

# Motivic construction of cohomological invariants

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## Abstract

The norm varieties and the varieties with special correspondences play a major role in the proof of the Bloch-Kato Conjecture by M. Rost and V. Voevodsky. In the present paper we show that a variety which possesses a special correspondence is a norm variety.

As an unexpected application we give a positive answer to a problem of J.-P. Serre about groups of type  $E_8$  over  $\mathbb{Q}$ . Apart from this we include the proof of the Voevodsky conjecture about  $\nu_n$ -varieties which is due to A. Vishik. This result plays an important role in our proofs.

## 1 Introduction

Let us recall some recent developments in the topics which are relevant for this paper.

### 1.1 Cohomological invariants

The study of cohomological invariants was initiated by J.-P. Serre in the 90'ies. Serre conjectured the existence of an invariant in  $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$  of  $G$ -torsors, where  $G$  is a simply-connected semisimple algebraic group over a field  $k$ . This invariant was constructed by M. Rost and is now called the *Rost invariant* of  $G$  (see [Inv, 31.B and pp. 448–449]).

Nowadays there exist numerous constructions and estimations of cohomological invariants for different classes of algebraic objects (see e.g. [GMS]). Nevertheless, the most constructions of cohomological invariants rely on a specific construction of the object under consideration. Unfortunately, for

many groups, like  $E_8$ , there is no classification and no general construction so far.

## 1.2 Chow motives of projective homogeneous varieties

Another direction which we discuss now is the theory of motives of projective homogeneous varieties. Typical examples of such varieties are projective quadrics and Severi-Brauer varieties.

The study of projective homogeneous varieties was initiated by Rost when he provided a motivic decomposition of a Pfister quadric used later in the proof of the Milnor conjecture (see [Ro98]). The motives of Severi-Brauer varieties were studied by Karpenko and the motives of general quadrics by Izuhboldin, Karpenko, Merkurjev, and Vishik.

A systematical theory of motives of general projective homogeneous varieties was developed in a series of our papers with a culmination in [PSZ] and [PS] where the structure of the motive of a generically split homogeneous variety was established. Moreover, in some cases we provided an alternative construction of *generalized Rost motives* as indecomposable direct summands of generically split varieties (see [PSZ, §7]).

The technique developed in our papers does not use cohomological invariants of algebraic groups and of projective homogeneous varieties at all, and our arguments don't rely on the classification or on a construction of algebraic groups.

## 1.3 The Bloch-Kato Conjecture

The Bloch-Kato conjecture (more precisely its proof) provides a bridge between cohomological invariants and motives.

The Bloch-Kato conjecture says that for any  $n$  and any prime number  $p$  the norm residue homomorphism

$$K_n^M(k)/p \rightarrow H^n(k, \mu_p^{\otimes n})$$
$$\{a_1, \dots, a_n\} \mapsto (a_1) \cup \dots \cup (a_n)$$

between the Milnor K-theory and the Galois cohomology is an isomorphism.

The case  $p = 2$  of the Bloch-Kato conjecture is known as the Milnor conjecture.

The proof of the Bloch-Kato conjecture goes as follows. Given a pure symbol  $u$  in  $H^n(k, \mu_p^{\otimes n})$  M. Rost constructs a splitting variety  $X_u$  for  $u$  with some additional properties (If  $p = 2$  one takes norm quadrics, for  $n = 2$  one takes Severi-Brauer varieties, for  $p = 3$  the construction is based on the Merkurjev-Suslin varieties (see [S08] for an explicit construction for  $p = n = 3$ ), for  $p \geq 5$  the construction is implicit). Then V. Voevodsky using the category of his motives and symmetric powers splits off a certain direct summand  $M_u$  from the motive of  $X_u$  (see [Vo03]). The motive  $M_u$  is called the *generalized Rost motive*.

Later Rost was able to simplify Voevodsky's construction and found the same motive  $M_u$  staying inside the category of classical Chow motives (see [Ro07]). His main idea was to produce an algebraic cycle on  $X_u$  called a *special correspondence* (see Definition 4.2), forget about the symbol  $u$  and use the special correspondence to construct the motive  $M_u$ . This motive is used then to construct exact triangles in the category of Voevodsky's motives. These triangles essentially involve mixed motives, in particular, deviate from the category of classical Chow motives, and are used (among other results of Rost and Voevodsky) to finish the proof of the Bloch-Kato conjecture.

The symbol  $u$  is a cohomological invariant of the variety  $X_u$  (see [Vo03, Theorem 7.3]). Apart from this, one can notice that the special correspondences of Rost resemble the algebraic cycles that we construct in the proof of our motivic decompositions (see [Ro07, Lemma 5.2] and [PSZ, Lemma 4.15]).

Summarizing: The technique developed by Rost and Voevodsky in the proof of the Bloch-Kato conjecture gives a way to produce algebraic cycles and motives out of cohomological invariants of varieties.

## 1.4 Conclusion

The discussion in the above subsections leads to the following conjecture. Namely, there should exist a way back from motives to cohomological invariants. Note that an evidence of this conjecture appears already in papers of O. Izhboldin and A. Vishik [IzhV00] and [Vi98], where the case of quadrics was treated.

In the present paper we show in a constructive way that the varieties which possess a special correspondence of Rost admit cohomological invariants.

This result has several unexpected applications in the theory of algebraic groups. In the present paper we discuss just one of them. Namely, we

construct a cohomological invariant in  $H^5(k, \mathbb{Z}/2)$  for any group of type  $E_8$  whose Rost invariant is trivial. In turn, this invariant allows to give a positive answer to a question of Serre about  $E_8$  (see the last section).

## 1.5 Structure of the paper

First we recall some results on algebraic cobordism of Levine-Morel and  $\nu_n$ -varieties. We need them to prove the Voevodsky conjecture. The proof belongs to A. Vishik, and we represent it here with his kind permission. Then we recall basic properties of the category of motives of Voevodsky and the Rost construction of special correspondences. Finally, we provide some general results which allow to “compress” varieties and finish the proofs of all main results of the present paper.

## 2 Algebraic cobordism and $\nu_n$ -varieties

**2.1** (Lazard ring). Fix a prime  $p$  and a field  $k$  with  $\text{char } k = 0$ . Consider the algebraic cobordism  $\Omega^*$  of Levine-Morel (see [LM]) and the Lazard ring  $\mathbb{L} = \Omega(\text{Spec } k)$ . This is a graded ring generated by the cobordism classes of maps  $[X] = [X \rightarrow \text{Spec } k] \in \mathbb{L}^{-\dim X}$ , where  $X$  is a smooth projective variety over  $k$ . It is known that this ring is isomorphic to the polynomial ring with integer coefficients on a countable set of variables.

**2.2** (Characteristic numbers). Given a partition  $J = (l_1, \dots, l_r)$  of arbitrary length  $r \geq 0$  with  $l_1 \geq l_2 \geq \dots \geq l_r > 0$  one can associated with it a characteristic class  $c_J(X) \in \text{CH}^{|J|}(X)$  ( $|J| = \sum_{i \geq 1} l_i$ ) of  $X$  as follows: Let  $P_J(x_1, \dots, x_r)$  be the smallest symmetric polynomial containing the monomial  $x_1^{l_1} \dots x_r^{l_r}$ . We can express  $P_J$  as a polynomial on the standard symmetric function  $\sigma_1, \dots, \sigma_r$  as  $P_J(x_1, \dots, x_r) = Q_J(\sigma_1, \dots, \sigma_r)$  for some polynomial  $Q_J$ . Let  $c_i = c_i(-T_X)$  denote the  $i$ -th Chern class of the virtual normal bundle of  $X$ . Then

$$c_J(X) = Q_J(c_1, \dots, c_r).$$

The degrees of the characteristic classes are called the characteristic numbers.

If  $J = (1, \dots, 1)$  ( $i$  times),  $c_J(X) = c_i(-T_X)$  is the usual Chern class. We denote  $b_i(X) := c_{(p-1, \dots, p-1)}(X)$  ( $i$  times). In particular, if  $p = 2$  then  $b_i(X) = c_i(-T_X)$ . If  $J = (p^n - 1)$ , we write  $c_J(X) = s_{p^n-1}(X)$ . The class

$s_d(X)$  is called the  $d$ -th *Milnor class* of  $X$ . Fortunately, if  $d := \dim X = p^n - 1$  we have

$$b_d(X) = \pm s_d(X) \bmod pI(X) + p^2\mathbb{Z}_{(p)},$$

where  $I(X) = \deg(\mathrm{CH}_0(X)) \subset \mathbb{Z}$  (see [Ro07, Lemma 9.13]).

**2.3 Definition** ( $\nu_n$ -varieties). Let  $p$  be a fixed prime.

1. A smooth projective variety  $X$  is called a  $\nu_n$ -variety if  $\dim X = p^n - 1$ , all characteristic numbers of  $X$  are divisible by  $p$ , and

$$s_{\dim X}(X) \neq 0 \bmod p^2.$$

2. A smooth projective variety  $X$  is called a  $\nu_{\leq n}$ -variety if for all  $0 \leq i \leq n$  there exists a  $\nu_i$ -variety  $Y_i$  and a morphism  $Y_i \rightarrow X$ .

**2.4 Example.** For  $p = 2$  a typical example of a  $\nu_n$ -variety is a smooth projective  $(2^n - 1)$ -dimensional quadric (see [Vo01, Proposition 3.4]).

**2.5** (Landweber-Novikov operations). Further on one can construct the Landweber-Novikov operations on the cobordism ring  $\Omega^*(V)$  for a smooth quasi-projective variety  $V$ . Given a partition  $J$  as above we write

$$S_{L-N}^J: \Omega^*(V) \rightarrow \Omega^{*+|J|}(V)$$

for the operation which maps the cobordism class  $[f: X \rightarrow V] \in \Omega^*(V)$  to  $f_*(Q_J(c_1^\Omega, \dots, c_r^\Omega))$  where  $c_i^\Omega = c_i^\Omega(-T_X + f^*(T_V))$  are the cobordism Chern classes (see [Vi07, Definition 2.12]).

Since  $\Omega^*$  is a universal cohomology theory, there exists a canonical map  $\mathrm{pr}: \Omega^* \rightarrow \mathrm{CH}^*$  which turns out to be surjective. For a particular partition  $J = (p-1, \dots, p-1)$  ( $i$  times) denote  $S_{L-N}^i := S_{L-N}^J$ . Then the following diagram commutes:

$$\begin{array}{ccccc} \Omega^*(V) & \xrightarrow{\mathrm{pr}} & \mathrm{CH}^*(V) & \longrightarrow & \mathrm{Ch}^*(V) \\ S_{L-N}^i \downarrow & & & & \downarrow S^i \\ \Omega^{*+(p-1)i}(V) & \xrightarrow{\mathrm{pr}} & \mathrm{CH}^{*+(p-1)i}(V) & \longrightarrow & \mathrm{Ch}^{*+(p-1)i}(V) \end{array}$$

where  $\mathrm{Ch}^* := \mathrm{CH}^*/p$  and  $S^i$  denotes the  $i$ -th Steenrod operation in the Chow theory (see [Br03]). In particular, since  $S^i|_{\mathrm{CH}^m(V)} = 0$  for  $i > m$ , the image

$$\mathrm{pr} \circ S_{L-N}^i(\Omega^m(V))$$

is divisible by  $p$  for such  $m$ . Moreover, the total operation  $S_{L-N} := \sum_i S_{L-N}^i$  is a ring homomorphism (see [Za08], [Vi07], and [LM]).

For an integer  $n$  let  $I(p, n)$  denote the ideal of  $\mathbb{L}$  generated by the varieties of dimension  $\leq p^n - 1$  with all characteristic numbers divisible by  $p$ . This is a prime ideal invariant under the action of the Landweber-Novikov operations on  $\mathbb{L}$ . By definition  $I(p, n-1) \subset I(p, n)$ . Moreover, as an  $\mathbb{L}$ -ideal  $I(p, n)$  is generated by  $I(p, n-1)$  and by any  $\nu_n$ -variety.

**2.6 Lemma** (Voevodsky's conjecture). *A  $\nu_n$ -variety is a  $\nu_{\leq n}$ -variety.*

*Proof.* (A. Vishik) It is well known that a smooth hypersurface of degree  $p$  in  $\mathbb{P}^{p^n}$  is a  $\nu_n$ -variety (see [Vo01, Proposition 3.4]). Denote its class in  $\mathbb{L}$  as  $u$ . Moreover, a direct computation shows that  $S_{L-N}^{p^{n-1}}(u)$  is a multiple of the class of a hypersurface of degree  $p$  in  $\mathbb{P}^{p^{n-1}}$  with a coefficient not divisible by  $p$ . In particular,  $S_{L-N}^{p^{n-1}}(u)$  is a  $\nu_{n-1}$ -element.

Let now  $v_n$  be a  $\nu_n$ -element in  $\mathbb{L}$ . Then  $(av_n - u) \in I(p, n-1)$  for some  $a \in \mathbb{Z}$  coprime to  $p$ . To conclude that  $v_n$  is a  $\nu_{n-1}$ -element, it suffices to show that  $S_{L-N}^{p^{n-1}}(y) \in I(p, n-2)$  for any  $y \in I(p, n-1)$ . We can write

$$y = v_{n-1}x_{n-1} + v_{n-2}x_{n-2} + \dots + v_0x_0$$

where  $v_i$  is a  $\nu_i$ -element and  $x_i \in \mathbb{L}$ . Since the total operation  $S_{L-N}$  is multiplicative, we see that

$$S_{L-N}^{p^{n-1}}(v_{n-2}x_{n-2} + \dots + v_0x_0) \in I(p, n-2).$$

Moreover,  $S_{L-N}^i(v_{n-1}) \in I(p, n-2)$  for all  $i > 0$ . Finally, we have the term

$$S_{L-N}^0(v_{n-1})S_{L-N}^{p^{n-1}}(x_{n-1}) = v_{n-1}S_{L-N}^{p^{n-1}}(x_{n-1}).$$

But  $S_{L-N}^{p^{n-1}}(x_{n-1})$  is divisible by  $p$ , since this Landweber-Novikov operation induces the Steenrod operation  $S^{p^{n-1}}$  on the Chow group. Thus,

$$S_{L-N}^{p^{n-1}}(y) \in I(p, n-2),$$

and we are done. □

### 3 Category DM

The base field  $k$  is assumed to have characteristic zero, and all varieties are assumed to be irreducible.

If  $A \rightarrow B$  is a morphism in a triangulated category, then there exists a unique up to isomorphism object  $C$  with an exact triangle of the form  $A \rightarrow B \rightarrow C$ . We write  $C = \text{Cone}(A \rightarrow B)$ .

The following lemma is a standard property of triangulated categories.

**3.1 Lemma.** *Assume we are given an exact (or distinguished) triangle*

$$A \rightarrow B \rightarrow C \rightarrow A[1]$$

*in a triangulated category. Then for any object  $U$  in this category the following induced sequences are exact:*

$$\dots \rightarrow \text{Hom}(C, U) \rightarrow \text{Hom}(B, U) \rightarrow \text{Hom}(B, U) \rightarrow \text{Hom}(C[-1], U) \rightarrow \dots$$

$$\dots \rightarrow \text{Hom}(U, A) \rightarrow \text{Hom}(U, B) \rightarrow \text{Hom}(U, C) \rightarrow \text{Hom}(U, A[1]) \rightarrow \dots$$

**3.2 Definition.** Let  $X$  be a smooth projective variety over  $k$ . As  $\check{C}(X)$  we denote the *standard simplicial scheme* associated with  $X$ :

$$X \rightrightarrows X \times X \rightleftarrows X \times X \times X \cdots$$

In the present paper we work in the category  $\text{DM} := \text{DM}_-^{eff}(k)$  of effective motives of V. Voevodsky over  $k$  with  $\mathbb{Z}/p$ -coefficients, where  $p$  is a prime number. It is well-known that there are functors from the category of smooth schemes over  $k$  and from the category of smooth simplicial schemes over  $k$  to DM. Therefore we can speak about the motives of smooth schemes and smooth simplicial schemes.

The category of classical Chow motives is a full subcategory of DM closed under direct summands. For an integer  $b$  we set  $\{b\} := (b)[2b]$ . For example, if  $M \in \text{DM}$ , then  $M\{b\}[1]$  means  $M(b)[2b+1]$  and  $M\{b\}$  is a *Tate twist* of  $M$ . For  $M \in \text{DM}$  its *motivic cohomology*:

$$H^{d,c}(M, \mathbb{Z}/p) = \text{Hom}(M, \mathbb{Z}/p(c)[d]).$$

We recall now some well-known properties of the category DM which are due to Voevodsky.

**3.3 Lemma** ([Vo95a, Proposition 4.2.9] and [Vo95b, Proposition 2.7]). *Let  $X$  be a smooth projective variety over  $k$ . Then*

$$\mathrm{Hom}(X, \mathbb{Z}/p(i)[j]) = \mathrm{CH}(X, 2i - j) \otimes \mathbb{Z}/p$$

*is the higher Chow group of  $X$  and*

$$\mathrm{Hom}(X, \mathbb{Z}/p(i)[2i]) = \mathrm{CH}^i(X) \otimes \mathbb{Z}/p =: \mathrm{Ch}^i(X)$$

*is the Chow group of  $X$ . If  $X = \mathrm{Spec} k$ , then additionally*

$$\mathrm{Hom}(X, \mathbb{Z}/p(i)[i]) = K_i^M(k)/p$$

*is the Milnor  $K$ -theory of  $k \bmod p$ .*

**3.4 Lemma** ([Vi98, Theorem 2.3.4]). *Let  $X$  be a smooth projective variety over  $k$  and let  $\mathcal{X}$  be the motive of  $\check{C}(X)$ . The natural morphism  $\mathcal{X} \rightarrow \mathrm{Spec} k$  is an isomorphism if and only if  $X$  has a zero-cycle of degree coprime to  $p$ .*

**3.5 Lemma** ([Vo96, Corollary 2.3]). *Let  $X$  be a smooth projective variety over  $k$  and  $M$  a direct summand of the motive of  $X$ . Then*

$$\mathrm{Hom}(M, \mathbb{Z}/p(c)[d]) = 0$$

*for  $d - c > \dim X$  and for  $d > 2c$ .*

**3.6 Lemma** (Cancellation; [Vo02, Corollary 4.10]). *Let  $A, B \in \mathrm{DM}$ . Then for all  $i, j \in \mathbb{Z}$ ,  $i \geq 0$ , we have*

$$\mathrm{Hom}(A, B) = \mathrm{Hom}(A(i)[j], B(i)[j]).$$

**3.7 Definition.** A localizing subcategory of  $\mathrm{DM}$  generated by an object  $A \in \mathrm{DM}$  is a minimal triangulated subcategory closed under direct summands, arbitrary direct sums, tensor products and Tate twists, and containing  $A$ .

**3.8 Lemma** ([Vo96, Lemma 3.8 and Lemma 4.7]). *Let  $X$  be a smooth projective variety over  $k$  and  $M$  an object of the localized subcategory of  $\mathrm{DM}$  generated by  $X$ . Denote as  $\mathcal{X}$  the motive of  $\check{C}(X)$ . Then*

1. *the natural morphism  $M \otimes \mathcal{X} \rightarrow M$  is an isomorphism,*
2. *the natural homomorphism  $\mathrm{Hom}(\mathbb{Z}/p, M) \rightarrow \mathrm{Hom}(\mathcal{X}, M)$  is an isomorphism,*

3. and the natural homomorphism  $\mathrm{Hom}(M, \mathcal{X}) \rightarrow \mathrm{Hom}(M, \mathbb{Z}/p)$  is also an isomorphism.

In particular, for all  $c, d$

$$\mathrm{Hom}(\mathcal{X}, \mathbb{Z}/p(c)[d]) \simeq \mathrm{Hom}(\mathcal{X}, \mathcal{X}(c)[d]) \simeq \mathrm{Hom}(\mathbb{Z}/p, \mathcal{X}(c)[d]).$$

In DM there is an action of the *Milnor operations* for smooth simplicial schemes  $U$

$$Q_i: \mathrm{Hom}(U, \mathbb{Z}/p(r)[s]) \rightarrow \mathrm{Hom}(U, \mathbb{Z}/p(r + p^i - 1)[s + 2p^i - 1]).$$

In the same way there is an action of the Milnor operations on the *reduced motivic cohomology*  $\widetilde{H}^{*,*}(U, \mathbb{Z}/p)$  for smooth pointed simplicial schemes  $U$ . They satisfy the following properties:

**3.9 Lemma** ([Vo96, Theorem 3.17]). *Let  $U$  be a smooth simplicial scheme (resp. smooth pointed simplicial scheme) and  $u, v \in H^{*,*}(U, \mathbb{Z}/p)$  (resp.  $u, v \in \widetilde{H}^{*,*}(U, \mathbb{Z}/p)$ ). Then*

1.  $Q_i^2 = 0$ ;
2.  $Q_i Q_j + Q_j Q_i = 0$ ;
3.  $Q_i(uv) = Q_i(u)v + uQ_i(v) + \sum \xi^{n_j} \varphi_j(u)\psi_j(v)$ , where  $n_j > 0$ ,  $\varphi_j$  and  $\psi_j$  are some homogeneous cohomological operations of bidegrees  $(*, *)$  with  $*' > 2* \geq 0$ , and  $\xi$  is the class of  $-1$  in  $\mathrm{Hom}(\mathbb{Z}/p, \mathbb{Z}/p(1)[1]) = k^*/k^{*p}$ . (In particular,  $\xi = 0$  if  $p$  is odd).

Thus, the following sequence is a complex:

$$\widetilde{H}^{*-2p^i+1, *'-p^i+1}(U, \mathbb{Z}/p) \xrightarrow{Q_i} \widetilde{H}^{*,*'}(U, \mathbb{Z}/p) \xrightarrow{Q_i} \widetilde{H}^{*+2p^i-1, *'+p^i-1}(U, \mathbb{Z}/p). \quad (1)$$

**3.10 Definition.** The cohomology groups of complex (1) are called *Margolis motivic cohomology groups* and are denoted as  $\widetilde{HM}_i^{*,*'}(U)$ .

The following lemma is of particular importance for us.

**3.11 Lemma.** *Let  $X$  be a  $\nu_n$ -variety over  $k$  and  $\mathcal{X}$  be the motive of  $\check{C}(X)$ . Denote as  $\widetilde{\mathcal{X}} = \mathrm{Cone}(\mathcal{X} \rightarrow \mathbb{Z}/p)$  the cone of the natural projection. Then for all  $0 \leq i \leq n$  and all  $*, *' \in \mathbb{Z}$*

$$\widetilde{HM}_i^{*,*'}(\widetilde{\mathcal{X}}) = 0.$$

*Proof.* The lemma immediately follows from [Vo03, Lemma 4.3] and Lemma 2.6.  $\square$

**3.12 Lemma** ([Vo96, Corollary 2.2]). *We have  $\mathrm{Hom}(\mathcal{X}, \mathbb{Z}/p(c)[d]) = 0$  if  $c < 0$ .*

**3.13 Lemma** ([IzhV00, Proof of Sublemma 6.2]). *Let  $X$  be a smooth projective variety and  $\mu' \in \mathrm{Hom}(\mathcal{X}, \mathcal{X}\{b\}[1])$  for some  $b$ . Let  $\mu \in \mathrm{Hom}(\mathcal{X}, \mathbb{Z}/p\{b\}[1])$  be the image of  $\mu'$  under the natural map of Lemma 3.8(3).*

*Then for all  $c, d$  the pull-back*

$$(\mu')^*: \mathrm{Hom}(\mathcal{X}, \mathbb{Z}/p(c)[d]) \rightarrow \mathrm{Hom}(\mathcal{X}, \mathbb{Z}/p(c+b)[d+2b+1])$$

*coincides with the multiplication by  $\mu$ .*

Die nächsten zwei Lemmata folgen aus der Bloch-Kato Vermutung.

**3.14 Lemma** ([Vo03, Lemma 6.6]). *Let  $X$  be a smooth projective variety over  $k$ , let  $\mathcal{X}$  be the motive of  $\check{C}(X)$ , and  $\tilde{\mathcal{X}} = \mathrm{Cone}(\mathcal{X} \rightarrow \mathrm{Spec} k)$ .*

*Then*

$$\tilde{H}^{d,c}(\tilde{\mathcal{X}}, \mathbb{Z}/p) = 0$$

*for all  $d \leq c + 1$ .*

**3.15 Lemma** ([Ro07, Lemma 2.1]). *Let  $X$  be a smooth projective (irreducible) variety over  $k$  and let  $\mathcal{X}$  be the motive of  $\check{C}(X)$ . Then there is a natural exact sequence:*

$$0 \rightarrow H^{n,n-1}(\mathcal{X}, \mathbb{Z}/p) \rightarrow H_{et}^n(k, \mu_p^{\otimes(n-1)}) \rightarrow H_{et}^n(k(X), \mu_p^{\otimes(n-1)}).$$

In particular, any element of  $H^{n,n-1}(\mathcal{X}, \mathbb{Z}/p)$  can be identified with an element of the Galois cohomology  $H_{et}^n(k, \mu_p^{\otimes(n-1)})$ .

## 4 Rost's construction

**4.1 Definition** (Norm varieties). Let  $u \in H_{et}^n(k, \mu_p^{\otimes n})$ . A variety  $X$  over  $k$  is called a norm variety for  $u$  if  $X$  is a  $\nu_{n-1}$ -variety and  $u_{k(X)} = 0$ .

If  $u$  is a pure symbol, i.e.,  $u = (a_1) \cup \dots \cup (a_n)$  for some  $a_i \in H^1(k, \mu_p) \simeq k^*/k^{*p}$ , then it was shown by M. Rost that a norm variety for  $u$  exists.

**4.2 Definition** (Rost). Let  $n$  be a positive integer,  $p$  a prime,  $b = \frac{p^{n-1}-1}{p-1}$ , and  $X$  a smooth projective variety over  $k$ . Consider the complex

$$\mathrm{CH}^b(X) \xrightarrow{\pi_0^* - \pi_1^*} \mathrm{CH}^b(X \times X) \xrightarrow{\pi_0^* - \pi_1^* + \pi_2^*} \mathrm{CH}^b(X \times X \times X)$$

where  $\pi_i$  is the  $i$ -th projection in the diagram

$$X \rightleftarrows X \times X \rightleftarrows X \times X \times X.$$

Let  $\rho \in \mathrm{CH}^b(X \times X)$ . Define

$$c(\rho) := (\pi_0)_*(\rho^{p-1}) \in \mathrm{CH}^0(X) = \mathbb{Z}.$$

The cycle  $\rho$  is called a *special correspondence* of type  $(n, p)$  for  $X$  if

$$(\pi_0^* - \pi_1^* + \pi_2^*)(\rho) = 0$$

and

$$c(\rho) \not\equiv 0 \pmod{p}.$$

M. Rost showed that a variety  $X$  which possesses a special correspondence and has no zero-cycles of degree coprime to  $p$  is a  $\nu_{n-1}$ -variety (see [Ro07, Section 9]).

**4.3 Example.** For  $p = 2$  let  $q = \langle\langle a_1, \dots, a_{n-1} \rangle\rangle \perp \langle -a_n \rangle$  for some  $a_i \in k^*$ . The projective quadric given by  $q = 0$  is a norm variety for the symbol  $(a_1) \cup \dots \cup (a_n)$  and possesses a special correspondence.

We recall now the construction of the special correspondences of Rost out of a symbol. Let  $n$  be an integer,  $p$  a prime,  $u \in H^n(k, \mu_p^{\otimes(n-1)})$  an element in the Galois cohomology of the field  $k$ , and  $X$  a smooth projective variety such that  $u_{k(X)} = 0$ .

By Lemma 3.15 the element  $u$  can be identified with an element  $\delta \in H^{n, n-1}(\mathcal{X}, \mathbb{Z}/p)$  of the motivic cohomology of the standard simplicial scheme  $\mathcal{X}$  associated with  $X$ .

Define now  $\mu = \tilde{Q}_0 \circ Q_1 \circ \dots \circ Q_{n-2}(\delta) \in H^{2b+1, b}(\mathcal{X}, \mathbb{Z})$ , where  $Q_i$  are the Milnor operations,  $\tilde{Q}_0$  is the integral valued Bockstein, and  $b = \frac{p^{n-1}-1}{p-1}$ .

Consider now the *skeleton filtration* on  $\mathcal{X}$ . Let  $Y = \mathrm{Cone}(X \rightarrow \mathcal{X})$  be the cone of the natural map. Then there is another natural map  $(X \times X)[1] \rightarrow Y$  such that the composition  $(X \times X)[1] \rightarrow Y \rightarrow X[1]$  equals  $\pi_0 - \pi_1$ .

Let  $Z = \text{Cone}((X \times X)[1] \rightarrow Y)$ . Then again there is a natural map  $(X \times X \times X)[2] \rightarrow Z$  such that the composition  $(X \times X \times X)[2] \rightarrow Z \rightarrow (X \times X)[2]$  equals  $\pi_0 - \pi_1 + \pi_2$ .

By Lemma 3.1 these exact triangles give the long exact sequences for motivic cohomology with integral coefficients:

$$H^{2b,b}(X) \rightarrow H^{2b+1,b}(Y) \rightarrow H^{2b+1,b}(X) \rightarrow H^{2b+1,b}(X)$$

and

$$H^{2b+1,b}(Y) \rightarrow H^{2b,b}(X \times X) \xrightarrow{\alpha} H^{2b+2,b}(Z).$$

Note now that  $H^{2b,b}(X) = \text{CH}^b(X)$ ,  $H^{2b+1,b}(X) = 0$ ,  $H^{2b,b}(X \times X) = \text{CH}^b(X \times X)$  and there is a map

$$H^{2b+2,b}(Z) \rightarrow H^{2b,b}(X \times X \times X) = \text{CH}^b(X \times X \times X)$$

such that the composition

$$\text{CH}^b(X \times X) \xrightarrow{\alpha} H^{2b+2,b}(Z) \rightarrow \text{CH}^b(X \times X \times X)$$

equals  $\pi_0^* - \pi_1^* + \pi_2^*$ .

Summarizing we see that the invariant  $\mu$  gives rise to an element  $\rho = \rho(\mu)$  in the homology of the complex

$$\text{CH}^b(X) \xrightarrow{\pi_0^* - \pi_1^*} \text{CH}^b(X \times X) \xrightarrow{\pi_0^* - \pi_1^* + \pi_2^*} \text{CH}^b(X \times X \times X).$$

If the element  $\rho$  (as a cycle in  $\text{CH}^b(X \times X)$ ) satisfies additionally the condition  $c(\rho) \neq 0 \pmod{p}$ , then  $\rho$  turns out to be a special correspondence on  $X$ .

## 5 Norm varieties and special correspondences

We continue to assume that  $\text{char } k = 0$ . The goal of this section is to prove the following theorem:

**5.1 Theorem.** *Let  $X$  be a smooth projective variety which possesses a special correspondence of type  $(n, p)$ . Then there exists an element  $u \in H^n(k, \mu_p^{\otimes(n-1)})$  such that  $X$  is a norm variety for  $u$ .*

*For any field extension  $K/k$  the invariant  $u_K = 0$  iff  $X_K$  has a zero cycle of degree coprime to  $p$ .*

First of all note that we may assume that  $I(X) := \deg(\mathrm{CH}_0(X)) \subset p\mathbb{Z}$ , since otherwise we can define  $u := 0$ . The rest of the proof goes as follows.

**5.2 Proposition** (M. Rost). *Let  $X$  be a smooth projective variety which possesses a special correspondence. Assume  $I(X) \subset p\mathbb{Z}$ . Then the motive of  $X$  considered with  $\mathbb{Z}_{(p)}$ -coefficients has a direct summand  $M$  such that*

$$M \otimes X \simeq \bigoplus_{i=0}^{p-1} X\{bi\}.$$

This Proposition is proved in [Ro07, Proposition 7.14]. The motive  $M$  is called a *generalized Rost motive*.

**5.3 Lemma.** *Let  $X$  be a smooth projective variety and  $R$  a motive in the localizing subcategory of  $\mathrm{DM}$  (with  $\mathbb{Z}_{(p)}$ - or  $\mathbb{Z}/p$ -coefficients) generated by  $X$  such that*

$$R \otimes X \simeq \bigoplus_{i=0}^{p-1} X\{bi\}.$$

*Then there exists the following filtration of  $R$  consisting of exact triangles:*

$$\begin{aligned} R_{p-2}\{b\} &\rightarrow R_{p-1} \rightarrow \mathcal{X} \\ R_{p-3}\{b\} &\rightarrow R_{p-2} \rightarrow \mathcal{X} \\ &\dots \\ R_0\{b\} &\rightarrow R_1 \rightarrow \mathcal{X} \\ 0 &\rightarrow R_0 \rightarrow \mathcal{X}, \end{aligned}$$

where  $R_{p-1} = R$  and  $\mathcal{X}$  denotes the motive of  $\check{C}(X)$ .

*Proof.* We write  $\mathbb{Z}/p$  for the coefficient ring. The proof goes by induction.

First of all, note that  $\mathcal{X}$  lies in the localizing subcategory generated by  $X$ . Let now  $M$  be a motive in the localizing subcategory generated by  $X$  such that  $M \otimes X \simeq \bigoplus_{i \in I} X\{i\}$  for some set  $I$  of non-negative integers containing 0. Then there is the induced map  $\alpha: M \otimes X \rightarrow X$ . We claim that there is a unique map  $\beta: M \rightarrow \mathbb{Z}/p$  such that  $\beta \otimes \mathrm{id} = \alpha$ .

Consider the skeleton filtration of  $\mathcal{X}$ . Let  $Y = \mathrm{Cone}(X \rightarrow \mathcal{X})$  be the cone of the canonical map  $X \rightarrow \mathcal{X}$  induced by the structural map  $X \rightarrow \mathrm{Spec} k$ . Then we have the following exact triangle:

$$M \otimes X \rightarrow M \otimes \mathcal{X} \rightarrow M \otimes Y. \tag{2}$$

By Lemma 3.8(1)  $M \otimes \mathcal{X} = M$ .

By Lemma 3.1 the above triangle induces the long exact sequence

$$\mathrm{Hom}(M \otimes Y, \mathbb{Z}/p) \rightarrow \mathrm{Hom}(M, \mathbb{Z}/p) \rightarrow \mathrm{Hom}(M \otimes X, \mathbb{Z}/p) \rightarrow \mathrm{Hom}((M \otimes Y)[-1], \mathbb{Z}/p).$$

The group  $\mathrm{Hom}(M \otimes Y, \mathbb{Z}/p) = 0$ , since  $\mathrm{Hom}(V[j], \mathbb{Z}/p) = 0$  for any variety  $V$  and  $j > 0$ , and since  $Y$  is an extension of  $X^i[i-1]$ ,  $i > 1$ .

Moreover, we have an injection

$$\mathrm{Hom}((M \otimes Y)[-1], \mathbb{Z}/p) \rightarrow \mathrm{Hom}(M \otimes X \otimes X, \mathbb{Z}/p)$$

induced by the map  $(X \otimes X)[1] \rightarrow Y$  with the property that the composition  $(X \otimes X)[1] \rightarrow Y \rightarrow X[1]$  is the difference of two projections  $\pi_0 - \pi_1$  (cf. Section 4). Therefore the image of  $\alpha$  in  $\mathrm{Hom}((M \otimes Y)[-1], \mathbb{Z}/p)$  is trivial, and we find  $\beta$  with the required property.

In particular, by Lemma 3.8(3) we have constructed a map  $R_{p-1} \rightarrow \mathcal{X}$ . Define  $R_{p-2}$  as  $\mathrm{Cone}[-1](R_{p-1} \rightarrow \mathcal{X})\{-b\}$ . Note that by construction  $R_{p-2} \otimes X \simeq \bigoplus_{i=0}^{p-2} X\{bi\}$ .

Therefore by induction we can construct the motives  $R_{p-3}, \dots, R_0, R_{-1}$  together with the above exact triangles and such that  $R_l \simeq \bigoplus_{i=0}^l X\{bi\}$  for all  $l$ . In particular,  $R_{-1} \otimes X = 0$ .

But then  $R_{-1} = R_{-1} \otimes \mathcal{X} = 0$ . We are done.  $\square$

Next we investigate the non-triviality of our construction of lemma 5.3. From now on we assume that  $X$  is a variety with a special correspondence  $\rho$  and with a motive given by Proposition 5.2 and its proof in [Ro07].

**5.4 Lemma.** *For  $i = 1, \dots, p-1$  if  $c_1 > c_2$  then*

$$\mathrm{Hom}(R_i(c_1)[d_1], \mathbb{Z}/p(c_2)[d_2]) = 0.$$

*Proof.* The statement follows from Lemma 5.3 and Lemma 3.12.  $\square$

**5.5 Lemma.** *Let  $X$  be a variety over  $k$  which possesses a special correspondence  $\rho$  of type  $(n, p)$ . Assume that  $I(X) \subset p\mathbb{Z}$ . Let  $K/k$  be a field extension. Then  $\mathrm{Hom}(\mathcal{X}_K, \mathbb{Z}/p\{b\}[1]) = 0$  iff  $X_K$  has a zero-cycle of degree coprime to  $p$ .*

*Proof.* Consider  $\mathrm{Hom}$  from the exact triangle  $R_{p-2}\{b\} \rightarrow R_{p-1} \rightarrow \mathcal{X}$  to  $\mathbb{Z}/p\{b\}$ . We get the following long exact sequence:

$$\begin{aligned} & \mathrm{Hom}(R_{p-1}, \mathbb{Z}/p\{b\}) \rightarrow \mathrm{Hom}(R_{p-2}\{b\}, \mathbb{Z}/p\{b\}) \\ & \rightarrow \mathrm{Hom}(\mathcal{X}[-1], \mathbb{Z}/p\{b\}) \rightarrow \mathrm{Hom}(R_{p-1}[-1], \mathbb{Z}/p\{b\}). \end{aligned}$$

Note that  $\mathrm{Hom}(R_{p-1}, \mathbb{Z}/p\{b\}) = \mathrm{Ch}^b(R_{p-1})$  (Lemma 3.3), since  $R_{p-1}$  is a pure motive,  $\mathrm{Hom}(R_{p-2}\{b\}, \mathbb{Z}/p\{b\}) = \mathrm{Hom}(R_{p-2}, \mathbb{Z}/p)$  (Lemma 3.6),  $\mathrm{Hom}(\mathcal{X}[-1], \mathbb{Z}/p\{b\}) = \mathrm{Hom}(\mathcal{X}, \mathbb{Z}/p\{b\}[1])$ , and  $\mathrm{Hom}(R_{p-1}[-1], \mathbb{Z}/p\{b\}) = 0$  (Lemma 3.3).

Moreover, it follows from the proof of Lemma 5.3 that the map

$$\mathrm{Hom}(R_{p-2}, \mathbb{Z}/p) \rightarrow \mathrm{Hom}(X, \mathbb{Z}/p) = \mathrm{Ch}^0(X)$$

coming from triangle (2) is an isomorphism. Thus, we get the following commutative diagram

$$\begin{array}{ccccccc} \mathrm{Hom}(R_{p-1}, \mathbb{Z}/p\{b\}) & \longrightarrow & \mathrm{Hom}(R_{p-2}\{b\}, \mathbb{Z}/p\{b\}) & \xrightarrow{g} & \mathrm{Hom}(\mathcal{X}, \mathbb{Z}/p\{b\}[1]) & \longrightarrow & 0 \\ \parallel & & \downarrow \simeq & & \parallel & & \\ \mathrm{Ch}^b(R_{p-1}) & \xrightarrow{f} & \mathrm{Ch}^b(X\{b\}) & \xrightarrow{g} & \mathrm{Hom}(\mathcal{X}, \mathbb{Z}/p\{b\}[1]) & \longrightarrow & 0 \end{array}$$

Moreover, by construction of the filtration for  $R_{p-1}$  given in Lemma 5.3 the map  $f$  is induced by the projection (in the category of classical Chow motives) onto the direct summand  $R_{p-1} \otimes X \rightarrow R_{p-2}\{b\} \otimes X$ . Therefore the Rost construction of the isomorphism of Proposition 5.2 given in [Ro07, Proposition 7.14] immediately implies that  $f$  is given by

$$\alpha \mapsto (\pi_0)_*((\alpha \times 1)\rho^{p-2}).$$

Therefore the lemma follows from [Ro07, Corollary 5.7].  $\square$

**5.6 Lemma.** *Define  $\mu \in \mathrm{Hom}(\mathcal{X}, \mathbb{Z}/p\{b\}[1])$  as the image of  $1 = [X] \in \mathrm{Ch}^0(X)$  under the map  $g$  given in the proof of Lemma 5.5. Then for  $0 \leq i \leq n-2$  we have  $Q_i(\mu) = 0$ .*

*Proof.* Since  $\mu$  is defined as the image of some element in  $\mathrm{Hom}(R_{p-2}, \mathbb{Z}/p)$  and the Milnor operations respect pull-backs, it suffices to show that

$$Q_i(\mathrm{Hom}(R_{p-2}, \mathbb{Z}/p)) = 0.$$

The exact triangle  $R_{p-1} \rightarrow \mathcal{X} \rightarrow R_{p-2}\{b\}[1]$  gives the exact sequence

$$\begin{aligned} \mathrm{Hom}(R_{p-2}(b)[2b+1], \mathbb{Z}/p(p^i-1)[2p^i-1]) &\rightarrow \mathrm{Hom}(\mathcal{X}, \mathbb{Z}/p(p^i-1)[2p^i-1]) \\ &\rightarrow \mathrm{Hom}(R_{p-1}, \mathbb{Z}/p(p^i-1)[2p^i-1]). \end{aligned}$$

The first group in this sequence is zero by Lemma 5.4, since  $b > p^i - 1$  by our assumptions. The last group is zero, since  $R_{p-1}$  is a pure motive and  $2p^i - 1 > 2(p^i - 1)$ . Therefore  $\mathrm{Hom}(\mathcal{X}, \mathbb{Z}/p(p^i-1)[2p^i-1]) = 0$ .

The exact triangle  $R_{p-3}\{b\} \rightarrow R_{p-2} \rightarrow \mathcal{X}$  gives the exact sequence

$$\begin{aligned} \mathrm{Hom}(\mathcal{X}, \mathbb{Z}/p(p^i - 1)[2p^i - 1]) &\rightarrow \mathrm{Hom}(R_{p-2}, \mathbb{Z}/p(p^i - 1)[2p^i - 1]) \\ &\rightarrow \mathrm{Hom}(R_{p-3}\{b\}, \mathbb{Z}/p(p^i - 1)[2p^i - 1]). \end{aligned}$$

The first and the last groups are zero by Lemma 5.4 and by the previous considerations. The lemma is proved.  $\square$

The following lemma is due to V. Voevodsky. We reproduce its proof for reader's convenience.

**5.7 Lemma** ([Vo03, Proof of Lemma 6.7]). *Let  $\delta \in H^{n,n-1}(\mathcal{X}, \mathbb{Z}/p)$ . If*

$$Q_{n-2} \dots Q_1 Q_0(\delta) = 0$$

*then  $\delta = 0$ .*

*Proof.* The exact triangle of pointed simplicial schemes

$$\mathcal{X}_+ \rightarrow \mathrm{Spec} k_+ \rightarrow \tilde{\mathcal{X}}$$

defines a monomorphism  $H^{d,c}(\mathcal{X}, \mathbb{Z}/p) \rightarrow \tilde{H}^{d+1,c}(\tilde{\mathcal{X}}, \mathbb{Z}/p)$  for any  $d > c$ . Thus, we can identify  $\delta$  with its image  $\tilde{\delta}$  in  $\tilde{H}^{n+1,n-1}(\tilde{\mathcal{X}}, \mathbb{Z}/p)$ .

Assume that  $\tilde{\delta} \neq 0$ . We want to show that  $Q_i Q_{i-1} \dots Q_0(\tilde{\delta}) \neq 0$  for all  $i \leq n-2$ . Assume by induction that  $Q_{i-1} \dots Q_0(\tilde{\delta}) \neq 0$ . If  $Q_i Q_{i-1} \dots Q_0(\tilde{\delta}) = 0$ , then by Lemma 3.11 there exists  $v$  such that  $Q_i(v) = Q_{i-1} \dots Q_0(\tilde{\delta})$ .

A straightforward computation shows that  $v \in \tilde{H}^{d,c}(\tilde{\mathcal{X}}, \mathbb{Z}/p)$  where  $c = n - i - p^i + 1 + \frac{p^i - p}{p-1}$  and  $d = n - i + 2 - 2p^i + 2 \cdot \frac{p^i - p}{p-1}$ .

But  $d - c \leq 1$ . Therefore by Lemma 3.14  $v = 0$ . Contradiction.  $\square$

**5.8 Lemma.** *Assume that  $p > 2$ . Then in the above notation there exists  $\delta \in \mathrm{Hom}(\mathcal{X}, \mathbb{Z}/p(n-1)[n])$  such that  $\mu = Q_{n-2} \dots Q_1 Q_0(\delta)$ .*

*Proof.* As in the previous lemma it suffices to prove this identity in the reduced motivic cohomology of  $\tilde{\mathcal{X}}$ . Let  $\eta \in \tilde{H}^{2b+2,b}(\tilde{\mathcal{X}}, \mathbb{Z}/p)$  be the image of  $\mu$ .

Assume we have constructed a sequence of elements  $\eta_{n-2}, \eta_{n-3}, \dots, \eta_{n-s}$  for some  $2 \leq s < n$  such that  $\eta_{n-i+1} = Q_{n-i}(\eta_{n-i})$  and

$$\eta = Q_{n-2} Q_{n-3} \dots Q_{n-s}(\eta_{n-s}).$$

For  $s = 2$  such an element  $\eta_{n-2}$  with  $\eta = Q_{n-2}(\eta_{n-2})$  exists by Lemma 3.11, since  $Q_{n-2}(\eta) = 0$  by Lemma 5.6.

We claim that there exists  $\eta_{n-s-1}$  satisfying the same conditions. To prove this it suffices to show that  $v := Q_{n-s-1}(\eta_{n-s})$  equals 0.

Consider

$$\begin{aligned} Q_{n-2}Q_{n-3}\cdots Q_{n-s}(v) &= Q_{n-2}Q_{n-3}\cdots Q_{n-s}Q_{n-s-1}(\eta_{n-s}) \\ &= \pm Q_{n-s-1}(Q_{n-2}Q_{n-3}\cdots Q_{n-s}(\eta_{n-s})) = \pm Q_{n-s-1}(\eta) = 0, \end{aligned}$$

since  $0 \leq n - s - 1 \leq n - 3$ .

A straightforward computation shows that

$$\eta_{n-s} \in \tilde{H}^{2b+s-2, \frac{p^{n-1}-p^{n-s}}{p-1}+1, b+s-1-\frac{p^{n-1}-p^{n-s}}{p-1}}(\tilde{\mathcal{X}}, \mathbb{Z}/p)$$

and

$$v \in \tilde{H}^{2b+s-2, \frac{p^{n-1}-p^{n-s}}{p-1}+2p^{n-s-1}, b+s-2-\frac{p^{n-1}-p^{n-s}}{p-1}+p^{n-s-1}}(\tilde{\mathcal{X}}, \mathbb{Z}/p).$$

Let  $2 \leq t \leq s$ . We claim that if  $Q_{n-t}(Q_{n-t-1}\cdots Q_{n-s}(v)) = 0$ , then  $Q_{n-t-1}\cdots Q_{n-s}(v) = 0$ . Indeed, if  $Q_{n-t}(Q_{n-t-1}\cdots Q_{n-s}(v)) = 0$ , then there exists  $w$  such that  $Q_{n-t}(w) = Q_{n-t-1}\cdots Q_{n-s}(v)$ . A straightforward computation shows that  $w \in \tilde{H}^{d,c}(\tilde{\mathcal{X}}, \mathbb{Z}/p)$ , where

$$c = \frac{p^{n-t}-1}{p-1} - 1 + p^{n-s-1} + t - p^{n-t}$$

and

$$d = 2 \cdot \frac{p^{n-t}-1}{p-1} + 2p^{n-s-1} + t - 2p^{n-t} + 1.$$

Another straightforward computation shows that  $d \leq c + 1$  if  $p > 2$ , and by Lemma 3.14  $w = 0$ . Therefore  $v = 0$ .  $\square$

*Proof of Theorem 5.1 for  $p > 2$ :* The element  $\delta$  constructed in Lemma 5.8 can be identified with an element  $u \in H_{et}^n(k, \mu_p^{\otimes(n-1)})$  by Lemma 3.15. It remains to show that for any field extension  $K/k$

$$u_K = 0 \text{ iff } X_K \text{ has a zero-cycle of degree coprime to } p.$$

By construction  $u_K = 0$  iff  $\delta_K = 0$ . By Lemma 5.7  $\delta_K = 0$  iff  $\mu_K = 0$ . Finally by Lemma 5.5  $\mu_K = 0$  iff  $X_K$  has a zero-cycle of degree coprime to  $p$ . The theorem is proved.

## 6 Binary motives

In this section we determine the structure of binary direct summands of smooth projective varieties.

**6.1 Theorem.** *Let  $X$  be a smooth projective variety over a field  $k$  with no zero-cycles of odd degree. Let  $M$  be a (binary) direct summand of  $X$  with  $M_{k(X)} \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2\{\dim X\}$ . Then*

1.  $\dim X = 2^{n-1} - 1$  for some  $n$ .
2. *There exists an element  $u \in H^n(k, \mathbb{Z}/2)$  such that for any field extension  $K/k$  we have  $u_K = 0$  iff  $X_K$  has a zero-cycle of odd degree.*

One can notice that Rost's proof ([Ro07, Proof of Theorem 9.1 and Lemma 9.10]) implies the following lemma. For reader's convenience we sketch Rost's proof below.

**6.2 Lemma.** *Let  $X$  be a smooth projective variety of dimension  $d$  with no zero-cycles of odd degree which possesses a cycle  $r \in \text{CH}_d(X \times X)$  such that  $r_{k(X)} \bmod 2$  is a projector and  $(X_{k(X)}, r_{k(X)}) \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2\{d\}$ .*

*Then  $d = 2^{n-1} - 1$  for some  $n$  and  $X$  is a  $\nu_{n-1}$ -variety.*

*Proof.* Let  $S_\bullet$  (resp.  $S^\bullet$ ) denote the homological (resp. cohomological) Steenrod operations in the Chow theory (see [Br03]).

Then  $S_\bullet(\alpha) = S^\bullet(\alpha)b_\bullet(X)$ ,  $\alpha \in \text{Ch}(X)$ , where  $b_\bullet$  is the (total) characteristic number defined in 2.2 for partition  $(p-1, \dots, p-1)$  (Since  $p=2$ , we have  $b_\bullet(X) = c_\bullet(-T_X)$ ).

One has  $b_d(X) = S_d([X])$ . By our assumptions the cycle  $r$  has the property  $(\pi_0)_*(r) = c(r)[X]$  with  $c(r) = 1 \bmod 2$ . Moreover,  $S_\bullet(r) = S^\bullet(r)b_\bullet(X^2)$ .

We show first that  $b_d(X) \neq 0 \bmod p^2$ . To do this it suffices to find an integral representation of the cycle  $S_d(r)$  with degree not divisible by 4.

We have

$$S_d(r) = \sum_{i=0}^d S^i(r)b_{d-i}(X^2). \quad (3)$$

The last term  $S^d(r) = r^2 \bmod 2$  and  $\deg r^2 = 2c(r)^2$  is not divisible by 4. Therefore it suffices to show that all other terms of this sum have integral representatives whose degrees are divisible by 4.

Let  $F$  be the function field of  $X$ . For the first term of (3) we have

$$r_F b_d(X^2) = H \times b_d(X) + b_d(X) \times H,$$

where  $H$  is the class of a rational point on  $X_F$ . This cycle has degree divisible by 4.

Let  $\pi$  be an integral representative of  $S^i(r)$ . We want to show that  $\deg(\pi b_{d-i}(X^2))$  is divisible by 4. By dimension reasons  $S^i(H) = 0$  for  $1 \leq i \leq d-1$ . Therefore  $S^i(r_F) = 0$  for such  $i$ 's. Thus, there exists a cycle  $\gamma$  over  $F$  such that  $\pi_F = 2\gamma$ . It remains to show that  $\deg(\gamma b_{d-i}(X^2))$  is divisible by 2.

We set  $X = X_0 = X_1$  and write  $X^2 = X_0 \times X_1$ . We have

$$b_{d-i}(X^2) = \sum_{j+r=d-i} b_j(X_0) b_r(X_1).$$

By assumptions  $j+r = d-i > 0$ . Therefore  $r > 0$  or  $j > 0$ . In the first case  $\deg(\gamma b_j(X_0) b_r(X_1)) = \deg((\pi_0)_*(\gamma b_j(X_0) b_r(X_1)))$ . Therefore it suffices to show that  $\deg(\alpha_F \beta)$  is divisible by 2 for all  $\alpha \in \text{CH}^r(X)$ ,  $\beta \in \text{CH}^{d-r}(X_F)$ .

Let  $\varphi$  be a preimage of  $\beta$  under the natural map

$$\text{CH}^{d-r}(X^2) \rightarrow \text{CH}^{d-r}(X \times \text{Spec } F).$$

Consider the cycle  $\omega = r(\alpha \times 1)\varphi \in \text{CH}_0(X^2)$ .

One has  $\omega_F = (H \times 1)(\alpha_F \times 1)\varphi_F + (1 \times H)(\alpha_F \times 1)\varphi_F = (1 \times H)(\alpha_F \times 1)\varphi_F$ . Therefore  $\deg \omega = \deg(\alpha_F \beta) = (\pi_0)_*((\alpha_F \times 1)\varphi_F) = c(r) \deg(\alpha_F \beta)$  is divisible by 2, since  $X^2$  has no zero-cycles of odd degree.

The case  $j > 0$  is similar.

Thus  $b_d(X) \neq 0 \pmod{p^2}$ . By [Ro07, Lemma 9.10]  $\dim X = 2^{n-1} - 1$  for some  $n$ . Finally, by [Ro07, Lemma 9.13]  $X$  is a  $\nu_{n-1}$ -variety.  $\square$

By Lemma 5.3 we have an exact triangle

$$\mathcal{X}\{b\} \rightarrow M \rightarrow \mathcal{X} \xrightarrow{\mu'} \mathcal{X}\{b\}[1].$$

Denote  $\mu \in \text{Hom}(\mathcal{X}, \mathbb{Z}/2(b)[2b+1])$  the pull-back of  $\mu'$  under the map of Lemma 3.8(3). Observe that the construction of  $\mu$  here is compatible with the construction of  $\mu$  of Section 5 for  $p = 2$ .

The following lemmas were proven by O. Izhboldin and A. Vishik in the case when  $X$  is a quadric. Nevertheless their proofs are general and do not use any specific of quadrics. For reader's convenience we reproduce them below.

**6.3 Lemma** ([IzhV00, Sublemma 6.3]). *The map*

$$\mathrm{Hom}(\mathcal{X}, \mathbb{Z}/p(c)[d]) \rightarrow \mathrm{Hom}(\mathcal{X}, \mathbb{Z}/p(c+b)[d+2b+1])$$

*induced by the multiplication by  $\mu$  is surjective for  $d \geq c$ .*

*Proof.* Consider the morphism from the exact triangle

$$M \rightarrow \mathcal{X} \xrightarrow{\mu'} \mathcal{X}(b)[2b+1]$$

to  $\mathbb{Z}/2(b+c)[2b+d+1]$ .

We get the long exact sequence

$$\begin{aligned} \mathrm{Hom}(\mathcal{X}(b)[2b+1], \mathbb{Z}/2(b+c)[2b+d+1]) &\xrightarrow{\mu'^*} \mathrm{Hom}(\mathcal{X}, \mathbb{Z}/2(b+c)[2b+d+1]) \\ &\rightarrow \mathrm{Hom}(M, \mathbb{Z}/2(b+c)[2b+d+1]). \end{aligned}$$

Note that  $\mathrm{Hom}(\mathcal{X}(b)[2b+1], \mathbb{Z}/2(b+c)[2b+d+1]) = \mathrm{Hom}(\mathcal{X}, \mathbb{Z}/2(c)[d])$  and by Lemma 3.13 the map  $\mu'^*$  coincides with the multiplication by  $\mu$ . Moreover,  $\mathrm{Hom}(M, \mathbb{Z}/2(b+c)[2b+d+1]) = 0$  by Lemma 3.5, since  $M$  is a pure motive and  $2b+d+1 - (b+c) = b+d-c+1 > b = \dim X$ .

Therefore the multiplication by  $\mu$  is surjective for  $d \geq c$ .  $\square$

Let  $\tilde{\mathcal{X}} = \mathrm{Cone}(\mathcal{X} \rightarrow \mathrm{Spec} k)$ .

**6.4 Lemma** ([IzhV00, Sublemma 6.6]). *For any  $i = 0, 1, \dots, n-2$  the Milnor operation*

$$Q_i: \tilde{H}^{d,c}(\tilde{\mathcal{X}}, \mathbb{Z}/2) \rightarrow \tilde{H}^{d+2^{i+1}-1, c+2^i-1}(\tilde{\mathcal{X}}, \mathbb{Z}/2)$$

*is injective if  $d - c = b + 2 + 2^i$ .*

*Proof.* Let  $v \in \tilde{H}^{d,c}(\tilde{\mathcal{X}}, \mathbb{Z}/2)$  with  $Q_i(v) = 0$ . We want to show that then  $v = 0$ .

Since  $Q_i(v) = 0$ , by Lemma 3.11 there exists

$$t \in \tilde{H}^{d-2^{i+1}+1, c-2^i+1}(\tilde{\mathcal{X}}, \mathbb{Z}/2)$$

such that  $v = Q_i(t)$ .

On the other hand, the exact triangle of pointed simplicial schemes

$$\tilde{\mathcal{X}}[-1] \xrightarrow{\tau} \mathcal{X}_+ \rightarrow \mathrm{Spec} k_+$$

induces an isomorphism

$$\tau^*: H^{d-2^{i+1}+1, c-2^i+1}(\mathcal{X}, \mathbb{Z}/2) \rightarrow \tilde{H}^{d-2^{i+1}+2, c-2^i+1}(\tilde{\mathcal{X}}, \mathbb{Z}/2).$$

Therefore there exists

$$w \in \text{Hom}(\mathcal{X}, \mathbb{Z}/2(c-2^i+1)[d-2^{i+1}+1])$$

with  $\tau^*(w) = t$ . Moreover, by Lemma 6.3 there exists

$$u \in \text{Hom}(\mathcal{X}, \mathbb{Z}/2(c-2^i+1-b)[d-2^{i+1}-2b])$$

such that  $w = \mu \cdot u$ .

By Lemma 3.9(3) we have  $Q_i(\mu \cdot u) = Q_i(\mu) \cdot u + \mu \cdot Q_i(\mu) + \sum \xi^{n_j} \varphi_j(\mu) \psi_j(u)$ , where  $\xi$  is the class of  $-1$  in  $\text{Hom}(\mathbb{Z}/2, \mathbb{Z}/2(1)[1])$ ,  $n_j > 0$  and  $\varphi_j, \psi_j$  are some homogeneous cohomological operations of some bidegrees  $(*)[*']$  with  $*' > 2* \geq 0$ .

Since  $e := d - 2^{i+1} - 2b = c - 2^i + 1 - b$  by assumptions, Lemma 3.14 implies an isomorphism  $\text{pr}^*: \text{Hom}(\mathbb{Z}/2, \mathbb{Z}/2(e)[e]) \rightarrow \text{Hom}(\mathcal{X}, \mathbb{Z}/2(e)[e])$ , where  $\text{pr}: \mathcal{X} \rightarrow \text{Spec } k$  is the structural map. Therefore there exists

$$u_0 \in \text{Hom}(\mathbb{Z}/2, \mathbb{Z}/2(e)[e])$$

with  $u = \text{pr}^*(u_0)$ .

On the other hand, since  $\text{Hom}(\mathbb{Z}/2, \mathbb{Z}/2(f)[g]) = 0$  for any  $g > f$ , we have  $Q_i(u_0) = 0$  and  $\psi_j(u_0) = 0$ . Since the Milnor operations commute with pull-backs, we also have  $Q_i(u) = 0$  and  $\psi_j(u) = 0$ .

Summarizing we obtain:

$$v = Q_i(t) = Q_i(\tau^*(w)) = \tau^*(Q_i(w)) = \tau^*(Q_i(\mu \cdot u)) = \tau^*(Q_i(\mu) \cdot u) = 0$$

the last equality by Lemma 5.6.  $\square$

**6.5 Lemma** ([IzhV00, Sublemma 6.7]). *Lemma 5.8 holds for  $p = 2$ , i.e., there exists  $\delta \in \text{Hom}(\mathcal{X}, \mathbb{Z}/2(n-1)[n])$  such that  $\mu = Q_{n-2} \dots Q_1 Q_0(\delta)$ .*

*Proof.* As in the proofs of Lemmas 5.7 and 5.8 we identify  $\mu$  with its image in the reduced motivic cohomology  $\tilde{H}^{2b+2, b}(\tilde{\mathcal{X}}, \mathbb{Z}/p)$  which we denote as  $\eta = \eta_{-1}$ .

Assume by induction that we have constructed an element  $\eta_m$ ,  $-1 \leq m \leq n-3$ , such that  $\eta = Q_m \dots Q_1 Q_0(\eta_m)$ . We want to show that there exists  $\eta_{m+1}$  with  $\eta_m = Q_{m+1}(\eta_{m+1})$ .

By Lemma 3.11 it suffices to show that  $v := Q_{m+1}(\eta_m) = 0$ . We have  $Q_m Q_{m-1} \dots Q_0(v) = Q_m Q_{m-1} \dots Q_0 Q_{m+1}(\eta_m) = Q_{m+1} Q_m Q_{m-1} \dots Q_0(\eta_m) = Q_{m+1}(\eta) = 0$  by Lemma 5.6.

A straightforward computation shows that for any  $0 \leq t \leq m$  the element  $Q_{t-1} \dots Q_0(v) \in \tilde{H}^{d,c}(\tilde{\mathcal{X}}, \mathbb{Z}/2)$  with  $d - c = b + 2 + 2^t$ . By Lemma 6.4  $Q_t$  is injective on  $\tilde{H}^{d,c}(\tilde{\mathcal{X}}, \mathbb{Z}/2)$  for such  $c$  and  $d$ . Therefore the equality  $Q_m Q_{m-1} \dots Q_0(v) = 0$  implies  $v = 0$ .  $\square$

*Proof of Theorem 6.1:* The first part of the Theorem follows from Lemma 6.2.

The proof of the second assertions repeats word by word the proof of Theorem 5.1 given at the end of the previous section with Lemma 5.8 replaced by Lemma 6.5.

*Proof of Theorem 5.1 for  $p = 2$ :* This is a particular case of Theorem 6.1.

## 7 Compression of varieties

The goal of this section is to provide a certain compression algorithm for smooth projective varieties. We assume that  $\text{char } k = 0$ .

**7.1 Lemma.** *Let  $X$  be a smooth projective variety and  $Y$  a closed subvariety of  $X$  of minimal dimension such that  $Y_{k(X)}$  has a zero-cycle of degree coprime to  $p$ .*

*Then there exists a smooth projective variety  $\tilde{Y}$  which is birational to  $Y$  such that  $\tilde{Y}_{k(X)}$  has a zero-cycle of degree coprime to  $p$ .*

*Proof.* Let  $U \subset Y$  be the open subvariety of smooth points in  $Y$  and  $V = Y \setminus U$  the singular locus. We claim that there is a zero-cycle of degree coprime to  $p$  supported in  $U_{k(X)}$ . Indeed, assume the contrary. Then all zero-cycles supported on  $U_{k(X)}$  have degrees divisible by  $p$ . Since there exists a zero-cycle on  $Y_{k(X)}$  of degree coprime to  $p$ , the variety  $V_{k(X)} \subset X_{k(X)}$  has a zero-cycle of degree coprime to  $p$ . Since  $\dim V < \dim Y$  and  $\dim Y$  is minimal by our assumptions, we come to a contradiction.

Using the standard algorithm of resolution of singularities due to Hironaka we can find a variety  $\tilde{Y}$  birational to  $Y$  which contains  $U$ . This  $\tilde{Y}$  has a zero-cycle of degree 1 mod  $p$  over  $k(X)$ .  $\square$

**7.2 Remark.** The proof of the previous lemma leads to the following problem. Let  $Y$  be a projective variety over  $k$ . Does there exist a resolution of

singularities  $\tilde{Y}$  of  $Y$  such that for all field extensions  $K/k$  if  $Y$  has a  $K$ -point then  $\tilde{Y}$  has a  $K$ -point? It is known that if  $\dim Y = 1$ , then the answer is in general negative. The case  $\dim Y > 1$  is to my knowledge open. Nevertheless, if for all smooth projective varieties  $X$  any morphism  $X \rightarrow Y$  has a pull-back in the Chow theory, then the answer is positive.

**7.3 Definition.** A motive  $M$  over  $k$  is called *generically split*, if there exists a smooth projective  $k$ -variety  $X$  and a number  $l$  such that  $M$  is a direct summand of the motive  $X\{l\}$  and  $M_{k(X)}$  is a direct sum of twisted Tate motives over  $k(X)$ .

**7.4 Corollary.** *Let  $X$  be a generically split smooth projective  $k$ -variety. Let  $Y$  be a closed subvariety of  $X$  of minimal dimension such that the class  $[Y_{k(X)}]$  in  $\text{Ch}(X_{k(X)})$  is non-zero. Then there exists a smooth projective variety  $\tilde{Y}$  birational to  $Y$  such that  $\tilde{Y}_{k(X)}$  has a zero-cycle of degree 1 mod  $p$ .*

*Proof.* Since  $X$  is generically split, by [KM06, Remark 5.6] there exists a closed subvariety  $Z \subset X_{k(X)}$  such that  $[Y_{k(X)}] \cdot [Z]$  has degree coprime to  $p$ . Since the product  $Y_{k(X)} \cdot Z$  in the Chow ring can be represented by a cycle on the intersection  $Y_{k(X)} \cap Z$ , the variety  $Y_{k(X)} \cap Z \subset Y_{k(X)}$  has a zero-cycle of degree coprime to  $p$ .

By Lemma 7.1 it suffices to show that  $Y$  has minimal dimension among all closed subvarieties of  $X$  such that  $Y_{k(X)}$  has a zero-cycle of degree coprime to  $p$ .

The proof of this claim copies the proof of [KM06, Theorem 5.8].

Let  $Y' \xrightarrow{in} X$  be a closed subvariety of  $X$  such that  $Y'_{k(X)}$  has a zero-cycle of degree coprime to  $p$ . It remains to show that  $\dim Y' \geq \dim Y$ .

Let  $\overline{\text{Ch}}$  denote the image of the restriction map  $\text{Ch}(-) \rightarrow \text{Ch}(-_{k(-)})$ . It suffices to show that  $\overline{\text{Ch}}_i(X)$  is non-zero for some  $i \leq \dim Y'$ . By [KM06, Corollary 5.4] it suffices to show that the push-forward

$$(in \times \text{id}_X)_* : \overline{\text{Ch}}(Y' \times X) \rightarrow \overline{\text{Ch}}(X \times X)$$

is non-zero.

Denote  $F = k(X)$ . By assumptions there exists a  $p$ -coprime field extension  $K/F$  such that  $k(Y') \hookrightarrow K$ . Let  $Z$  be the closure of the image of the morphism  $K \rightarrow Y' \times X$ . Then the cycle  $(in \times \text{id}_X)_*([Z])$  is non-zero, since for the second projection  $\pi_1 : X^2 \rightarrow X$  we have

$$(\pi_1)_*(in \times \text{id}_X)_*([Z]) = [K : k(X)] \cdot 1 \neq 0.$$

□

**7.5 Definition.** Let  $X$  be a smooth projective variety and  $R = (X, \pi)$  a direct summand of the motive of  $X$ . We say that  $R$  supports 0-cycles of  $X$  if  $\deg(\pi_*(\text{CH}_0(X))) \neq 0$  ( $\pi_*$  denotes the realization of  $\pi$ ).

**7.6 Definition.** Let  $M$  be a pure motive over  $k$ . We say that *Rost nilpotence* holds for  $M$  if for any field extension  $K/k$  the kernel of the natural map

$$\text{End}_k(M) \rightarrow \text{End}_K(M_K)$$

consists of nilpotent correspondences.

According to [ViZ08, Proposition 3.1] Rost nilpotence holds for all generically split motives.

Finally, we need the following lemma.

**7.7 Lemma.** *Let  $X$  and  $Y$  be smooth projective varieties. Consider the category of Chow motives with  $\mathbb{Z}/p$ -coefficients. Assume that  $X$  has a zero-cycle of degree 1 mod  $p$  over  $k(Y)$  and  $Y$  has a zero-cycle of degree 1 mod  $p$  over  $k(X)$ . Let  $R$  be an indecomposable generically split direct summand of the motive of  $X$  which supports zero-cycles. Then  $R$  is an indecomposable direct summand of  $Y$  which supports zero-cycles.*

*Proof.* Since  $X$  has a zero-cycle of degree 1 mod  $p$  over  $k(Y)$ , there exists by [PSZ, Lemma 1.6] a cycle  $\varphi_1 \in \text{Ch}_{\dim X}(X \times Y)$  such that  $\varphi_1: pt_X \mapsto pt_Y$  over  $k(Y)$ .

By symmetry there exists a cycle  $\psi_1 \in \text{Ch}_{\dim Y}(Y \times X)$  such that  $\psi_1: pt_Y \mapsto pt_X$  over  $k(X)$ .

Let  $r$  be a projector defining  $R$ , i.e.,  $R = (X, r)$ . Consider the cycles  $\varphi_2 = \varphi_1 \circ r$  and  $\psi_2 = r \circ \psi_1$ . The composition  $\psi_2 \circ \varphi_2 \in \text{End}(R)$  and is non-zero, since  $\psi_2 \circ \varphi_2: pt_X \mapsto pt_X$  over  $k(X)$ .

Consider  $\text{End}(R_{k(X)})$ . Since  $R$  is generically split and indecomposable, it follows that the image of the natural map  $\text{End}(R) \rightarrow \text{End}(R_{k(X)})$  equals  $\mathbb{Z}/p \cdot r_{k(X)}$ . Therefore the cycle  $(\psi_2 \circ \varphi_2)_{k(X)}$  equals  $c \cdot r_{k(X)}$  with  $c \in (\mathbb{Z}/p)^\times$ . Multiplying  $\psi_2$  or  $\varphi_2$  by  $c^{-1}$  we may assume that  $c = 1$ .

By [ViZ08, Proposition 3.1] Rost nilpotence holds for  $R$ . Therefore  $\psi_2 \circ \varphi_2 = r + n$ , where  $n$  is some nilpotent element in  $\text{End}(R)$ . Since  $n$  is nilpotent,  $r + n$  is invertible and  $(r + n)^{-1} \circ \psi_2 \circ \varphi_2 = r$ . Define now  $\psi_3 = (r + n)^{-1} \circ \psi_2$ . Then  $\psi_3 \circ \varphi_2 = r$ , and therefore  $\varphi_2 \circ \psi_3$  is a projector on  $Y$  and  $(Y, \varphi_2 \circ \psi_3) \simeq (X, r) = R$  with mutually inverse isomorphisms  $\varphi_2$  and  $\psi_3$ . □

## 8 Serre's question about $E_8$

Let  $G$  be a simple adjoint linear algebraic group over  $k$  with  $\text{char } k = 0$ . We briefly sketch the definitions of two invariants of  $G$ : the  $J$ -invariant and the Rost invariant. See [PSZ] and [Inv, §31] for formal definitions.

**8.1** ( $J$ -invariant). Let  $p$  be a prime number.

**8.2 Definition.** Let  $R$  be a motive which is a finite direct sum of Tate motives. The polynomial

$$P(R, t) = \sum_{i \geq 0} a_i t^i \in \mathbb{Z}[t],$$

where  $a_i = \dim \text{Ch}_i(R)$  is called the Poincaré polynomial of  $R \bmod p$ .

Informally speaking, the  $J$ -invariant of a simple group  $G$  is a discrete invariant which measures the motivic decomposition of the variety of Borel subgroups of  $G$ .

Consider the motive of the variety of complete  $G$ -flags (variety of Borel subgroups) with  $\mathbb{Z}/p$ - or  $\mathbb{Z}_{(p)}$ -coefficients. It turns out that it is isomorphic to a direct sum of the same indecomposable generically split motive  $R = R_p(G)$  such that over any splitting field  $K$  of  $G$  the Poincaré polynomial of  $R_K$  equals

$$P(R_K, t) = \prod_{i=1}^r \frac{t^{d_i p^{j_i}} - 1}{t^{d_i} - 1} \tag{4}$$

for some given numbers  $r$  and  $d_1 \leq d_2 \leq \dots \leq d_r$  which depend only on the Dynkin type of  $G$  and some  $r$ -tuple  $(j_1, \dots, j_r)$  of non-negative integers (see [PSZ, Definition 4.5 and Theorem 4.9]).

The tuple  $J_p(G) = (j_1, \dots, j_r)$  arising from this decomposition is called the  $J$ -invariant of  $G$  modulo  $p$ . We leave the problem of correctness of such a definition aside and refer to [PSZ] for more details.

An important application of the  $J$ -invariant which is essentially used in the proof of Serre's problem described below is the classification of generically split projective homogeneous varieties given in [PS, Theorems 5.3 and 5.4].

Note also that formula (4) implies that the variety of Borel subgroups of  $G$  has a binary direct summand if  $J_2(G) = (0, \dots, 0, 1, 0, \dots, 0)$ . In Lemma 8.6 we'll provide a concrete example.

**8.3 Example.** If  $G$  has type  $E_8$ , then  $r = 4$ ,  $d_1 = 3$ ,  $d_2 = 5$ ,  $d_3 = 9$ , and  $d_4 = 15$ . Moreover,  $j_4 = 0$  or  $1$ , and if  $j_1 = 0$  then  $j_2 = j_3 = 0$  (see [PSZ, §6, Table]).

**8.4 (Rost invariant).** Let  $G$  be a simply connected simple algebraic group over  $k$ . The Rost invariant of  $G$  is an invariant of  $G$ -torsors which takes values in  $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ .

Formally: consider the abelian group  $\text{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))$  of natural transformations of functors  $H^1(-, G) \rightarrow H^3(-, \mathbb{Q}/\mathbb{Z}(2))$  defined on the category of field extensions of  $k$ .

It turns out that it is a finite cyclic group with a canonical generator called the *Rost invariant* of  $G$ . If  $G$  is adjoint and simply-connected, then one can identify  $H^1(k, G)$  with the (pointed) set of isomorphism classes of the twisted forms of  $G$  over  $k$ . In particular, in this case one can associate with each twisted form of  $G$  an element in  $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$  (the Rost invariant).

**8.5 Example.** Let  $G$  be a group of type  $E_8$ . It is well-known that it is adjoint and simply-connected, and the Rost invariant of  $G$  takes values in  $H^3(k, \mathbb{Z}/4) \oplus H^3(k, \mathbb{Z}/3) \oplus H^3(k, \mu_5^{\otimes 2})$ , i.e., consists of a mod-4, mod-3, and mod-5 component. If the group  $G$  splits by a field extension of degree coprime to  $l$ , then the  $l$ -component of its Rost invariant is zero ( $l \in \mathbb{Z}$ ).

**8.6 Lemma.** *Let  $G$  be a group of type  $E_8$  and  $X$  the variety of Borel subgroups of  $G$ . If the even component of the Rost invariant of  $G$  is trivial, then the motive of  $X$  with  $\mathbb{Z}/2$ - (or  $\mathbb{Z}_{(2)}$ -)coefficients is a direct sum of twisted copies of a motive  $R$  whose Poincaré polynomial over any splitting field of  $G$  equals  $1 + t^{15} \in \mathbb{Z}[t]$ .*

*Proof.* Since we are talking about the even component of the Rost invariant and about the motives with  $\mathbb{Z}/2$ -coefficients we may assume that the base field  $k$  does not have any proper finite field extension of odd degree.

Let  $X_i$  be a projective  $G$ -homogeneous variety of maximal parabolic subgroups of  $G$  of type  $i$ . If the Rost invariant of  $G$  is trivial, then so is the Rost invariant of  $G_{k(X_i)}$ . On the other hand, since the semisimple anisotropic kernel of  $G_{k(X_i)}$  has rank at most 7, the group  $G_{k(X_i)}$  must be split (see e.g. [Ga01, Theorem 0.5]). Thus,  $X_i$  is a generically split variety for all  $i = 1, \dots, 8$ .

The classification of generically split varieties of type  $E_8$  (see [PS, Theorem 5.5(7)]) and the classification of the  $J$ -invariants for groups of type  $E_8$

(see [PSZ, §6, Table]) immediately imply that the  $J$ -invariant of  $G$  modulo 2 equals either  $J_2(G) = (0, 0, 0, 1)$  or  $(0, 0, 0, 0)$ .

In the first case we are done by formula (4). In the second case the variety of Borel subgroups is a direct sum of twisted Tate motives and the present lemma is trivial.  $\square$

**8.7 Corollary.** *Let  $k$  be a field with  $\text{char } k = 0$ . Let  $G$  be a group of type  $E_8$  over  $k$  with the trivial even component of the Rost invariant. Then there exists an invariant  $u \in H^5(k, \mathbb{Z}/2)$  of  $G$  such that for any field extension  $K/k$  the invariant  $u_K = 0$  iff  $G$  splits over a field extension of  $K$  of odd degree.*

*Proof.* Let  $X$  be the variety of Borel subgroups of  $G$ . It is generically split, since over  $k(X)$  the group  $G$  splits. By Lemma 8.6 the motive of  $X$  contains a binary direct summand  $R$  which supports zero-cycles.

Consider the image  $\overline{\text{Ch}}(X)$  of the natural map  $\text{Ch}(X) \rightarrow \text{Ch}(X_{k(X)})$ . By [PSZ, Proposition 6.4] and [KM06, Theorem 5.8]

$$\min\{i \mid \overline{\text{Ch}}_i(X) \neq 0\} = 15.$$

By Corollary 7.4 there exists a smooth projective variety  $\tilde{Y}$  of dimension 15 birational to some closed subvariety  $Y$  of  $X$  such that  $\tilde{Y}_{k(X)}$  has a zero-cycle of odd degree. The converse obviously holds, i.e.,  $X_{k(\tilde{Y})}$  has a zero-cycle of odd degree.

Therefore by Lemma 7.7 the variety  $\tilde{Y}$  has the same direct summand  $R$ . We are done by Theorem 6.1.  $\square$

We come now to Serre's problem. Let  $G$  be a group of type  $E_8$  over  $\mathbb{Q}$  such that  $G_{\mathbb{R}}$  is a compact group (In fact, such  $G$  is unique). Let now  $K/\mathbb{Q}$  be a field extension and  $q = \langle\langle -1, -1, -1, -1, -1 \rangle\rangle$  a 5-fold Pfister form. One can show that if  $q_K$  is hyperbolic, then  $G_K$  is split (see below). J.-P. Serre posed in a letter to M. Rost in 1999 the following problem:

Assume  $G_K$  is split. Is it true that  $q_K$  is hyperbolic?

In what follows we give a positive answer to Serre's question.

**8.8 Theorem.** *Serre's question has a positive answer.*

*Proof.* This group  $G$  arises from the Pfister form  $q = \langle\langle -1, -1, -1, -1, -1 \rangle\rangle$  using the construction given in [Ga08, Example 5.9]. In particular, it has trivial Rost invariant. Therefore we can consider the invariant  $u \in H^5(\mathbb{Q}, \mathbb{Z}/2)$  of  $G$  constructed above in Corollary 8.7.

We show first that  $u$  is non-trivial. Indeed, if  $u = 0$  then  $u_{\mathbb{R}} = 0$ . Then by Corollary 8.7  $G_{\mathbb{R}}$  splits over a field extension of odd degree. Since  $\mathbb{R}$  has only one field extension of odd degree,  $G_{\mathbb{R}}$  is split, a contradiction.

Let  $\tilde{Y}$  be a 15-dimensional norm variety for  $u$ . Its existence follows from the proof of Corollary 8.7 (where it was denoted also as  $\tilde{Y}$ ). It is indeed a norm variety by Lemma 6.2.

Let  $Q$  be the 15-dimensional projective quadric given by the quadratic form  $\langle\langle -1, -1, -1, -1 \rangle\rangle \perp \langle 1 \rangle$ . It is well-known (see e.g. [Vo96, Lemma 4.3]) that for any field extension  $K/\mathbb{Q}$  the quadric  $Q_K$  has a zero-cycle of odd degree iff  $q_K$  is isotropic.

It follows from [Ga08, §15] that  $G$  splits over  $k(Q)$ . Therefore  $u$  also splits over  $k(Q)$ , and, thus,  $\tilde{Y}_{k(Q)}$  has a zero-cycle of odd degree.

On the other hand, since  $\tilde{Y}$  and  $Q$  are  $\nu_4$ -varieties with no zero-cycles of odd degree, the converse holds by the *Rost degree formula* ([Me03, Theorem 7.2(2a)]).

Therefore  $Q$  is an invariant of  $G$  which behaves with respect to scalar extensions in the same way as  $\tilde{Y}$  (more precisely, as the invariant  $u$ ) does. We are done.  $\square$

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