$ of a duality (D, δ) consists of classes of Witt equivalence of symmetric bilinear spaces with the operation $\oplus$ induced by direct sum.

In the case of vector spaces over a field F of the characteristic different from 2 every metabolic form is hyperbolic and the Witt ring of duality of example **1** is usual Witt ring $W(F)$ of the field F. In the case of example **2**, of dualization $DV = V\hat{}$ of vector bundles, there arise usual Witt ring $W(X)$ of a scheme X, introduced by Knebusch in [5]. The third example is less known (but not new - e.g. [3]) - it yields the Witt group $W(X, \mathcal{L})$ of (Witt classes of) $\mathcal{L}$-valued symmetric bilinear forms of X. Note that the usual Witt rings of Severi-Brauer varieties are known ([15]) as well as the Witt groups of $\mathcal{L}$-valued symmetric bilinear forms ([16]).

2.3. **Description of $K_1(\mathfrak{M})$.** A double short exact sequence - shortly: a d.s.e.s. - is a pair of admissible short exact sequences with the same objects:

$$\begin{pmatrix} C \overset{\beta}{\twoheadleftarrow} B \overset{\alpha}{\leftarrowtail} A \\ C \overset{\delta}{\twoheadleftarrow} B \overset{\gamma}{\leftarrowtail} A \end{pmatrix}.$$

We will abbreviate the notation for a d.s.e.s. to

$$(*) \qquad C \underset{\delta}{\overset{\beta}{\twoheadleftarrow}} B \underset{\gamma}{\overset{\alpha}{\leftarrowtail}} A$$

and refer to $C \overset{\beta}{\twoheadleftarrow} B \overset{\alpha}{\leftarrowtail} A$ as the upper short exact sequence of this d.s.e.s., and to $C \overset{\delta}{\twoheadleftarrow} B \overset{\gamma}{\leftarrowtail} A$ as the lower short exact sequence of the d.s.e.s. (*). We use a quite unusual convention: if the sequence is depicted vertically, with arrows going from top to bottom, the upper exact sequence is on the left side of the arrow, as if arrow rotated like a rigid body together with its upper and lower sides.

A d.s.e.s. (*) defines a path in the K-theory space of the category $\mathfrak{M}$, e.g. in the G-construction of $\mathfrak{M}$ from (A, A) to (B, B) and, together with the d.s.e.s's

$$A \underset{1_A}{\overset{1_A}{\twoheadleftarrow}} A \leftarrowtail 0\,, \qquad B \underset{1_B}{\overset{1_B}{\twoheadleftarrow}} B \leftarrowtail 0$$

a loop $\ell = \ell(\alpha, \gamma; \beta, \delta)$

$$\ell : \quad \begin{array}{ccc} (A, A) & \xrightarrow{(*)} & (B, B) \\ & \nwarrow \quad \nearrow & \\ & (0, 0) & \end{array} .$$

Theorem 2.6. *$K_1(\mathfrak{M})$ may be described as follows.*

(a) *Every element of $K_1(\mathfrak{M})$ is represented by the loop $\ell = \ell(\alpha, \gamma; \beta, \delta)$ of a d.s.e.s.;*

(b) *$K_1(\mathfrak{M})$ is an abelian group generated by all d.s.e.s. in $\mathfrak{M}$, subject to relations:*

(i) *The class of (the loop of) the d.s.e.s. with equal the upper and the lower short exact sequences is zero;*

(ii) (3×3-lemma) *for any diagram of six d.s.e.s.*

$$\begin{array}{ccccc}
A'' & \underset{b'}{\overset{b}{\twoheadleftarrow}} & A & \underset{a'}{\overset{a}{\leftarrowtail}} & A' \\
{\scriptstyle i}\downarrow{\scriptstyle i'} & & {\scriptstyle h}\downarrow{\scriptstyle h'} & & {\scriptstyle g}\downarrow{\scriptstyle g'} \\
B'' & \underset{d'}{\overset{d}{\twoheadleftarrow}} & B & \underset{c'}{\overset{c}{\leftarrowtail}} & B' \\
{\scriptstyle l}\downarrow{\scriptstyle l'} & & {\scriptstyle k}\downarrow{\scriptstyle k'} & & {\scriptstyle j}\downarrow{\scriptstyle j'} \\
C'' & \underset{f'}{\overset{f}{\twoheadleftarrow}} & C & \underset{e'}{\overset{e}{\leftarrowtail}} & C'
\end{array}$$

such that the diagram of upper short exact sequences commutes, and the diagram of lower exact sequences commutes, the alternate sums of rows and columns coincide:

$$\ell(a, a'; b, b') - \ell(c, c'; d, d') + \ell(e, e'; f, f')$$
$$= \ell(g, g'; j, j') - \ell(h, h'; k, k') + \ell(i, i'; l, l').$$

Proof. (a) - [10, Theorem 2.1]; (b) - [11, Theorem]. □

Given an object A of $\mathfrak{M}$ and $\alpha \in \mathrm{Aut}(A)$ we put

$$\ell(\alpha) = \ell(0 \to A, 0 \to A; A \xrightarrow{1} A, A \xrightarrow{\alpha} A).$$

Remark 2.4. In fact, a pair $\alpha, \alpha' \in \mathrm{Aut}(A)$ gives rise to the two double short exact sequences:

$$A \underset{\alpha'}{\overset{\alpha}{\twoheadleftarrow}} A \leftarrowtail 0 \tag{2}$$

and

$$0 \twoheadleftarrow A \underset{\alpha'}{\overset{\alpha}{\leftarrowtail}} A \tag{3}$$

which classes are opposite to each other in $K_1(\mathfrak{M})$. There is a choice of signs: we choose the class of $(\alpha')^{-1}\alpha$ in $K_1(\mathfrak{M})$ to be given by (3).

Lemma 2.7. *Given an object A of $\mathfrak{M}$,*

i) *the class $\ell(\alpha)$ of the automorphism* $\alpha = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathrm{Aut}(A \oplus A)$ *in $K_1(\mathfrak{M})$ vanishes;*

ii) *the class of the d.s.e.s of the form*

$$A \underset{[-1,0]}{\overset{[0,1]}{\twoheadleftarrow}} A \oplus A \underset{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\leftarrowtail}} A$$

vanishes in $K_1(\mathfrak{M})$.

Proof. [11, Lemma 3.2] □

2.4. **The Nenashev's K-theory space $T(\mathfrak{M})$.** Since its vital to this investigation, we restate here notions and results of [12, sect. 2.4], and an unpublished result of Nenashev, the 3×4-lemma.

Definition of the loop corresponding to a four-term double exact sequence may also be stated in terms of self-dual K-theory space $T(\mathfrak{M})$ introduced by A. Nenashev as a bisimplicial set, a mixture of $G(\mathfrak{M})$ and $G(\mathfrak{M}^{op})$: a $(p,0)$-simplex of $T(\mathfrak{M})$ is a p-simplex of $G(\mathfrak{M})$ and a $(0,q)$-simplex of $T(\mathfrak{M})$ is a q-simplex of $G(\mathfrak{M}^{op})$. Both these embedding $G(\mathfrak{M}) \to T(\mathfrak{M})$, $G(\mathfrak{M}^{op}) \to T(\mathfrak{M})$ are homotopy equivalences. More precisely $T(\mathfrak{M})$ is the result of the Nenashev mapping cone construction $C^{(1,-1)}$ applied to the square

$$\begin{array}{ccc} \mathcal{M} & \xleftarrow{\ \mathrm{Id}\ } & \mathcal{M} \\ {\scriptstyle \mathrm{diag}}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ \mathcal{M} \times \mathcal{M} & \xleftarrow{\ \mathrm{diag}\ } & \mathcal{M} \end{array}$$

(see [12, sect. 2.4]).

Let $\mathfrak{M}$ be an exact category. $T(\mathfrak{M})$ is following bisimplicial set.

A (p,q)-simplex is given by five families of objects:

$$A_{i,k},\ A'_{i,k},\ A_{j/i,k},\ A_{i,l/k},\ A_{j/i,l/k}$$

for $i, j = 0, 1, \ldots, p,\ i < j,\ k, l = 0, 1, \ldots, q,\ k < l$ and six families of admissible short exact sequences:

$$\begin{aligned}
A_{i,k} &\rightarrowtail A_{j,k} \twoheadrightarrow A_{j/i,k},\ A_{i,l/k} \rightarrowtail A_{i,k} \twoheadrightarrow A_{i,l},\\
A'_{i,k} &\rightarrowtail A'_{j,k} \twoheadrightarrow A_{j/i,k},\ A_{i,l/k} \rightarrowtail A'_{i,k} \twoheadrightarrow A'_{i,l},\\
A_{i,l/k} &\rightarrowtail A_{j,k} \twoheadrightarrow A_{i,l},\ A_{i,l/k} \rightarrowtail A'_{j,k} \twoheadrightarrow A'_{i,l},\\
A_{i,l/k} &\rightarrowtail A_{j,l/k} \twoheadrightarrow A_{j/i,l/k},\ A_{j/i,l/k} \rightarrowtail A_{j/i,k} \twoheadrightarrow A_{j/i,l},
\end{aligned}$$

such that all diagrams

$$\begin{array}{ccccc}
A_{i,l/k} & \rightarrowtail & A_{j,l/k} & \twoheadrightarrow & A_{j/i,l/k}\\
\downarrow & & \downarrow & & \downarrow\\
A_{i,k} & \rightarrowtail & A_{j,k} & \twoheadrightarrow & A_{j/i,k}\\
\downarrow & & \downarrow & & \downarrow\\
A_{i,l} & \rightarrowtail & A_{j,l} & \twoheadrightarrow & A_{j/i,l}
\end{array}
\qquad
\begin{array}{ccccc}
A_{i,l/k} & \rightarrowtail & A_{j,l/k} & \twoheadrightarrow & A_{j/i,l/k}\\
\downarrow & & \downarrow & & \downarrow\\
A'_{i,k} & \rightarrowtail & A'_{j,k} & \twoheadrightarrow & A_{j/i,k}\\
\downarrow & & \downarrow & & \downarrow\\
A'_{i.l} & \rightarrowtail & A'_{j,l} & \twoheadrightarrow & A_{j/i,l}
\end{array}$$

commute.

In other words a (p,q)-simplex is a pair of admissible filtrations-cofiltrations

$$
(4)\quad \left(
\begin{array}{ccccccccc}
A_{0,0} & \rightarrowtail & A_{1,0} & \rightarrowtail & A_{2,0} & \rightarrowtail & \cdots & \rightarrowtail & A_{p,0} \\
\twoheaddownarrow & & \twoheaddownarrow & & \twoheaddownarrow & & & & \twoheaddownarrow \\
A_{0,1} & \rightarrowtail & A_{1,1} & \rightarrowtail & A_{2,1} & \rightarrowtail & \cdots & \rightarrowtail & A_{p,1} \\
\twoheaddownarrow & & \twoheaddownarrow & & \twoheaddownarrow & & & & \twoheaddownarrow \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\twoheaddownarrow & & \twoheaddownarrow & & \twoheaddownarrow & & & & \twoheaddownarrow \\
A_{0,q} & \rightarrowtail & A_{1,q} & \rightarrowtail & A_{2,q} & \rightarrowtail & \cdots & \rightarrowtail & A_{p,q}
\end{array}
\quad
\begin{array}{ccccccccc}
A'_{0,0} & \rightarrowtail & A'_{1,0} & \rightarrowtail & A'_{2,0} & \rightarrowtail & \cdots & \rightarrowtail & A'_{p,0} \\
\twoheaddownarrow & & \twoheaddownarrow & & \twoheaddownarrow & & & & \twoheaddownarrow \\
A'_{0,1} & \rightarrowtail & A'_{1,1} & \rightarrowtail & A'_{2,1} & \rightarrowtail & \cdots & \rightarrowtail & A'_{p,1} \\
\twoheaddownarrow & & \twoheaddownarrow & & \twoheaddownarrow & & & & \twoheaddownarrow \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\twoheaddownarrow & & \twoheaddownarrow & & \twoheaddownarrow & & & & \twoheaddownarrow \\
A'_{0,q} & \rightarrowtail & A'_{1,q} & \rightarrowtail & A'_{2,q} & \rightarrowtail & \cdots & \rightarrowtail & A'_{p,q}
\end{array}
\right)
$$

with fixed common: subquotients $A_{j/i,k}$, subkernels $A_{i,l/k}$, induced cofiltrations of subquotients and induced filtrations of subkernels.

Degeneracy maps are defined by duplicating a row or a column in both diagrams (4) and reindexing. Boundaries are defined by deleting a row or a column in both diagrams (4) and reindexing.

Obviously all $(p,0)$-simplexes for $p = 0, 1, 2, \ldots$ form a simplicial subset isomorphic to $G(\mathfrak{M})$ and all $(0,p)$-simplexes for $p = 0, 1, 2, \ldots$ form a simplicial subset isomorphic to $G(\mathfrak{M}^{op})$. Moreover, Nenashev proved following theorem.

Theorem 2.8. *The embeddings*

$$G(\mathfrak{M}) = T_{\cdot,0}(\mathfrak{M}) \hookrightarrow T(\mathfrak{M}) \textit{ and } G(\mathfrak{M}^{op}) = T_{0,\cdot}(\mathfrak{M}) \hookrightarrow T(\mathfrak{M})$$

are homotopy equivalences.

Proof. [12, Theorem 2.5]. □

Thus geometric realization of $T(\mathfrak{M})$ is a K-theory space: $K_n(\mathfrak{M}) = \pi_n(T(\mathfrak{M}))$. The main advantage of this particular K-theory space is it's self-duality. Namely, for any bisimplicial set X let $\overline{X}$ be the bisimplicial set $\overline{X}_{m,n} = X_{n,m}$. There is a canonical homeomorphism of their geometric realizations $\phi_X : |\overline{X}| \to |X|$ which takes $x \times \Delta^m \times \Delta^n$ to $x \times \Delta^n \times \Delta^m$ for any $x \in X_{n,m}$. The $T(\mathfrak{M})$ is self-dual in the sense that $T(\mathfrak{M}^{op}) = \overline{T(\mathfrak{M})}$ which yields a homeomorphism $|T(\mathfrak{M}^{op})| = \left|\overline{T(\mathfrak{M})}\right| \to |T(\mathfrak{M})|$.

A duality functor $D : \mathfrak{M} \to \mathfrak{M}^{op}$ induces a bisimplicial map $T(\mathfrak{M}) \to T(\mathfrak{M}^{op}) = \overline{T(\mathfrak{M})}$ which maps a (p,q)-simplex (4) onto (q,p)-simplex

$$\left(\begin{array}{ccccccc}
DA_{0,0} & \rightarrowtail & DA_{0,1} & \rightarrowtail & DA_{0,2} & \rightarrowtail \cdots \rightarrowtail & DA_{0,q} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
DA_{1,0} & \rightarrowtail & DA_{1,1} & \rightarrowtail & DA_{1,2} & \rightarrowtail \cdots \rightarrowtail & DA_{1,q} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
DA_{p,0} & \rightarrowtail & DA_{p,1} & \rightarrowtail & DA_{p,2} & \rightarrowtail \cdots \rightarrowtail & DA_{p,q}
\end{array}\right.$$

$$\left.\begin{array}{ccccccc}
DA'_{0,0} & \rightarrowtail & DA'_{0,1} & \rightarrowtail & DA'_{0,2} & \rightarrowtail \cdots \rightarrowtail & DA'_{0,q} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
DA'_{1,0} & \rightarrowtail & DA'_{1,1} & \rightarrowtail & DA'_{1,2} & \rightarrowtail \cdots \rightarrowtail & DA'_{1,q} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
DA'_{p,0} & \rightarrowtail & DA'_{p,1} & \rightarrowtail & DA'_{p,2} & \rightarrowtail \cdots \rightarrowtail & DA'_{p,q}
\end{array}\right)$$

with subquotients $DA_{i,l/k}$ and subkernels $DA_{j/i,k}$. Therefore $G(\mathfrak{M}) = T_{\cdot,0}(\mathfrak{M})$ is mapped onto $G(\mathfrak{M}^{op}) = T_{0,\cdot}(\mathfrak{M})$.

There is a commutative diagram of spaces

$$\begin{array}{ccccc} |T(\mathfrak{M})| & \xrightarrow{D} & \left|\overline{T(\mathfrak{M})}\right| & \xrightarrow{\phi_{T(\mathfrak{M})}} & |T(\mathfrak{M})| \\ \uparrow & & & & \uparrow \\ |G(\mathfrak{M})| & & \xrightarrow{\qquad D \qquad} & & |G(\mathfrak{M}^{op})| \end{array} \quad .$$

A vertex (i.e. a $(0,0)$-simplex) of $T(\mathfrak{M})$ is a pair of objects (P, P') of $\mathfrak{M}$. Thus, given a pair of objects, think of it as a vertex in the triangulation of the geometric realization $|T(\mathfrak{M})|$. We display this situation as $\overset{(P,P')}{\bullet}$.

Given two vertices (P_0, P_0') and (P_1, P_1'), there may generally be edges of two types connecting (P_0, P_0') to (P_1, P_1'), namely $(1,0)$- and $(0,1)$-simplices; imagine something like

$$\overset{(P_0, P_0')}{\bullet} \longrightarrow \overset{(P_1, P_1')}{\bullet}$$

in both cases. The $(1,0)$-edges are in one-to-one correspondence with the pairs of short exact sequences of the form

$$(P_0 \rightarrowtail P_1 \twoheadrightarrow P_{1/0},\ P_0' \rightarrowtail P_1' \twoheadrightarrow P_{1/0}), \tag{5}$$

and it is essential that the cokernel objects in them are identical. The $(1,0)$-simplices in $T(\mathfrak{M})$ are therefore the same as the edges in the G-construction. In fact, the $(-,0)$-part of $T(\mathfrak{M})$ is isomorphic to $G(\mathfrak{M})$, and it is how we get an embedding $G(\mathfrak{M}) \subset T(\mathfrak{M})$.

The $(0,1)$-simplices connecting (P_0, P_0') to (P_1, P_1') are given by pairs of short exact sequences

$$(P_0 \twoheadleftarrow P_1 \leftarrowtail P_{0\backslash 1},\ P_0' \twoheadleftarrow P_1' \leftarrowtail P_{0\backslash 1}) \tag{6}$$

in $\mathfrak{M}$ with identical kernel objects. This is the same as the edges in $G(\mathfrak{M}^{op})$, and in fact, $G(\mathfrak{M}^{op})$ embeds into $T(\mathfrak{M})$ as its $(0,-)$-part.

A $(1,1)$-simplex (a cell of square shape in the geometric realization) is defined by a pair of diagrams

(7)
$$
\begin{array}{ccccc}
A_{0,1/0} & \rightarrowtail & A_{1,1/0} & \twoheadrightarrow & A_{1/0,1/0} \\
\downarrow & & \downarrow & & \downarrow \\
A_{0,1} & \rightarrowtail & A_{1,1} & \twoheadrightarrow & A_{1/0,1} \\
\downarrow & & \downarrow & & \downarrow \\
A_{0,0} & \rightarrowtail & A_{1,0} & \twoheadrightarrow & A_{1/0,0}
\end{array}
\qquad
\begin{array}{ccccc}
A_{0,1/0} & \rightarrowtail & A_{1,1/0} & \twoheadrightarrow & A_{1/0,1/0} \\
\downarrow & & \downarrow & & \downarrow \\
A'_{0,1} & \rightarrowtail & A'_{1,1} & \twoheadrightarrow & A_{1/0,1} \\
\downarrow & & \downarrow & & \downarrow \\
A'_{0,0} & \rightarrowtail & A'_{1,0} & \twoheadrightarrow & A_{1/0,0}
\end{array}
$$

with identical upper horizontal and right vertical short exact sequences. The vertices of this 'square' are the pairs $(A_{i,j}, A'_{i,j})$ with $i, j \in \{0,1\}$. Its four edges are given by the corresponding pairs of short exact sequences in these diagrams.

The map $D : |T(\mathfrak{M})| \to |T(\mathfrak{M})|$ takes a vertex (P, P') to (DP, DP'). If a $(1,0)$-edge e from (P_0, P'_0) to (P_1, P'_1) is given by (5), then

$$De = (DP_0 \twoheadleftarrow DP_1 \leftarrowtail DP_{1/0},\ DP'_0 \twoheadleftarrow DP'_1 \leftarrowtail DP_{1/0}),$$

which is a $(0,1)$-simplex.

Recall that by theorem 2.6 given a double short exact sequence $u = \left(C \underset{\delta}{\overset{\beta}{\twoheadleftarrow}} B \underset{\gamma}{\overset{\alpha}{\leftarrowtail}} A \right)$, we may associate to it a 3-edge loop

$$\ell(u) = \left((0,0) \xrightarrow{e(A)} (A,A) \xrightarrow{e(u)} (B,B) \xleftarrow{e(B)} (0,0) \right)$$

in $G(\mathfrak{M}) = T_{\cdot,0}(\mathfrak{M})$, where $e(A) = \left(0 \to A \xrightarrow{1} A, 0 \to A \xrightarrow{1} A\right)$ is the canonical edge from the base point $(0,0)$ to (A,A) (the same for B), and $e(u)$ is $(1,0)$-simplex given by u.

The map $D : |T(\mathfrak{M})| \to |T(\mathfrak{M})|$ takes the loop $\ell(u)$ to the loop

$$D\ell(u) = \left((0,0) \overset{De(A)}{\dashrightarrow} (DA, DA) \overset{De(u)}{\dashrightarrow} (DB, DB) \overset{De(B)}{\dashleftarrow} (0,0) \right)$$

in $G(\mathfrak{M}^{op}) = T_{0,\cdot}(\mathfrak{M})$, where $De(A) = \left(0 \overset{0}{\leftarrow} DA \overset{1}{\leftarrow} DA, 0 \overset{0}{\leftarrow} DA \overset{1}{\leftarrow} DA\right)$ (the same for B), and

$$De(u) = \left(DA \overset{D\alpha}{\twoheadleftarrow} DB \overset{D\beta}{\leftarrowtail} DC, \qquad DA \overset{D\gamma}{\twoheadleftarrow} DB \overset{D\delta}{\leftarrowtail} DC \right).$$

There is also the dual d.s.e.s.

$$Du = \left(DA \underset{D\gamma}{\overset{D\alpha}{\twoheadleftarrow}} DB \underset{D\delta}{\overset{D\beta}{\leftarrowtail}} DC \right)$$

and its loop

$$\ell(Du) = \left(\begin{array}{ccccccc} (0,0) & \xrightarrow{e(DC)} & (DC,DC) & \xrightarrow{e(Du)} & (DB,DB) & \xleftarrow{e(DB)} & (0,0) \end{array} \right)$$

We restate here Proposition 3.1, Corollary 3.2 and Lemma 3.3 of [12]:

Proposition 2.9. *$D\ell(u)$ is homotopic to $\ell(Du)$ in $T(\mathfrak{M})$.*

Corollary 2.10. *If $m(u)$ is the class of $\ell(u)$ in $K_1(\mathfrak{M})$, then $Dm(u) = m(Du)$.*

Lemma 2.11. *For any object X in $\mathfrak{M}$ put $e^{op}(X) = \left(0 \overset{0}{\leftarrow} X \overset{1}{\leftarrow} X, 0 \overset{0}{\leftarrow} X \overset{1}{\leftarrow} X\right) \in T_{0,1}(\mathfrak{M})$. Then the loop*

$$\left(\begin{array}{ccccc} (0,0) & \xrightarrow{e(X)} & (X,X) & \xleftarrow{e^{op}(X)} & (0,0) \end{array} \right)$$

in $T(\mathfrak{M})$ is contractible.

Proof of the lemma. A contracting 2-cell is given by the $(1,1)$-simplex

$$\begin{array}{ccccc} 0 & \rightarrowtail & X & \overset{1}{\twoheadrightarrow} & X \\ \downarrow & & \downarrow 1 & & \downarrow 1 \\ 0 & \rightarrowtail & X & \overset{1}{\twoheadrightarrow} & X \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrowtail & 0 & \twoheadrightarrow & 0 \end{array} \qquad \begin{array}{ccccc} 0 & \rightarrowtail & X & \overset{1}{\twoheadrightarrow} & X \\ \downarrow & & \downarrow 1 & & \downarrow 1 \\ 0 & \rightarrowtail & X & \overset{1}{\twoheadrightarrow} & X \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrowtail & 0 & \twoheadrightarrow & 0 \end{array}$$

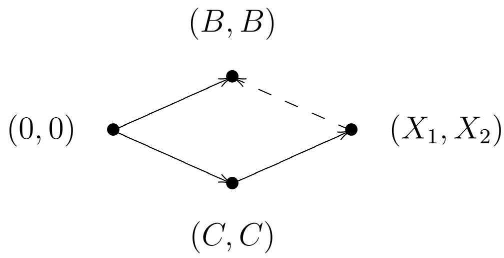
□

Proof of the proposition. By the lemma, we can replace the edges $De(A) = e^{op}(DA)$ and $De(B) = e^{op}(DB)$ in $D\ell(u)$ by $e(DA)$ and $e(DB)$ respectively, and the edge $e(DC)$ in $\ell(Du)$ by $e^{op}(DC)$. It then follows that $D\ell(u)\ell(Du)^{-1}$ is homotopic to the contour of the $(1,1)$-simplex

$$\begin{array}{ccccc} DC & \xrightarrow{1} & DC & \twoheadrightarrow & 0 \\ {\scriptstyle 1}\downarrow & & {\scriptstyle D\beta}\downarrow & & \downarrow \\ DC & \xrightarrow{D\beta} & DB & \xrightarrow{D\alpha} & DA \\ \downarrow & & {\scriptstyle D\alpha}\downarrow & & {\scriptstyle 1}\downarrow \\ 0 & \rightarrowtail & DA & \underset{1}{\twoheadrightarrow} & DA \end{array} \qquad \begin{array}{ccccc} DC & \xrightarrow{1} & DC & \twoheadrightarrow & 0 \\ {\scriptstyle 1}\downarrow & & {\scriptstyle D\delta}\downarrow & & \downarrow \\ DC & \xrightarrow{D\delta} & DB & \xrightarrow{D\gamma} & DA \\ \downarrow & & {\scriptstyle D\gamma}\downarrow & & {\scriptstyle 1}\downarrow \\ 0 & \rightarrowtail & DA & \underset{1}{\twoheadrightarrow} & DA \end{array}$$

□

Definition 2.11. Given a four-term double exact sequence $D \overset{c_1}{\underset{c_2}{\twoheadleftarrow}} C \overset{b_1}{\underset{b_2}{\leftarrow}} B \overset{a_1}{\underset{a_2}{\leftarrowtail}} A$, let $B \overset{u_1}{\twoheadrightarrow} X_1$ be $\operatorname{Coker} a_1$, $B \overset{u_2}{\twoheadrightarrow} X_2$ be $\operatorname{Coker} a_2$, $X_1 \overset{v_1}{\rightarrowtail} C$ be $\operatorname{Ker} c_1$ and $X_2 \overset{v_2}{\rightarrowtail} C$ be $\operatorname{Ker} c_2$. A corresponding element $\ell\left(D \overset{c_1}{\underset{c_2}{\twoheadleftarrow}} C \overset{b_1}{\underset{b_2}{\leftarrow}} B \overset{a_1}{\underset{a_2}{\leftarrowtail}} A \right)$ of $K_1(\mathfrak{M})$ is a class of the loop



consisting of paths:

- $e(B)$ from $(0,0)$ to (B,B) given by the $(1,0)$-simplex $B \overset{1}{\underset{1}{\twoheadleftarrow}} B \leftarrowtail 0$;
- from (X_1, X_2) to (B,B) given by the $(0,1)$-simplex

$$\left(\begin{array}{c} X_1 \overset{u_1}{\twoheadleftarrow} B \overset{a_1}{\leftarrowtail} A \\ X_2 \overset{u_2}{\twoheadleftarrow} B \overset{a_2}{\leftarrowtail} A \end{array} \right);$$

- from (C, C) to (X_1, X_2) given by the $(1,0)$-simplex

$$\begin{pmatrix} D \overset{c_1}{\twoheadleftarrow} C \overset{v_1}{\leftarrowtail} X_1 \\ D \overset{c_2}{\twoheadleftarrow} C \overset{v_2}{\leftarrowtail} X_2 \end{pmatrix};$$

- $e(C)$ from $(0,0)$ to (C,C) given by the $(1,0)$-simplex $C \overset{1}{\underset{1}{\twoheadleftarrow}} C \leftarrowtail 0$.

Remark 2.5. There is another choice of signs: given a double short exact sequence (*),

$$\ell(\alpha, \gamma; \beta, \delta) = \ell \left(C \overset{\beta}{\underset{\delta}{\twoheadleftarrow}} B \overset{\alpha}{\underset{\gamma}{\leftarrowtail}} A \leftarrowtail 0 \right)$$

and $\ell(\alpha, \gamma; \beta, \delta)$ is opposite to $\ell \left(0 \twoheadleftarrow C \overset{\beta}{\underset{\delta}{\twoheadleftarrow}} B \overset{\alpha}{\underset{\gamma}{\leftarrowtail}} A \right)$.

Here are some standard ways to replace a pair of adjoint edges in a combinatorial path by another pair of edges and get a homotopic path as a result.

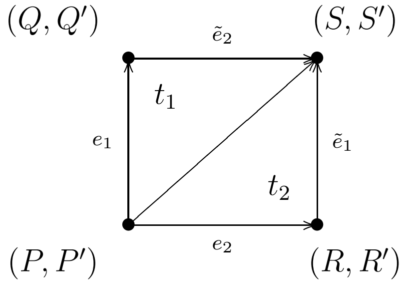
- Suppose we are given a pair of $(1,0)$-simplices sharing the source:

(8)
$$(Q, Q') \overset{e_1}{\longleftarrow} (P, P') \overset{e_2}{\longrightarrow} (R, R')$$

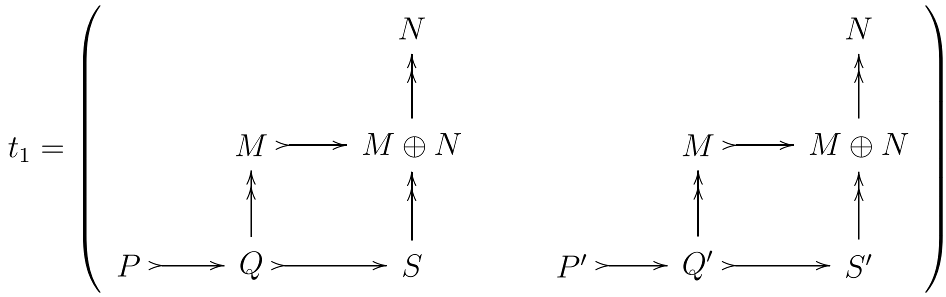
Then e_1 and e_2 are given by data of the form

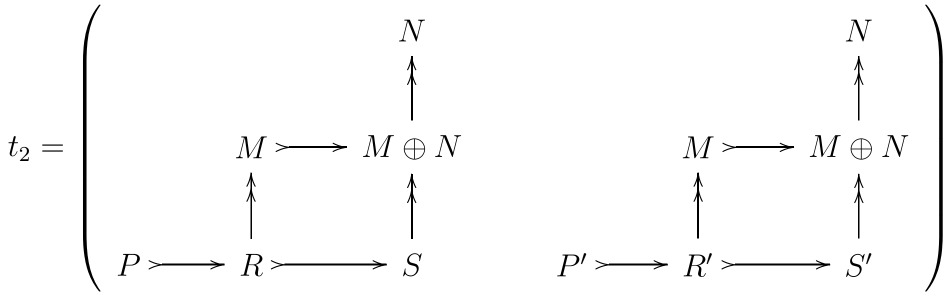
$$e_1 = \quad (P \rightarrowtail Q \twoheadrightarrow M,\ P' \rightarrowtail Q' \twoheadrightarrow M)$$
$$e_2 = \quad (P \rightarrowtail R \twoheadrightarrow N,\ P' \rightarrowtail R' \twoheadrightarrow N)$$

Following [2] we choose push out objects $S = Q\coprod_P R$, $S' = Q'\coprod_{P'} R'$ and consider the two $(2,0)$-simplices

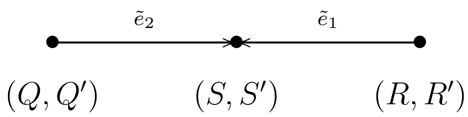


given by the diagrams





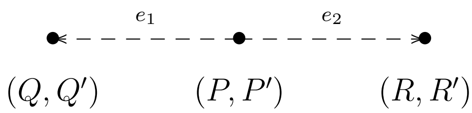
This enables us to replace the 2-edge path (8) by the homotopic path



Note that everything here is happening in the $G(\mathfrak{M})$-part of $T(\mathcal{M})$.

- Given a pair of $(0,1)$-simplices with a common source

(9)



$$e_1 = \quad (P \twoheadleftarrow Q \leftarrowtail M,\ P' \twoheadleftarrow Q' \leftarrowtail M')$$

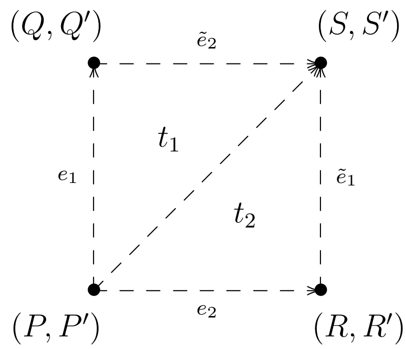
$$e_2 = \quad (P \twoheadleftarrow Q \leftarrowtail N,\ P' \twoheadleftarrow Q' \leftarrowtail N')$$

take the pullbacks $S = Q \underset{P}{\times} R,\ S' = Q' \underset{P'}{\times} R'$ and form the two $(0,2)$-simplices

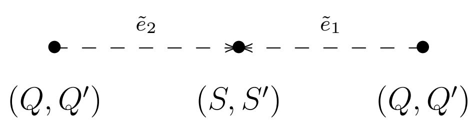
$$t_1 = \left(\begin{array}{ccccc} & & & & N \\ & & & & \downarrow \\ & & M & \twoheadleftarrow & M \oplus N \\ & & \downarrow & & \downarrow \\ P & \twoheadleftarrow & Q & \twoheadleftarrow & S \end{array} \qquad \begin{array}{ccccc} & & & & N \\ & & & & \downarrow \\ & & M & \twoheadleftarrow & M \oplus N \\ & & \downarrow & & \downarrow \\ P' & \twoheadleftarrow & Q' & \twoheadleftarrow & S' \end{array} \right)$$

$$t_2 = \left(\begin{array}{ccccc} & & & & N \\ & & & & \downarrow \\ & & M & \twoheadleftarrow & M \oplus N \\ & & \downarrow & & \downarrow \\ P & \twoheadleftarrow & R & \twoheadleftarrow & S \end{array} \qquad \begin{array}{ccccc} & & & & N \\ & & & & \downarrow \\ & & M & \twoheadleftarrow & M \oplus N \\ & & \downarrow & & \downarrow \\ P' & \twoheadleftarrow & R' & \twoheadleftarrow & S' \end{array} \right).$$

The whole picture has the form



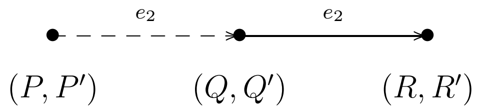
It follows that the path



is homotopic to the given one. Everything here lies in the $G(\mathfrak{M}^{op})$-part of $T(\mathcal{M})$.

- Suppose we are given a $(0,1)$-simplex followed by a $(1,0)$-simplex:

(10)



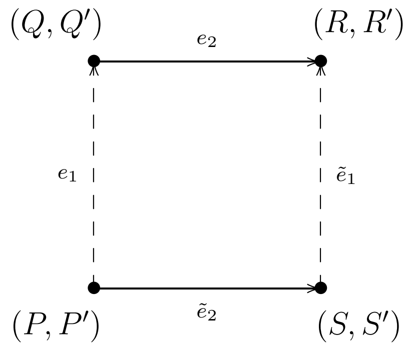
$$e_1 = (P \twoheadleftarrow Q \leftarrowtail M,\ P' \twoheadleftarrow Q' \leftarrowtail M)$$

$$e_2 = (Q \rightarrowtail R \twoheadrightarrow N,\ Q' \rightarrowtail R' \twoheadrightarrow N)$$

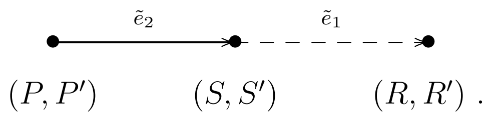
Choose pushout objects $S = P\coprod_Q R$, $S' = P'\coprod_{Q'} R'$ and consider the $(1,1)$-simplex given by the diagrams

$$\begin{array}{ccccc} M & \rightarrowtail & M & \twoheadrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ Q & \rightarrowtail & R & \twoheadrightarrow & N \\ \downarrow & & \downarrow & & \downarrow \\ P & \rightarrowtail & S & \twoheadrightarrow & N \end{array} \qquad \begin{array}{ccccc} M' & \rightarrowtail & M' & \twoheadrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ Q' & \rightarrowtail & R' & \twoheadrightarrow & N' \\ \downarrow & & \downarrow & & \downarrow \\ P' & \rightarrowtail & S' & \twoheadrightarrow & N' \end{array} .$$

It looks like

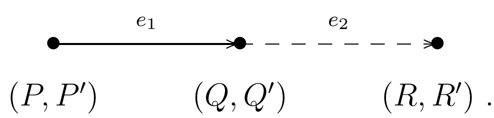


which enables us to replace the path (10) by the homotopic path



- Given a path of the form

(11)



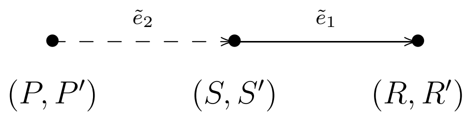
where

$$\begin{aligned} e_1 &= (P \rightarrowtail Q \twoheadrightarrow M,\ P' \rightarrowtail Q' \twoheadrightarrow M) \\ e_2 &= (Q \twoheadleftarrow R \qquad N,\ Q' \twoheadleftarrow R' \qquad N) \end{aligned}$$

choose pullbacks $S = P \underset{Q}{\times} R$ and $S' = P' \underset{Q'}{\times} R'$ and consider the $(1,1)$-simplex

$$\begin{array}{ccccc} N & \rightarrowtail & N & \twoheadrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ S & \rightarrowtail & R & \twoheadrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ P & \rightarrowtail & Q & \twoheadrightarrow & M \end{array} \qquad \begin{array}{ccccc} N' & \rightarrowtail & N' & \twoheadrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ S' & \rightarrowtail & R' & \twoheadrightarrow & M' \\ \downarrow & & \downarrow & & \downarrow \\ P' & \rightarrowtail & Q' & \twoheadrightarrow & M' \end{array} .$$

Thus we can replace (11) by



where

$$\begin{aligned} \tilde{e}_1 &= (S \rightarrowtail R \twoheadrightarrow M,\ S' \rightarrowtail R' \twoheadrightarrow M) \\ \tilde{e}_2 &= (Q \twoheadleftarrow R \leftarrowtail N,\ Q' \twoheadleftarrow R' \leftarrowtail N) . \end{aligned}$$

The main technical tool for computations with four-term double exact sequences is the following proposition proved by A. Nenashev:

Proposition 2.12 (3 × 4 lemma)**.** *Suppose we are given a diagram*

(12)
$$\begin{array}{ccccccc} D' & \overset{h_1'}{\underset{h_2'}{\twoheadleftarrow}} & C' & \overset{g_1'}{\underset{g_2'}{\longleftarrow}} & B' & \overset{f_1'}{\underset{f_2'}{\leftarrowtail}} & A' \\ d_1' \downarrow d_2' & & c_1' \downarrow c_2' & & b_1' \downarrow b_2' & & a_1' \downarrow a_2' \\ D & \overset{h_1}{\underset{h_2}{\twoheadleftarrow}} & C & \overset{g_1}{\underset{g_2}{\longleftarrow}} & B & \overset{f_1}{\underset{f_2}{\leftarrowtail}} & A \\ d_1 \downarrow d_2 & & c_1 \downarrow c_2 & & b_1 \downarrow b_2 & & a_1 \downarrow a_2 \\ D'' & \overset{h_1''}{\underset{h_2''}{\twoheadleftarrow}} & C'' & \overset{g_1''}{\underset{g_2''}{\longleftarrow}} & B'' & \overset{f_1''}{\underset{f_2''}{\leftarrowtail}} & A'' \end{array}$$

which consists of three double 4-term exact sequences and four d.s.e.s, such that each of the two diagrams with indices $i = 1$ or 2 commutes. Let q', q, and q'' denote the horizontal double four-term exact sequences and l_A, l_B, l_C, l_D denote the vertical double short exact sequences. Then the equality

$$\ell(q') - \ell(q) + \ell(q'') = -\ell(l_A) + \ell(l_B) - \ell(l_C) + \ell(l_D).$$

holds in $K_1(\mathfrak{M})$.

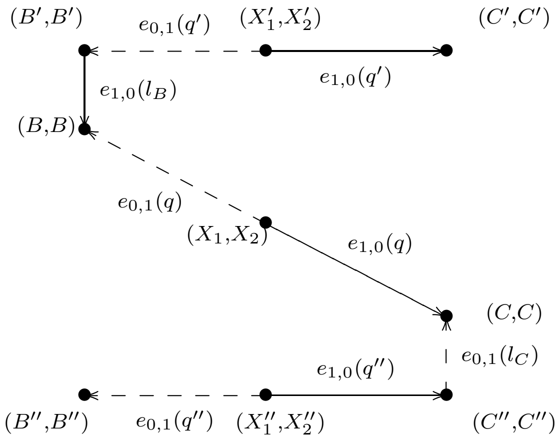
Proof. We simplify the things and do not display the unnecesary parts of loops, in the following sense. Let $\widetilde{T}(\mathcal{M})$ denote the bisimplicial subset of $T(\mathcal{M})$ which consists of diagonal bi-simplices, i.e. the bi-simplices given by pairs of identical diagrams. It is easily seen to be contractible and we can therefore compute $K_1(\mathfrak{M}) = \pi_1(T(\mathcal{M}))$ as the relative fundamental group: $\pi_1(T(\mathcal{M})) = \pi_1\left(T(\mathcal{M}), \widetilde{T}(\mathcal{M})\right)$. In this context, we can omit the side edges of the type $e(X)$ in the definition of $\ell(q)$ and $\ell(l)$ and work with relative loops; $\ell(l)$ (resp. $\ell(q)$) amounts then to the single edge $e(l)$ (resp. the pair of edges $(e_{0,1}(q), e_{1,0}(q))$).

With the notation of Definition 2.11, the given 3×4-diagram may be cut into two 3×3-diagrams:

$$\text{(13)} \qquad \begin{array}{ccccc}
D' & \overset{h'_i}{\twoheadleftarrow} & C' & \overset{v'_i}{\leftarrowtail} & X'_i \\
{\scriptstyle d'_i}\downarrow & & \downarrow{\scriptstyle c'_i} & & \downarrow{\scriptstyle x'_i} \\
D & \overset{h_i}{\twoheadleftarrow} & C & \overset{v_i}{\leftarrowtail} & X_i \\
{\scriptstyle d_i}\downarrow & & \downarrow{\scriptstyle c_i} & & \downarrow{\scriptstyle x_i} \\
D'' & \overset{h''_i}{\twoheadleftarrow} & C'' & \overset{v''_i}{\leftarrowtail} & X''_i
\end{array}
\qquad
\begin{array}{ccccc}
X'_i & \overset{u'_i}{\twoheadleftarrow} & B' & \overset{f'_i}{\leftarrowtail} & A' \\
{\scriptstyle x'_i}\downarrow & & \downarrow{\scriptstyle b'_i} & & \downarrow{\scriptstyle a'_i} \\
X_i & \overset{u_i}{\twoheadleftarrow} & B & \overset{f_i}{\leftarrowtail} & A \\
{\scriptstyle x_i}\downarrow & & \downarrow{\scriptstyle b_i} & & \downarrow{\scriptstyle a_i} \\
X''_i & \overset{u''_i}{\twoheadleftarrow} & B'' & \overset{f''_i}{\leftarrowtail} & A''
\end{array}
\qquad (i = 0, 1)$$

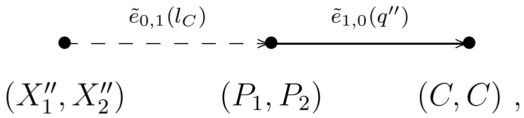
The same argument as in the proof of Proposition 2.9 lets us replace the $(1,0)$-edge $e(l_C)$ from (C', C') to (C, C) by homotopic (rel. $\widetilde{T}(\mathcal{M})$) $(0,1)$-edge from (C'', C'') to (C, C). The element $\ell(q'') + \ell(l_C) - \ell(q) - \ell(l_B) + \ell(q')$ can be thus represented by the following path

(14)



We will now construct several homotopies in order to show that the above expression equals $-\ell(l_A) + \ell(l_D)$.

1. Denote by P_i a pullback object for $X_i'' \overset{v_i''}{\rightarrowtail} C'' \overset{c_i}{\twoheadleftarrow} C$, $i = 1, 2$ and apply (11) to the edges $\bullet \xrightarrow{e_{1,0}(q'')} \bullet \overset{e_{0,1}(l_C)}{- - \twoheadrightarrow} \bullet$ in (14). We get two other edges



the homotopy being provided by the $(1,1)$-simplex

(15)
$$\begin{array}{ccccc} C' & \overset{1}{\rightarrowtail} & C' & \twoheadrightarrow & 0 \\ \downarrow{\scriptstyle \tilde{c}_i'} & & \downarrow{\scriptstyle c_i'} & & \downarrow \\ P_i & \overset{\tilde{v}_i''}{\rightarrowtail} & C & \overset{\tilde{h}_i''}{\twoheadrightarrow} & D'' \\ \downarrow{\scriptstyle \tilde{c}_i} & & \downarrow{\scriptstyle c_i} & & \downarrow{\scriptstyle 1} \\ X_i'' & \overset{v_i''}{\rightarrowtail} & C'' & \overset{h_i''}{\twoheadrightarrow} & D'' \end{array} \qquad (i = 1, 2)$$

The arrows $\tilde{c}_i'$, $\tilde{c}_i$, $\tilde{v}_i''$, and $\tilde{h}_i''$ in this diagrams are induced by the arrows c_i', c_i, v_i'', and h_i'' of (13).

2. As $c_i v_i = v_i'' x_i$, there is a unique arrow $X_i \xrightarrow{p_i} P_i$ with $\tilde{c}_i p_i = \tilde{x}_i$ and $\tilde{v}_i'' p_i = v_i$ $(i = 1, 2)$. We leave it to the reader to deduce from Quillen's third axiom of an exact category that

this arrow is an admissible monomorphism (see the example after the definition 2.1) and we have the following pair of diagrams

(16)
$$\begin{array}{ccccc}
X_i & \overset{p_i}{\rightarrowtail} & P_i & \overset{r_i}{\twoheadrightarrow} & D' \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \tilde v''_i} & & \downarrow{\scriptstyle d'_i} \\
X_i & \overset{v_i}{\rightarrowtail} & C & \overset{h_i}{\twoheadrightarrow} & D \\
\downarrow & & \downarrow{\scriptstyle d_i h_i} & & \downarrow{\scriptstyle d_i} \\
0 & \rightarrowtail & D'' & \overset{1}{\twoheadrightarrow} & D''
\end{array} \qquad (i = 1, 2)$$

The arrows r_i here are induced by the other arrows in the diagrams.

3. Denote by U_i a pushout object for $B \overset{b'_i}{\twoheadleftarrow} B' \overset{u'_i}{\rightarrowtail} X'_i$, $i = 1, 2$ and apply (10) to the edges $\bullet \overset{e_{1,0}(l_B)}{\longleftarrow} \bullet \overset{e_{0,1}(q')}{\twoheadleftarrow - - - -} \bullet$ in (14). We get then two other edges

$$\underset{(B,B)}{\bullet} \overset{\tilde e_{0,1}(q')}{\twoheadleftarrow - - - - - -} \underset{(U_1,U_2)}{\bullet} \overset{\tilde e_{1,0}(l_B)}{\longleftarrow} \underset{(X'_1,X'_2)}{\bullet} \ ,$$

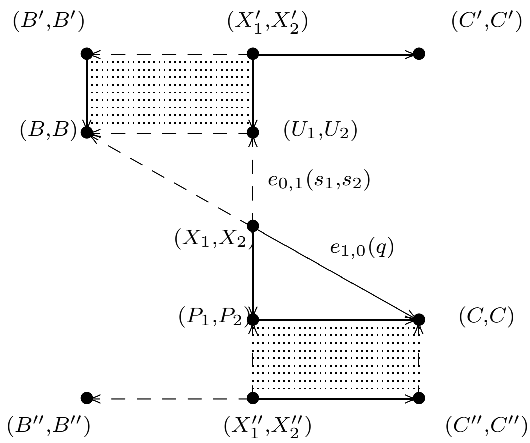
the homotopy being provided by the $(1,1)$-simplex

(17)
$$\begin{array}{ccccc}
A' & \overset{1}{\rightarrowtail} & A' & \twoheadrightarrow & 0 \\
{\scriptstyle f'_i}\downarrow & & \downarrow{\scriptstyle \tilde f'_i} & & \downarrow \\
B' & \overset{b'_i}{\rightarrowtail} & B & \overset{b''_i}{\twoheadrightarrow} & B'' \\
{\scriptstyle u'_i}\downarrow & & \downarrow{\scriptstyle \tilde u'_i} & & \downarrow{\scriptstyle 1} \\
X'_i & \overset{\tilde b'_i}{\rightarrowtail} & U_i & \overset{\tilde b''_i}{\twoheadrightarrow} & B''
\end{array} \qquad (i = 1, 2)$$

4. Dualizing 2, we get admissible epimorphisms $U_i \overset{s_i}{\twoheadrightarrow} X_i$ with $s_i\tilde{u}'_i = u_i$ and $s_i\tilde{b}'_i = x'_i$ $(i = 1, 2)$ and the diagrams

(18)
$$\begin{array}{ccccc}
A' & \overset{1}{\rightarrowtail} & A' & \twoheadrightarrow & 0 \\
{\scriptstyle a'_i}\downarrow & & \downarrow{\scriptstyle f_i a'_i} & & \downarrow \\
A & \overset{f_i}{\rightarrowtail} & B & \overset{u_i}{\twoheadrightarrow} & X_i \\
{\scriptstyle a''_i}\downarrow & & \downarrow{\scriptstyle \tilde{u}'_i} & & \downarrow{\scriptstyle 1} \\
A'' & \overset{t_i}{\rightarrowtail} & U_i & \overset{s_i}{\twoheadrightarrow} & X_i
\end{array} \qquad (i = 1, 2)$$

The arrows t_i here are induced by the other arrows in these diagrams. This results with the following figure



in which:

- the horizontal arrow from (P_1, P_2) to (C, C) is $\tilde{e}_{1,0}(q'')$;
- the horizontal arrow from (X''_1, X''_2) to (C'', C'') is $e_{1,0}(q'')$;
- the vertical arrow from (X_1, X_2) to (P_1, P_2) is

$$e_{1,0}(p_1, p_2) = \qquad (X_1 \overset{p_1}{\rightarrowtail} P_1 \overset{r_1}{\twoheadrightarrow} D' \ , \ X_2 \overset{p_2}{\rightarrowtail} P_2 \overset{r_2}{\twoheadrightarrow} D') \ ,$$

and the shadowed areas are $(1, 1)$-simplices. The triangular contour with vertices $(X_1, X_2), (P_1, P_2), (C, C)$ is an admissible triple of edges in the G-part of $T(\mathcal{M})$, in the sense of [11, Definition, p. 207]. By [11, Prop. 4.5], the diagrams (16) show that the associated d.s.e.s. is l_D. Dually, the triangular contour with vertices $(X_1, X_2), (U_1, U_2), (B, B)$ is a (co)admissible triple in the G^{op}-part (we leave it to the reader to dualize the definition and related arguments), the associated d.s.e.s. being l_A. Thus by [11, Prop. 4.5] and its dual version, the path

$$(19) \qquad \underset{(B'',B'')}{\bullet} \xleftarrow{e_{0,1}(q'')} \underset{(X_1'',X_2'')}{\bullet} \xrightarrow{\tilde{e}_{1,0}(l_C)} \underset{(P_1,P_2)}{\bullet} \xleftarrow{e_{1,0}(p_1,p_2)} \underset{(X_1,X_2)}{\bullet} \xrightarrow{e_{0,1}(s_1,s_2)} \underset{(U_1,U_2)}{\bullet} \xleftarrow{\tilde{e}_{1,0}(l_B)} \underset{(X_1',X_2')}{\bullet} \xrightarrow{e_{1,0}(q')} \underset{(C',C')}{\bullet}$$

represents the element $\ell(v') - \ell(v) + \ell(v'') + \ell(l_C) - \ell(l_B) + \ell(l_A) - \ell(l_D)$ in $\pi_1(T(\mathcal{M}), \widetilde{T}(\mathcal{M}))$, and it remains to show that this path is contractible.

5. We leave it to the reader to check the following elementary assertion

Lemma 2.13. *In an abelian category, given a commutative diagram*

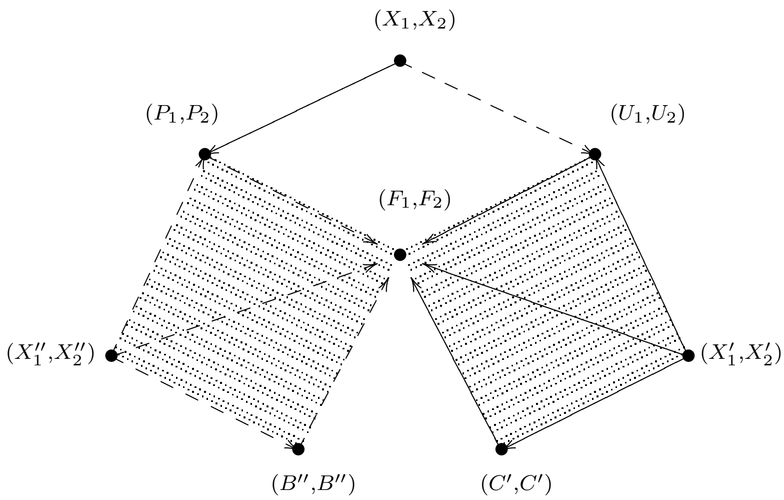
$$\begin{array}{ccccc} B' & \rightarrowtail & B & \twoheadrightarrow & B'' \\ \downarrow & & \downarrow & & \downarrow \\ C' & \rightarrowtail & C & \twoheadrightarrow & C'' \end{array}$$

whose rows are s.e.s's, there is a natural map $C' \amalg_{B'} B \to C \times_{C''} B''$ and this map is an isomorphism. ☐

Let F_i, $i = 1, 2$ denote a choice for the pullback / pushout object of the lemma applied to the i-th arrows in the B-C-part of diagram (12). Then

$$\begin{aligned} C' \amalg_{X_i'} U_i &\cong C' \amalg_{X_i'} (X_i' \amalg_{B'} B) \cong C' \amalg_{B'} B \cong F_i, \\ P_i \times_{X_i''} B'' &\cong (C \times_{C''} X_i'') \times_{X_i''} B'' \cong C \times_{C''} B'' \cong F_i \end{aligned}$$

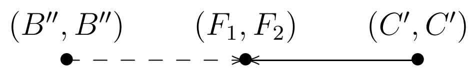
and we get the following picture



where the shadowed rectangles are as in (9) and (8) respectively. The remaining rectangular contour can be filled by the $(1,1)$-simplex

$$\begin{array}{ccccc} A'' & \overset{1}{\rightarrowtail} & A'' & \twoheadrightarrow & 0 \\ {\scriptstyle t_i}\downarrow & & \downarrow & & \downarrow \\ U_i & \rightarrowtail & F_i & \twoheadrightarrow & D' \\ {\scriptstyle s_i}\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} \\ X_i & \overset{p_i}{\rightarrowtail} & P_i & \overset{r_i}{\twoheadrightarrow} & D' \end{array} \qquad (i=1,2)$$

Thus the path (19) is homotopic to the two-edge path



The latter can be contracted to $\widetilde{T}(\mathcal{M})$ by means of the $(1,1)$-simplex

$$\begin{array}{ccccc} C' & \overset{1}{\rightarrowtail} & C' & \twoheadrightarrow & 0 \\ {\scriptstyle 1}\downarrow & & \downarrow & & \downarrow \\ C' & \rightarrowtail & F_i & \twoheadrightarrow & B'' \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} \\ 0 & \rightarrowtail & B'' & \overset{1}{\twoheadrightarrow} & B'' \end{array} \qquad (i=1,2)$$

□

Corollary 2.14.

$$\ell\left(D \underset{c'}{\overset{c}{\twoheadleftarrow}} C \underset{b'}{\overset{b}{\longleftarrow}} B \underset{a'}{\overset{a}{\leftarrowtail}} A\right) + \ell\left(H \underset{h'}{\overset{h}{\twoheadleftarrow}} G \underset{g'}{\overset{g}{\longleftarrow}} F \underset{f'}{\overset{f}{\leftarrowtail}} E\right) =$$

$$\ell\left(D \oplus H \underset{\left[\begin{smallmatrix} c' & 0 \\ 0 & h' \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} c & 0 \\ 0 & h \end{smallmatrix}\right]}{\twoheadleftarrow}} C \oplus G \underset{\left[\begin{smallmatrix} b' & 0 \\ 0 & g' \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} b & 0 \\ 0 & g \end{smallmatrix}\right]}{\longleftarrow}} B \oplus F \underset{\left[\begin{smallmatrix} a' & 0 \\ 0 & f' \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} a & 0 \\ 0 & f \end{smallmatrix}\right]}{\leftarrowtail}} A \oplus E \right) \qquad \square$$

Corollary 2.15.

$$\ell\left(D \underset{c'}{\overset{c}{\twoheadleftarrow}} C \underset{b'}{\overset{b}{\longleftarrow}} B \underset{a'}{\overset{a}{\leftarrowtail}} A\right) + \ell\left(D \underset{c}{\overset{c'}{\twoheadleftarrow}} C \underset{b}{\overset{b'}{\longleftarrow}} B \underset{a}{\overset{a'}{\leftarrowtail}} A\right) = 0 \qquad \square$$

Lemma 2.16. *For arbitrary automorphisms $\alpha, \alpha' \in \operatorname{Aut}(A)$ and arbitrary exact sequences $A \overset{b}{\twoheadleftarrow} B \overset{c}{\leftarrowtail} C$, $F \overset{f}{\twoheadleftarrow} E \overset{e}{\leftarrowtail} A$ the equality*

$$\ell\left(0 \twoheadleftarrow A \underset{\alpha'}{\overset{\alpha}{\longleftarrow}} A \leftarrowtail 0\right) = \ell\left(F \underset{f}{\overset{f}{\twoheadleftarrow}} E \underset{e\circ\alpha'\circ b}{\overset{e\circ\alpha\circ b}{\longleftarrow}} B \underset{c}{\overset{c}{\leftarrowtail}} C\right)$$

holds in $K_1(\mathfrak{M})$.

Proof. Apply 3×3 lemma and 3×4 lemma to diagrams:

$$\begin{array}{ccccc}
0 & \twoheadleftarrow & A & \underset{\alpha'}{\overset{\alpha}{\leftarrowtail}} & A \\
\downarrow & & {\scriptstyle e}\downarrow{\scriptstyle e} & & {\scriptstyle 1}\downarrow{\scriptstyle 1} \\
F & \underset{f}{\overset{f}{\twoheadleftarrow}} & E & \underset{e\circ\alpha'}{\overset{e\circ\alpha}{\leftarrowtail}} & A \\
{\scriptstyle 1}\downarrow{\scriptstyle 1} & & {\scriptstyle f}\downarrow{\scriptstyle f} & & \downarrow \\
F & \underset{1}{\overset{1}{\twoheadleftarrow}} & F & \leftarrowtail & 0
\end{array}
\qquad
\begin{array}{ccccccc}
0 & \twoheadleftarrow & 0 & \longleftarrow & C & \underset{1}{\overset{1}{\leftarrowtail}} & C \\
\downarrow & & \downarrow & & {\scriptstyle a}\downarrow{\scriptstyle a} & & {\scriptstyle 1}\downarrow{\scriptstyle 1} \\
F & \underset{f}{\overset{f}{\twoheadleftarrow}} & E & \underset{e\circ\alpha'\circ b}{\overset{e\circ\alpha\circ b}{\longleftarrow}} & B & \underset{c}{\overset{c}{\leftarrowtail}} & C \\
{\scriptstyle 1}\downarrow{\scriptstyle 1} & & {\scriptstyle 1}\downarrow{\scriptstyle 1} & & {\scriptstyle b}\downarrow{\scriptstyle b} & & \downarrow \\
F & \underset{f}{\overset{f}{\twoheadleftarrow}} & E & \underset{e\circ\alpha'}{\overset{e\circ\alpha}{\longleftarrow}} & A & \leftarrowtail & 0
\end{array}
\qquad \square$$

Lemma 2.17.

$$\ell\left(D \underset{c'}{\overset{c}{\twoheadleftarrow}} C \underset{b'}{\overset{b}{\longleftarrow}} B \underset{a'}{\overset{a}{\leftarrowtail}} A\right) + \ell\left(D \underset{c''}{\overset{c'}{\twoheadleftarrow}} C \underset{b''}{\overset{b'}{\longleftarrow}} B \underset{a''}{\overset{a'}{\leftarrowtail}} A\right) =$$

$$\ell\left(D \underset{c''}{\overset{c}{\twoheadleftarrow}} C \underset{b''}{\overset{b}{\longleftarrow}} B \underset{a''}{\overset{a}{\leftarrowtail}} A\right).$$

Proof. Apply 3×4 lemma to the diagram

$$
\begin{array}{ccccccc}
D & \underset{c''}{\overset{c}{\twoheadleftarrow}} & C & \underset{b''}{\overset{b}{\longleftarrow}} & B & \underset{a''}{\overset{a}{\leftarrowtail}} & A \\
{\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \downarrow \; \downarrow {\scriptstyle \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & & {\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \downarrow \; \downarrow {\scriptstyle \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & & {\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \downarrow \; \downarrow {\scriptstyle \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & & {\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \downarrow \; \downarrow {\scriptstyle \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \\
D \oplus D & \underset{\left[\begin{smallmatrix}c' & 0\\0 & c''\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}c & 0\\0 & c'\end{smallmatrix}\right]}{\twoheadleftarrow}} & C \oplus C & \underset{\left[\begin{smallmatrix}b' & 0\\0 & b''\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}b & 0\\0 & b'\end{smallmatrix}\right]}{\twoheadleftarrow}} & B \oplus B & \underset{\left[\begin{smallmatrix}a' & 0\\0 & a''\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}a & 0\\0 & a'\end{smallmatrix}\right]}{\twoheadleftarrow}} & A \oplus A \\
{\scriptstyle [0\,1]} \downarrow \; \downarrow {\scriptstyle [-1\,0]} & & {\scriptstyle [0\,1]} \downarrow \; \downarrow {\scriptstyle [-1\,0]} & & {\scriptstyle [0\,1]} \downarrow \; \downarrow {\scriptstyle [-1\,0]} & & {\scriptstyle [0\,1]} \downarrow \; \downarrow {\scriptstyle [-1\,0]} \\
D & \underset{c'}{\overset{c'}{\twoheadleftarrow}} & C & \underset{b'}{\overset{b'}{\longleftarrow}} & B & \underset{a'}{\overset{a'}{\leftarrowtail}} & A
\end{array}
$$

□

3. The group $E^1(X)$ and the invariant e^1

For the sake of completeness we state the following obvious fact:

Lemma 3.1. *Let $\mathfrak{M}$ be a small exact category. For $A, B \in \mathrm{Ob}(\mathfrak{M})$ the following conditions are equivalent:*

i) the equality $[A] = [B]$ in $K_0(\mathfrak{M})$ holds;

ii_1) there exist $P, Q, R \in \mathrm{Ob}(\mathfrak{M})$ and $\alpha, \beta, \alpha', \beta' \in \mathrm{Mor}(\mathfrak{M})$ such that the sequences

$$0 \leftarrow R \overset{\beta}{\leftarrow} Q \overset{\alpha}{\leftarrow} A \oplus P \leftarrow 0$$

$$0 \leftarrow R \overset{\beta'}{\leftarrow} Q \overset{\alpha'}{\leftarrow} B \oplus P \leftarrow 0$$

are exact admissible;

ii_2) there exist $P, Q, R \in \mathrm{Ob}(\mathfrak{M})$ and $\gamma, \mu, \gamma', \mu' \in \mathrm{Mor}(\mathfrak{M})$ such that the sequences

$$0 \leftarrow A \oplus P \overset{\mu}{\leftarrow} Q \overset{\gamma}{\leftarrow} R \leftarrow 0 \tag{20}$$

$$0 \leftarrow B \oplus P \overset{\mu'}{\leftarrow} Q \overset{\gamma'}{\leftarrow} R \leftarrow 0$$

are exact admissible.

Moreover, if $(\mathfrak{M}, D, \delta)$ is an exact category with duality, then one may assume that in each case P carries a hyperbolic form $\chi : P \to DP$.

Proof. It is obvious that each of conditions ii$_1$), ii$_2$) implies equality $[A]=[B]$ in $K_0(\mathfrak{M})$. Assume that i) holds. By [17, Theorem 1] $K_0(\mathfrak{M})=\mathcal{A}/\mathcal{B}$ is a factor group of free abelian group $\mathcal{A}$ generated by classes of isomorphic objects in $\mathrm{Ob}(\mathfrak{M})$ modulo the subgroup $\mathcal{B}$ generated by expressions

$$[Y]-[X]-[Z]$$

for all admissible exact sequences $Z \twoheadleftarrow Y \leftarrowtail X$. Thus if $[A]=[B]$, then there exist two admissible exact sequences

$$\begin{array}{ccc} Z & \twoheadleftarrow & Y \overset{i}{\leftarrowtail} X \\ W & \twoheadleftarrow & V \overset{j}{\leftarrowtail} U \end{array}$$

and an isomorphism

$$\rho : B\oplus X\oplus Z\oplus V \overset{\sim}{\to} A\oplus Y\oplus U\oplus W.$$

Thus for

$$\begin{array}{rcl} P & = & X\oplus U, \\ Q & = & A\oplus Y\oplus U\oplus W \cong B\oplus X\oplus Z\oplus V, \\ R & = & Z\oplus W, \end{array}$$

$$\alpha = \begin{bmatrix} 1_A & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1_U \\ 0 & 0 & 0 \end{bmatrix} : A\oplus X\oplus U \to A\oplus Y\oplus U\oplus W,$$

$$\alpha' = \rho \circ \begin{bmatrix} 1_B & 0 & 0 \\ 0 & 1_X & 0 \\ 0 & 0 & 0 \\ 0 & 0 & j \end{bmatrix} : B\oplus X\oplus U \to A\oplus Y\oplus U\oplus W$$

ii_1) holds. Analogously i) implies ii_2) for the same Q,

$$P = Z \oplus W, \quad R = X \oplus U$$

and suitable morphisms.

If there is a duality in $\mathfrak{M}$, then substitution of $P \oplus DP$ for P, $Q \oplus DP$ for Q and $\alpha \oplus 1_{DP}$ for α, etc., yields an analogous exact sequence with a hyperbolic form χ defined on P. □

We will refer to the pair of exact sequences $A \oplus P \twoheadleftarrow Q \leftarrowtail R$, $B \oplus P \twoheadleftarrow Q \leftarrowtail R$ with hyperbolic P as a *common resolution* of $A \oplus P$, $B \oplus P$. Let us rephrase the last result in more convenient form:

Corollary 3.2. *Given two symmetric bilinear maps $\varphi : A \to DA$, $\psi : B \to DB$ such that the equality $[A] = [B]$ in $K_0(\mathfrak{M})$ holds, there exist:*

- *a hyperbolic form (P, χ),*
- *an object Q,*
- *admissible epimorphisms $\pi_A : Q \to A \oplus P$, and $\pi_B : Q \to B \oplus P$ with kernels $R \rightrightarrows Q$ having common source R,*
- *symmetric bilinear maps $D\pi_A \circ (\varphi \oplus \chi) \circ \pi_A : Q \to DQ$ and $D\pi_B \circ (\psi \oplus \chi) \circ \pi_B : Q \to DQ$ Witt equivalent to φ and ψ respectively* □

Definition 3.1. Let $\mathfrak{M}$ be an exact category with a duality (D, δ). With symmetric bilinear maps $\alpha : Q \to DQ$, $\beta : Q \to DQ$ with respective kernels $\gamma, \gamma' : R \rightarrowtail Q$, we associate a selfdual double exact sequence

$$DR \underset{D\gamma'}{\overset{D\gamma}{\twoheadleftarrow}} DQ \underset{\beta}{\overset{\alpha}{\longleftarrow}} Q \underset{\gamma'}{\overset{\gamma}{\leftarrowtail}} R \ . \tag{21}$$

In particular, with symmetric bilinear forms $\varphi : A \xrightarrow{\sim} DA$, $\psi : B \xrightarrow{\sim} DB$ such that the equality $[A] = [B]$ in $K_0(\mathfrak{M})$ holds, and a common resolution $A \oplus P \twoheadleftarrow Q \leftarrowtail R$,

$B \oplus P \twoheadleftarrow Q \leftarrowtail R$ with hyperbolic (P, χ) we associate a selfdual double exact sequence

$$DR \underset{D\gamma'}{\overset{D\gamma}{\twoheadleftarrow}} DQ \underset{D\mu'\circ(\psi\oplus\chi)\circ\mu'}{\overset{D\mu\circ(\varphi\oplus\chi)\circ\mu}{\longleftarrow}} Q \underset{\gamma'}{\overset{\gamma}{\leftarrowtail}} R \tag{22}$$

with admissible γ, γ'. We will refer to this double exact sequence as to *gluing a common resolution with its dual.*

We want to refine this to a map defined on classes of Witt equivalence, so we shall define an appropriate target group.

3.1. E-**groups.** An exact dualization functor $D : \mathfrak{M} \to \mathfrak{M}$ defines an action of the two element group $\{1, D\}$ on K-groups of $\mathfrak{M}$. One of objectives of this paper is the study of following groups for $n = 1$:

Definition 3.2. For $D : K_n(\mathfrak{M}) \to K_n(\mathfrak{M})$ the E^n-groups of an exact category $\mathfrak{M}$ with a duality D are

$$\begin{aligned} E^n(\mathfrak{M}; D) &= E^n_+(\mathfrak{M}; D) = \operatorname{Ker}(1 - D)/\operatorname{Im}(1 + D), \\ E^n_-(\mathfrak{M}; D) &= \operatorname{Ker}(1 + D)/\operatorname{Im}(1 - D). \end{aligned}$$

Equivalently one may define E^n-groups as homology groups of the complex

$$\cdots \xrightarrow{1+D} K_n(\mathfrak{M}) \xrightarrow{1-D} K_n(\mathfrak{M}) \xrightarrow{1+D} K_n(\mathfrak{M}) \xrightarrow{1-D} \cdots \tag{23}$$

or Tate cohomology groups

$$\begin{aligned} E^n(\mathfrak{M}; D) &= \widehat{H}^{2p}\left(\{1, D\}, K_n(\mathfrak{M})\right), \\ E^n_-(\mathfrak{M}; D) &= \widehat{H}^{2p-1}\left(\{1, D\}, K_n(\mathfrak{M})\right). \end{aligned}$$

Note that the case $n = 0$ was used extensively in [20], [22] and [21]. Let us recall that in these papers the map

$$e^0 : W(\mathfrak{M}) \to E^0(\mathfrak{M})$$

induced by the forgetful functor, and more general maps $e^0 : W^{\pm}(\mathfrak{M}, D, \delta) \to E^0(\mathfrak{M}, D)$ were used. The group

$$E^0_-(\mathfrak{M}, D) = \{x \in K_0(\mathfrak{M}) : Dx = -x\}/\{y - Dy : y \in K_0(\mathfrak{M})\}$$

also occurred. Each pair of hyperbolic spaces $(M \oplus DM, \left[\begin{smallmatrix} 0 & 1 \\ \delta_M & 0 \end{smallmatrix}\right])$, $(N \oplus DN, \left[\begin{smallmatrix} 0 & 1 \\ \delta_N & 0 \end{smallmatrix}\right])$ such that in $K_0(\mathfrak{M})$ the equality

$$[M \oplus DM] = [N \oplus DN]$$

holds, defines an element $[M] - [N]$ of $E^0_-(\mathfrak{M}; D)$.

Now we are interested in the case when for two symmetric bilinear forms (A, φ), (B, ψ) the equality

$$e^0(A, \varphi) = e^0(B, \psi)$$

holds. By corollary 3.2

$$e^0(A, \varphi) = e^0(B, \psi) \text{ iff there exist } M, N \in \mathrm{Ob}(\mathfrak{M})$$
$$\text{such that } [A \oplus M \oplus DM] = [B \oplus N \oplus DN] \text{ in } K_0(\mathfrak{M}).$$

Definition 3.3. Given two symmetric bilinear forms (A, φ), (B, ψ) and admissible exact sequences

$$A \overset{\beta}{\twoheadleftarrow} P \overset{\alpha}{\leftarrowtail} Q$$
$$B \overset{\nu}{\twoheadleftarrow} P \overset{\mu}{\leftarrowtail} Q$$

denote

$$\epsilon^1 \begin{pmatrix} A \overset{\beta}{\twoheadleftarrow} P \overset{\alpha}{\leftarrowtail} Q, \varphi \\ B \overset{\nu}{\twoheadleftarrow} P \overset{\mu}{\leftarrowtail} Q, \psi \end{pmatrix} = \ell \left(DQ \underset{D\mu}{\overset{D\alpha}{\twoheadleftarrow}} DP \underset{D\nu \circ \psi \circ \nu}{\overset{D\beta \circ \varphi \circ \beta}{\longleftarrow}} P \underset{\mu}{\overset{\alpha}{\leftarrowtail}} Q \right).$$

Theorem 3.3. *Given two symmetric bilinear forms* (A, φ), (B, ψ) *such that* $[A] = [B]$ *in* $K_0(\mathfrak{M})$, *and two common resolutions*

$$A \overset{\beta}{\twoheadleftarrow} P \overset{\alpha}{\hookleftarrow} Q, \qquad A \overset{b}{\twoheadleftarrow} R \overset{a}{\hookleftarrow} S,$$

$$B \overset{\nu}{\twoheadleftarrow} P \overset{\mu}{\hookleftarrow} Q, \qquad B \overset{d}{\twoheadleftarrow} R \overset{c}{\hookleftarrow} S,$$

we get

$$\epsilon^1 \begin{pmatrix} A \overset{\beta}{\twoheadleftarrow} P \overset{\alpha}{\hookleftarrow} Q, \varphi \\ B \overset{\nu}{\twoheadleftarrow} P \overset{\mu}{\hookleftarrow} Q, \psi \end{pmatrix} - \epsilon^1 \begin{pmatrix} A \overset{b}{\twoheadleftarrow} R \overset{a}{\hookleftarrow} S, \varphi \\ B \overset{d}{\twoheadleftarrow} R \overset{c}{\hookleftarrow} S, \psi \end{pmatrix} \in (1 + D)K_1(\mathfrak{M}).$$

Proof. The class

$$\epsilon^1 \begin{pmatrix} A \overset{\beta}{\twoheadleftarrow} P \overset{\alpha}{\hookleftarrow} Q, \varphi \\ B \overset{\nu}{\twoheadleftarrow} P \overset{\mu}{\hookleftarrow} Q, \psi \end{pmatrix} - \epsilon^1 \begin{pmatrix} A \overset{b}{\twoheadleftarrow} R \overset{a}{\hookleftarrow} S, \varphi \\ B \overset{d}{\twoheadleftarrow} R \overset{c}{\hookleftarrow} S, \psi \end{pmatrix} =$$

$$\epsilon^1 \begin{pmatrix} A \overset{\beta}{\twoheadleftarrow} P \overset{\alpha}{\hookleftarrow} Q, \varphi \\ B \overset{\nu}{\twoheadleftarrow} P \overset{\mu}{\hookleftarrow} Q, \psi \end{pmatrix} + \epsilon^1 \begin{pmatrix} B \overset{d}{\twoheadleftarrow} R \overset{c}{\hookleftarrow} S, \psi \\ A \overset{b}{\twoheadleftarrow} R \overset{a}{\hookleftarrow} S, \varphi \end{pmatrix}$$

corresponds to the double long exact sequence:

$$DQ \oplus DS \underset{\left[\begin{smallmatrix} D\mu & 0 \\ 0 & Da \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} D\alpha & 0 \\ 0 & Dc \end{smallmatrix}\right]}{\twoheadleftarrow}} DP \oplus DR \underset{\left[\begin{smallmatrix} \psi & 0 \\ 0 & \underline{\varphi} \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} \overline{\varphi} & 0 \\ 0 & \overline{\psi} \end{smallmatrix}\right]}{\longleftarrow}} P \oplus R \underset{\left[\begin{smallmatrix} \mu & 0 \\ 0 & a \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} \alpha & 0 \\ 0 & c \end{smallmatrix}\right]}{\hookleftarrow}} Q \oplus S$$

where

$$\overline{\varphi} = D\beta \circ \varphi \circ \beta, \quad \overline{\psi} = Dd \circ \psi \circ d,$$

$$\underline{\varphi} = Db \circ \varphi \circ b, \quad \underline{\psi} = D\nu \circ \psi \circ \nu.$$

It follows that if

$$
\begin{aligned}
t &= \ell\left(DQ\oplus DS \overset{\left[\begin{smallmatrix} D\alpha & 0 \\ 0 & Dc \end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix} D\mu & 0 \\ 0 & Da \end{smallmatrix}\right]}{\twoheadleftarrow}} DP\oplus DR \xleftarrow[\left[\begin{smallmatrix} \overline{\psi} & 0 \\ 0 & \underline{\varphi} \end{smallmatrix}\right]]{\left[\begin{smallmatrix} \overline{\varphi} & 0 \\ 0 & \underline{\psi} \end{smallmatrix}\right]} P\oplus R \overset{\left[\begin{smallmatrix} \alpha & 0 \\ 0 & c \end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix} \mu & 0 \\ 0 & a \end{smallmatrix}\right]}{\leftarrowtail}} Q\oplus S \right), \\
u &= \ell\left(DQ\oplus DS \overset{\left[\begin{smallmatrix} D\alpha & 0 \\ 0 & Dc \end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix} D\mu & 0 \\ 0 & Da \end{smallmatrix}\right]}{\twoheadleftarrow}} DP\oplus DR \overset{\left[\begin{smallmatrix} D\beta & 0 \\ 0 & Dd \end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix} 0 & D\nu \\ Db & 0 \end{smallmatrix}\right]}{\leftarrowtail}} DA\oplus DB \right), \\
v &= \ell\left(DQ\oplus DS \overset{\left[\begin{smallmatrix} D\alpha & 0 \\ 0 & Dc \end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix} D\mu & 0 \\ 0 & Da \end{smallmatrix}\right]}{\twoheadleftarrow}} DP\oplus DR \overset{\left[\begin{smallmatrix} D\beta\circ\varphi & 0 \\ 0 & Dd\circ\psi \end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix} 0 & D\nu\circ\psi \\ Db\circ\varphi & 0 \end{smallmatrix}\right]}{\leftarrowtail}} A\oplus B \right),
\end{aligned}
$$

then $u = v$ (since these d.s.e.s. are isomorphic) and $t = u + Dv = u + Du$ by the 3×4 lemma, since there is a commutative diagram

$$
\begin{array}{ccccccc}
 & & DA\oplus DB & \overset{\left[\begin{smallmatrix} \varphi\circ\beta & 0 \\ 0 & \psi\circ d \end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix} 0 & \varphi\circ b \\ \psi\circ\nu & 0 \end{smallmatrix}\right]}{\twoheadleftarrow}} & P\oplus R & \overset{\left[\begin{smallmatrix} \alpha & 0 \\ 0 & c \end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix} \mu & 0 \\ 0 & a \end{smallmatrix}\right]}{\leftarrowtail}} & Q\oplus S \\
 & & {\scriptstyle \left[\begin{smallmatrix} D\beta & 0 \\ 0 & Dd \end{smallmatrix}\right]}\Big\downarrow\Big\downarrow{\scriptstyle \left[\begin{smallmatrix} 0 & D\nu \\ Db & 0 \end{smallmatrix}\right]} & & \Big\downarrow & & \Big\downarrow \\
DQ\oplus DS & \overset{\left[\begin{smallmatrix} D\alpha & 0 \\ 0 & Dc \end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix} D\mu & 0 \\ 0 & Da \end{smallmatrix}\right]}{\twoheadleftarrow}} & DP\oplus DR & \xleftarrow[\left[\begin{smallmatrix} \underline{\varphi} & 0 \\ 0 & \underline{\psi} \end{smallmatrix}\right]]{\left[\begin{smallmatrix} \overline{\varphi} & 0 \\ 0 & \overline{\psi} \end{smallmatrix}\right]} & P\oplus R & \overset{\left[\begin{smallmatrix} \alpha & 0 \\ 0 & c \end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix} \mu & 0 \\ 0 & a \end{smallmatrix}\right]}{\leftarrowtail}} & Q\oplus S \\
\Big\downarrow & & {\scriptstyle \left[\begin{smallmatrix} D\alpha & 0 \\ 0 & Dc \end{smallmatrix}\right]}\Big\downarrow\Big\downarrow{\scriptstyle \left[\begin{smallmatrix} D\mu & 0 \\ 0 & Da \end{smallmatrix}\right]} & & & & \\
DQ\oplus DS & \longleftarrow & DQ\oplus DS & & & &
\end{array}
$$

with exact rows and columns. ☐

Definition 3.4. The *relative discriminant* $\varepsilon^1(\varphi \div \psi)$ of a pair $(A,\varphi),(B,\psi)$ of forms such that $[A]=[B]$ in $K_0(\mathfrak{M})$ is

$$
\varepsilon^1(\varphi\div\psi) = \epsilon^1 \begin{pmatrix} A \overset{\beta}{\twoheadleftarrow} P \overset{\alpha}{\leftarrowtail} Q,\varphi \\ B \overset{\nu}{\twoheadleftarrow} P \overset{\mu}{\leftarrowtail} Q,\psi \end{pmatrix} \operatorname{mod}(1+D)K_1(\mathfrak{M})
$$

This is well defined by theorem 3.3. Clearly

$$\varepsilon^1 (\varphi \div \psi) \in \operatorname{Ker}(1 - D) \tag{24}$$

by the very construction.

Lemma 3.4. *If* (A, φ) *is metabolic and* $\iota : L \rightarrowtail A$ *is a Lagrangian, then*

$$\varepsilon^1 \left(\varphi \div \begin{bmatrix} 0 & 1 \\ \delta_L & 0 \end{bmatrix} \right) = 0.$$

Proof. If $L \xrightarrow{\iota} A$ is a Lagrangian, then the sequence

$$DL \xleftarrow{D\iota\circ\varphi} A \xleftarrow{\iota} L$$

is exact. Thus one may form exact sequences

$$A \xleftarrow{[1,0]} A \oplus L \xleftarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} L$$

$$DL \oplus L \xleftarrow[\left[\begin{smallmatrix} D\iota\circ\varphi & 0 \\ 0 & 1 \end{smallmatrix}\right]]{} A \oplus L \xleftarrow[\left[\begin{smallmatrix} \iota \\ 0 \end{smallmatrix}\right]]{} L.$$

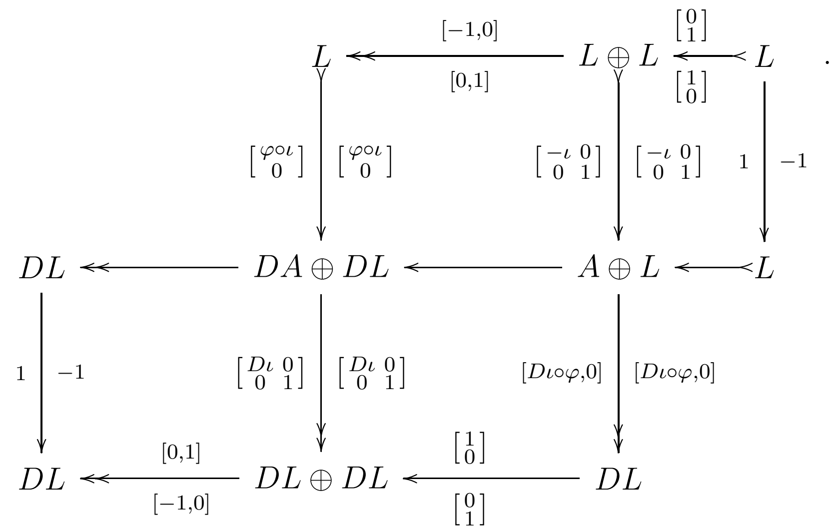
Since

$$D\varphi \circ D^2\iota \circ \delta_L = D\varphi \circ \delta_A \circ \iota = \varphi \circ \iota,$$

gluing them with their duals yields a double exact sequence

$$DL \underset{[D\iota,0]}{\overset{[0,1]}{\longleftarrow}} DA \oplus DL \underset{\left[\begin{smallmatrix} 0 & \varphi\circ\iota \\ D\iota\circ\varphi & 0 \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} \varphi & 0 \\ 0 & 0 \end{smallmatrix}\right]}{\longleftarrow}} A \oplus L \underset{\left[\begin{smallmatrix} \iota \\ 0 \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]}{\longleftarrow}} L$$

which may be "resolved" as follows: apply the 3×4 lemma to the commutative diagram



Then lemma 2.7 b) applied to the upper and the lower d.s.e.s. shows that

$$\epsilon^1\left(\varphi \div \begin{bmatrix} 0 & 1 \\ \delta_L & 0 \end{bmatrix}\right) = \{-1\}([L] + [DL]) = (1 + D)(\{-1\}[L])$$

(here $\{-1\} \in K_1(F)$), so a metabolic form and its hyperbolic form produce exactly 0 in $E^1(\mathfrak{M}, D)$. ☐

Proposition 3.5. *If* $(K \oplus DK, \left[\begin{smallmatrix} 0 & 1 \\ \delta_K & 0 \end{smallmatrix}\right])$ *and* $(L \oplus DL, \left[\begin{smallmatrix} 0 & 1 \\ \delta_L & 0 \end{smallmatrix}\right])$ *are hyperbolic spaces such that* $[K \oplus DK] = [L \oplus DL]$ *in* $K_0(\mathfrak{M})$*, and*

$$[K] - [L] \equiv 0 \bmod (1 - D)K_0(\mathfrak{M}),$$

then

$$\varepsilon^1\left(\left[\begin{smallmatrix} 0 & 1 \\ \delta_K & 0 \end{smallmatrix}\right] \div \left[\begin{smallmatrix} 0 & 1 \\ \delta_L & 0 \end{smallmatrix}\right]\right) \equiv 0 \bmod (1 + D)K_1(\mathfrak{M}).$$

Proof. Denote $H(X) = X \oplus DX$ for an object X. The second assumption implies the existence of objects X, Y of $\mathfrak{M}$ such that $[K] - [L] = ([X] - [DX]) - ([Y] - [DY])$, i.e.

$$[K \oplus DX \oplus Y] = [L \oplus X \oplus DY].$$

The common resolution may be chosen in a special way - as a direct sum

$$H\left(K\oplus X\oplus Y\oplus R\oplus S\right)\twoheadleftarrow P\oplus P'\oplus DR\oplus DS\leftarrowtail Q\oplus Q'$$

$$H\left(L\oplus X\oplus Y\oplus R\oplus S\right)\twoheadleftarrow P\oplus P'\oplus DR\oplus DS\leftarrowtail Q\oplus Q'$$

of exact sequences

$$(K\oplus DX\oplus Y)\oplus R \xtwoheadleftarrow{\left[\begin{smallmatrix}\kappa\\ \rho\end{smallmatrix}\right]} P \xleftarrowtail{\alpha} Q$$

$$(L\oplus X\oplus DY)\oplus R \xtwoheadleftarrow{\left[\begin{smallmatrix}l\\ r\end{smallmatrix}\right]} P \xleftarrowtail{a} Q$$

and

$$(DK\oplus X\oplus DY)\oplus S \xtwoheadleftarrow{\left[\begin{smallmatrix}\kappa'\\ \sigma\end{smallmatrix}\right]} P' \xleftarrowtail{\alpha'} Q'$$

$$(DL\oplus DX\oplus Y)\oplus S \xtwoheadleftarrow{\left[\begin{smallmatrix}l'\\ s\end{smallmatrix}\right]} P' \xleftarrowtail{a'} Q'$$

with added $DR \overset{1}{\twoheadleftarrow} DR \leftarrowtail 0$ and $DS \overset{1}{\twoheadleftarrow} DS \leftarrowtail 0$ respectively.

This common resolution glued with its dual yields the double exact sequence

$$DQ\oplus DQ' \xtwoheadleftarrow[\left[\begin{smallmatrix}D\alpha & 0 & 0 & 0\\ 0 & D\alpha' & 0 & 0\end{smallmatrix}\right]]{\left[\begin{smallmatrix}Da & 0 & 0 & 0\\ 0 & Da' & 0 & 0\end{smallmatrix}\right]} DP\oplus DP'\oplus R\oplus S$$

$$\xleftarrow[\left[\begin{smallmatrix}0 & v & D\rho & 0\\ Dv & 0 & 0 & Ds\\ \delta\circ r & 0 & 0 & 0\\ 0 & \delta\circ s & 0 & 0\end{smallmatrix}\right]]{\left[\begin{smallmatrix}0 & u & D\rho & 0\\ Du & 0 & 0 & D\sigma\\ \delta\circ\rho & 0 & 0 & 0\\ 0 & \delta\circ\sigma & 0 & 0\end{smallmatrix}\right]} P\oplus P'\oplus DR\oplus DS \xleftarrowtail[\left[\begin{smallmatrix}a & 0\\ 0 & a'\\ 0 & 0\\ 0 & 0\end{smallmatrix}\right]]{\left[\begin{smallmatrix}\alpha & 0\\ 0 & \alpha'\\ 0 & 0\\ 0 & 0\end{smallmatrix}\right]} Q\oplus Q'$$

which is a direct sum of the double exact sequence

$$DQ' \xtwoheadleftarrow[[Da',0]]{[D\alpha',0]} DP'\oplus R \xleftarrow[\left[\begin{smallmatrix}Dv & Ds\\ \delta\circ r & 0\end{smallmatrix}\right]]{\left[\begin{smallmatrix}Du & D\sigma\\ \delta\circ\rho & 0\end{smallmatrix}\right]} P\oplus DS \xleftarrowtail[\left[\begin{smallmatrix}a\\ 0\end{smallmatrix}\right]]{\left[\begin{smallmatrix}\alpha\\ 0\end{smallmatrix}\right]} Q$$

with the one isomorphic to its dual. □

The assumption on the class in the group $E^0_-(\mathfrak{M}; D)$ is necessary:

Example 3.1. Let X be a projective line $X = \mathbf{Proj}\, F[x, y]$ over a field F, x, y - homogeneous coordinates. Consider vector bundles on X with the dualization

$$DA = A\hat{} = \mathcal{H}\mathrm{om}(A, \mathcal{O}_X)$$

and the canonical isomorphism $A \to A\hat{}\,\hat{}$ as δ_A. The equality

$$[\mathcal{O}_X(-1)] + [\mathcal{O}_X(1)] = 2\,[\mathcal{O}_X]$$

in $K_0(X)$ follows from the exactness of the sequence

$$\mathcal{O}_X(1) \xleftarrow{[x,y]} \mathcal{O}_X \oplus \mathcal{O}_X \xleftarrow{\left[\begin{smallmatrix}-y\\ x\end{smallmatrix}\right]} \mathcal{O}_X(-1)\ .$$

Consider the hyperbolic forms

$$\varphi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : \mathcal{O}_X(-1) \oplus \mathcal{O}_X(1) \to \mathcal{O}_X(1) \oplus \mathcal{O}_X(-1),$$

$$\psi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : \mathcal{O}_X \oplus \mathcal{O}_X \to \mathcal{O}_X \oplus \mathcal{O}_X.$$

The common resolution

$$\begin{array}{ccccc} \mathcal{O}_X \oplus \mathcal{O}_X & \xleftarrow{\left[\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}\right]} & \mathcal{O}_X(-1) \oplus \mathcal{O}_X \oplus \mathcal{O}_X & \xleftarrow{\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]} & \mathcal{O}_X(-1) \\ \mathcal{O}_X(-1) \oplus \mathcal{O}_X(1) & \xleftarrow{\left[\begin{smallmatrix}1&0&0\\0&x&y\end{smallmatrix}\right]} & \mathcal{O}_X(-1) \oplus \mathcal{O}_X \oplus \mathcal{O}_X & \xleftarrow{\left[\begin{smallmatrix}0\\-y\\x\end{smallmatrix}\right]} & \mathcal{O}_X(-1) \end{array}$$

glued with its dual yields the double exact sequence

$$\mathcal{O}_X(1) \underset{[0,-y,x]}{\overset{[1,0,0]}{\longleftarrow}} \mathcal{O}_X(1) \oplus {\mathcal{O}_X}^2 \underset{\left[\begin{smallmatrix}0&x&y\\x&0&0\\y&0&0\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}0&0&0\\0&0&1\\0&1&0\end{smallmatrix}\right]}{\longleftarrow}} \mathcal{O}_X(-1) \oplus {\mathcal{O}_X}^2 \underset{\left[\begin{smallmatrix}0\\-y\\x\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]}{\longleftarrow}} \mathcal{O}_X(-1)\ . \tag{25}$$

Since the short double exact sequence

$$\mathcal{O}_X(-1) \oplus {\mathcal{O}_X}^2 \xleftarrow[1]{\left[\begin{smallmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ x & 0 & 1 \end{smallmatrix}\right]} \mathcal{O}_X(-1) \oplus {\mathcal{O}_X}^2 \longleftarrow 0$$

produces 0 in $K_1(X)$, the class of (3.1) in $K_1(X)$ is - by the 3×4-lemma - the same as the class of the double exact sequence

$$\mathcal{O}_X(1) \xleftarrow[{[0,-y,x]}]{[1,0,0]} \mathcal{O}_X(1) \oplus {\mathcal{O}_X}^2 \xleftarrow[\left[\begin{smallmatrix} 0 & x & y \\ x & 0 & 0 \\ y & 0 & 0 \end{smallmatrix}\right]]{\left[\begin{smallmatrix} 0 & 0 & 0 \\ x & 0 & 1 \\ y & 1 & 0 \end{smallmatrix}\right]} \mathcal{O}_X(-1) \oplus {\mathcal{O}_X}^2 \xleftarrow[\left[\begin{smallmatrix} 0 \\ -y \\ x \end{smallmatrix}\right]]{\left[\begin{smallmatrix} 1 \\ -y \\ -x \end{smallmatrix}\right]} \mathcal{O}_X(-1)$$

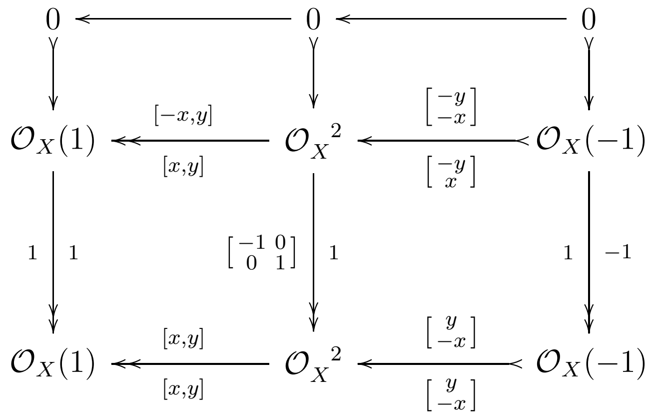
which in turn is a middle row of the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{O}_X(1) & \xleftarrow[{[y,x]}]{[y,x]} & {\mathcal{O}_X}^2 & \xleftarrow[\left[\begin{smallmatrix} -x \\ y \end{smallmatrix}\right]]{\left[\begin{smallmatrix} -x \\ y \end{smallmatrix}\right]} & \mathcal{O}_X(-1) & & \\
\big\downarrow & & \left[\begin{smallmatrix} y & x \\ -1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \big\downarrow \left[\begin{smallmatrix} y & x \\ -1 & 0 \\ 0 & 1 \end{smallmatrix}\right] & & \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right] \big\downarrow \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right] & & \\
\mathcal{O}_X(1) & \longleftarrow & \mathcal{O}_X(1) \oplus {\mathcal{O}_X}^2 & \longleftarrow & \mathcal{O}_X(-1) \oplus {\mathcal{O}_X}^2 & \longleftarrow & \mathcal{O}_X(-1) \\
 & & {[1,y,-x]} \big\downarrow {[1,y,-x]} & & \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right] \big\downarrow \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right] & & \big\downarrow \\
 & & \mathcal{O}_X(1) & \xleftarrow[{[x,y]}]{[-x,y]} & {\mathcal{O}_X}^2 & \xleftarrow[\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]]{\left[\begin{smallmatrix} -y \\ -x \end{smallmatrix}\right]} & \mathcal{O}_X(-1)
\end{array}$$

It follows that the relative discriminant is the class of the short double exact sequence

$$\mathcal{O}_X(1) \xleftarrow[{[x,y]}]{[-x,y]} {\mathcal{O}_X}^2 \xleftarrow[\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]]{\left[\begin{smallmatrix} -y \\ -x \end{smallmatrix}\right]} \mathcal{O}_X(-1),$$

which occurs in the commutative diagram



It follows from theorem 2.6 that the relative discriminant equals

$$-\{-1\}[\mathcal{O}_X] - \{-1\}[\mathcal{O}_X(-1)] = \{-1\}\left([\mathcal{O}_X] - [\mathcal{O}_X(-1)]\right),$$

($\{-1\}$ has order 2 in $K_1(F)$) which is not 0.

Remark 3.1. It is easy to see that if the classes of symmetric bilinear forms φ, ψ are equal in a Grothendieck group of symmetric bilinear spaces, then $\epsilon^1(\varphi \div \psi) = 0$. It follows that the Grothendieck ring of symmetric bilinear spaces of a projective line over a field F differs from the Grothendieck ring of symmetric bilinear spaces of the field F, contrary to the case of Witt rings.

As an example shows, to define an invariant of the Witt equivalence one must factor out relative discriminants of pairs of hyperbolic forms.

Definition 3.5. Let $\mathcal{H}(\mathfrak{M}; D, \delta)$ be the subgroup of $E^1(\mathfrak{M}; D)$ generated by classes of all relative discriminants $\varepsilon^1(\mu \div \nu)$ of pairs of hyperbolic spaces (the class of hyperbolic spaces depends on δ). The k_1-group of $(\mathfrak{M}; D, \delta)$ is the factor group

$$k_1(\mathfrak{M}; D) = E^1(\mathfrak{M}; D)/\mathcal{H}(\mathfrak{M}; D, \delta).$$

For applications it is important - by proposition 3.5 - that to compute the group $\mathcal{H}(\mathfrak{M}; D, \delta)$ it is enough to compute relative discriminants of pairs of hyperbolic spaces corresponding to nonzero elements of $E^0_-(\mathfrak{M}; D)$.

Corollary 3.6. *There is a natural surjective homomorphism* $E^0_-(\mathfrak{M}; D) \to \mathcal{H}(\mathfrak{M}; D, \delta)$*. In particular* $\mathcal{H}(\mathfrak{M}; D, \delta) = 0$ *whenever* $E^0_-(\mathfrak{M}; D) = 0$*.*

Proof. Consider the pull back $\mathfrak{X} = \mathfrak{X}(\mathfrak{M}; D, \delta)$

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & K_0(\mathcal{M}) \\ \downarrow & & \downarrow{\scriptstyle 1+D} \\ K_0(\mathcal{A}) & \xrightarrow{1+D} & K_0(\mathcal{M}) \end{array}$$

There are two homomorphisms defined on $\mathfrak{X}$:

- $([K], [L]) \mapsto [K] - [L] \bmod (1 - D)K_0(\mathfrak{M}) \in E^0_-(\mathfrak{M}; D)$
- $([K], [L]) \mapsto \varepsilon^1(H(K), H(L)) \in E^1(\mathfrak{M}; D)$

Proposition 3.5 states that the kernel of the first one is contained in the kernel of the second one, so there is a homomorphism $E^0_-(\mathfrak{M}; D) \to E^1(\mathfrak{M}; D)$.

Whenever two hyperbolic forms $H(A)$, $H(B)$ such that $[H(A)] = [H(B)]$ in $K_0(\mathcal{A})$ are given,

$$[A] + D[A] = [B] + D[B]$$

$$[A] - [B] = D[B] - D[A] = -D([A] - [B])$$

so $[A]-[B]$ defines an element of $E^0_-(\mathfrak{M}; D)$, which maps onto the class $\epsilon^1\,(H(A) \div H(B))$. □

For instance, for the projective line $X = \mathbb{P}^1_F$ from example 3.1, the group

$$\mathcal{H}(X) = \mathcal{H}(\mathfrak{M}; D, \delta) = \mu_2(F) \cdot E^0_-(X) \tag{26}$$

has two elements: 0 and $\{-1\}\,([\mathcal{O}_X] - [\mathcal{O}_X(-1)])$.

Corollary 3.7. *If for symmetric bilinear spaces (A, φ) and (B, ψ) there are metabolic forms (M, μ), (M', μ'), (N, ν) and (N', ν') with common resolutions*

$$A \oplus M \twoheadleftarrow P \leftarrowtail Q, \qquad B \oplus N \twoheadleftarrow P \leftarrowtail Q,$$

$$A \oplus M' \twoheadleftarrow R \leftarrowtail S, \qquad B \oplus N' \twoheadleftarrow R \leftarrowtail S,$$

then the difference

$$\varepsilon^1 (\varphi \oplus \mu \div \psi \oplus \nu) - \varepsilon^1 (\varphi \oplus \mu' \div \psi \oplus \nu')$$

belongs to $\mathcal{H}(\mathfrak{M}; D, \delta)$.

Proof. This is a formal consequence of theorem 3.3 and the proposition: for double exact sequences

$$DQ \twoheadleftarrow DP \longleftarrow P \leftarrowtail Q \ ,$$

$$DS \twoheadleftarrow DR \longleftarrow R \leftarrowtail S$$

the difference between them is the class of the direct sum of the first one and the upside-down of the second one:

$$DQ \oplus DS \twoheadleftarrow DP \oplus DR \longleftarrow P \oplus R \leftarrowtail Q \oplus S.$$

Denote by u its class in $K_1(\mathfrak{M})$. Next choose common a resolution

$$M' \oplus N \oplus W \twoheadleftarrow U \leftarrowtail T, \qquad M \oplus N' \oplus W \twoheadleftarrow U \leftarrowtail T$$

for some metabolic (W, ω). Since

$$\varepsilon^1 (\mu' \oplus \nu \oplus \omega \div \mu \oplus \nu' \oplus \omega) \equiv 0 \bmod \mathcal{H}(\mathfrak{M}; D, \delta),$$

the class v in $K_1(\mathfrak{M})$ defined by the direct sum

$$DQ \oplus DS \oplus DT \twoheadleftarrow DP \oplus DR \oplus DU \longleftarrow P \oplus R \oplus U \leftarrowtail Q \oplus S \oplus T$$

is congruent to $u \bmod \mathcal{H}(\mathfrak{M}; D, \delta)$. But this is the class corresponding to the common resolution of isomorphic forms

$$(X, \xi) = (A \oplus M \oplus B \oplus N' \oplus M' \oplus N \oplus W, \varphi \oplus \mu \oplus \psi \oplus \nu' \oplus \mu' \oplus \nu \oplus \omega)$$

and

$$(Y, \upsilon) = (B \oplus N \oplus A \oplus M' \oplus M \oplus N' \oplus W, \psi \oplus \nu \oplus \varphi \oplus \mu' \oplus \mu \oplus \nu' \oplus \omega)\,.$$

By theorem 3.3 this class is congruent $\bmod \mathcal{H}(\mathfrak{M}; D, \delta)$ to the class of

$$0 \twoheadleftarrow DX \underset{\xi}{\overset{\xi}{\leftarrow}} X \leftarrowtail 0\ .$$

It follows that $u \equiv 0 \bmod \mathcal{H}(\mathfrak{M}; D, \delta)$. □

The goal of this paper is to define the discriminant map:

Definition 3.6. Let $I(\mathfrak{M}; D, \delta) = \operatorname{Ker} e^0$. The discriminant map

$$e^1 : I(\mathfrak{M}; D, \delta) \to k_1(\mathfrak{M}; D)$$

is defined as follows

$$\begin{aligned} \text{if } [A \oplus M \oplus DM] &= [N \oplus DN] \text{ in } K_0(\mathfrak{M}), \\ \text{then } e^1(A, \varphi) &= \varepsilon^1\left(\varphi \oplus \mu \div \nu\right) \bmod \mathcal{H}(\mathfrak{M}; D, \delta) \end{aligned}$$

where $\mu : M \oplus DM \to DM \oplus M$, $\nu : N \oplus DN \to DN \oplus N$ are hyperbolic forms.

Let us state several elementary properties of these notions.

Proposition 3.8. *For arbitrary exact categories with duality* $(\mathfrak{M}, (D, \delta))$, $(\mathfrak{N}, (D', \delta'))$,
i) E^n*-groups are elementary* 2*-groups (groups of exponent* 2*).*

ii) if $f : \mathfrak{M} \to \mathfrak{N}$ is an exact functor which commutes with the duality, then f induces homomorphisms

$$\begin{aligned} f &: W(\mathfrak{M}; D, \delta) \to W(\mathfrak{N}; D', \delta') \text{ and } f : I(\mathfrak{M}; D, \delta) \to I(\mathfrak{N}; D', \delta'), \\ f &: E^n(\mathfrak{M}; D, \delta) \to E^n(\mathfrak{N}; D', \delta') \text{ and } f : E^n_-(\mathfrak{M}; D, \delta) \to E^n_-(\mathfrak{N}; D', \delta') \\ f &: k_1(\mathfrak{M}; D, \delta) \longrightarrow k_1(\mathfrak{N}; D, \delta) \end{aligned}$$

such that diagrams

$$\begin{array}{ccc} W(\mathfrak{M}; D, \delta) & \xrightarrow{e^0} & E^0(\mathfrak{M}, D) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ W(\mathfrak{N}, D', \delta') & \xrightarrow{e^0} & E^0(\mathfrak{N}, D') \end{array} \qquad \begin{array}{ccc} I(\mathfrak{M}; D, \delta) & \xrightarrow{e^1} & k_1(\mathfrak{M}, D) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ I(\mathfrak{N}; D', \delta') & \xrightarrow{e^1} & k_1(\mathfrak{N}, D') \end{array}$$

commute.

Proof. i) If $(1 \pm D)a = 0$, then $2a = (1 \mp D)a \equiv 0 \,(\mathrm{mod}\, \mathrm{Im}(1 \mp D))$. □

The following example provides a motivation for the notation e^1 and $I(\mathfrak{M}; D, \delta)$, and shows that the above defined notion generalizes usual notion of the discriminant of a quadratic form.

Example 3.2. Let $\mathfrak{M}$ be the category of vector spaces over a field F with the usual dualization. In this case it is obvious that e^0 coincides with the usual dimension index, so $I(\mathfrak{M}) = I(F)$. Moreover, $E^0_-(F) = 0$, so $k_1(\mathfrak{M}) = E^1(\mathfrak{M})$. Note that D acts trivially on $K_1(F)$, so

$$E^1(F) \stackrel{\mathrm{df}}{=} E^1(\mathfrak{M}) = K_1(F)/2K_1(F) = k_1(F),$$

hence the new notation $k_1(F)$ is consistent with the one introduced in [8].

$$E^1_-(F) \stackrel{\mathrm{df}}{=} E^1_-(\mathfrak{M}) = \mu_2(F) = \{1, -1\}.$$

The group $E^1(F) \cong F^*/F^{*2}$ is usually denoted by $g(F)$ - the square classes group - in the theory of quadratic forms. If $e^0(A, \varphi) = 0$, then A is an even-dimensional vector space, $\dim(A) = 2k$, so there exists an isomorphism $\rho : B \oplus B^* \to A$ of a space $B \oplus B^*$ supporting a hyperbolic form χ, with A. Thus one may choose the exact sequences (20) in a special way:

$$A \overset{\rho}{\leftarrow} B \oplus B^* \leftarrow 0 \leftarrow 0,$$

$$B \oplus B^* \overset{1}{\leftarrow} B \oplus B^* \leftarrow 0 \leftarrow 0.$$

The double exact sequence (22)

$$0 \Lleftarrow B \oplus B^* \underset{\rho^* \circ \varphi \circ \rho}{\overset{\chi}{\Longleftarrow}} B \oplus B^* \Lleftarrow 0$$

defines the element

$$\det(\varphi)\det(\chi) = (-1)^k \det(\varphi) = (-1)^{2k(2k-1)/2} \det(\varphi) \bmod 2K_1(F),$$

which is exactly the discriminant of (A, φ).

The additional E^1-group

$$E^1_-(F) = \mu_2(F) = \{1, -1\}$$

has no direct interpretation yet.

Example 3.3. If $\mathfrak{M}$ is the category of vector bundles on a scheme X with the usual dualization ($D = \hat{} = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(-, \mathcal{O}_X)$), then we write

$$W(X) = W(\mathfrak{M}), \quad I(X) = I(\mathfrak{M}), \quad E^n(X) = E^n(\mathfrak{M}), \quad k_1(X) = k_1(\mathfrak{M}).$$

Let $f : Y \to X$ be a morphism of schemes, and $\mathfrak{N}$ be the category of vector bundles on Y. The exact functor $f^* : \mathfrak{M} \to \mathfrak{N}$ commutes with the dualization, so it induces homomorphisms on E-groups and Witt groups and homomorphisms of E^0 and W commute with e^0. Thus the homomorphism of Witt groups maps $I(X)$ in $I(Y)$, and this homomorphism commutes with e^1.

This means that if, in particular, Y is a point, then the above defined discriminant reduces to the usual discriminant on fibres.

It is known that exist symmetric bilinear forms of even rank (and hence with trivial e^0 on fibres) which have nontrivial global e^0, say on split projective quadrics of even dimension. So there is no reason to expect in general that local discriminant determine the value of e^1.

The following property of e^1 provides a motivation for adopting the classical notation $I(X)$ for the notion defined above:

Theorem 3.9. *Consider the category $\mathcal{P}_X$ of locally free sheaves of finite rank on a variety X over a field F with a line bundle L (L may be trivial), the usual duality functor $DA = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(A, L)$ and δ either the canonical isomorphism, or the opposite to the canonical one. Let moreover $c \in \dot{F}$ be a nonzero constant. Then for arbitrary form (A, φ) such that $e^0(\varphi) = 0$ the equality*

$$e^1(c \cdot \varphi) = e^1(\varphi)$$

in $k_1(X)$ holds.

Proof. Choose metabolic (M, μ), (N, ν) such that $A \oplus M$ and N have a common resolution:

$$A \oplus M \xleftarrow{\pi} P \xleftarrow{\rho} Q$$

$$N \xleftarrow{\pi'} P \xleftarrow{\rho'} Q$$

Note, that $(M, c \cdot \mu)$ and $(N, c \cdot \nu)$ are metabolic forms, and the four-term double exact sequences needed to compute discriminants are rows of the commutative diagram:

$$\begin{array}{ccccccc}
DQ & \underset{D\rho'}{\overset{D\rho}{\longleftarrow\!\!\longleftarrow}} & DP & \underset{cD\pi'\circ(\varphi\oplus\mu)\circ\pi'}{\overset{cD\pi\circ(\varphi\oplus\mu)\circ\pi}{\longleftarrow\!\!\longleftarrow}} & P & \underset{D\rho'}{\overset{D\rho}{\longleftarrow\!\!\longleftarrow}} & Q \\
{\scriptstyle 1}\big\downarrow{\scriptstyle 1} & & {\scriptstyle 1}\big\downarrow{\scriptstyle 1} & & {\scriptstyle c}\big\downarrow{\scriptstyle c} & & {\scriptstyle c}\big\downarrow{\scriptstyle c} \\
DQ & \underset{\rho'}{\overset{\rho}{\longleftarrow\!\!\longleftarrow}} & DP & \underset{D\pi'\circ(\varphi\oplus\mu)\circ\pi'}{\overset{D\pi\circ(\varphi\oplus\mu)\circ\pi}{\longleftarrow\!\!\longleftarrow}} & P & \underset{\rho'}{\overset{\rho}{\longleftarrow\!\!\longleftarrow}} & Q
\end{array}$$

with isomorphic vertical arrows. It follows from the 3×4 lemma that both rows define the same class in $K_1(X)$, so $e^1(c \cdot \varphi) = e^1(\varphi)$. ☐

Corollary 3.10. *Under assumptions of the theorem,* $I(F) \cdot I(X) \subset \mathrm{Ker}(e^1)$. ☐

4. The map $e^1 : I(X) \to k_1(X)$ for certain projective variety X.

Here we assume that the dualization (D, δ) is fixed, and suppress it in the notation. It is easy to compute the E^1-groups in the following particular case:

Theorem 4.1. *Assume that X is a quasiprojective variety over a field F such that $K_1(X) = K_0(X) \otimes_{\mathbb{Z}} K_1(F)$. Then*

$$\begin{aligned} E^1(X) &\cong E^1(F) \otimes E^0(X) \oplus E^1_-(F) \otimes E^0_-(X), \\ E^1_-(X) &\cong E^1(F) \otimes E^0_-(X) \oplus E^1_-(F) \otimes E^0(X). \end{aligned}$$

Proof. E-groups are Tate cohomology groups of the group $\{1, D\}$:

$$\begin{aligned} E^1(X) &= \hat{H}^0(\{1, D\}, K_1(F) \otimes K_0(X)) \\ E^1_-(X) &= \hat{H}^1(\{1, D\}, K_1(F) \otimes K_0(X)). \end{aligned}$$

By the universal coefficients formula there are exact sequences

$$0 \to K_1(F) \otimes \hat{H}^i\left(\{1, D\}, K_0(X)\right) \to E^1_{\pm}(X) \to$$
$$\to \mathrm{Tor}_1\Big(K_1(F), \hat{H}^{i+1}\left(\{1, D\}, K_0(X)\right)\Big) \to 0,$$

i.e. exact sequences

$$\begin{aligned} 0 &\to K_1(F) \otimes E^0(X) \to E^1(X) \to \mathrm{Tor}_1\left(K_1(F), E^0_-(X)\right) \to 0 \\ 0 &\to K_1(F) \otimes E^0_-(X) \to E^1_-(X) \to \mathrm{Tor}_1\left(K_1(F), E^0(X)\right) \to 0. \end{aligned}$$

$E^0(X)$ and $E^0_-(X)$ are elementary 2-groups (groups of exponent 2), so

$$\begin{aligned} K_1(F)\otimes E^0_\pm(X) &= g(F)\otimes E^0_\pm(X) \\ \mathrm{Tor}_1\big(K_1(F), E^0_\pm(X)\big) &= \mu_2(F)\otimes E^0_\pm(X) \end{aligned}$$

and the assertion is proved. □

It is obvious that $\mathcal{H}(X)=\mathcal{H}(\mathfrak{M})\subset \mu_2(F)\otimes E^0_-(X)$.

Conjecture 4.2. *Under assumptions of theorem 4.1*

$$\begin{aligned} \mathcal{H}(X) &= E^1_-(F)\otimes E^0_-(X) \\ k_1(X) &= E^1(F)\otimes E^0(X). \end{aligned}$$

Example 4.1. The projective space $X=\mathbb{P}^n_F=\mathrm{Proj}F[x_0,x_1,\dots,x_n]$ with the usual dualization functor ($D=\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(-,\mathcal{O}_X)$) meets the assumption of theorem 4.1. In this case the E^0-groups are known:

$$E^0(X)=\mathbb{Z}/2\mathbb{Z}\,[\mathcal{O}_X] \qquad \text{and} \qquad E^0_-(X)=\begin{cases} 0 & \text{for even } n \\ \mathbb{Z}/2\mathbb{Z}H^n & \text{for odd } n \end{cases};$$

where $H=1-[\mathcal{O}_X(-1)]$ is the class of a hyperplane section ([20, Prop. 2.1.3], [22, Prop. 5.1], [21, Cor. 3.4]). Thus:

$$\begin{aligned} E^1(X) &= \begin{cases} g(F)\cdot[\mathcal{O}_X] & \text{for even } n \\ g(F)\cdot[\mathcal{O}_X]\oplus \mu_2(F)\cdot H^n & \text{for odd } n \end{cases}, \\ \mathcal{H}(X) &= \begin{cases} 0 & \text{for even } n \\ \mu_2(F)\cdot H^n & \text{for odd } n \end{cases}, \\ E^1_-(X) &= \begin{cases} \mu_2(F)\cdot[\mathcal{O}_X] & \text{for even } n \\ \mu_2(F)\cdot[\mathcal{O}_X]\oplus g(F)H^n & \text{for odd } n \end{cases}. \end{aligned}$$

It is known that the canonical map $W(F) \to W(X)$ induced by the inverse image functor $V \mapsto V \otimes_F \mathcal{O}_X$ for the structure map $X \to \mathrm{Spec}(F)$ is an isomorphism ([1, Satz]). Therefore $I(X) = I(F) \otimes_F \mathcal{O}_X$ and the map $e^1 : I(X) \to k_1(X)$ is surjective for even n.

For a field F of characteristic different from 2, the classes of hyperbolic spaces form a cyclic direct summand of even order in the Grothendieck group of symmetric bilinear spaces generated by hyperbolic plane ([23, Th. 1.1]). It is not so for the odd-dimensional projective spaces. First of all, there is an infinite sequence of hyperbolic planes:

$$\mathfrak{h}_k = \mathcal{O}_X(k) \oplus \mathcal{O}_X(-k) \text{ for } k = 0, 1, 2, \dots$$

Corollary 4.3. *On the projective space $X = \mathbb{P}^n_F$ of odd dimension n there exists a hyperbolic space, which class in a Grothendieck group of symmetric bilinear spaces is not a multiple of the standard hyperbolic plane $\mathfrak{h}_0 = \left(\mathcal{O}_X^2, \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\right)$.*

Proof. For the case $n = 1$ see example 3.1. For $n > 1$ there exists a pair of hyperbolic spaces with relative discriminant $\{-1\}H^n \neq 0$, since $\mathcal{H}$ is nontrivial.

Let $n = 2k - 1$,

$$A = \sum_{i=0}^{k-1} \binom{2k-1}{2i} \mathfrak{h}_i \tag{27}$$

$$B = \sum_{i=0}^{k-1} \binom{2k-1}{2i+1} \mathfrak{h}_i \, . \tag{28}$$

The identity $H^{n+1} = 0$ yields that A and B define equal classes in $K_0(X)$. The proof of corollary 3.6 shows that $\varepsilon^1(A, B)$ is nonzero. Thus it is impossible that each of A, B is a multiple of the standard hyperbolic plane $\mathfrak{h}_0$. □

4.1. **A conic.**

We compute here E_0-groups and E_1-groups of a projective conic X:

$$q = {z_0}^2 - a{z_1}^2 - b{z_2}^2$$

for anisotropic quadratic form $\langle 1, -a, -b\rangle$ over a field F defined on the space V over F with base v_0, v_1, v_2 (z_0, z_1, z_2 is the dual base of the dual space V^*), so that the projective conic

$$X = \mathbf{Proj}\, S(V^*)/(q)$$

has no rational points.

The Clifford algebra $C(q)$ has a base given by

$$1, v_0, v_1, v_2, v_0v_1, v_0v_2, v_1v_2, v_0v_1v_2,$$

and

$$v_iv_j = -v_jv_i \text{ for } i \neq j,$$
$$v_0{}^2 = 1, v_1{}^2 = -a, v_2{}^2 = -b.$$

The elements $1, v_0v_1, v_0v_2, v_2v_1$ form a base of the even Clifford algebra C_0, which is isomorphic to the quaternion algebra $\left(\frac{a, b}{F}\right)$:

$$\begin{aligned} (v_0v_1)^2 &= a, & (v_0v_2)(v_2v_1) &= -(v_2v_1)(v_0v_2),\\ (v_0v_2)^2 &= b, & (v_0v_1)(v_2v_1) &= -(v_1v_2)(v_0v_1),\\ (v_2v_1)^2 &= -ab, & (v_0v_1)(v_0v_2) &= -(v_0v_2)(v_0v_1) = v_2v_1. \end{aligned}$$

We denote as usual

$$i = v_0v_1, \quad j = v_0v_2, \quad k = v_2v_1.$$

Let

$$\varphi = z_0v_0 + z_1v_1 + z_2v_2 \in \Gamma(X, \mathcal{O}_X(1) \otimes C_1)$$

be the "generic zero vector" of q.

$$\varphi^2 = (z_0v_0 + z_1v_1 + z_2v_2)^2 = z_0{}^2 - az_1{}^2 - bz_2{}^2 = 0.$$

The complex

$$\cdots \xrightarrow{\varphi\cdot} \mathcal{O}_X(-n)\otimes_F C_{n+2} \xrightarrow{\varphi\cdot} \mathcal{O}_X(1-n)\otimes_F C_{n+1}$$
$$\xrightarrow{\varphi\cdot} \mathcal{O}_X(2-n)\otimes_F C_n \xrightarrow{\varphi\cdot} \cdots$$

(subscripts in C_n are taken $\mod 2$) is exact and splits locally ([19, Prop. 8.2.(a)]). It may be written explicitely - for the bases $1, i, j, k$ of C_0 and $v_0, v_1, v_2, v_2v_1v_0$ of C_1 the maps $\varphi\cdot -$ have the matrices

$$\begin{bmatrix} z_0 & -az_1 & -bz_2 & 0 \\ -z_1 & z_0 & 0 & bz_2 \\ -z_2 & 0 & z_0 & -az_1 \\ 0 & z_2 & -z_1 & z_0 \end{bmatrix} \text{for } \mathcal{O}_X(2s-1)\otimes_F C_1 \xrightarrow{\varphi\cdot} \mathcal{O}_X(2s)\otimes C_0,$$

$$\begin{bmatrix} z_0 & az_1 & bz_2 & 0 \\ z_1 & z_0 & 0 & -bz_2 \\ z_2 & 0 & z_0 & az_1 \\ 0 & -z_2 & z_1 & z_0 \end{bmatrix} \text{for } \mathcal{O}_X(2s)\otimes_F C_0 \xrightarrow{\varphi\cdot} \mathcal{O}_X(2s+1)\otimes C_1.$$

Definition 4.1. The Swan sheaf $\mathcal{U}$ is defined as the cokernel

$$\mathcal{U} = \mathcal{U}_0 = \mathrm{Coker}(\mathcal{O}_X(-2)\otimes_F C_0 \xrightarrow{\varphi\cdot} \mathcal{O}_X(-1)\otimes_F C_1).$$

Let $E_0^{\pm}(X)$, $E_0^{\pm}(X, L)$ be E-groups of the dualization

$$A\hat{} = \mathcal{H}\mathrm{om}(A, \mathcal{O}_X), \quad A^{\hat{}L} = \mathcal{H}\mathrm{om}(A, \mathcal{O}_X(-1)),$$

with canonical δ, δ_L respectively.

Proposition 4.4. *E-groups of X are:*

$$E_0(X) = (\mathbb{Z}/2)\cdot[\mathcal{O}_X], \quad E_0^-(X) = (\mathbb{Z}/2)\cdot(2-[\mathcal{U}]),$$
$$E_0(X, L) = 0, \qquad E_0^-(X, L) = 0.$$

Proof. There is an exact sequence

$$0 \leftarrow \mathcal{O}_X(1) \leftarrow {\mathcal{O}_X}^3 \leftarrow \mathcal{O}_X(-1)^3 \leftarrow \mathcal{O}_X(-2) \leftarrow 0$$

inherited from the projective plane. Now to use results of [19], note that the sheaf $\mathcal{O}_X(1)$ has the truncated canonical resolution

$$0 \leftarrow \mathcal{O}_X(1) \leftarrow {\mathcal{O}_X}^3 \leftarrow \mathcal{U} \leftarrow 0. \tag{29}$$

The Swan sheaf $\mathcal{U}$ is a right module over

$$\mathrm{End}_X(\mathcal{U}) = C_0(\langle 1, -a, -b\rangle) \cong \left(\frac{a,b}{F}\right).$$

The quaternion algebra $\mathcal{D} = \left(\frac{a,b}{F}\right)$ is a skew field, so $\mathcal{U}$ is an indecomposable vector bundle. The sheaf $\mathcal{U}$ has rank 2, so there is nonsingular skew symmetric pairing $\mathcal{U} \otimes_{\mathcal{O}_X} \mathcal{U} \to \bigwedge^2 \mathcal{U}$ given by the exterior multiplication. Thus

$$\mathcal{U} \cong \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{U}, \bigwedge^2 \mathcal{U}).$$

Taking the highest exterior powers in the truncated canonical resolution yields

$$\begin{aligned} \mathcal{O}_X(1) \otimes \bigwedge^2 \mathcal{U} &\cong \mathcal{O}_X \\ \bigwedge^2 \mathcal{U} &\cong \mathcal{O}_X(-1). \end{aligned}$$

It follows that

$$\mathcal{U} \cong \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{U}, \mathcal{O}_X(-1)) \cong \mathcal{U}\hat{} \otimes \mathcal{O}_X(-1),$$

The formula for determinant of tensor product yields that in general

$$\bigwedge^2 \mathcal{U}(n) = \bigwedge^2 (\mathcal{U} \otimes \mathcal{O}_X(n)) \cong \left(\bigwedge^2 \mathcal{U}\right) \otimes (\mathcal{O}_X(n))^{\otimes 2} \cong \mathcal{O}_X(2n-1)$$

and

$$\mathcal{U}(n) \cong \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{U}(n), \mathcal{O}_X(2n-1)) \cong \mathcal{U}(n)\hat{} \otimes \mathcal{O}_X(2n-1),$$

so $\mathcal{U}(n)$ carries a skew symmetric $\mathcal{O}_X(2n-1)$-valued bilinear form.

It follows that $K_0(X)$ is a free abelian group of rank 2 with a free base $1 = [\mathcal{O}_X]$, $[\mathcal{U}]$ and identities

$$H^3 = 0$$

$$[\mathcal{O}_X(1)] - 3 + [\mathcal{U}] = 0 \text{ or } [\mathcal{O}_X(1)]H = 2 - [\mathcal{U}]$$

where $H = 1 - [\mathcal{O}_X(-1)]$ is the class of hyperplane section, as above. By [20, corollary 3.4.3] there is an additional identity

$$[\mathcal{U}] + [\mathcal{U}\hat{}\,] = 4 \text{ or } [\mathcal{U}] + [\mathcal{U}(1)] = 4. \tag{30}$$

Thus

$$[\mathcal{U}] + [\mathcal{U}(1)] = 3 - [\mathcal{O}_X(1)] + 3[\mathcal{O}_X(1)] - [\mathcal{O}_X(2)] = 4,$$

$$1 - 2[\mathcal{O}_X(1)] + [\mathcal{O}_X(2)] = 0 \text{ or } H^2 = 0.$$

The same result - that $H^2 = 0$ - may be obtained easily by checking locally that the sequence

$$\begin{aligned} 0 \ &\leftarrow \ \mathcal{O}_X(1) \overset{\alpha}{\leftarrow} \mathcal{O}_X{}^2 \overset{\beta}{\leftarrow} \mathcal{O}_X(-1) \leftarrow 0 \\ \alpha \ &= \ [z_0, z_1], \qquad \beta = \begin{bmatrix} z_1 \\ -z_0 \end{bmatrix} \end{aligned}$$

is exact. Here $z_i \in \Gamma(X, \mathcal{O}_X(1)) = \mathrm{Hom}_X(\mathcal{O}_X(k), \mathcal{O}_X(k+1))$ are global morphisms of $\mathcal{O}_X$-modules under consideration.

The condition $H^2 = 0$ implies $2 - [\mathcal{U}] = [\mathcal{O}_X(1)]H = H$, and

$$0 = (2 - [\mathcal{U}])^2$$

It follows that

$$\begin{aligned}[\mathcal{U}]^2 &= 4[\mathcal{U}]-4\\ [\mathcal{O}_X(1)] &= 3-[\mathcal{U}] = 1+(2-[\mathcal{U}])\\ [\mathcal{O}_X(-1)] &= [\mathcal{U}]-1 = 1-(2-[\mathcal{U}]).\end{aligned}$$

There is a ring isomorphism $K_0(X) \cong \mathbb{Z}[u]/(u^2)$ which maps $2-[\mathcal{U}]$ onto coset of u and - by (30) -

$$1\hat{} = 1, \qquad u\hat{} = -u.$$

Denote $L = \mathcal{O}_X(-1)$. With respect to the base $1, (2-[\mathcal{U}])$ the involutions $\hat{}$ and $\hat{}^L$ have matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

respectively.

Now the assertion follows immediately. □

Note that the discriminant map e^1 is defined on the whole $W(X, L)$.

The atgument used in the argument above proof can be rwformulated as follows:

Corollary 4.5. *In $K_0(X)$ the action of involutions induced by duality functors $\hat{}$, $\hat{}^L$ respectively, is given by formulae:*

$$\begin{aligned}(x+y[\mathcal{U}])\hat{} &= (x+4y)-y[\mathcal{U}]\\ (x+y[\mathcal{U}])\hat{}^L &= -x+(x+y)[\mathcal{U}].\end{aligned}$$

More precisely: the map $\hat{}$ on $K_0(X)$ is induced by functors:

- the identity from F-modules to F-modules (the summand x),
- the forgetful functor from $\mathcal{D}$-modules to F-modules (the summand $4y$),
- the identity functor from $\mathcal{D}$-modules to $\mathcal{D}$-modules (the summand $y[\mathcal{U}]$),

while the map $\hat{}^L$ is induced by functors:

- the identity from F-modules to F-modules (the summand x)
- the natural functor from F-modules to $\mathcal{D}$-modules (the summand $x[\mathcal{U}]$)
- the identity functor from $\mathcal{D}$-modules to $\mathcal{D}$-modules (the summand $y[\mathcal{U}]$).

Proposition 4.4 states that $E_0^-(X, L) = 0$. Hence the group $\mathcal{H}$ is trivial and $k_1(X, L) = E_1(X, L)$.

To compute E^1-groups let us recall some facts on K-theory of quaternion algebras.

The reduced norm is an injective homomorphism

$$\mathrm{Nrd} : K_1(\mathcal{D}) \to K_1(F)$$

such that for any splitting field E ($E \otimes_F \mathcal{D} \cong M_2(E)$ is the matrix algebra over E) if E/F is finite, then the diagram

(31)
$$\begin{array}{ccc} K_1(M_2(E)) & = & K_1(E \otimes_F \mathcal{D}) \\ \sim \uparrow & & \downarrow N \\ K_1(E) & & K_1(\mathcal{D}) \\ N_{E/F} \downarrow & \swarrow \mathrm{Nrd} & \\ K_1(F) & & \end{array}$$

commutes ([7]). It follows that $K_1(\mathcal{D}) = \dot{\mathcal{D}}/[\dot{\mathcal{D}}, \dot{\mathcal{D}}]$ is isomorphic to the group B of values of quadratic form $\langle 1, -a, -b, ab \rangle$ in $K_1 F = \dot{F}$ and the involution $\hat{}$ acts trivially on $K_1(\mathcal{D})$.

Note that the composition $K_1(F) \to K_1(\mathcal{D}) \xrightarrow{\mathrm{Nrd}} K_1(F)$ is the map $x \mapsto x^2$. We will identify $K_1(\mathcal{D})$ with B via the map Nrd.

For example:

- if the involution $\hat{} : K_0(X) \to K_0(X)$ maps $x + y[\mathcal{U}] \in K_0(X)$ onto $x + 4y - y[\mathcal{U}]$, then $\hat{} : K_1(X) \to K_1(X)$ maps $x + y[\mathcal{U}] \in K_1(X)$ onto $x + N_{\mathcal{D}/F}(y) - y[\mathcal{U}]$; if $x = \{\xi\} \in K_1(F)$, $y = \{\zeta\} \in K_1(\mathcal{D})$, then

$$x + N_{\mathcal{D}/F}(y) = \{\xi N_{\mathcal{D}/F}(\zeta)\} = \{\xi \mathrm{Nrd}(\zeta)^2\} \in K_1(F)$$

$$-y = \{\zeta^{-1}\} \in K_1(\mathcal{D})$$

and we write $\mathrm{Nrd}(\zeta^{-1}) \in B$ instead $\{\zeta^{-1}\} \in K_1(\mathcal{D})$.

- if the involution $\hat{}^L : K_0(X) \to K_0(X)$ maps $x + y[\mathcal{U}] \in K_0(X)$ onto $-x + (x+y)[\mathcal{U}]$, then $\hat{}^L : K_1(X) \to K_1(X)$ maps $x + y[\mathcal{U}] \in K_1(X)$ onto

$$-x + (r(x) + y)[\mathcal{U}] = \{\xi^{-1}\} + \{\xi\zeta\}[\mathcal{U}],$$

where $r : K_1(F) \to K_1(D)$ is the canonical map; after identification of $K_1(\mathcal{D})$ with B we write $\mathrm{Nrd}(\xi\zeta) = \xi^2\mathrm{Nrd}(\zeta) \in B$ instead $\{\xi\zeta\} \in K_1(\mathcal{D})$.

It follows from corollary 4.5 that in $K_1(X) = K_1(F) \oplus K_1(\mathcal{D})[\mathcal{U}] = \dot{F} \oplus B \cdot [\mathcal{U}]$ for $t \in \dot{F}$, $u \in B$ the equalities:

$$\begin{aligned}(t, u[\mathcal{U}])\hat{} &= (tu^2, u^{-1}[\mathcal{U}])\\ (t, u[\mathcal{U}])\hat{}^L &= (t^{-1}, t^2u[\mathcal{U}])\end{aligned}$$

hold. Thus it is obvious that the following result holds.

Theorem 4.6. *For a projective conic X: $x_0^2 - ax_1^2 - bx_2^2 = 0$ given by anisotropic quadratic form $\langle 1, -a, -b\rangle$ over a field F and line bundle $L = \mathcal{O}_X(-1)$ let $E^1_\pm(X) = E^1_\pm(\mathcal{P}_X; \hat{}, \delta)$ and $E^1_\pm(X, L) = E^1_\pm(\mathcal{P}_X; \hat{}^L, \delta_L)$. Then*

$$E^1(X) = g(F)[\mathcal{O}_X] \oplus (\mu_2(F) \cap B)\,[\mathcal{U}], \qquad E^-_1(X) = \mu_2(F)[\mathcal{O}_X] \oplus B/B^2,$$

$$E^1(X, L) \cong \mu_2(F) \oplus B/\dot{F}^2[\mathcal{U}], \qquad E^-_1(X, L) \cong \mu_2(F) \oplus B/\dot{F}^2. \quad \square$$

The following computation looks strange, but will be used bellow, in the proof of Proposition 4.13.

Consider two $\mathcal{O}_X(-1)$-valued bilinear forms: the first one is **skew symmetric** form

$$\pi : \mathcal{U} \to \mathcal{U}\hat{} \otimes \mathcal{O}_X(-1) = \mathcal{U}\hat{}^L$$

given by the exterior multiplication $\mathcal{U}\times\mathcal{U}\to\bigwedge^2\mathcal{U}=\mathcal{O}_X(-1)$, and the second one is the **symmetric** hyperbolic form

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}:\mathcal{O}_X\oplus\mathcal{O}_X(-1)\to\mathcal{O}_X(-1)\oplus\mathcal{O}_X=(\mathcal{O}_X\oplus\mathcal{O}_X(-1))^{\hat{}L}.$$

Then use common resolution

$$\begin{array}{l}0\leftarrow\mathcal{O}_X(1)\xleftarrow{[z_0,-z_1,z_2]}{\mathcal{O}_X}^3\xleftarrow{f}\mathcal{U}\leftarrow 0\\ 0\leftarrow\mathcal{O}_X(1)\xleftarrow{[z_0,-z_1,0]}{\mathcal{O}_X}^3\xleftarrow{\left[\begin{smallmatrix}0&z_1\\0&z_0\\1&0\end{smallmatrix}\right]}\mathcal{O}_X\oplus\mathcal{O}_X(-1)\leftarrow 0\end{array}$$

to produce four-term double exact sequence.

Proposition 4.7. *The class of the double exact sequence*

$$\mathcal{O}_X(1)\;\underset{[z_0,-z_1,0]}{\overset{[z_0,-z_1,z_2]}{\twoheadleftarrow}}\;{\mathcal{O}_X}^3\;\underset{\left[\begin{smallmatrix}0&0&z_1\\0&0&z_0\\z_1&z_0&0\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}0&-z_2&-z_1\\z_2&0&-z_0\\z_1&z_0&0\end{smallmatrix}\right]}{\longleftarrow}}\;\mathcal{O}_X(-1)^3\;\underset{\left[\begin{smallmatrix}z_0\\-z_1\\0\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}z_0\\-z_1\\z_2\end{smallmatrix}\right]}{\leftarrowtail}}\;\mathcal{O}_X(-2)\;. \tag{32}$$

in $K_1(X)=K_1(F)[\mathcal{O}_X]\oplus K_1(\mathcal{D})[\mathcal{U}]$ *equals to* $\{-1\}[\mathcal{O}_X]+\{1\}[\mathcal{U}]$.

Proof. Two resolutions

$$\begin{array}{l}0\leftarrow\mathcal{O}_X(1)\xleftarrow{[z_0,-z_1,z_2]}{\mathcal{O}_X}^3\xleftarrow{f}\mathcal{U}\leftarrow 0\\ 0\leftarrow\mathcal{O}_X(1)\xleftarrow{[z_0,-z_1,0]}{\mathcal{O}_X}^3\xleftarrow{\left[\begin{smallmatrix}0&z_1\\0&z_0\\1&0\end{smallmatrix}\right]}\mathcal{O}_X\oplus\mathcal{O}_X(-1)\leftarrow 0\end{array}$$

glued with their $\hat{}^L$-duals along π and $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$ give the double exact sequence

$$\mathcal{O}_X(1)\;\underset{[z_0,-z_1,0]}{\overset{[z_0,-z_1,z_2]}{\twoheadleftarrow}}\;{\mathcal{O}_X}^3\;\underset{\left[\begin{smallmatrix}0&0&z_1\\0&0&z_0\\z_1&z_0&0\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}0&-z_2&-z_1\\z_2&0&-z_0\\z_1&z_0&0\end{smallmatrix}\right]}{\longleftarrow}}\;\mathcal{O}_X(-1)^3\;\underset{\left[\begin{smallmatrix}z_0\\-z_1\\0\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}z_0\\-z_1\\z_2\end{smallmatrix}\right]}{\leftarrowtail}}\;\mathcal{O}_X(-2)\;. \tag{33}$$

This double exact sequence may be resolved as follows: in the commutative diagram

$$
\begin{array}{ccccccc}
\mathcal{O}_X(1) & \xleftarrow[{[z_0,-z_1]}]{{[z_0,-z_1]}} & {\mathcal{O}_X}^2 & \xleftarrow[\left[\begin{smallmatrix} z_1 \\ z_0 \end{smallmatrix}\right]]{\left[\begin{smallmatrix} z_1 \\ z_0 \end{smallmatrix}\right]} & \mathcal{O}_X(-1) & \longleftarrow & 0 \\
{\scriptstyle 1}\Big\downarrow{\scriptstyle 1} & & \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\Big\downarrow\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] & & \left[\begin{smallmatrix} 0 \\ 0 \\ -1 \end{smallmatrix}\right]\Big\downarrow\left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right] & & \Big\downarrow \\
\mathcal{O}_X(1) & \longleftarrow & {\mathcal{O}_X}^3 & \longleftarrow & \mathcal{O}_X(-1)^3 & \longleftarrow & \mathcal{O}_X(-2) \\
\Big\downarrow & & {\scriptstyle [0,0,1]}\Big\downarrow{\scriptstyle [0,0,1]} & & \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]\Big\downarrow\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right] & & {\scriptstyle 1}\Big\downarrow{\scriptstyle 1} \\
0 & \longleftarrow & \mathcal{O}_X & \xleftarrow[{[z_1,z_0]}]{{[z_1,z_0]}} & \mathcal{O}_X(-1)^2 & \xleftarrow[\left[\begin{smallmatrix} z_0 \\ -z_1 \end{smallmatrix}\right]]{\left[\begin{smallmatrix} z_0 \\ -z_1 \end{smallmatrix}\right]} & \mathcal{O}_X(-2)
\end{array}
$$

all displayed d.s.e.s. but one have equal upper and lower part. Thus the double exact sequence (33) defines the same element of $K_1(X)$ as the one defined by the split d.s.e.s.

$$
\mathcal{O}_X(-1)^2 \xleftarrow[\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]]{\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]} \mathcal{O}_X(-1)^3 \xleftarrow[\left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]]{\left[\begin{smallmatrix} 0 \\ 0 \\ -1 \end{smallmatrix}\right]} \mathcal{O}_X(-1).
$$

The same element of $K_1(X)$ is defined by the automorphism of taking opposite on $\mathcal{O}_X(-1)$, since there is a commutative diagram of d.s.e.s.'s:

$$
\begin{array}{ccccc}
0 & \longleftarrow & \mathcal{O}_X(-1) & \xleftarrow[1]{-1} & \mathcal{O}_X(-1) \quad . \\
\Big\downarrow & & \Big\downarrow & & {\scriptstyle 1}\Big\downarrow{\scriptstyle 1} \\
\mathcal{O}_X(-1)^2 & \xleftarrow[\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]]{\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]} & \mathcal{O}_X(-1)^3 & \xleftarrow[\left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]]{\left[\begin{smallmatrix} 0 \\ 0 \\ -1 \end{smallmatrix}\right]} & \mathcal{O}_X(-1), \\
{\scriptstyle 1}\Big\downarrow{\scriptstyle 1} & & \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]\Big\downarrow\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right] & & \Big\downarrow \\
\mathcal{O}_X(-1)^2 & \xleftarrow[1]{1} & \mathcal{O}_X(-1)^2 & \longleftarrow & 0
\end{array}
$$

Since

$$\{-1\}[\mathcal{O}_X(-1)] = \{-1\}\left([\mathcal{U}]-1\right) = \{\mathrm{Nrd}(-1)\}[\mathcal{U}] + \{-1\}[\mathcal{O}_X]$$

in $K_1(X)$, the assertion follows. □

This computation is important for determining discriminants of generators of the Witt group $W(X,L)$ found by Susane Pumplün in [16, Cor. 4.4].

A (skew) symmetric isomorphism $\phi : \mathcal{U} \to D\mathcal{U}(-1)$ yields an involution $* = *^{\phi}$ on $\mathrm{End}_X(\mathcal{U})$: if $\psi \in \mathrm{End}_X(\mathcal{U})$ then

$$\psi^* = \phi^{-1} \circ \left(D\psi \otimes 1_{\mathcal{O}_X(-1)}\right) \circ \phi.$$

Proposition 4.8. *The involution $*^{\pi}$ coincides with the conjugation $\alpha \mapsto \overline{\alpha}$ on the quaternion algebra* $\mathrm{End}_X(\mathcal{U}) = C_0 = \left(\frac{a,b}{F}\right)$.

$$(q_0 + q_1 i + q_2 j + q_3 k)^{*\pi} = q_0 - q_1 i - q_2 j - q_3 k.$$

Proof. Replace $\mathcal{U}$ with isomorphic $\varphi\mathcal{U} \subset \mathcal{O}_X \otimes C_0$. The exact sequence

$$\varphi \cdot \mathcal{U} \hookrightarrow \mathcal{O}_X \otimes C_0 \twoheadrightarrow \mathcal{U}(1)$$

splits locally, so it yields an exact sequence on each of the principal affine open subsets

$$D(z_m) = \{(z_0 : z_1 : z_2) : z_m \neq 0\} \text{ for } m = 0,1,2.$$

Locally, in $D(z_m)$, the module $\varphi{\cdot}\mathcal{U}(D(z_m))$ as a submodule of $\mathcal{O}(D(z_m)) \otimes_F C_0$ is generated by products of φ times $v_0, v_1, v_2, v_2v_1v_0$:

$$\begin{aligned} &\tfrac{1}{z_m}(z_0 - z_1 i - z_2 j), && \tfrac{1}{z_m}(-a z_1 + z_0 i + z_2 k) \\ &\tfrac{1}{z_m}(-b z_2 + z_0 j - z_1 k) && \tfrac{1}{z_m}(b z_2 i - a z_1 j + z_0 k). \end{aligned} \tag{34}$$

The module $\mathcal{U}(1)|_{D(x_m)} = \mathcal{U}|_{D(z_m)}$ is generated by cosets of $1, i, j, k$ modulo $\varphi \cdot \mathcal{U}|_{D(z_m)}$. Thus the module $\bigwedge^2 \mathcal{U}|_{D(z_m)}$ is generated by the exterior products

$$1 \wedge i, \quad 1 \wedge j, \quad 1 \wedge k, \quad i \wedge j, \quad i \wedge k, \quad j \wedge k$$

subject to twelve identities of the form

$$(\text{generator }(34)) \wedge (\text{one of } 1, i, j, k) = 0,$$

e.g.

$$\begin{aligned}
&\tfrac{1}{z_m}(z_1 1 \wedge i + z_2 1 \wedge j) = 0, \qquad \tfrac{1}{z_m}(z_0 1 \wedge i + z_2 1 \wedge k) = 0,\\
&\tfrac{1}{z_m}(z_0 1 \wedge i + z_2 i \wedge j) = 0, \qquad \tfrac{1}{z_m}(a z_1 i \wedge j + z_0 i \wedge k) = 0,\\
&\tfrac{1}{z_m}(b z_2 1 \wedge j - z_1 j \wedge k) = 0.
\end{aligned}$$

Thus the exterior multiplication

$$\wedge : \mathcal{U} \otimes \mathcal{U} \to \bigwedge^2 \mathcal{U} = \mathcal{O}_X(-1)$$

has in $D(z_m)$ the "Gram matrix"

$$\frac{1}{z_m}\begin{bmatrix} 0 & z_2 & -z_1 & -z_0 \\ -z_2 & 0 & -z_0 & -az_1 \\ z_1 & z_0 & 0 & -bz_2 \\ z_0 & az_1 & bz_2 & 0 \end{bmatrix}.$$

Now it is easy to see that

$$\alpha i \wedge \beta = -\alpha \wedge \beta i, \quad \alpha j \wedge \beta = -\alpha \wedge \beta j, \quad \alpha k \wedge \beta = -\alpha \wedge \beta k$$

for all $\alpha, \beta \in \mathcal{U}(D(z_m))$, so $(q_0 + q_1 i + q_2 j + q_3 k)^{*\pi} = q_0 - q_1 i - q_2 j - q_3 k$ holds in every $D(z_m)$, $m = 0, 1, 2$, and hence holds globally. □

Corollary 4.9. *For every pure quaternion* $\gamma \in \left(\dfrac{a, b}{F}\right)$ *the map*

$$\pi \circ (\cdot\gamma) : \mathcal{U} \to \mathcal{U}^{\wedge}(-1)$$

defines a nonsingular symmetric bilinear form with values in $\mathcal{O}_X(-1)$.

Proof. Locally

$$\begin{aligned}(\pi \circ (\cdot\gamma))(\alpha)(\beta) &= \pi(\alpha\cdot\gamma)(\beta) = (\alpha\cdot\gamma)\wedge\beta = -\alpha\wedge(\beta\cdot\gamma)\\ &= (\beta\cdot\gamma)\wedge\alpha = (\pi\circ(\cdot\gamma))(\beta)(\alpha). \quad \square\end{aligned}$$

Susane Pumplün proved following fact:

Proposition 4.10. *The Witt group $W(X, L)$ is generated by the classes of the forms*

$$p = \left(\mathrm{tr}_{F'/F}(\mathcal{O}_{X'}(-1)), \mathrm{tr}_{F'/F}(c\,\mathrm{id}\right), c \in F'. \tag{35}$$

Here F'/F is a quadratic extension which splits $\langle 1, -a, -b\rangle$ (splits $\left(\frac{a,b}{F}\right)$).

Proof. [16, Cor. 4.4]; the canonical bundle $\mathcal{I}$ is our $\mathcal{U}$. $\square$

We will refer to forms of the kind (35) as Pumplün generators.

Corollary 4.11. *For every pure quaternion $\gamma \in \left(\frac{a,b}{F}\right)$ there exists a constant $c \in \dot{F}$ such that*

$$\pi \circ (\cdot\gamma) = c \cdot p$$

for a suitable Pumplün generator p.

Proof. The bundle $\mathcal{U}$ is indecomposable, so the assertion follows immediately from [16, Prop. 4.1]. $\square$

Note that if $\pi \circ (\cdot\gamma) = c \cdot p$, then $\pi \circ (\cdot c^{-1}\gamma) = p$. Moreover the quadratic extension F' needed to define the Pumplün generator p is $F' = F(\gamma) \cong F\left[\sqrt{-\mathrm{Nrd}(\gamma)}\right]$.

Proposition 4.12. *For every Pumplün generator p there exist a pure quaternion $\eta \in \left(\frac{a,b}{F}\right)$ and a constant $c \in F^{\cdot}$ such that*

$$p \circ (\cdot\eta) = c\pi.$$

Proof. By the Noether-Skolem theorem every involution of the algebra $\left(\frac{a,b}{F}\right)$ is a composition of the canonical involution with an inner automorphism. Let $\gamma \in \left(\frac{a,b}{F}\right)$ be a quaternion such that

$$(\cdot\xi)^{*p} = \overline{\gamma^{-1}\xi\gamma}$$

for all $\xi \in \left(\frac{a,b}{F}\right)$. A priori there are two cases. If $\overline{\gamma} = \gamma$, then $*^p$ is the conjugation, and η is any pure quaternion. If $\overline{\gamma} \neq \gamma$, then $\eta = \overline{\gamma} - \gamma$ is a pure quaternion and

$$\eta^{*p} = \overline{\gamma^{-1}\delta\gamma} = -\eta$$

and in any case $p \circ (\cdot\eta)$ is skew symmetric, since locally

$$(p \circ (\cdot\eta))\,(\alpha)(\beta) = p(\alpha\eta)(\beta) = p(\alpha)(-\beta\eta) = -p(\beta\eta)(\alpha) = -\,(p \circ (\cdot\eta))\,(\beta)(\alpha).$$

Hence $p \circ (\cdot\eta)$ must be a scalar multiple of π. $\square$

Remark 4.1. It follows that the first case - when $*^p$ is the conjugation - never occurs: if $p = \pi \circ (\cdot\eta^{-1})$, for a pure quaternion η, then

$$(36)\quad p(\alpha\gamma)(\beta) = \pi(\alpha\gamma\eta^{-1})(\beta) = \pi(\alpha)(\beta\overline{\gamma\eta^{-1}}) = -\pi(\alpha)(\beta\eta^{-1}\overline{\gamma}) =$$

$$-\,\pi(\alpha)(\beta\eta^{-1}\overline{\gamma}\eta\eta^{-1}) = -\pi(\alpha\overline{\eta^{-1}})(\beta\eta^{-1}\overline{\gamma}\eta) = \pi(\alpha\eta^{-1})(\beta\eta^{-1}\overline{\gamma}\eta) = p(\alpha)(\beta\eta^{-1}\overline{\gamma}\eta)$$

$$\gamma^{*p} = \eta^{-1}\overline{\gamma}\eta = \overline{\eta\gamma\eta^{-1}}$$

$$\pi^{-1} \circ \alpha^{\hat{}L} \circ \pi = -\alpha.$$

The composition $\pi \circ \alpha : \mathcal{U} \to \mathcal{U}^{\hat{}L}$ is a symmetric bilinear form.

Proposition 4.13. *If α is an invertible anti-selfadjoint endomorphism of the sheaf $\mathcal{U}$, then*

$$e^1(\pi \circ \alpha) = (-1, \mathrm{Nrd}(\alpha)\dot{F}^2) \in \mu_2(F) \oplus B/\dot{F}^2.$$

Proof. The commutative diagram

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \twoheadleftarrow & \mathcal{U}^{\wedge L} & \overset{\pi\circ\alpha}{\underset{\pi}{\longleftarrow}} & \mathcal{U} & \leftarrowtail & 0 \\
 & & {\scriptstyle 1}\downarrow{\scriptstyle 1} & & {\scriptstyle \alpha}\downarrow{\scriptstyle 1} & & \\
0 & \twoheadleftarrow & \mathcal{U}^{\wedge L} & \overset{\pi}{\underset{\pi}{\longleftarrow}} & \mathcal{U} & \leftarrowtail & 0
\end{array} \tag{37}
$$

yields that $\ell\left(0 \twoheadleftarrow \mathcal{U} \overset{\pi\circ\alpha}{\underset{\pi}{\longleftarrow}} \mathcal{U} \leftarrowtail 0\right)$ is the element of $K_1(D)$ corresponding to $\alpha \in \mathrm{Aut}(\mathcal{U})$, which is obviously the reduced norm $\mathrm{Nrd}(\alpha)$. By lemma 2.16 element

$$\ell\left(0 \twoheadleftarrow \mathcal{U} \overset{\pi\circ\alpha}{\underset{\pi}{\longleftarrow}} \mathcal{U} \leftarrowtail 0\right)$$

coincides with the element defined by

$$
\mathcal{O}_X(1) \overset{[z_0,-z_1,z_2]}{\underset{[z_0,-z_1,0]}{\twoheadleftarrow}} {\mathcal{O}_X}^3 \overset{f\circ\pi\circ\alpha\circ Df}{\underset{f\circ\pi\circ Df}{\longleftarrow}} \mathcal{O}_X(-1)^3 \overset{\left[\begin{smallmatrix} z_0 \\ -z_1 \\ z_2 \end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix} z_0 \\ -z_1 \\ 0 \end{smallmatrix}\right]}{\leftarrowtail}} \mathcal{O}_X(-2)\ . \tag{38}
$$

Moreover by lemma 2.17 the sum of the element of $K_1(X)$ defined by (38) and the element defined by (33) coincides with the element defined by double exact sequence

$$
\mathcal{O}_X(1) \overset{[z_0,-z_1,z_2]}{\underset{[z_0,-z_1,0]}{\twoheadleftarrow}} {\mathcal{O}_X}^3 \overset{f\circ\pi\circ\alpha\circ Df}{\underset{\left[\begin{smallmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_0 \\ z_1 & z_0 & 0 \end{smallmatrix}\right]}{\longleftarrow}} \mathcal{O}_X(-1)^3 \overset{\left[\begin{smallmatrix} z_0 \\ -z_1 \\ z_2 \end{smallmatrix}\right]}{\underset{\left[\begin{smallmatrix} z_0 \\ -z_1 \\ 0 \end{smallmatrix}\right]}{\leftarrowtail}} \mathcal{O}_X(-2)
$$

which represents $e^1(\pi \circ \alpha)$. Thus

$$e^1(\pi\circ\alpha) = \varepsilon^1\left(\pi\circ\alpha \div \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = (-1, \mathrm{Nrd}(\alpha)).$$ □

Remark 4.2. The group B is generated by reduced norms of pure quaternions; in fact, every nonzero quaternion either is a pure quaternion or is a product of two pure quaternions.

Corollary 4.14. *If* $\mathrm{Nrd}(\alpha\beta)$ *is not a square, then the forms* $\pi \circ \alpha$ *and* $\pi \circ \beta$ *are not Witt equivalent.*

This corollary may be partially reversed:

Proposition 4.15. *If* γ, δ *are pure quaternions such that* $\mathrm{Nrd}(\gamma) = \mathrm{Nrd}(\delta)$, *then for some constant* $c \in \dot{F}$ *the spaces* $(\mathcal{U}, \pi \circ (\cdot\gamma))$ *and* $(\mathcal{U}, c \cdot \pi \circ (\cdot\delta))$ *are isomorphic.*

Proof. If $\gamma + \delta = 0$ then $c = -1$. The condition $\mathrm{Nrd}(\gamma) = \mathrm{Nrd}(\delta)$ yields

$$\gamma^2 = -\mathrm{Nrd}(\gamma) = -\mathrm{Nrd}(\delta) = \delta^2.$$

If $\gamma + \delta \neq 0$ then for $\lambda = \gamma + \delta$

$$\gamma\lambda = \lambda\delta \quad \text{i.e.} \quad \gamma = \lambda\delta\lambda^{-1}.$$

Thus locally

$$\begin{aligned}(\pi \circ (\cdot\gamma))(\alpha)(\beta) &= \pi(\alpha\gamma)(\beta) = \pi(\alpha\lambda\delta\lambda^{-1})(\beta) \\ &= \frac{1}{\mathrm{Nrd}(\lambda)}\pi(\alpha\lambda\delta\overline{\lambda})(\beta) = \frac{1}{\mathrm{Nrd}(\lambda)}\pi(\alpha\lambda\delta)(\beta\lambda) \\ &= \frac{1}{\mathrm{Nrd}(\lambda)}(\pi \circ (\cdot\delta))(\alpha\lambda)(\beta\lambda)\end{aligned}$$

which means that $\cdot\lambda \in \mathrm{End}_X(\mathcal{U})$ is an isomorphism of $\mathrm{Nrd}(\lambda)\pi \circ (\cdot\gamma)$ and $\pi \circ (\cdot\delta)$. □

Corollary 4.16. *The discriminant map* $e^1 : W(X, L) \to \mu_2(F) \oplus B/\dot{F}^2$ *is given by the formula:*

$$e^1\left(\sum_{m=1}^{n} \pi \circ (\cdot\alpha_m)\right) = ((-1)^n, \mathrm{Nrd}(\alpha_1 \cdot \cdots \cdot \alpha_n))$$

Corollary 4.17. *If* n *is odd or* $\mathrm{Nrd}(\alpha_1 \cdot \cdots \cdot \alpha_n)$ *is not a square, then the element* $\sum_{m=1}^{n} \pi \circ (\cdot\alpha_m)$ *is not zero in* $W(X, L)$.

References

[1] J. Kr. Arason, *Der Wittring projektiver Räume*, Math. Annalen **253** (1980), 205–212.

[2] H. Gillet, D. Grayson, *On the loop space of the Q-construction*, Illinois J. Math. 31, 4 (1987), 574-598.

[3] W. Bichsel, M.-A. Knus, *Quadratic forms with values in line bundles*, in: *Recent advances in real algebraic geometry and quadratic forms* (Berkeley, CA, 1990/1991; San Francisco, CA, 1991), Contemp. Math., **155** (1994), 293–306, Providence, RI

[4] B. Keller, *Chain complexes and stable categories*, Manuscripta Math. **67** (1990), pp. 379-417

[5] M. Knebusch, *Symmetric bilinear forms over algebraic varieties*, Conference on quadratic forms 1976, ed. G. Orzech, Queens papers in pure and applied math. **46**(1977), 103–283.

[6] M.-A. Knus, R. Parimala, *Quadratic forms associated with projective modules over quaternion algebras*, Crelle's J. Reine und Angew. Math. B **318** (1980) 20-31

[7] A.S. Merkuriev, A.A. Suslin, *K-cohomology of of Severi-Brauer varieties an norm-residue homomorphism*, Izv. AN USSR 46 N^o5 (1982), 1011-1046.

[8] J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1970), 318-344.

[9] J. Milnor, D. Husemoller, *Symmetric bilinear forms*, Springer V. Berlin 1973

[10] A. Nenashev, *Double Short Exact Sequences Produce All Elements of Quillen's K_1*, Algebraic K-theory (Poznań Conference on algebraic K-theory, 1995) Contemp. Math. **199** (1996), 151-160

[11] A. Nenashev, *K_1 by generators and relations.* J. Pure Appl. Alg. **131** (1998), 195-212

[12] A. Nenashev, *Categorical and simplicial constructions and dualization in algebraic K-theory*, K-theory **24** (2001), no. 4, 327–340.

[13] A. Nenashev, M. Szyjewski, *On the generalized discriminant for quadratic forms over schemes*, in preparation.

[14] R. Parimala, *Failure of a quadratic analogue of Serre's Conjecture*, Bull. Amer. Math. Soc. **82** (1976), 5-14.

[15] S. Pumplün, *The Witt ring of a Brauer-Severi variety*, Manuscripta Math. **97**(1998), 93-108

[16] S. Pumplün *The Witt group of symmetric bilinear forms over a Brauer-Severi variety with values in a line bundle*, K-Theory 18(1999), 255-265.

[17] D. Quillen, *Higher algebraic K-theory I*, Lecture Notes in Math. **341**, Springer, Berlin 1973.

[18] W. Scharlau, *Quadratic and Hermitian Forms*, Springer V. Berlin 1985.

[19] R. G. Swan, *K-theory of quadric hypersurfaces*, Annals of Math. **122** (1985), 113–153.

[20] M. Szyjewski, *An invariant of quadratic forms over schemes*, Documenta Math. J. DMV **1** (1996), 449-478; http://www.mathematik.uni-bielefeld.de/documenta.

[21] M. Szyjewski, *On the Witt ring of a relative projective line* Colloquium Math. No 1, **75**(1998), 53-78.

[22] M. Szyjewski, *Witt rings of Grassmann varieties* Algebraic K-theory (Poznań, 1995), 185–210, Contemp. Math., **199**, Amer. Math. Soc., Providence, RI, 1996.

[23] K. Szymiczek, *A structure theorem for Grothendieck group of quadratic forms over a field* Colloquia Mathematica Soc. János Bolyai vol. 13, Topics in Number Theory, Debrecen (Hungary) 1974, pp. 389-397

[24] M. Weiss, B. Williams, *Duality in Waldhausen categories*, Forum Math. **10**(1998) no. 5, 533-603.

[25] Witt, E. *Theorie der quadratischen Formen in beliebigen Körpern.* J. reine angew. Math. No **176** (1937), 31-44

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